Goal Driven Optimization

Wenqing Chen[*] Melvyn Sim[†]

May 2006

Abstract

Achieving a targeted objective, goal or aspiration level are relevant aspects of decision making under uncertainties. We develop a goal driven stochastic optimization model that takes into account an aspiration level. Our model maximizes the *shortfall aspiration level criterion*, which encompasses the probability of success in achieving the goal and an expected level of under-performance or shortfall. The key advantage of the proposed model is its tractability. We show that proposed model is reduced to solving a small collections of stochastic linear optimization problems with objectives evaluated under the popular conditional-value-at-risk (CVaR) measure. Using techniques in robust optimization, we propose a decision rule based deterministic approximation of the goal driven optimization problem by solving a polynomial number of second order cone optimization problems (SOCP) with respect to the desired accuracy. We compare the numerical performance of the deterministic approximation with sampling approximation and report the computational insights.

[*]NUS Business School, National University of Singapore. Email: chenwenqing@gmail.com

[†]NUS Business School, National University of Singapore and Singapore MIT Alliance (SMA). Email: dscsimm@nus.edu.sg. The research of the author is supported by SMA, NUS academic research grants R-314-000-066-122 and R-314-000-068-122.

1 Introduction

Stochastic optimization is an adopted framework for modeling optimization problems that involve uncertainty. In a standard stochastic optimization problem, one seeks to minimize the aggregated expected cost over a multiperiod planning horizon, which corresponds to decision makers who are risk neutral; see for instance, Birge and Louveaux [7]. However, optimization of an expectation assumes that the decision can be repeated a great number of times under identical conditions. Such assumptions may not be widely applicable in practice. The framework of stochastic optimization can also be adopted to address downside risk by optimizing over an expected utility or more recently, a mean risk objective; see chapter 2 of Birge and Louveaux [7], Ahmed [1] and Ogryczak and Ruszczynski [22]. In such a model, the onus is on the decision maker to articulate his/her utility function or to determine the right parameter for the mean-risk functional. This can be rather subjective and difficult to obtain in practice.

Recent research in decision theory suggests a way of comprehensively and rigorously discussing decision theory without using utility functions; see Castagnoli and LiCalzi [9] and Bordley and LiCalzi [6]. With the introduction of an *aspiration level* or the targeted objective, the decision risk analysis focuses on making decisions so as to maximize the probability of reaching the aspiration level. As a matter of fact, aspirations level plays an important role in daily decision making. Lanzillotti's study [18], which interviewed the officials of 20 large companies, verified that the managers are more concerned about a target return on investment. In another study, Payne et al. [23, 24] illustrated that managers tend to disregard investment possibilities that are likely to under perform against their target. Simon [31] also argued that most firms' goals are not maximizing profit but attaining a target profit. In an empirical study by Mao [20], managers were asked to define what they considered as risk. From their responses, Mao concluded that "risk is primarily considered to be the prospect of not meeting some target rate of return".

Based on the motivations from decision analysis, we study a two stage stochastic optimization model that takes into account an aspiration level. This work is closely related to Charnes et al.'s P-model [10, 11] and Bereanu's [5] optimality criterion of maximizing the probability of getting a profit above a targeted level. However, maximizing the probability of achieving a target is generally not a computationally tractable model. As such, studies along this objective have been confined to simple problems such as the Newsvendor problem; see Sankarasubramanian and Kumaraswamy [29], Lau and Lau [17], Li et al. [19] and Parlar and Weng [25].

Besides its computational intractability, maximizing the success probability assumes that the modeler is indifferent to the level of losses. It does not address how catastrophic these losses can be expected when the "bad", small probability events occur. However, studies have suggested that subjects are not completely insensitive to these losses; see for instance Payne et al [23]. Diecidue and van de Ven [14] argue that a model that solely maximizes the success probability is "too crude to be normatively or descriptively relevant." They suggested an objective that takes into account of a weighted combination of the success probability as well as an expected utility. However, such a model remains computationally intractable when applied to the stochastic optimization framework.

Our goal driven optimization model maximizes the *shortfall aspiration level criterion*, which takes into account of the probability of success in achieving the goal and an expected level of under-performance or shortfall. A key advantage of the proposed model over maximizing the success probability is its tractability. We show that proposed model is reduced to solving a small collections of stochastic optimization problems with objectives evaluated under the popular Conditional-Value-at-Risk (CVaR) measure proposed Rockafellar and Uryasev [27]. This class of stochastic optimization problems with mean risk objectives have recently been studied by Ahmed [1] and Riis and Schultz [26]. They proposed decomposition methods that facilitate sampling approximations.

The quality of sampling approximation of a stochastic optimization problem depends on several issues; the confidence of the approximation around the desired accuracy, the size of the problem, the type of recourse and the variability of the objective; see Shaprio and Nemirovski [30]. Even in a two stage model, the number of sampled scenarios required to approximate the solution to reasonable accuracy can be astronomical large, for instance, in the presence of rare but catastrophic scenarios or in the absence of relatively complete recourse. Moreover, sampling approximation of stochastic optimization problems requires complete probability descriptions of the underlying uncertainties, which are almost never available in real world environments. Hence, it is conceivable that models that are heavily tuned to an assumed distribution may perform poorly in practice.

Motivated from recent development in robust optimization involving multiperiod decision process (see Ben-Tal et al. [3], Chen, Sim and Sun [12] and Chen et al. [13]), we propose a new decision rule based deterministic approximation of the stochastic optimization problems with CVaR objectives. In line with robust optimization, we require only modest assumptions on distributions, such as known means and bounded supports, standard deviations and the *forward and backward deviations* introduced by Chen, Sim and Sun [12]. We adopt a comprehensive model of uncertainty that incorporates both models of Chen, Sim and Sun [12] and Chen et al. [13]. We also introduce new bounds on the CVaR measures and expected positivity of a weighted sum of random variables, both of which are integral in achieving a tractable approximation in the form of second order cone optimization problem (SOCP); see Ben-Tal and Nemirovski [4]. This allows us to leverage on the state-of-the-art SOCP solvers, which are increasingly becoming more powerful, efficient and robust. Finally, we compare the performance of the deterministic with sampling approximation on a class of project management problem that maximizes the shortfall aspiration level criterion.

The structure of the paper is as follows. In Section 2, we introduce the goal driven model and propose the shortfall aspiration level criteria. We show that the goal driven optimization problem can be reduced to solving a sequence of stochastic optimization problems with CVaR objectives. Using techniques in robust optimization, we develop in Section 3, a deterministic approximation of the stochastic optimization problem with CVaR objective. In Section 4, we report some computational results and insights. Finally, Section 5 concludes this paper.

Notations We denote a random variable, $\tilde{x}$, with the tilde sign. Bold face lower case letters such as $\boldsymbol{x}$ represent vectors and the corresponding upper case letters such as $\boldsymbol{A}$ denote matrices. In addition, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. The same operations can be used on vectors, such as $\boldsymbol{y}^+$ and $\boldsymbol{z}^-$ in which corresponding operations are performed componentwise.

2 A Goal Driven Optimization Model

We consider a two stage decision process in which the decision maker first selects a feasible solution $\boldsymbol{x} \in \Re^{n_1}$, or so-called *here and now* solution in the face of uncertain outcomes that may influence the optimization model. Upon realization of $\tilde{\boldsymbol{z}}$, which denotes the vector of N random variables whose realizations correspond to the various scenarios, we select an optimal *wait-and-see* solution or recourse action. We also call $\tilde{\boldsymbol{z}}$ as the vector of primitive uncertainties, which consolidate all underlying uncertainties in the stochastic model. Given the solution, $\boldsymbol{x}$ and a realization of scenario, $\boldsymbol{z}$, the optimal wait-and-see objective we consider is given by

$$f(\boldsymbol{x},\boldsymbol{z}) = \boldsymbol{c}(\boldsymbol{z})'\boldsymbol{x} + \begin{array}{ll} \min & \boldsymbol{d_u}'\boldsymbol{u} + \boldsymbol{d_y}'\boldsymbol{y} \\ \text{s.t.} & \boldsymbol{T}(\boldsymbol{z})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{u} + \boldsymbol{Y}\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{z}) \\ & \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{u} \text{ free}, \end{array} \tag{1}$$

where $\boldsymbol{d_u} \in \Re^{n_2}$ and $\boldsymbol{d_y} \in \Re^{n_3}$ are known vectors, $\boldsymbol{U} \in \Re^{m_2 \times n_2}$ and $\boldsymbol{Y} \in \Re^{m_2 \times n_3}$ are known matrices, $\boldsymbol{c}(\tilde{\boldsymbol{z}}) \in \Re^{n_1}$, $\boldsymbol{T}(\tilde{\boldsymbol{z}}) \in \Re^{m_2 \times n_1}$ and $\boldsymbol{h}(\tilde{\boldsymbol{z}}) \in \Re^{m_2}$ are random data as function mapping of $\tilde{\boldsymbol{z}}$. In the language of stochastic optimization, this is a fixed recourse model in which the matrices $\boldsymbol{U}$ and $\boldsymbol{Y}$ associated with the recourse actions, are not influenced by uncertainties; see Birge and Louveaux [7]. The model (1) represents a rather general fixed recourse framework characterized in classical stochastic optimization formulations. Using the convention of stochastic optimization, if the model (1) is infeasible, the function $f(\boldsymbol{x},\boldsymbol{z})$ will be assigned an infinite value.

We denote $\tau(\tilde{\boldsymbol{z}})$ as the target level or aspiration level, which, in the most general setting, depends on the primitive uncertainties, $\tilde{\boldsymbol{z}}$; see Bordley and LiCalzi [6]. The wait-and-see objective $f(\boldsymbol{x},\tilde{\boldsymbol{z}})$ is a random variable with probability distribution as as function of $\boldsymbol{x}$. Under the *aspiration level criterion*, which we will subsequently define, we examine the following model:

$$\begin{array}{ll} \max & \beta\Big(f(\boldsymbol{x},\tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big) \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \epsilon \geq 0, \end{array} \tag{2}$$

where $\boldsymbol{b} \in \Re^{m_1}$ and $\boldsymbol{A} \in \Re^{m_1 \times n_1}$ are known. We use the phrase *aspiration level prospect* to represent the random variable, $f(\boldsymbol{x},\tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})$, which evaluates the shortfall of the wait-and-see objective from the target level. The functional $\beta(\cdot)$ is the aspiration level criterion, which evaluates the chance of exceeding the target level of performance.

Definition 1 *Given an aspiration level prospect, $\tilde{v}$, the aspiration level criterion is defined as*

$$\beta(\tilde{v}) \triangleq \mathrm{P}(\tilde{v} \leq 0). \tag{3}$$

We adopt the same definition as used in Diecidue and van de Ven [14] and in Canada et al. [8], chapter 5. We can equivalently express the aspiration level criterion as

$$\beta(\tilde{v}) = 1 - \mathrm{P}(\tilde{v} > 0) = 1 - \mathrm{E}(\mathcal{H}(\tilde{v})) \tag{4}$$

where $\mathcal{H}(\cdot)$ is a heavy-side utility function defined as

$$\mathcal{H}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

2.1 Shortfall aspiration level criterion

The aspiration level criterion has several drawbacks from the computational and modeling perspectives. The lack of any form of structural convexity leads to computational intractability. Moreover, it is evident from Equation (4) that the aspiration level criterion does not take account into the shortfall level and may equally value a catastrophic event with low probability over a mild violation with the same probability. In view of the deficiencies of the aspiration level criterion, we introduce the shortfall aspiration level criterion.

Definition 2 *Given an aspiration level prospect, $\tilde{v}$ with the following conditions:*

$$\begin{aligned} \mathrm{E}(\tilde{v}) &< 0 \\ \mathrm{P}(\tilde{v} > 0) &> 0, \end{aligned} \tag{5}$$

the shortfall aspiration level criterion is defined as

$$\alpha(\tilde{v}) \triangleq 1 - \inf_{a>0} \left(\mathrm{E}\left(\mathcal{S}(\tilde{v}/a)\right)\right) \tag{6}$$

where we define the shortfall utility function as follows:

$$\mathcal{S}(x) = (x+1)^+.$$

We present the properties of the shortfall aspiration level criterion in the following theorem.

Theorem 1 *Let $\tilde{v}$ be an aspiration level prospect satisfying the inequalities (5). The shortfall aspiration level criterion has the following properties*

(a)

$$\alpha(\tilde{v}) \geq \beta(\tilde{v})$$

(b)

$$\alpha(\tilde{v}) \in (0,1).$$

Moreover, there exists a finite $a^* > 0$, such that

$$\alpha(\tilde{v}) = 1 - \mathrm{E}\left(\mathcal{S}\left(\tilde{v}/a^*\right)\right)$$

(c)

$$\alpha(\tilde{v}) = \sup\{1-\gamma \ : \ \psi_{1-\gamma}(\tilde{v}) \leq 0, \gamma \in (0,1)\}$$

where

$$\psi_{1-\gamma}(\tilde{v}) \triangleq \min_{v} \left(v + \frac{\mathrm{E}\left((\tilde{v}-v)^+\right)}{\gamma}\right) \tag{7}$$

is the popular risk measure known as Conditional-Value-at-Risk (CVaR) popularized by Rockafellar and Uryasev [27].

(d) *Suppose for all $\boldsymbol{x} \in X$, $\tilde{v} = \tilde{v}(\boldsymbol{x})$ is normally distributed, then the feasible solution that maximizes the shortfall aspiration level criterion also maximizes the aspiration level criterion.*

Proof : **(a)** Observe that for all $a > 0$, $\mathcal{S}(x/a) \geq \mathcal{H}(x)$, hence, we have

$$\begin{aligned} \mathrm{P}(\tilde{v} > 0) &= \mathrm{E}\left(\mathcal{H}(\tilde{v})\right) \\ &\leq \inf_{a>0} \mathrm{E}\left(\mathcal{S}(\tilde{v}/a)\right) \\ &= 1 - \alpha(\tilde{v}). \end{aligned}$$

Therefore,

$$\beta(\tilde{v}) = \mathrm{P}(\tilde{v} \leq 0) = 1 - \mathrm{P}(\tilde{v} > 0) \geq \alpha(\tilde{v}).$$

(b) Since $\mathrm{P}(\tilde{v} > 0) > 0$, from (a), we have $\alpha(\tilde{v}) \leq 1 - \mathrm{P}(\tilde{v} > 0) < 1$. To show that $\alpha(\tilde{v}) > 0$, it suffices to find a $b > 0$ such that $\mathrm{E}(\mathcal{S}(\tilde{v}/b)) < 1$. Observe that

$$\mathrm{E}(\mathcal{S}(\tilde{v}/a)) = 1 + \frac{\mathrm{E}(\tilde{v}) + \mathrm{E}\left((\tilde{v}+a)^-\right)}{a}.$$

As $\mathrm{E}(\tilde{v}) < 0$ and $\mathrm{E}\left((\tilde{v}+a)^-\right)$ is nonnegative, continuous in a and converges to zero as a approaches infinity, there exists a $b > 0$, such that $\mathrm{E}(\tilde{v}) + \mathrm{E}\left((\tilde{v}+b)^-\right) < 0$. Hence,

$$\alpha(\tilde{v}) = 1 - \inf_{a>0} \frac{\mathrm{E}\left((\tilde{v}+a)^+\right)}{a} \geq 1 - \frac{\mathrm{E}\left((\tilde{v}+b)^+\right)}{b} > 0.$$

Since $\mathrm{P}(\tilde{v} > 0) > 0$ implies $\mathrm{E}(\tilde{v}^+) > 0$, we also observe that

$$\lim_{a\downarrow 0} \mathrm{E}(\mathcal{S}(\tilde{v}/a)) = \lim_{a\downarrow 0} \frac{\mathrm{E}\left((\tilde{v}+a)^+\right)}{a} \geq \lim_{a\downarrow 0} \frac{\mathrm{E}\left(\tilde{v}^+\right)}{a} = \infty.$$

Moreover,

$$\lim_{a\to\infty} \mathrm{E}(\mathcal{S}(\tilde{v}/a)) = 1.$$

We have also shown that $\inf_{a>0} \mathrm{E}(\mathcal{S}(\tilde{v}/a)) \in (0,1)$, hence, the infimum cannot be achieved at the limits of $a = 0$ and $a = \infty$. Moreover, due to the continuity of the function $\mathrm{E}(\mathcal{S}(\tilde{v}/a))$ over $a > 0$, the infimum is achieved at a finite $a > 0$.

(c) Using the observations in (b), we have

$$\begin{aligned} & 1 - \inf_{a>0} \mathrm{E}(\mathcal{S}(\tilde{v}/a)) \\ = \;& \sup_{v<0} \left(1 + \tfrac{\mathrm{E}\left((\tilde{v}-v)^+\right)}{v}\right) \\ = \;& \sup\left\{1-\gamma \;:\; 1-\gamma \leq 1 + \tfrac{\mathrm{E}\left((\tilde{v}-v)^+\right)}{v}, v < 0, \gamma \in (0,1)\right\} \\ = \;& \sup\left\{1-\gamma \;:\; v + \tfrac{\mathrm{E}\left((\tilde{v}-v)^+\right)}{\gamma} \leq 0, v < 0, \gamma \in (0,1)\right\} \\ = \;& \sup\left\{1-\gamma \;:\; v + \tfrac{\mathrm{E}\left((\tilde{v}-v)^+\right)}{\gamma} \leq 0, \gamma \in (0,1)\right\} \qquad \text{With } \mathrm{E}(\tilde{v}^+) > 0,\ v < 0 \text{ is implied} \\ = \;& \sup\{1-\gamma \;:\; \psi_{1-\gamma}(\tilde{v}) \leq 0, \gamma \in (0,1)\}. \end{aligned}$$

(d) Observe that

$$\max\left\{\beta\Big(\tilde{v}(\boldsymbol{x})\Big) \;:\; \boldsymbol{x} \in \mathcal{X}\right\} \tag{8}$$

is equivalent to

$$\max\left\{1-\epsilon \;:\; \mathrm{P}\Big(\tilde{v}(\boldsymbol{x}) \leq 0\Big) \geq 1-\epsilon, \boldsymbol{x} \in \mathcal{X}\right\}.$$

Let $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ be the mean and standard deviation of $\tilde{v}(\boldsymbol{x})$. The constraint $\mathrm{P}\Big(\tilde{v}(\boldsymbol{x}) \leq 0\Big) \geq 1-\epsilon$ is equivalent to

$$\tau - \mu(\boldsymbol{x}) \geq \Phi^{-1}(1-\epsilon)\sigma(\boldsymbol{x}),$$

where $\Phi(\cdot)$ is the distribution function of a standard normal. Since $\mathrm{E}(\tilde{v}(\boldsymbol{x})) < 0$, the optimal objective satisfies $1-\epsilon > 1/2$ and hence, $\Phi^{-1}(1-\epsilon) > 0$. Noting that $\Phi^{-1}(1-\epsilon)$ is a decreasing function in ϵ, the optimal solution in Model (8) corresponds to maximizing the following ratio:

$$\begin{array}{ll} \max & \dfrac{\tau - \mu(\boldsymbol{x})}{\sigma(\boldsymbol{x})} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X}. \end{array} \tag{9}$$

This relation was observed by Dragomirescu [15]. Using the result in (c), we can express the maximization of the shortfall aspiration level criterion as follows:

$$\begin{array}{ll} \max & 1-\gamma \\ \text{s.t.} & \psi_{1-\gamma}(\tilde{v}(\boldsymbol{x})) \leq 0 \\ & \boldsymbol{x} \in \mathcal{X}, \gamma \in (0,1) \end{array} \tag{10}$$

Under normal distribution, we can also evaluate the CVaR measure in close form as follows:

$$\psi_{1-\gamma}(\tilde{v}(\boldsymbol{x})) = \mu(\boldsymbol{x}) - \tau + \underbrace{\frac{\phi(\Phi^{-1}(\gamma))}{\gamma}}_{s_\gamma}\sigma(\boldsymbol{x})$$

where ϕ is the density of a standard normal. Moreover, s_γ is also a decreasing function in γ. Therefore, the optimum solution of Model (10) is identical to Model (9). ■

We now propose the following goal driven optimization problem.

$$\begin{array}{ll} \max & \alpha\Big(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big) \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{11}$$

Theorem 1(a) implies that an optimal solution of Model (11), $\boldsymbol{x}^*$ can achieve the following success probability,

$$\mathrm{P}(f(\boldsymbol{x}^*, \tilde{\boldsymbol{z}}) \leq \tau(\tilde{\boldsymbol{z}})) \geq \alpha\Big(f(\boldsymbol{x}^*, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big).$$

The optimal parameter, a^* within the shortfall aspiration level criterion is chosen to attain the tightest bound in meeting the success probability. The aspiration level criterion of (4) penalizes the shortfall with an heavy-side utility function that is insensitive to the magnitude of violation. In contract, the shortfall aspiration level criterion,

$$\alpha(f(\boldsymbol{x}^*, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})) = 1 - \frac{1}{a^*}\mathrm{E}\left((f(\boldsymbol{x}^*, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}}) + a^*)^+\right) \qquad \text{for some } a^* > 0$$

has an expected utility component that penalizes an expected level of "near" shortfall when the aspiration level prospect raises above $-a^*$. Speaking intuitively, given aspiration level prospects, $\tilde{v}_1$ and

$\tilde{v}_2$ with the same aspiration level criteria defined in (3), suppose $\tilde{v}_2$ incurs greater expected shortfall, the shortfall aspiration level criterion will rank $\tilde{v}_1$ higher than $\tilde{v}_2$. Nevertheless, Theorem 1(d) suggests that if the distribution of the objective is "fairly normally distributed", we expect the solution that maximizes the shortfall aspiration level criterion would also maximize the aspiration level criterion.

We now discuss the conditions of (5) with respect to the goal driven optimization model. The first condition implies that the aspiration level should be strictly achievable in expectation. Hence, the goal driven optimization model appeals to decision makers who are risk averse and are not unrealistic in setting their goals. The second condition implies that there does not exist a feasible solution, which always achieves the aspiration level. In other words, the goal driven optimization model is used in problem instances where the risk of under-performance is inevitable. Hence, it appeals to decision makers who are not too apathetic in setting their goals.

Theorem 1(c) shows the connection between the shortfall aspiration level criterion with the CVaR measure. The CVaR measure satisfies four desirable properties of financial risk measures known as *coherent risk*. A coherent risk measure or functional, $\varphi(\cdot)$ satisfies the following *Axioms of coherent risk measure*:

(i) Translation invariance: For all $a \in \Re$, $\varphi(\tilde{v} + a) = \varphi(\tilde{v}) + a$.

(ii) Subadditivity: For all random variables $\tilde{v}_1, \tilde{v}_2$, $\varphi(\tilde{v}_1 + \tilde{v}_2) \leq \varphi(\tilde{v}_1) + \varphi(\tilde{v}_2)$.

(iii) Positive homogeneity: For all $\lambda \geq 0$, $\varphi(\lambda\tilde{v}) = \lambda\varphi(\tilde{v})$.

(iv) Monotonicity: For all $\tilde{v} \leq \tilde{w}$, $\varphi(\tilde{v}) \leq \varphi(\tilde{w})$.

The four axioms were presented and justified in Artzner et al. [2]. The first axiom ensures that $\varphi(\tilde{v} - \varphi(\tilde{v})) = 0$, so that the risk of $\tilde{v}$ after compensation with $\varphi(\tilde{v})$ is zero. It means that reducing the cost by a fixed amount of a simply reduces the risk measure by a. The subadditivity axiom states that the risk associated with the sum of two financial instruments is not more than the sum of their individual risks. It appears naturally in finance - one can think equivalently of the fact that "a merger does not create extra risk," or of the "risk pooling effects" observed in the sum of random variables. The positive homogeneity axiom implies that the risk measure scales proportionally with its size. The final axiom is an obvious criterion, but it rules out the classical mean-standard deviation risk measure.

A byproduct of a risk measure that satisfies these axioms is the preservation of convexity; see for instance Ruszczynski and Shapiro [28]. Hence, the function $\psi_{1-\gamma}(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}}))$ is therefore convex in $\boldsymbol{x}$. Using the connection with the CVaR measure, we express the goal driven optimization model (11), equivalently as follows:

$$
\begin{array}{ll}
\max & 1 - \gamma \\
\text{s.t.} & \psi_{1-\gamma}(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})) \leq 0 \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0} \\
& \gamma \in (0, 1).
\end{array} \tag{12}
$$

2.2 Reduction to stochastic optimization problems with CVaR objectives

For a fixed γ, the first constraint in Model (12) is convex in the decision variable $\boldsymbol{x}$. However, the Model is not jointly convex in γ and $\boldsymbol{x}$. Nevertheless, we can still obtain the optimal solution by solving a sequence of subproblems in the form of stochastic optimization problems with CVaR objectives as follows:

$$\begin{aligned} Z(\gamma) = \quad \min \quad & \psi_{1-\gamma}\Big(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{13}$$

or equivalently,

$$\begin{aligned} Z(\gamma) = \quad \min \quad & \psi_{1-\gamma}\Big(\boldsymbol{c}(\tilde{\boldsymbol{z}})'\boldsymbol{x} + \boldsymbol{d_u}'\boldsymbol{u}(\tilde{\boldsymbol{z}}) + \boldsymbol{d_y}'\boldsymbol{y}(\tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{u}(\tilde{\boldsymbol{z}}) + \boldsymbol{Y}\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}(\tilde{\boldsymbol{z}}) \\ & \boldsymbol{y}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y}(\cdot) \in \mathcal{Y}, \end{aligned} \tag{14}$$

where $\boldsymbol{u}(\tilde{\boldsymbol{z}})$ and $\boldsymbol{y}(\tilde{\boldsymbol{z}})$ correspond to the second stage or recourse variables in the space of measurable functions, $\mathcal{Y}$.

Algorithm 1 *(Binary Search)*
Input: *A routine that solves Model (13) optimally and* $\zeta > 0$
Output: $\boldsymbol{x}$

1. *Set* $\gamma_1 := 0$ *and* $\gamma_2 := 1$.
2. *If* $\gamma_2 - \gamma_1 < \zeta$, *stop. Output:* $\boldsymbol{x}$
3. *Let* $\gamma := \frac{\gamma_1+\gamma_2}{2}$. *Compute* $Z(\gamma)$ *from Model (13) and obtain the corresponding optimal solution* $\boldsymbol{x}$.
4. *If* $Z(\gamma) \leq 0$, *update* $\gamma_2 := \gamma$. *Otherwise, update* $\gamma_1 := \gamma$
5. *Go to Step 2.*

Proposition 1 *Suppose Model (12) is feasible. Algorithm 1 finds a solution,* $\boldsymbol{x}$ *with objective* γ^{**} *satisfying* $|\gamma^{**} - \gamma^*| < \zeta$ *in at most* $\lceil \log_2(1/\zeta) \rceil$ *computations of the subproblem (13), where* $1 - \gamma^*$ *being the optimal objective of Model (12).*

Proof : Observe that each looping in Algorithm 1 reduces the gap between γ_2 and γ_1 by half. We now show the correctness of the binary search. Suppose $Z(\gamma) \leq 0$, γ is feasible in Model (12) and hence $\gamma^* \leq \gamma$. Otherwise, γ would be infeasible in Model (12). In this case, we claim that the optimal feasible solution, γ^* must be greater than γ. Suppose not, we have $\gamma^* \leq \gamma$. We know the optimal solution $\boldsymbol{x}^*$ of Model (12) satisfies

$$\psi_{1-\gamma^*}\Big(f(\boldsymbol{x}^*, \tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big) \leq 0.$$

Then since $\gamma^* \le \gamma$, we have

$$Z(\gamma) \le \psi_{1-\gamma}\Big(f(\boldsymbol{x}^*,\tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big) \le \psi_{1-\gamma^*}\Big(f(\boldsymbol{x}^*,\tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}})\Big) \le 0,$$

contradicting that $Z(\gamma) > 0$. ■

If $\tilde{\boldsymbol{z}}$ takes values from $\boldsymbol{z}^k$, $k = 1,\dots,K$ with probability p_k, we can formulate the subproblem of (13) as a linear optimization problem as follows:

$$\begin{array}{lll}
\min & \omega + \frac{1}{\gamma}\sum_{k=1}^K y_k p_k & \\
\text{s.t.} & y_k \ge \boldsymbol{c}(\boldsymbol{z}^k)'\boldsymbol{x} + \boldsymbol{d_u}'\boldsymbol{u}^k + \boldsymbol{d_y}'\boldsymbol{y}^k - \tau(\boldsymbol{z}^k) - \omega & k = 1,\dots,K \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \\
& \boldsymbol{T}(\boldsymbol{z}^k)\boldsymbol{x} + \boldsymbol{U}\boldsymbol{u}^k + \boldsymbol{Y}\boldsymbol{y}^k = \boldsymbol{h}(\boldsymbol{z}^k) & k = 1,\dots,K \\
& \boldsymbol{x} \ge \boldsymbol{0}, \boldsymbol{y} \ge \boldsymbol{0} &
\end{array}$$

Unfortunately, the number of possible recourse decisions increases proportionally with the number of possible realization of the random vector $\tilde{\boldsymbol{z}}$, which could be extremely large or even infinite. Nevertheless, under relatively complete recourse, the two stage stochastic optimization model can be solved rather effectively using sampling approximation. In such problems, the second stage problem is always feasible regardless of the choice of feasible first stage variables. Decomposition techniques has been studied in Ahmed [1] and Riis and Schultz [26] to enable efficient computations of the stochastic optimization problem with CVaR objective.

In the absence of relatively complete recourse, the solution obtained from sampling approximation may not be meaningful. Even though the objective of the sampling approximation could be finite, in the actual performance, the second stage problem can be infeasible, in which case the actual objective is infinite. Indeed, a two stage stochastic optimization is generally intractable. For instance, checking whether the first stage decision $\boldsymbol{x}$ gives rise to feasible recourse for all realization of $\tilde{\boldsymbol{z}}$ is already an NP-hard problem; see Ben-Tal et al. [3]. Moreover, with the assumption that the stochastic parameters are independently distributed, Dyer and Stougie [16] show that two-stage stochastic programming problems are #P-hard. Under the same assumption they show that certain multi-stage stochastic programming problems are PSPACE-hard. We therefore pursue an alterative method of approximating the stochastic optimization problem, that could at least guarantee the feasibility of the solution, and determine an upper bound of the objective function.

3 Deterministic Approximations via Robust Optimization

We have shown that solving the goal driven optimization model (11) involves solving a sequence of stochastic optimization problems with CVaR objectives in the form of Model (14). Hence, we devote this section to formulating a tractable deterministic approximation of Model (14).

One of the central problems in stochastic models is how to properly account for data uncertainty. Unfortunately, complete probability descriptions are almost never available in real world environments. Following the recent development of robust optimization such as Ben-Tal et al. [3], Chen, Sim and

Sun [12] and Chen et al. [13], we relax the assumption of full distributional knowledge and modify the representation of data uncertainties with the aim of producing a computationally tractable model. We adopt the parametric uncertainty model in which the data uncertainties are affinely dependent on the primitive uncertainties.

Affine Parametric Uncertainty: We assume that the uncertain input data to the model $\boldsymbol{c}(\tilde{\boldsymbol{z}})$, $\boldsymbol{T}(\tilde{\boldsymbol{z}})$, $\boldsymbol{h}(\tilde{\boldsymbol{z}})$ and $\tau(\tilde{\boldsymbol{z}})$ are affinely dependent on the primitive uncertainties $\tilde{\boldsymbol{z}}$ as follows:

$$\begin{aligned}
\boldsymbol{c}(\tilde{\boldsymbol{z}}) &= \boldsymbol{c}^0 + \sum_{j=1}^{N} \boldsymbol{c}^j \tilde{z}_j, \\
\boldsymbol{T}(\tilde{\boldsymbol{z}}) &= \boldsymbol{T}^0 + \sum_{j=1}^{N} \boldsymbol{T}^j \tilde{z}_j, \\
\boldsymbol{h}(\tilde{\boldsymbol{z}}) &= \boldsymbol{h}^0 + \sum_{j=1}^{N} \boldsymbol{h}^j \tilde{z}_j, \\
\tau(\tilde{\boldsymbol{z}}) &= \tau^0 + \sum_{j=1}^{N} \tau^j \tilde{z}_j.
\end{aligned}$$

This parametric uncertainty representation is useful for relating multivariate random variables across different data entries through the shared primitive uncertainties.

Since the assumption of having exact probability distributions of the primitive uncertainties is unrealistic, as in the spirit of robust optimization, we adopt a modest distributional assumption on the primitive uncertainties, such as known means, supports, subset of independently distributed random variables and some aspects of deviations. Under the affine parametric uncertainty, we can translate the primitive uncertainties so that their means are zeros. For the subset of independently distributed primitive uncertainties, we will use the forward and backward deviations, which were recently introduced by Chen, Sim and Sun [12].

Definition 3 *Given a random variable $\tilde{z}$ with zero mean, the forward deviation is defined as*

$$\sigma_f(\tilde{z}) \triangleq \sup_{\theta>0} \left\{ \sqrt{2\ln(\mathrm{E}(\exp(\theta\tilde{z})))/\theta^2} \right\} \tag{15}$$

and backward deviation is defined as

$$\sigma_b(\tilde{z}) \triangleq \sup_{\theta>0} \left\{ \sqrt{2\ln(\mathrm{E}(\exp(-\theta\tilde{z})))/\theta^2} \right\}. \tag{16}$$

Given a sequence of independent samples, we can essentially estimate the magnitude of the deviation measures from (15) and (16). Some of the properties of the deviation measures include:

Proposition 2 *(Chen, Sim and Sun [12])*
Let σ, p and q be respectively the standard, forward and backward deviations of a random variable, $\tilde{z}$ with zero mean.

(a) *Then* $p \geq \sigma$ *and* $q \geq \sigma$. *If* $\tilde{z}$ *is normally distributed, then* $p = q = \sigma$.

(b)

$$\mathrm{P}(\tilde{z} \geq \Omega p) \leq \exp(-\Omega^2/2);$$
$$\mathrm{P}(\tilde{z} \leq -\Omega q) \leq \exp(-\Omega^2/2).$$

(c) *For all* $\theta \geq 0$,

$$\ln \mathrm{E}(\exp(\theta\tilde{z})) \leq \frac{\theta^2 p^2}{2};$$
$$\ln \mathrm{E}(\exp(-\theta\tilde{z})) \leq \frac{\theta^2 q^2}{2}.$$

Proposition 2(a) shows that the forward and backward deviations are no less than the standard deviation of the underlying distribution, and under normal distribution, these two values coincide with the standard deviation. As exemplified in Proposition 2(b), the deviation measures provide an easy bound on the distributional tails. Chen, Sim and Sun ([12]) show that new deviation measures provide tighter approximation of probabilistic bounds compared to standard deviations. These information, whenever available, enable us to improve upon the solutions of the approximation.

Model of Primitive Uncertainty, U: We assume that the primitive uncertainties $\{\tilde{z}_j\}_{j=1:N}$ are zero mean random variables, with covariance $\mathbf{\Sigma}$ and support $\tilde{\boldsymbol{z}} \in \mathcal{W} = [-\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$. Of the N primitive uncertainties, the first I random variables, that is, $\tilde{z}_j$, $j = 1, \ldots, I$ are stochastically independent. Moreover, the corresponding forward and backward deviations given by $p_j = \sigma_f(\tilde{z}_j)$ and $q_j = \sigma_b(\tilde{z}_j)$ respectively for $j = 1, \ldots, I$.

Similar uncertainty models have been defined in Chen, Sim and Sun [12] and Chen et al. [13]. While the uncertainty model proposed in the former focuses on only independent primitive uncertainties with known support, forward and backward deviation measures, the uncertainty model proposed in the latter disregards independence and assumes known support and covariance of the primitive uncertainties. Hence, the model of primitive uncertainty, U encompasses both of the models discussed in Chen, Sim and Sun [12] and Chen et al. [13].

Under the model of primitive uncertainty, U, it is evident that $\boldsymbol{h}^0$, for instance, represents the mean of $\boldsymbol{h}(\tilde{\boldsymbol{z}})$ and $\boldsymbol{h}^j$ represents the magnitude and direction associated with the primitive uncertainty, $\tilde{z}_j$. The model of primitive uncertainty, U, provides a flexility of incorporating mutually a subset of independent random variables, which, as we shall see, leads to tighter approximation. For instance, if $\tilde{\boldsymbol{h}}$ is multivariate normally distributed with mean $\boldsymbol{h}^0$ and covariance, $\mathbf{\Sigma}$, then $\tilde{\boldsymbol{h}}$ can be decomposed into primitive uncertainties that are stochastically independent. For instance, we have

$$\tilde{\boldsymbol{h}} = \boldsymbol{h}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}^0 + \mathbf{\Sigma}^{1/2}\tilde{\boldsymbol{z}}.$$

To fit into the affine parametric uncertainty and the model of primitive uncertainty, U, we can assign the vector $\boldsymbol{h}^j$ to the jth column of $\mathbf{\Sigma}^{1/2}$. Moreover, $\tilde{\boldsymbol{z}}$ has stochastically independent entries with covariance equals the identity matrix, infinite support and unit forward and backward deviations; see Proposition 2(a).

3.1 Approximation of the CVaR measure

Having define the uncertainty structure, we next show how we can approximate the CVaR measure. Even though the CVaR measure leads to convex constraints, it is not necessarily easy to evaluate its value accurately. Our aim here is to approximate the CVaR measure by solving a tractable convex problem, primarily in the form of second order conic constraints. We first consider the approximation of $\psi_{1-\gamma}(y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}})$ and draws upon this insight in building a tractable approximation of Model (13) in the form of SOCP. In particular, we are interested in upper bounds of $\psi_{1-\gamma}(y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}})$. If the function $g_{1-\gamma}(y^0,\boldsymbol{y})$ satisfies $\psi_{1-\gamma}(y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}}) \leq g_{1-\gamma}(y_0,\boldsymbol{y})$ for all γ and $(y_0,\boldsymbol{y})$, then the constraints $g_{1-\gamma}(y_0,\boldsymbol{y}) \leq 0$ is sufficient to guarantee $\psi_{1-\gamma}(y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}}) \leq 0$.

Using different aspects of the distributional information prescribed in the model of primitive uncertainty, U, we can obtain different bounds on $\psi_{1-\gamma}(y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}})$ as follows:

Proposition 3 *Under the Model of Uncertainty, U, the following functions are upper bounds of* $\psi_{1-\gamma}(y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}})$, *for all* $(y_0,\boldsymbol{y})$ *and* $\gamma \in (0,1)$.

(a)

$$\begin{aligned} g^1_{1-\gamma}(y_0,\boldsymbol{y}) &\triangleq y_0 + \max_{\boldsymbol{z}\in\mathcal{W}} \boldsymbol{y}'\boldsymbol{z} \\ &= y_0 + \min\{\boldsymbol{s}'\bar{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}} \ : \ \boldsymbol{s}-\boldsymbol{t}=\boldsymbol{y}, \boldsymbol{s},\boldsymbol{t} \geq \boldsymbol{0}\} \end{aligned}$$

(b)

$$g^2_{1-\gamma}(y_0,\boldsymbol{y}) \triangleq y_0 + \sqrt{\frac{1-\gamma}{\gamma}}\sqrt{\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}$$

(c)

$$g^3_{1-\gamma}(y_0,\boldsymbol{y}) \triangleq \begin{cases} y_0 + \sqrt{-2\ln(\gamma)}\|\boldsymbol{u}\|_2 & \textit{if } y_j = 0, \forall j = I+1,\ldots,N \\ +\infty & \textit{otherwise} \end{cases}$$

where $u_j = \max\{p_j y_j, -q_j y_j\}$.

(d)

$$g^4_{1-\gamma}(y_0,\boldsymbol{y}) \triangleq \begin{cases} y_0 + \frac{1-\gamma}{\gamma}\sqrt{-2\ln(1-\gamma)}\|\boldsymbol{v}\|_2 & \textit{if } y_j = 0, \forall j = I+1,\ldots,N \\ +\infty & \textit{otherwise;} \end{cases}$$

where $v_j = \max\{-p_j y_j, q_j y_j\}$.

Proof :

(a) Let $w = y_0 + \max_{\boldsymbol{z}\in\mathcal{W}} \boldsymbol{y}'\boldsymbol{z}$. We have

$$\begin{aligned} & \min_{\omega}\left\{\omega + \frac{1}{\gamma}\mathrm{E}\left((y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}}-\omega)^+\right)\right\} \\ \leq\ & w + \tfrac{1}{\gamma}\mathrm{E}\left(\Big(\underbrace{y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}}-w}_{\leq 0}\Big)^+\right) \\ =\ & w \\ =\ & y_0 + \max_{\boldsymbol{z}\in\mathcal{W}} \boldsymbol{y}'\boldsymbol{z} \\ =\ & y_0 + \min\{\boldsymbol{s}'\bar{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}} \ : \ \boldsymbol{s}-\boldsymbol{t}=\boldsymbol{y}, \boldsymbol{s},\boldsymbol{t} \geq \boldsymbol{0}\}, \end{aligned}$$

where the last equality follows from strong LP duality, with $\mathcal{W} = [-\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$.

(b) Observe that

$$
\begin{aligned}
& \min_{\omega} \left\{ \omega + \frac{1}{\gamma} \mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} - \omega)^+ \right) \right\} & \\
= \;& \min_{\omega} \left\{ \omega + \frac{1}{\gamma} \mathrm{E}\left((\boldsymbol{y}'\tilde{\boldsymbol{z}} - \omega)^+ \right) \right\} + y_0 & \text{Translation Invariance} \\
= \;& \min_{\omega} \left\{ \omega + \frac{1}{2\gamma} \Big(-\omega + \mathrm{E}(|\boldsymbol{y}'\tilde{\boldsymbol{z}} - \omega|) \Big) \right\} + y_0 & \text{Since } x^+ = (x + |x|)/2 \\
\leq \;& \min_{\omega} \left\{ \omega + \frac{1}{2\gamma} \Big(-\omega + \sqrt{\mathrm{E}((\boldsymbol{y}'\tilde{\boldsymbol{z}} - \omega)^2)} \Big) \right\} + y_0 & \text{Jensen inequality} \\
= \;& \min_{\omega} \left\{ \omega + \frac{1}{2\gamma} \Big(-\omega + \sqrt{\omega^2 + \boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}} \Big) \right\} + y_0 & \\
= \;& y_0 + \sqrt{\frac{1-\gamma}{\gamma}} \sqrt{\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}, &
\end{aligned}
$$

where the last equality follows from choosing the tightest ω^*, that is

$$\omega^* = \frac{\sqrt{\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}(1 - 2\gamma)}{2\sqrt{\gamma(1-\gamma)}}.$$

(c) The bound is trivially true if there exists $y_j \neq 0$ for any $j > I$. Henceforth, we assume $y_j = 0, \forall j = I+1, \ldots, N$. The key idea of the inequality comes from the observation that

$$x^+ \leq \mu \exp(x/\mu - 1) \qquad \forall \mu > 0.$$

Hence, $\mathrm{E}(x^+) \leq \mu \mathrm{E}(\exp(x/\mu - 1))$ for all $\mu > 0$. Let $\tilde{y} = y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}$. Observe that

$$
\begin{aligned}
& \min_{\omega} \left\{ \omega + \frac{1}{\gamma} \mathrm{E}\left((\tilde{y} - \omega)^+ \right) \right\} & \\
\leq \;& \min_{\omega} \left\{ \omega + \frac{\mu}{\gamma} \mathrm{E}(\exp((\tilde{y} - \omega)/\mu - 1)) \right\} & \forall \mu > 0 \\
= \;& \omega^* + \frac{\mu}{\gamma} \mathrm{E}(\exp((\tilde{y} - \omega^*)/\mu - 1)) & \forall \mu > 0,
\end{aligned}
$$

where

$$\omega^* = \mu \ln \left(\frac{\mathrm{E}(\exp(\tilde{y}/\mu))}{\gamma e} \right).$$

Substituting, we have

$$\psi_{1-\gamma}(\tilde{y}) \leq \mu \ln \mathrm{E}(\exp(\tilde{y}/\mu)) - \mu \ln(\gamma) \qquad \forall \mu > 0.$$

This relation was first shown in Nemirovski and Shapiro [21]. Using the deviation measures of Chen, Sim and Sun [12], and Proposition 2(c), we have

$$\ln(\mathrm{E}(\exp(y_j \tilde{z}_j/\mu))) \leq \begin{cases} {y_j}^2 {p_j}^2/(2\mu^2) & \text{if } y_j \geq 0 \\ {y_j}^2 {q_j}^2/(2\mu^2) & \text{otherwise.} \end{cases} \tag{17}$$

Since p_j and q_j are nonnegative, we have

$$\ln(\mathrm{E}(\exp(y_j\tilde{z}_j/\mu))) \leq \frac{(\max\{y_jp_j, -y_jq_j\})^2}{2\mu^2} = \frac{u_j^2}{2\mu^2}. \tag{18}$$

Therefore,

$$\begin{aligned}
& \psi_{1-\gamma}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}) \\
\leq\ & \inf_{\mu>0}\left\{\mu \ln \mathrm{E}(\exp((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})/\mu)) - \mu\ln(\gamma)\right\} \\
=\ & y_0 + \inf_{\mu>0}\left\{\mu\sum_{j=1}^{I}\ln(\mathrm{E}(\exp(y_j\tilde{z}_j/\mu))) - \mu\ln(\gamma)\right\} \quad \text{Independence of } \tilde{z}_j,\ j = 1,\ldots,I \\
\leq\ & y_0 + \inf_{\mu>0}\left\{\frac{\|\boldsymbol{u}\|_2^2}{2\mu} - \mu\ln(\gamma)\right\} \\
=\ & y_0 + \sqrt{-2\ln(\gamma)}\|\boldsymbol{u}\|_2.
\end{aligned}$$

(d) Again, we assume $y_j = 0, \forall j = I+1,\ldots,N$. Noting that $x^+ = x + (-x)^+$, we have

$$\begin{aligned}
& \min_{\omega}\left\{\omega + \frac{\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} - \omega)^+\right)}{\gamma}\right\} \\
=\ & \min_{\omega}\left\{\omega + \frac{y_0}{\gamma} - \frac{\omega}{\gamma} + \frac{\mathrm{E}\left((-y_0 - \boldsymbol{y}'\tilde{\boldsymbol{z}} + \omega)^+\right)}{\gamma}\right\} \\
=\ & \frac{y_0}{\gamma} + \min_{\omega}\left\{\frac{\gamma-1}{\gamma}\omega + \frac{\mathrm{E}\left((-y_0 - \boldsymbol{y}'\tilde{\boldsymbol{z}} + \omega)^+\right)}{\gamma}\right\} \\
=\ & \frac{y_0}{\gamma} + \min_{\omega}\left\{\omega + \frac{\mathrm{E}\left((-y_0 - \boldsymbol{y}'\tilde{\boldsymbol{z}} - \frac{\gamma}{1-\gamma}\omega)^+\right)}{\gamma}\right\} \\
=\ & \frac{y_0}{\gamma} + \min_{\omega}\left\{\omega + \frac{\mathrm{E}\left((\frac{1-\gamma}{\gamma}(-y_0 - \boldsymbol{y}'\tilde{\boldsymbol{z}}) - \omega)^+\right)}{1-\gamma}\right\} \\
\leq\ & \frac{y_0}{\gamma} - \frac{1-\gamma}{\gamma}y_0 + \frac{1-\gamma}{\gamma}\sqrt{-2\ln(1-\gamma)}\|\boldsymbol{v}\|_2 \\
=\ & y_0 + \frac{1-\gamma}{\gamma}\sqrt{-2\ln(1-\gamma)}\|\boldsymbol{v}\|_2.
\end{aligned}$$

The inequality follows the result of (c). ■

Remark : The first bound in Proposition 3 sets the limit of the CVaR measure using the support information of the primitive uncertainties and is independent of the parameter γ. The second bound is derived from the covariance of the primitive uncertainties. The last two bounds are more interesting, as they act upon primitive uncertainties that are stochastically independent.

To understand the conservativeness of the approximation, we compare the bounds of $\psi_{1-\gamma}(\tilde{z})$, where $\tilde{z}$ is standard normally distributed. Figure 1 compares the approximation ratios given by

$$\rho_i(\gamma) = \frac{g^i_{1-\gamma}(0,1) - \psi_{1-\gamma}(\tilde{z})}{\psi_{1-\gamma}(\tilde{z})}, \qquad i = 2,3,4$$

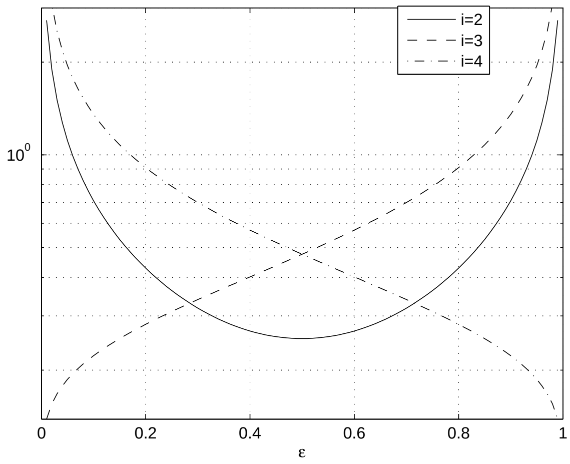

Figure 1: Plot of $\rho_i(\gamma)$ against γ for $i = 2, 3$ and 4, defined in Proposition 3

It is clear that none of the bounds dominate another across $\gamma \in (0, 1)$. Hence, this motivate us to integrate the best of all bounds to achieve the tightest approximation. The unified approximation will achieve a worst case approximation error of 33% at $\gamma = 0.2847$ and $\gamma = 0.7153$.

Theorem 2 *Let*

$$\begin{aligned} g_{1-\gamma}(y_0, \boldsymbol{y}) \triangleq \ \min \ & g^1_{1-\gamma}(y_{10}, \boldsymbol{y}_1) + g^2_{1-\gamma}(y_{20}, \boldsymbol{y}_2) + g^3_{1-\gamma}(y_{30}, \boldsymbol{y}_3) + g^4_{1-\gamma}(y_{40}, \boldsymbol{y}_4) \\ \text{s.t.} \ & y_{10} + y_{20} + y_{30} + y_{40} = y_0 \\ & \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 = \boldsymbol{y} \\ & y_{0i}, \boldsymbol{y}_i \ \textit{free} \qquad i = 1, 2, 3, 4. \end{aligned}$$

Then for all $(y_0, \boldsymbol{y})$ *and* $\gamma \in (0, 1)$

$$\psi_{1-\gamma}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}) \leq g_{1-\gamma}(y_0, \boldsymbol{y}) \leq \min\{g^1_{1-\gamma}(y_0, \boldsymbol{y}), g^2_{1-\gamma}(y_0, \boldsymbol{y}), g^3_{1-\gamma}(y_0, \boldsymbol{y}), g^4_{1-\gamma}(y_0, \boldsymbol{y})\} \tag{19}$$

Moreover, the constraint $g_{1-\gamma}(y_0, \boldsymbol{y}) \leq s$ *is second order cone representable (see Ben-Tal and Nemirovski*

[4]) as follows:

$$
\begin{array}{ll}
\exists \delta_i, y_{0i} \in \Re, \boldsymbol{y}_i, \boldsymbol{s}, \boldsymbol{t} \in \Re^N, i = 1, \ldots, 4, \boldsymbol{u}, \boldsymbol{v} \in \Re^I \ \textit{such that} & \\
\delta_1 + \delta_2 + \delta_3 + \delta_4 \leq s & \\
y_{10} + \boldsymbol{s}'\bar{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}} \leq \delta_1 & \\
\boldsymbol{s}, \boldsymbol{t} \geq 0 & \\
\boldsymbol{s} - \boldsymbol{t} = \boldsymbol{y}_1 & \\
y_{20} + \sqrt{\frac{1-\gamma}{\gamma}} \|\boldsymbol{\Sigma}^{1/2} \boldsymbol{y}_2\|_2 \leq \delta_2 & \\
y_{30} + \sqrt{-2\ln(\gamma)} \|\boldsymbol{u}\|_2 \leq \delta_3 & \\
u_j \geq p_j y_{3j}, u_j \geq -q_j y_{3j} & \forall j = 1, \ldots, I \\
y_{3j} = 0 & \forall j = I+1, \ldots, N \\
y_{40} + \frac{1-\gamma}{\gamma} \sqrt{-2\ln(1-\gamma)} \|\boldsymbol{v}\|_2 \leq \delta_4 & \\
v_j \geq q_j y_{4j}, v_j \geq -p_j y_{4j} & \forall j = 1, \ldots, I \\
y_{4j} = 0 & \forall j = I+1, \ldots, N \\
y_{10} + y_{20} + y_{30} + y_{40} = y_0 & \\
\boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 = \boldsymbol{y}. &
\end{array}
$$

Proof : To show the upper bound, we note that

$$
\begin{array}{lll}
& g^1_{1-\gamma}(y_{10}, \boldsymbol{y}_1) + g^2_{1-\gamma}(y_{20}, \boldsymbol{y}_2) + g^3_{1-\gamma}(y_{30}, \boldsymbol{y}_3) + g^4_{1-\gamma}(y_{40}, \boldsymbol{y}_4) & \\
\geq & \psi_{1-\gamma}(y_{10} + \boldsymbol{y}_1'\tilde{\boldsymbol{z}}) + \psi_{1-\gamma}(y_{20} + \boldsymbol{y}_2'\tilde{\boldsymbol{z}}) + \psi_{1-\gamma}(y_{30} + \boldsymbol{y}_3'\tilde{\boldsymbol{z}}) + \psi_{1-\gamma}(y_{40} + \boldsymbol{y}_4'\tilde{\boldsymbol{z}}) & \text{Proposition 3} \\
\geq & \psi_{1-\gamma}(y_{10} + y_{20} + y_{30} + y_{40} + (\boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4)'\tilde{\boldsymbol{z}}) & \text{Subadditvity} \\
= & \psi_{1-\gamma}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}). &
\end{array}
$$

Finally, to show that $g_{1-\gamma}(y_0, \boldsymbol{y}) \leq g^i_{1-\gamma}(y_0, \boldsymbol{y})$, $i = 1, \ldots, 4$, let

$$
(y_{r0}, \boldsymbol{y}_r) = \begin{cases} (y_0, \boldsymbol{y}) & \text{if } r = i \\ (0, \boldsymbol{0}) & \text{otherwise} \end{cases} \qquad \text{for } r = 1, \ldots, 4.
$$

Hence,

$$
g^r_{1-\gamma}(y_{r0}, \boldsymbol{y}_r) = \begin{cases} g^r_{1-\gamma}(y_0, \boldsymbol{y}) & \text{if } r = i \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } r = 1, \ldots, 4,
$$

and therefore

$$
g_{1-\gamma}(y_0, \boldsymbol{y}) \leq g^1_{1-\gamma}(y_{10}, \boldsymbol{y}_1) + g^2_{1-\gamma}(y_{20}, \boldsymbol{y}_2) + g^3_{1-\gamma}(y_{30}, \boldsymbol{y}_3) + g^4_{1-\gamma}(y_{40}, \boldsymbol{y}_4) = g^i_{1-\gamma}(y_0, \boldsymbol{y}).
$$

■

3.2 Decision rule approximation of recourse

Depending on the distribution of $\tilde{\boldsymbol{z}}$, the second stage recourse decisions, $\boldsymbol{u}(\tilde{\boldsymbol{z}})$ and $\boldsymbol{y}(\tilde{\boldsymbol{z}})$ can be very large or even infinite. Moreover, since we do not specify the exact distributions of the primitive uncertainties, it would not be possible to obtain an optimal recourse decision. To enable us to formulate a tractable problem in which we could derive an upper bound of Model (14), we first adopt the linear decision rule used in Ben-Tal et al. [3] and Chen, Sim, and Sun [12]. We restrict $\boldsymbol{u}(\tilde{\boldsymbol{z}})$ and $\boldsymbol{y}(\tilde{\boldsymbol{z}})$ to be affinely dependent on the primitive uncertainties, that is

$$\boldsymbol{u}(\tilde{\boldsymbol{z}}) = \boldsymbol{u}^0 + \textstyle\sum_{j=1}^N \boldsymbol{u}^j \tilde{z}_j,\ \boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{y}^0 + \textstyle\sum_{j=1}^N \boldsymbol{y}^j \tilde{z}_j. \tag{20}$$

Under linear decision rule, the following constraint

$$\boldsymbol{T}^j\boldsymbol{x} + \boldsymbol{U}\boldsymbol{u}^j + \boldsymbol{Y}\boldsymbol{y}^j = \boldsymbol{h}^j \qquad j = 0,\ldots,N$$

is a sufficient condition to satisfy the affine constraint involving recourse variables in Model (14). Moreover, since the support of $\tilde{\boldsymbol{z}}$ is $\mathcal{W} = [-\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$, an inequality constraint $y_i(\tilde{\boldsymbol{z}}) \geq 0$ in Model (14) is the same as the robust counterpart

$$y_i^0 + \sum_{j=1}^N y_i^j z_j \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W},$$

which is representable by the following linear inequalities

$$y_i^0 \geq \sum_{j=1}^N (\underline{z}_j s_j^i + \bar{z}_j t_j^i)$$

for some $\boldsymbol{s}^i, \boldsymbol{t}^i \geq \boldsymbol{0}$ satisfying $s_j^i - t_j^i = y_i^j$, $j = 1,\ldots,N$. As for the aspiration level prospect, we let

$$\xi(\tilde{\boldsymbol{z}}) = \xi_0 + \sum_{j=1}^N \xi_j \tilde{z}_j, \tag{21}$$

where

$$\xi_j = \boldsymbol{c}^{j\prime}\boldsymbol{x} + \boldsymbol{d_u}'\boldsymbol{u}^j + \boldsymbol{d_y}'\boldsymbol{y}^j - \tau^j \qquad j = 0,\ldots,N, \tag{22}$$

so that

$$\xi(\tilde{\boldsymbol{z}}) = \boldsymbol{c}(\tilde{\boldsymbol{z}})'\boldsymbol{x} + \boldsymbol{d_u}'\boldsymbol{u}(\tilde{\boldsymbol{z}}) + \boldsymbol{d_y}'\boldsymbol{y}(\tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}}).$$

Hence, applying the bound on the CVaR measure at the objective function, we have

$$\psi_{1-\gamma}(\xi(\tilde{\boldsymbol{z}})) \leq g_{1-\gamma}(\xi_0, \boldsymbol{\xi})$$

where we use $\boldsymbol{\xi}$ to denote the vector with elements ξ_j, $j = 1,\ldots,N$. Putting these together, we solve the following problem, which is an SOCP.

$$\begin{array}{lll} Z_{LDR}(\gamma) = \min & g_{1-\gamma}(\xi_0, \boldsymbol{\xi}) & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \\ & \xi_j = \boldsymbol{c}^{j\prime}\boldsymbol{x} + \boldsymbol{d_u}'\boldsymbol{u}^j + \boldsymbol{d_y}'\boldsymbol{y}^j - \tau^j & j = 0,\ldots,N. \\ & \boldsymbol{T}^j\boldsymbol{x} + \boldsymbol{U}\boldsymbol{u}^j + \boldsymbol{Y}\boldsymbol{y}^j = \boldsymbol{h}^j & j = 0,\ldots,N. \\ & y_i^0 + \sum_{j=1}^N y_i^j z_j \geq \boldsymbol{0} & \forall \boldsymbol{z} \in \mathcal{W}, i = 1,\ldots,n_3 \\ & \boldsymbol{x} \geq 0. & \end{array} \tag{23}$$

Theorem 3 *Let $\boldsymbol{x}$, $\boldsymbol{u}^0,\ldots,\boldsymbol{u}^N$, $\boldsymbol{y}^0,\ldots,\boldsymbol{y}^N$ be the optimal solutions of Model (23). The solution $\boldsymbol{x}$ and linear decision rules $\boldsymbol{u}(\tilde{\boldsymbol{z}})$ and $\boldsymbol{y}(\tilde{\boldsymbol{z}})$ defined in the equations (20), are feasible in the subproblem (14). Moreover,*

$$Z(\gamma) \leq Z_{LDR}(\gamma).$$

Deflected linear decision rule

We adopt a similar idea as in Chen et al. [13], where they proposed *deflected linear decision rules* as an improvement over linear decision rules. Without any loss of generality, we split the matrix $\boldsymbol{Y}$ into two matrices $\boldsymbol{Y} = [\boldsymbol{V}\ \boldsymbol{W}]$. Likewise, we have $\boldsymbol{y} = \binom{\boldsymbol{v}}{\boldsymbol{w}}$, $\boldsymbol{v} \in \Re^{n_4}$, $\boldsymbol{w} \in \Re^{n_5}$, $n_5 = n_3 - n_4$ so that

$$\boldsymbol{Y}\boldsymbol{y} = \boldsymbol{V}\boldsymbol{v} + \boldsymbol{W}\boldsymbol{w}.$$

Similarly, we split the vector $\boldsymbol{d}_y = \begin{pmatrix} \boldsymbol{d}_v \\ \boldsymbol{d}_w \end{pmatrix}$ so that $\boldsymbol{d}_y'\boldsymbol{y} = \boldsymbol{d}_v'\boldsymbol{v} + \boldsymbol{d}_w'\boldsymbol{w}$. The recourse variables $\boldsymbol{v}$ and $\boldsymbol{w}$ are characterized from the feasibility of the following sets

$$X_i^w = \left\{ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} : \begin{array}{l} \boldsymbol{U}\boldsymbol{u} + \boldsymbol{V}\boldsymbol{v} + \boldsymbol{W}\boldsymbol{w} = \boldsymbol{0} \\ w_i = 1 \\ \boldsymbol{w} \geq \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0}, \boldsymbol{v} \text{ free} \end{array} \right\}$$

for all $i = 1, \ldots, n_5$ and the infeasibility of the following sets

$$X_j^v = \left\{ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} : \begin{array}{l} \boldsymbol{U}\boldsymbol{u} + \boldsymbol{V}\boldsymbol{v} + \boldsymbol{W}\boldsymbol{w} = \boldsymbol{0} \\ v_j = 1 \\ \boldsymbol{w} \geq \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0}, \boldsymbol{v} \text{ free} \end{array} \right\}$$

for all $j = 1, \ldots, n_4$. We define the vector $\bar{\boldsymbol{d}}$ with elements

$$\begin{array}{ll} \bar{d}_i = \min & \boldsymbol{d_u}'\boldsymbol{u} + \boldsymbol{d_v}'\boldsymbol{v} + \boldsymbol{d_w}'\boldsymbol{w} \\ \text{s.t.} & \boldsymbol{U}\boldsymbol{u} + \boldsymbol{V}\boldsymbol{v} + \boldsymbol{W}\boldsymbol{w} = \boldsymbol{0} \\ & w_i = 1 \\ & \boldsymbol{w} \geq \boldsymbol{0}, \boldsymbol{v} \geq \boldsymbol{0}, \boldsymbol{u} \text{ free}, \end{array}$$

and $\bar{\boldsymbol{u}}^i$, $\bar{\boldsymbol{v}}^i$ and $\bar{\boldsymbol{w}}^i$ are the corresponding optimal solutions of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$, for $i = 1, \ldots, n_5$. Note that if $\bar{d}_i < 0$, then given any feasible solution $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$, the solution $\boldsymbol{u} + \kappa\bar{\boldsymbol{u}}^i$, $\boldsymbol{v} + \kappa\bar{\boldsymbol{v}}^i$ and $\boldsymbol{w} + \kappa\bar{\boldsymbol{w}}^i$ will also be feasible, and that the objective will be reduced by $|\kappa\bar{d}_i|$. Hence, whenever a second stage decision is feasible, its objective will be unbounded from below. Therefore, it is reasonable to assume that $\bar{\boldsymbol{d}} \geq \boldsymbol{0}$.

Before we can proceed with the approximation, we introduce a new bound on $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right)$. Note that in the proposal of Chen et al. [13], the bound is derived from the covariance matrix and support of $\tilde{\boldsymbol{z}}$. We provide a tighter bound based on the model of primitive uncertainty, U, which contains richer distributional information.

Proposition 4 *Under the model of primitive uncertainty, U, the following functions are upper bounds of* $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right)$.

(a)

$$\begin{array}{rll} l^1(y_0,\boldsymbol{y}) \triangleq & \min & r^0 \\ & \text{s.t.} & r^0 \geq y_0 + \max\limits_{\boldsymbol{z}\in\mathcal{W}} (\boldsymbol{y}-\boldsymbol{r})'\boldsymbol{z} \\ & & r^0 \geq \max\limits_{\boldsymbol{z}\in\mathcal{W}} (-\boldsymbol{r}'\boldsymbol{z}) \\ = & \min & r^0 \\ & \text{s.t.} & r^0 \geq y_0 + \boldsymbol{s}'\overline{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}} \\ & & r^0 \geq \boldsymbol{d}'\overline{\boldsymbol{z}} + \boldsymbol{h}'\underline{\boldsymbol{z}} \\ & & \boldsymbol{s}-\boldsymbol{t} = \boldsymbol{y}-\boldsymbol{r} \\ & & \boldsymbol{d}-\boldsymbol{h} = -\boldsymbol{r} \\ & & \boldsymbol{s},\boldsymbol{t},\boldsymbol{d},\boldsymbol{h} \geq \boldsymbol{0}. \end{array}$$

Moreover, the bound is tight whenever $y_0 + \boldsymbol{y}'\boldsymbol{z} \geq 0$ *or* $y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0$ *for all* $\boldsymbol{z} \in \mathcal{W}$.

(b)

$$l^2(y_0,\boldsymbol{y}) \triangleq \tfrac{1}{2}y_0 + \tfrac{1}{2}\sqrt{{y_0}^2 + \boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}.$$

(c)

$$l^3(y_0,\boldsymbol{y}) \triangleq \begin{cases} \inf_{\mu>0}\left\{\frac{\mu}{e}\exp\left(\frac{y_0}{\mu} + \frac{\|\boldsymbol{u}\|_2^2}{2\mu^2}\right)\right\} & \textit{if } y_j = 0\ \forall j = I+1,\dots,N \\ +\infty & \textit{otherwise} \end{cases}$$

where $u_j = \max\{p_j y_j, -q_j y_j\}$.

(d)

$$l^4(y_0,\boldsymbol{y}) \triangleq \begin{cases} y_0 + \inf_{\mu>0}\left\{\frac{\mu}{e}\exp\left(-\frac{y_0}{\mu} + \frac{\|\boldsymbol{v}\|_2^2}{2\mu^2}\right)\right\} & \textit{if } y_j = 0\ \forall j = I+1,\dots,N \\ +\infty & \textit{otherwise} \end{cases}$$

where $v_j = \max\{-p_j y_j, q_j y_j\}$.

Proof :

(a) We consider $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right)$ as a two stage stochastic optimization problem. Then

$$\begin{array}{rll} \mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) = & \min & \mathrm{E}(r(\tilde{\boldsymbol{z}})) \\ & \text{s.t.} & r(\tilde{\boldsymbol{z}}) \geq y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \\ & & r(\tilde{\boldsymbol{z}}) \geq 0 \end{array}$$

To obtain an upper bound, we restrict $r(\tilde{\boldsymbol{z}})$ to follow linear decision rule, that is, $r(\tilde{\boldsymbol{z}}) = r_0 + \sum_{j=1}^N r_j\tilde{z}_j$. Hence,

$$\begin{array}{rll} \mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \leq & \min & r_0 \\ & \text{s.t.} & r_0 \geq y_0 + \max\limits_{\boldsymbol{z}\in\mathcal{W}} (\boldsymbol{y}-\boldsymbol{r})'\boldsymbol{z} \\ & & r_0 \geq \max\limits_{\boldsymbol{z}\in\mathcal{W}} (-\boldsymbol{r}'\boldsymbol{z}) \\ = & \min & r_0 \\ & \text{s.t.} & r_0 \geq y_0 + \boldsymbol{s}'\overline{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}} \\ & & r_0 \geq \boldsymbol{d}'\overline{\boldsymbol{z}} + \boldsymbol{h}'\underline{\boldsymbol{z}} \\ & & \boldsymbol{s}-\boldsymbol{t} = \boldsymbol{y}-\boldsymbol{r} \\ & & \boldsymbol{d}-\boldsymbol{h} = -\boldsymbol{r} \\ & & \boldsymbol{s},\boldsymbol{t},\boldsymbol{d},\boldsymbol{h} \geq \boldsymbol{0}, \end{array}$$

where the last equality follows from strong LP duality, with $\mathcal{W} = [-\underline{\boldsymbol{z}}, \overline{\boldsymbol{z}}]$.
If $y_0 + \boldsymbol{y}'\boldsymbol{z} \geq 0,\ \forall \boldsymbol{z} \in \mathcal{W}$, then $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) = y_0$. Moreover, $y_0 + (\boldsymbol{y} - \boldsymbol{r})'\boldsymbol{z} \geq -\boldsymbol{r}'\boldsymbol{z}$ for all $\boldsymbol{z} \in \mathcal{W}$. Hence,

$$\begin{array}{rcll}
\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) & \leq & l^1(y_0, \boldsymbol{y}) & \\
& = & \min\limits_{\boldsymbol{r}} \left\{ y_0 + \max\limits_{\boldsymbol{z} \in \mathcal{W}} (\boldsymbol{y} - \boldsymbol{r})'\boldsymbol{z} \right\} & \\
& \leq & y_0 & \text{Choosing } \boldsymbol{r} = \boldsymbol{y} \\
& = & \mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right). &
\end{array}$$

Similarly, if $y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0,\ \forall \boldsymbol{z} \in \mathcal{W}$ then $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) = 0$. Moreover, we have $y_0 + (\boldsymbol{y} - \boldsymbol{r})'\boldsymbol{z} \leq -\boldsymbol{r}'\boldsymbol{z}$ for all $\boldsymbol{z} \in \mathcal{W}$. Hence,

$$\begin{array}{rcll}
\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) & \leq & l^1(y_0, \boldsymbol{y}) & \\
& = & \min\limits_{\boldsymbol{r}} \left\{ \max\limits_{\boldsymbol{z} \in \mathcal{W}} (-\boldsymbol{r}'\boldsymbol{z}) \right\} & \\
& \leq & 0 & \text{Choosing } \boldsymbol{r} = \boldsymbol{0} \\
& = & \mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right). &
\end{array}$$

(b),(c) and **(d)** The derivation of these bounds can be found within the proofs of Proposition 3(b), (c) and (d) respectively. ■

Theorem 4 *Let*

$$\begin{array}{rll}
l(y_0, \boldsymbol{y}) \triangleq & \min & l^1(y_{10}, \boldsymbol{y}_1) + l^2(y_{20}, \boldsymbol{y}_2) + l^3(y_{30}, \boldsymbol{y}_3) + l^4(y_{40}, \boldsymbol{y}_4) \\
& \text{s.t.} & y_{10} + y_{20} + y_{30} + y_{40} = y_0 \\
& & \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 = \boldsymbol{y} \\
& & y_{0i}, \boldsymbol{y}_i \ \textit{free} \qquad i = 1, 2, 3, 4.
\end{array}$$

Then for all $(y_0, \boldsymbol{y})$

$$\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \leq l(y_0, \boldsymbol{y}) \leq \min\{l^1(y_0, \boldsymbol{y}), l^2(y_0, \boldsymbol{y}), l^3(y_0, \boldsymbol{y}), l^4(y_0, \boldsymbol{y})\} \tag{24}$$

The constraint $l(y_0,\boldsymbol{y}) \le s$ can be expressed as follows:

$$
\begin{array}{ll}
\exists \delta_i, y_{0i} \in \Re, \boldsymbol{y}_i, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{td}, \boldsymbol{h} \in \Re^N, i=1,\ldots,4, \boldsymbol{u},\boldsymbol{v} \in \Re^I, \text{ such that} & \\
\delta_1+\delta_2+\delta_3+\delta_4 \le s & \\
y_{10} + \boldsymbol{s}'\bar{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}} \le \delta_1 & \\
\boldsymbol{d}'\bar{\boldsymbol{z}} + \boldsymbol{h}'\underline{\boldsymbol{z}} \le \delta_1 & \\
\boldsymbol{s},\boldsymbol{t},\boldsymbol{d},\boldsymbol{h} \ge 0 & \\
\boldsymbol{s}-\boldsymbol{t} = \boldsymbol{y}_1 - \boldsymbol{r} & \\
\boldsymbol{d}-\boldsymbol{h} = -\boldsymbol{r} & \\
y_{20} + \frac{1}{2}\|(y_{20}, \boldsymbol{\Sigma}^{1/2}\boldsymbol{y}_2)\|_2 \le \delta_2 & \\
\inf_{\mu>0} \frac{\mu}{e}\exp\left(\frac{y_0}{\mu} + \frac{\|\boldsymbol{u}\|_2^2}{2\mu^2}\right) \le \delta_3 & \\
u_j \ge p_j y_{3j}, u_j \ge -q_j y_{3j} & \forall j=1,\ldots,I \\
y_{3j} = 0 & \forall j = I+1,\ldots,N \\
y_0 + \inf_{\mu>0} \frac{\mu}{e}\exp\left(-\frac{y_0}{\mu} + \frac{\|\boldsymbol{v}\|_2^2}{2\mu^2}\right) \le \delta_4 & \\
v_j \ge q_j y_{4j}, v_j \ge -p_j y_{4j} & \forall j=1,\ldots,I \\
y_{4j} = 0 & \forall j = I+1,\ldots,N \\
y_{10}+y_{20}+y_{30}+y_{40} = y_0 & \\
\boldsymbol{y}_1+\boldsymbol{y}_2+\boldsymbol{y}_3+\boldsymbol{y}_4 = \boldsymbol{y}. &
\end{array}
\qquad (25)
$$

Proof : We note the subadditivity of the functional $\mathrm{E}((\cdot)^+)$, and that $l^i(0,\boldsymbol{0}) = 0$, $i=1,\ldots,4$. Therefore, the proof is similar to the proof of Theorem 2. ■

Remark : Due to the presence of the constraint, $\inf_{\mu>0} \mu \exp\left(\frac{a}{\mu} + \frac{b^2}{\mu^2}\right) \le c$, the set of constraints in (25) is not exactly second order cone representable. Fortunately, using a few number second order cones, we can accurately approximate such constraint to within the precision of the solver. We present the second order cone approximation in the Appendix.

Next, we present the model that achieves a better bound than Model (23).

$$
\begin{array}{lll}
Z_{DLDR}(\gamma) = \min & g_{1-\gamma}(\xi_0, \boldsymbol{\xi}) + \sum_{i=1}^{n_5}\Big(r_i + \frac{1}{\gamma} l(-w_i^0 - r_i, -\boldsymbol{w}_i)\Big)\bar{d}_i & \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \\
& \xi_j = \boldsymbol{c^j}'\boldsymbol{x} + \boldsymbol{d_u}'\boldsymbol{u}^j + \boldsymbol{d_v}'\boldsymbol{v}^j + \boldsymbol{d_w}'\boldsymbol{w}^j - \tau^j & j = 0,\ldots,N. \\
& \boldsymbol{T}^j\boldsymbol{x} + \boldsymbol{U}\boldsymbol{u}^j + \boldsymbol{V}\boldsymbol{v}^j + \boldsymbol{W}\boldsymbol{w}^j = \boldsymbol{h}^j & j = 0,\ldots,N. \\
& v_i^0 + \sum_{j=1}^N v_i^j z_j \ge 0 & \forall \boldsymbol{z} \in \mathcal{W}, i = 1,\ldots,n_4 \\
& \boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{r} \ge \boldsymbol{0} &
\end{array}
\qquad (26)
$$

Theorem 5 *Let $\boldsymbol{x}$, $\boldsymbol{u}^0,\ldots,\boldsymbol{u}^N$, $\boldsymbol{v}^0,\ldots,\boldsymbol{v}^N$, $\boldsymbol{w}^0,\ldots,\boldsymbol{w}^N$ be the optimal solutions of Model (26). The solution $\boldsymbol{x}$ and the corresponding* deflected linear decision rule

$$
\begin{array}{rcl}
\boldsymbol{u}(\tilde{\boldsymbol{z}}) & = & \boldsymbol{u}^0 + \sum_{j=1}^N \boldsymbol{u}^j \tilde{z}_j + \sum_{i=1}^{n_5} \bar{\boldsymbol{u}}^i (w_i^0 + \boldsymbol{w}_i'\tilde{\boldsymbol{z}})^- \\
\boldsymbol{y}(\tilde{\boldsymbol{z}}) & = & \begin{pmatrix}\boldsymbol{v}^0 \\ \boldsymbol{w}^0\end{pmatrix} + \sum_{j=1}^N \begin{pmatrix}\boldsymbol{v}^j \\ \boldsymbol{w}^j\end{pmatrix} \tilde{z}_j + \sum_{i=1}^{n_5} \begin{pmatrix}\bar{\boldsymbol{v}}^i \\ \bar{\boldsymbol{w}}^i\end{pmatrix} (w_i^0 + \boldsymbol{w}_i'\tilde{\boldsymbol{z}})^-,
\end{array}
\qquad (27)
$$

in which $\boldsymbol{w}_i$ denotes the vector with elements w_i^j, $j = 1, \ldots, N$, is feasible in the subproblem (14). Moreover,

$$Z(\gamma) \leq Z_{DLDR}(\gamma) \leq Z_{LDR}(\gamma).$$

Proof : Noting that

$$\boldsymbol{U}\bar{\boldsymbol{u}}^i + \boldsymbol{V}\bar{\boldsymbol{v}}^i + \boldsymbol{W}\bar{\boldsymbol{w}}^i = \boldsymbol{0},$$

it is straight forward to verify that the recourse with deflected linear decision rule satisfies the affine constraints in Model (14). Besides, since $\bar{\boldsymbol{v}}^j \geq \boldsymbol{0}$, $v_i^0 + \sum_{j=1}^N v_i^j \tilde{z}_j \geq 0$, $\bar{\boldsymbol{w}}^i \geq \boldsymbol{0}$ and $\bar{w}_i^i = 1$, the nonnegative constraints

$$\boldsymbol{y}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0}$$

are also satisfied. To show the bound, $Z(\gamma) \leq Z_{DLDR}(\gamma)$, we note that $\bar{d}_i = \boldsymbol{d}_u\bar{\boldsymbol{u}}_i + \boldsymbol{d}_v\bar{\boldsymbol{v}}_i + \boldsymbol{d}_w\bar{\boldsymbol{w}}_i$. Under the deflected linear decision rule, the aspiration level prospect becomes

$$\begin{aligned} & \boldsymbol{c}(\tilde{\boldsymbol{z}})'\boldsymbol{x} + \boldsymbol{d_u}'\boldsymbol{u}(\tilde{\boldsymbol{z}}) + \boldsymbol{d_y}'\boldsymbol{y}(\tilde{\boldsymbol{z}}) - \tau(\tilde{\boldsymbol{z}}) \\ = \; & \xi(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{n_5} \bar{d}_i (w_i^0 + \boldsymbol{w}_i'\tilde{\boldsymbol{z}})^-, \end{aligned}$$

where $\xi(\tilde{\boldsymbol{z}})$ is defined in Equations (21) and (22). We now evaluate the objective of Model (14) under the deflected linear decision rule as follows:

$$\begin{aligned} & \psi_{1-\gamma}\Big(\xi(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{n_5} \bar{d}_i (w_i^0 + \boldsymbol{w}_i'\tilde{\boldsymbol{z}})^-\Big) \\ = \; & \min_{\omega} \left\{ \omega + \frac{1}{\gamma}\mathrm{E}\left(\Big(\xi(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{n_5} \bar{d}_i (w_i^0 + \boldsymbol{w}_i'\tilde{\boldsymbol{z}})^- - \omega \Big)^+ \right) \right\} \\ = \; & \min_{\omega} \min_{\boldsymbol{r} \geq \boldsymbol{0}} \left\{ \omega + \frac{1}{\gamma}\mathrm{E}\left(\left(\xi(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{n_5} \bar{d}_i \left(r_i + (-w_i^0 - r_i - \boldsymbol{w}_i'\tilde{\boldsymbol{z}})^+ \right) - \omega \right)^+ \right) \right\} \\ \leq \; & \min_{\omega} \min_{\boldsymbol{r} \geq \boldsymbol{0}} \left\{ \omega + \frac{1}{\gamma}\mathrm{E}\left(\Big(\xi(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{n_5} \bar{d}_i r_i - \omega \Big)^+ \right) + \sum_{i=1}^{n_5} \frac{1}{\gamma}\mathrm{E}\left(\Big(-w_i^0 - r_i - \boldsymbol{w}_i'\tilde{\boldsymbol{z}} \Big)^+ \right) \bar{d}_i \right\} \\ = \; & \min_{\omega} \left\{ \omega + \frac{1}{\gamma}\mathrm{E}\left((\xi(\tilde{\boldsymbol{z}}) - \omega)^+ \right) \right\} + \min_{\boldsymbol{r} \geq \boldsymbol{0}} \left\{ \sum_{i=1}^{n_5} \left(r_i + \frac{1}{\gamma}\mathrm{E}\left(\Big(-w_i^0 - r_i - \boldsymbol{w}_i'\tilde{\boldsymbol{z}} \Big)^+ \right) \right) \bar{d}_i \right\} \\ = \; & \psi_{1-\gamma}\left(\xi(\tilde{\boldsymbol{z}})\right) + \min_{\boldsymbol{r} \geq \boldsymbol{0}} \left\{ \sum_{i=1}^{n_5} \left(r_i + \frac{1}{\gamma}\mathrm{E}\left(\Big(-w_i^0 - r_i - \boldsymbol{w}_i'\tilde{\boldsymbol{z}} \Big)^+ \right) \right) \bar{d}_i \right\} \\ \leq \; & g_{1-\gamma}(\xi_0, \boldsymbol{\xi}) + \min_{\boldsymbol{r} \geq \boldsymbol{0}} \left\{ \sum_{i=1}^{n_5} \left(r_i + \frac{1}{\gamma} l(-w_i^0 - r_i, -\boldsymbol{w}_i) \right) \bar{d}_i \right\} \\ = \; & Z_{DLDR}(\gamma), \end{aligned} \tag{28}$$

where the second equality and the first inequality are due to $(x + a)^+ \leq (x)^+ + a$, for all $a \geq 0$, and that $\bar{\boldsymbol{d}} \geq \boldsymbol{0}$. The third equality is due to the translation invariance of the CVaR measure. The last inequality is due to Theorems 2 and 4.

To prove the improvement over Model (23), we now consider an optimal solution of Model (23), $\boldsymbol{x}$, $\boldsymbol{u}^0, \ldots, \boldsymbol{u}^N$, $\boldsymbol{y}^0, \ldots, \boldsymbol{y}^N$. Accordingly, we separate $\boldsymbol{y}^0, \ldots, \boldsymbol{y}^N$ to the respective variables, $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^N$ and $\boldsymbol{w}^0, \ldots, \boldsymbol{w}^N$. Clearly, the solution, together with $\boldsymbol{r} = \boldsymbol{0}$ are feasible in the constraints of Model

(26). From Proposition 4 and Theorem 4, the constraint $w_i^0 + \sum_{j=1}^N w_i^j z_j \geq 0, \ \forall \boldsymbol{z} \in \mathcal{W}$ enforced in of Model (23) ensures that

$$0 \leq l(-w_i^0, -\boldsymbol{w}_i) \leq l^1(-w_i^0, -\boldsymbol{w}_i) = 0.$$

Therefore, the solution of Model (23) and $\boldsymbol{r} = \boldsymbol{0}$ yield the same objective as Model (26). Hence, $Z_{DLDR}(\gamma) \leq Z_{LDR}(\gamma)$. ■

4 Computation Studies

We apply the goal driven optimization model to a project management problem with uncertainty activity completion times. Project management is a well known problem which can be described with a directed graph having m arcs and n nodes. The arc set is denoted as $\mathcal{E}$, $|\mathcal{E}| = m$. Each arc (i,j) represents an activity which has uncertain completion time $\tilde{t}_{ij}$. It is affinely dependent on the additional amount of resource $x_{ij} \in [0, \bar{x}_{ij}]$ and a primitive uncertainty $\tilde{z}_{ij}$, as follows:

$$\tilde{t}_{ij} = (1 + \tilde{z}_{ij})b_{ij} - a_{ij}x_{ij}$$

where $\tilde{z}_{ij} \in [-\underline{z}_{ij}, \overline{z}_{ij}], \underline{z}_{ij} \leq 1, \ (i,j) \in \mathcal{E}$ is an independent random variable with zero mean, standard deviation σ_{ij}, forward and backward deviations, p_{ij} and q_{ij} respectively. The completion time adheres to precedent constraints. For instance, activity e_1 precedes activity e_2 if activity e_1 must be completed before activity e_2. Each node on the graph represents an event marking the completion of a particular subset of activities. For simplicity, we use node 1 as the start event and node n as the end event. The cost of using each unit of resource on activity (i,j) is c_{ij} and the total cost is limited to a budget B. Our goal is to find a resource allocation to each activity $(i,j) \in \mathcal{E}$ that maximize the shortfall aspiration level criterion in achieving a fixed targeted completion time, τ. We formulate the goal driven optimization model as follows:

$$\begin{array}{ll}
\max & \alpha(u_n(\tilde{\boldsymbol{z}}) - \tau) \\
& u_j(\tilde{\boldsymbol{z}}) - u_i(\tilde{\boldsymbol{z}}) - w_{ij}(\tilde{\boldsymbol{z}}) = (1 + \tilde{z}_{ij})b_{ij} - a_{ij}x_{ij} \quad \forall (i,j) \in \mathcal{E} \\
& u_1(\tilde{\boldsymbol{z}}) = 0 \\
& \boldsymbol{c}'\boldsymbol{x} \leq B \\
& \boldsymbol{0} \leq \boldsymbol{x} \leq \overline{\boldsymbol{x}}, \boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0} \\
& \boldsymbol{x} \in \Re^m, \ \boldsymbol{u}(\cdot), \boldsymbol{w}(\cdot) \in \mathcal{Y},
\end{array} \tag{29}$$

where $u_i(\tilde{\boldsymbol{z}})$ is the second stage decision vector, representing the completion time at node i when the uncertain parameters $\tilde{\boldsymbol{z}}$ are realized. The recourse $w_{ij}(\tilde{\boldsymbol{z}})$ represents the slack at the arc (i,j). Using Algorithm 1, we reduce the problem (29) to solving a sequence of subproblems in the form of stochastic optimization problems with CVaR objectives. Since the project management problem has complete recourse, accordingly, we use sampling approximation to obtain solutions to the subproblem as follows:

$$\begin{array}{rlll}\tilde{Z}^s_K(\gamma) = & \min & \omega + \frac{1}{\gamma K}\sum_{k=1}^K t_k & \\ & \text{s.t.} & t_k \geq u^k_n - \tau - \omega & \forall k = 1,\ldots,K \\ & & u^k_j - u^k_i \geq (1+\tilde{z}^k_{ij})b_{ij} - a_{ij}x_{ij} & \forall (i,j)\in\mathcal{E}, k=1,\ldots,K \\ & & u^k_1 = 0 & \forall k=1,\ldots,K \\ & & \boldsymbol{c}'\boldsymbol{x} \leq B & \\ & & \boldsymbol{t} \geq \boldsymbol{0}, \boldsymbol{0} \leq \boldsymbol{x} \leq \overline{\boldsymbol{x}} & \\ & & \boldsymbol{x} \in \Re^m,\ \boldsymbol{u}\in\Re^{n\times K},\ \boldsymbol{t}\in\Re^K & \end{array} \tag{30}$$

where $\tilde{\boldsymbol{z}}^1,\ldots,\tilde{\boldsymbol{z}}^K$ are K independent samples of $\tilde{\boldsymbol{z}}$. We use the same samples throughout the iterations of Algorithm 1.

To derive a deterministic approximation of Model (26), we note that the following linear program

$$\begin{array}{lll}\bar{d}_{ij} = \min & u_n & \\ \text{s.t.} & u_j - u_i - w_{ij} = 0 & \forall (i,j) \\ & u_1 = 0, w_{ij} = 1 & \\ & \boldsymbol{w} \geq 0, \boldsymbol{u}\in\Re^n, \boldsymbol{w}\in\Re^m. & \end{array}$$

achieves the optimum value at $\bar{d}_{ij} = 1$. Accordingly, we formulate the deterministic approximation of the subproblem as follows:

$$\begin{array}{rlll}Z^d(\gamma) = & \min & g_{1-\gamma}(u^0_n - \tau, \boldsymbol{u}_n) + \sum_{(i,j)\in\mathcal{E}}\left(r_{ij} + \frac{1}{\gamma}l(-w^0_{ij} - r_{ij}, -\boldsymbol{w}_{ij})\right) & \\ & \text{s.t.} & u^0_j - u^0_i - b_{ij} + a_{ij}x_{ij} - w^0_{ij} = 0 & \forall (i,j)\in\mathcal{E} \\ & & u^{kl}_j - u^{kl}_i - b_{ij} + a_{ij}x_{ij} - w^{kl}_{ij} = 0 & \forall (i,j),(k,l)\in\mathcal{E} \\ & & u^0_1 = 0, u^{kl}_1 = 0 & \forall (k,l)\in\mathcal{E} \\ & & \boldsymbol{c}'\boldsymbol{x} \leq B & \\ & & \boldsymbol{0}\leq\boldsymbol{x}\leq\overline{\boldsymbol{x}},\ \boldsymbol{r}\geq 0. & \end{array} \tag{31}$$

We formulate Model (31) using an in-house developed software, *PROF* (Platform for Robust Optimization Formulation). The Matlab based software is essentially an SOCP modeling environment that contains reusable functions for modeling multiperiod robust optimization using decision rules. We have implemented bounds for the CVaR measure and expected positivity of a weighted sum of random variables. The software calls upon CPLEX 9.1 to solve the underlying SOCP.

We use the fictitious project introduced in Chen, Sim and Sun [12] as an experiment. We create a 6 by 4 grid (See Figure 2) as the activity network. There are in total 24 nodes and 38 arcs in the activity network. The first node lies at the bottom left corner and the last node lies at the right upper corner. Each arc proceeds either towards the right node or the upper node. Every activity $(i,j)\in\mathcal{E}$ has independent and identically distributed completion time with distribution at

$$\mathrm{P}(\tilde{z}_{ij} = z) = \begin{cases} 0.9 & \text{if } z = -25/900 \\ 0.1 & \text{if } z = 25/100. \end{cases}$$

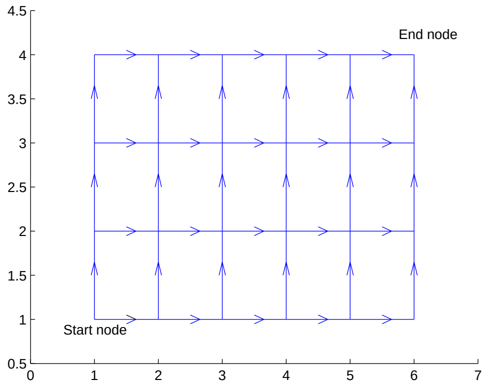

Figure 2: Activity grid 6 by 4

From the distribution of each arc, we can easily calculate the support and deviation information, that is, $\underline{z}_{ij} = 25/900$, $\overline{z}_{ij} = 25/100$, $\sigma_{ij} = 0.0833$, $p_{ij} = 0.1185$, $q_{ij} = 0.0833$. For all activities, we let $a_{ij} = c_{ij} = 1$, $\overline{x}_{ij} = 24$ and $b_{ij} = 100$. We choose an aspiration level of $\tau = 800$. The total cost of resource is kept under the budget B. We compare the performance of the sampling approximation model (30) against the deterministic approximation model (31). After deciding the allocation of the resource, we use $M = 500,000$ samples to obtain a sampled distribution of the actual completion time $u_n^1, \ldots, u_n^M$. Using these samples we determine the sampled shortfall aspiration level criterion as follows:

$$\hat{\alpha} = 1 - \inf_{a>0} \frac{1}{aM} \sum_{k=1}^{M} (u_n^M - \tau + a)^+.$$

We denote $\hat{\alpha}_K^s$ as the sampled shortfall aspiration level criterion when Model (30) is used to approximate the subproblem. Likewise, we denote $\hat{\alpha}^d$ as the sampled shortfall aspiration level criterion when Model (31) in the approximation. By adjusting the budget level, B from 240 to 640, we show the results in Table 1 and Figure 3. In both the deterministic and the sampling approximations, we observe that the shortfall aspiration level criterion decreases with increasing budget levels.

It is evident that when the number of samples are limited, sampling approximation can perform poorly. Moreover, due to the variability of sampling approximation, the performance does not necessarily improve with more samples; see Table 1 with $B = 440, 560, 600, 640$. We note that despite the modest distributional assumption and the non-optimal recourse, the performance of the deterministic approximation is rather comparable with the performance of the sampling approximation where sufficient number of samples are used.

5 Conclusions

We propose a new framework for modeling stochastic optimization problem that takes into account of an aspiration level. We also introduce the shortfall aspiration level criterion, which factors into the

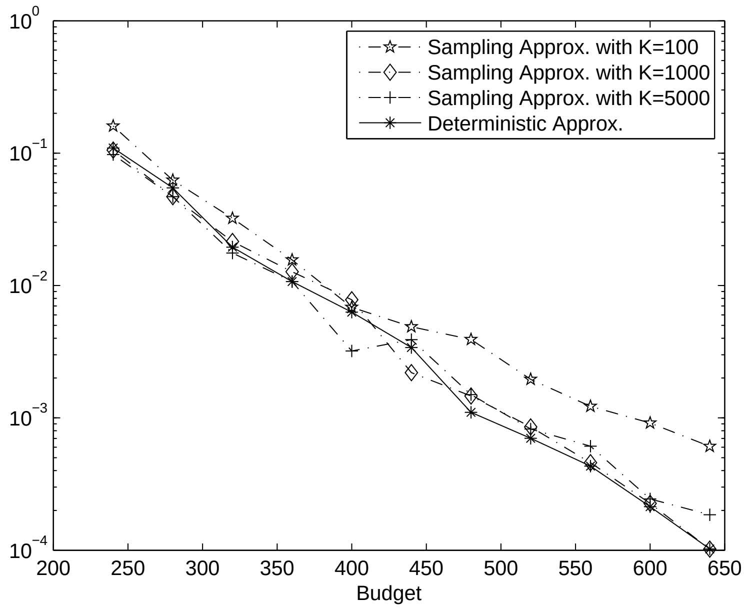

Figure 3: Comparison of the deterministic and sampling models

B	$1-\hat{\alpha}_K^d$	$1-\hat{\alpha}_{100}^s$	$1-\hat{\alpha}_{1000}^s$	$1-\hat{\alpha}_{5000}^s$
240	0.1094	0.1602	0.1055	0.0977
280	0.0547	0.0625	0.0469	0.0469
320	0.0195	0.0322	0.0215	0.0176
360	0.0107	0.0156	0.0127	0.0107
400	0.0063	0.0068	0.0078	0.0032
440	0.0034	0.0049	0.0022	0.0039
480	0.0011	0.0039	0.0015	0.0015
520	7.02×10^{-4}	0.0020	8.54×10^{-4}	8.24×10^{-4}
560	4.31×10^{-4}	0.0012	4.58×10^{-4}	6.10×10^{-4}
600	2.14×10^{-4}	9.16×10^{-4}	2.26×10^{-4}	2.44×10^{-4}
640	1.02×10^{-4}	6.10×10^{-4}	1.02×10^{-4}	1.86×10^{-4}

Table 1: Comparison of the deterministic and sampling models.

success probability and the adversity of under-performance. Moreover, the goal driven optimization model that maximizes the shortfall aspiration level criteria is analytically tractable.

We also propose two methods of solving the goal driven optimization problem, one using sampling approximations, while the other using deterministic approximations. Although the exposition in this paper is confined to a two period model, the deterministic approximation via decision rule can easily be extended to multiperiod modeling; see for instance Chen et al. [13]. This has immense advantage over sampling approximation.

A Approximation of a conic exponential quadratic constraint

Our aim to is show that the following conic exponential quadratic constraint,

$$\mu \exp\left(\frac{a}{\mu} + \frac{b^2}{\mu^2}\right) \leq c$$

for some $\mu > 0, a, b$ and c, can be approximately represented in the form of second order cones. Note with $\mu > 0$, the constraint

$$\mu \exp\left(\frac{a}{\mu} + \frac{b^2}{\mu^2}\right) \leq c$$

is equivalent to

$$\mu \exp\left(\frac{x}{\mu}\right) \leq c$$

for some variables x and d satisfying

$$\begin{aligned} &b^2 \leq \mu d \\ &a + d \leq x. \end{aligned}$$

To approximate the conic exponential constraint, we use the method described in Ben-Tal and Nemirovski [4]. Using Taylor's series expansion, we have

$$\exp(x) = \exp\left(\frac{x}{2^L}\right)^{2^L} \approx \left(1 + \frac{x}{2^L} + \frac{1}{2}\left(\frac{x}{2^L}\right)^2 + \frac{1}{6}\left(\frac{x}{2^L}\right)^3 + \frac{1}{24}\left(\frac{x}{2^L}\right)^4\right)^{2^L}.$$

Using the approximation, the following constraint

$$\mu\left(1 + \frac{x/\mu}{2^L} + \frac{1}{2}\left(\frac{x/\mu}{2^L}\right)^2 + \frac{1}{6}\left(\frac{x/\mu}{2^L}\right)^3 + \frac{1}{24}\left(\frac{x/\mu}{2^L}\right)^4\right)^{2^L} \leq c$$

is equivalent to

$$\mu\left(\frac{1}{24}\left(23 + 20\frac{x/\mu}{2^L} + 6\left(\frac{x/\mu}{2^L}\right)^2 + \left(1 + \frac{x/\mu}{2^L}\right)^4\right)\right)^{2^L} \leq c,$$

which is equivalent to the following set of constraints

$$\begin{aligned} &y = \tfrac{x}{2^L} \\ &z = \mu + \tfrac{x}{2^L} \\ &y^2 \leq \mu f, \quad z^2 \leq \mu g, \quad g^2 \leq \mu h \\ &\tfrac{1}{24}(23\mu + 20y + 6f + h) \leq v_1 \\ &v_i^2 \leq \mu v_{i+1} && \forall i = 1, \ldots, L-1 \\ &v_L^2 \leq \mu c \end{aligned}$$

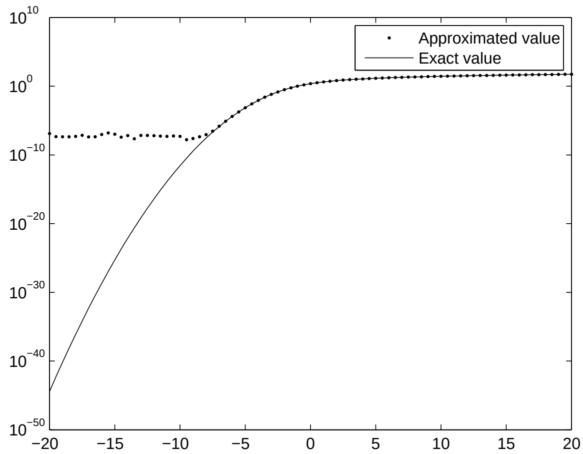

Figure 4: Evaluation of approximation of $\inf_{\mu>0} \mu \exp\left(\frac{a}{\mu} + \frac{1}{\mu^2}\right)$.

for some variables $y, z \in \Re, f, g, h \in \Re_+, \boldsymbol{v} \in \Re_+^L$. Finally, using the well known result that

$$w^2 \leq st, \quad s, t \geq 0$$

is second order cone representable as

$$\left\| \begin{bmatrix} w \\ (s-t)/2 \end{bmatrix} \right\|_2 \leq \frac{s+t}{2},$$

we obtain an approximation of the conic exponential quadratic constraint that is second order cone representable.

To test the approximation, we plot in Figure 4, the exact and approximated values of the function $f(a)$ defined as follows:

$$f(a) = \inf_{\mu>0} \mu \exp\left(\frac{a}{\mu} + \frac{1}{\mu^2}\right).$$

We obtain the exact solution by substituting $\mu^* = \frac{a+\sqrt{a^2+8}}{2}$ and the approximated solution by solving the SOCP approximation with $L = 4$. We solve the SOCP using CPLEX 9.1, with precision level of 10^{-7}. The relative errors for $a \geq -3$ is less than 10^{-7}. The approximation is poor when the actual value of $f(a)$ falls below the precision level, which is probably not a major concern in practice.

References

[1] Ahmed, S. (2006): Convexity and decomposition of mean-risk stochastic programs, *Mathematical Progamming*, 106, 433-446.

[2] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D. (1999): Coherent Risk Measures, *Mathematical Finance*, 9(3), 203-228.

[3] Ben-Tal, A., A. Goryashko, E. Guslitzer and A. Nemirovski. (2002): Adjusting Robust Solutions of Uncertain Linear Programs, *Working Paper*, Technion.

[4] Ben-Tal, A., Nemirovski, A. (2001): Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, *MPR-SIAM Series on Optimization*, SIAM, Philadelphia.

[5] Bereanu, B. (1964): Programme de risque minimal en programmation linéaire stochastique, *C.R. Acad. Sci. (Paris)*, tom 259, nr. 5, 981-983.

[6] Bordley, R., LiCalzi, M. (2000): Decision analysis using targets instead of utility functions. *Decisions in Economics anf Finance*, 23, 53-74.

[7] Birge, J.R., Louveaux, F. (1997): Introduction to Stochastic Programming, *Springer*, New York.

[8] Canada, J.R., Sullivan, W.G., Kulonda, D.J., White, J.A. (2005): Capital investment analysis for engineering and management, *Prentice Hall* 3rd ed.

[9] Castagnoli, E., R., LiCalzi, M. (1996): Expected utility without utility, *Theory and Decision*, 41, 281-301.

[10] Charnes, A.,Cooper, W. W., Symonds, G. H. (1958): Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil, *Management Science* 4, 235-263.

[11] Charnes, A.,Cooper, W. W. (1963): Deterministic equivalents for optimizing and satisficing under chance constraints, *Operations research*, 11, 18-39.

[12] Chen, X., Sim, M., Sun, P. (2005): A robust optimization perspective of stochastic programming, *Working Paper*, National University of Singapore.

[13] Chen, X., Sim, M., Sun, P, Zhang, J. (2006): A tractable approximation of stochastic programming via robust optimization *Working Paper, Revised June 2006*, National University of Singapore.

[14] Diecidue, E., van de Ven, J. Aspiration level, Probability of Success, and Expected Utility. INSEAD working paper.

[15] Dragomirescu, M, (1972): An algorithm for the minimum-risk problem of stochastic programming, *Operations Research*, 20(1), 154-164.

[16] Dyer, M., Stougie, L. (2005): Computational complexity of stochastic programming problems, *Mathematical Programming*, to appear.

[17] Lau, H.S., Lau, A.H.L., (1988): Maximizing the probability of achieving a target profit in a two-product newsboy problem, *Decision Sciences*, 19, 392-408.

[18] Lanzillotti, R.F., (1958): Pricing objectives in large companies, *American Economic Review*, 48,921-940.

[19] Li, J., , H.S., Lau, A.H.L., (1991): A two-product newsboy problem with satisficing objective and independent exponential demands, *IIE Transactions*, 23(1), 29-39.

[20] Mao, J. C. T. (1970): Survey of Capital Budgeting: Theory and Practice, *Journal of Finance*, 25, 349-360.

[21] Nemirovski, A. Shapiro, A., (2004): Convex approximation of chance constrained programs, *Working Paper*, Georgia Tech, Atlanta.

[22] Ogryczak, W., Ruszczynski, A., (2002): Dual stochastic dominance and related mean-risk model, *SIAM Journal of Optimization*, 13, 60-78.

[23] Payne, J.W., Laughhunn, D.J. and Crum, R., (1980): Tanslation of gambles and aspiration level effects in risky choice behaviour, *Management Science*, 26, 1039-1060.

[24] Payne, J.W., Laughhunn, D.J. and Crum, R., (1980): Further tests of aspiration level effects in risky choice behaviour, *Management Science*, 27, 953-958.

[25] Parlar, M. and Weng, K. (2003): Balancing desirable but conflicting objectives in the newsvendor problem, *IIE Transactions*, 35, 131142.

[26] Riis, M., Schultz, R., (2003): Applying the minimum risk criterion in stochastic recourse programs, *Computational Optimization and Applications*, 24, 267-287.

[27] Rockafellar, R.T., Uryasev, S., (2002): Conditional value-at-risk for general loss distributuions, *Journal of banking and finance*, 26

[28] Ruszczynski, A., A. Shapiro. 2004. Optimization of convex risk functions. *Working Paper*, available from http://ideas.repec.org/p/wpa/wuwpri/0404001.html.

[29] Sankarasubramanian, E., Kumaraswamy, S., (1983): Optimal ordering quantity for pre-determined level of profit, *Management Science*, 29, 512-514.

[30] Shapiro, A. and Nemirovski, A. (2004): On Complexity of stochastic programming Problems, *Working Paper*, Georgia Tech, Atlanta.

[31] Simon, H.A., (1959): Theories of decision-making in economics and behavioral science, *The American Economic Review*,49(3),253-283.