Stability and Sensitivity Analysis for Optimal Control Problems with a First-order State Constraint having (nonessential) Touch Points

J. Frédéric Bonnans* Audrey Hermant*

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Abstract

The paper deals with an optimal control problem with a scalar first-order state constraint and a scalar control. In presence of (nonessential) touch points, the arc structure of the trajectory is not stable. We show how to perform a sensitivity analysis that predicts which touch points will, under a small perturbation, become inactive, remain touch points or switch into boundary arcs. The main tools are the study of a quadratic tangent problem and the notion of strong regularity. The results can be interpreted as an extension of the shooting algorithm to the case when touch points occur for first-order state constraints. An illustrative example is given.

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Introduction

This paper deals with an optimal control problem (of an ordinary differential equation) with a scalar first-order state constraint and a scalar control. It is well-known that for *first-order* state constraints, touch points (locally unique times where the constraint is active) are nonessential (the associated jump of the multiplier is null) (see e.g. [10, 9]). Situations where touch points are present may be encountered, for instance, when solving the optimal control problem by indirect approaches using an homotopy method in order to guess the arc structure of the trajectory, see e.g. the famous example in [1]. Therefore it is of interest to study sensitivity of solutions around touch

*Address: INRIA, BP 105, 78153 Le Chesnay, France.
Email: frederic.bonnans@inria.fr, audrey.hermant@inria.fr.

points, when the constraint becomes active. Under a small perturbation, several events may occur. Among them, the constraint may locally become inactive, the touch point may remain a touch point, or it may give rise to a boundary arc. Our main result is that, under natural hypotheses, these are the only three possibilities, and that the boundary arcs have a length of order of the perturbation, and satisfy a "strict complementarity" hypothesis. In addition, we show how to compute a first-order expansion of the solution. The analysis uses in a critical way a certain tangent quadratic problem, and at the same time is in the spirit of the shooting approach, in the sense that touch points are converted into boundary arcs of zero length, and we compute the first-order expansion of all entry and exit points. The proof applies the notion of "strong regularity" in the sense of Robinson [16] to a system that happens to be equivalent to the optimality conditions of the tangent quadratic problem. Our formulation of the corresponding shooting formulation (of which all entry and exit times are variables, in addition to the initial costate and jumps of the alternative multiplier at entry times) allows exit times to be lower than entry times; however, we check that the solution of the shooting formulation is such that entry times are lower than or equal to corresponding exit times.

This paper is related to our previous work: the study of no-gap second-order optimality conditions in [2], and the shooting formulation, allowing nonessential touch points for state constraints of order greater than one, and for which we also use the notion of strong regularity [3]. In both papers we assume also the state constraint and the control to be scalar-valued. In a forthcoming paper we will extend all these results to the case of vector-valued state constraints and control.

Two ways have been explored to obtain stability results for optimal control problems: on the one hand, implicit mapping theorems in infinite dimensional spaces (see e.g. [17, 11]), and one the other hand, shooting formulations that reduce locally the problem into a finite-dimensional one (a two-(or multi) point boundary value problem; see e.g. [18, 15]). With first-order state constraints, L^2-stability of solutions was obtained, under strong second-order sufficient conditions, by Malanowski [12], using an infinite-dimensional implicit function theorem based on two-norms approach, and by Dontchev and Hager [6], using an implicit function theorem in metric spaces. In Malanowski [12], directional differentiability of solutions in L^2 was established, using the results on differentiability of projection onto a closed convex cone in Hilbert spaces [8]. In Dontchev and Hager [6], L^∞-stability of solutions was derived, under an additional assumption on the contact set ("contact separation"). As for finite dimensional approaches, Malanowski and Maurer obtained in [13] differentiability of solutions in L^∞ by application of the implicit function theorem to the shooting mapping, under stronger assumptions (finitely many nontangential junction points, and strict complementarity). The present work can be seen as an extension of [13], to the case when (nonessential) touch points are present. We obtain Fréchet directional derivatives as solution of an inequality-constrained linear quadratic problem, under a *minimal* second-order sufficient condition, weakening that of [13].

The paper is organized at follows. Section 1 recalls basic definitions and the alternative formulation. In section 2, the main result is stated. In section 3, the problem is reduced to a generalized finite-dimensional equation, with a complementarity constraint. Robinson's strong regularity theory is applied to the latter in section 4, where the main result is proved. Section 5 deals with directional differentiability of solutions. In section 6, a basic illustrative example is presented. Finally, in the Appendix, a complement of proof is given.

1 Preliminaries and Optimality Conditions

Let $\mathcal{U} := \mathrm{L}^\infty(0,T)$ (resp. $\mathcal{Y} := \mathrm{W}^{1,\infty}(0,T;\mathbb{R}^n)$) denote the control (resp. state) space. Let M be a Banach space (the space of perturbations parameter) and, for $\mu \in M$, the cost function $\ell^\mu : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, final cost function $\phi^\mu : \mathbb{R}^n \to \mathbb{R}$, dynamics $f^\mu : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, state constraint $g^\mu : \mathbb{R}^n \to \mathbb{R}$, initial condition $y_0^\mu \in \mathbb{R}^n$, and (fixed) final time $T > 0$. We consider the following optimal control problem:

$$(\mathcal{P}^\mu) \qquad \min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \int_0^T \ell^\mu(u(t),y(t))\mathrm{d}t + \phi^\mu(y(T)) \tag{1}$$

$$\text{subject to} \qquad \dot{y}(t) = f^\mu(u(t),y(t)) \text{ for a.a. } t\in[0,T]\ ;\ y(0)=y_0^\mu, \tag{2}$$

$$g^\mu(y(t)) \le 0, \text{ for all } t\in[0,T]. \tag{3}$$

We study perturbations of problem $(\mathcal{P}^\mu)$ around a given value of parameter $\mu_0 \in M$, and we sometimes omit the superscript μ when we refer to the problem and data associated with μ_0, i.e. $(\mathcal{P}) := (\mathcal{P}^{\mu_0})$ and $(\ell,\phi,f,g,y_0) := (\ell^{\mu_0},\phi^{\mu_0},f^{\mu_0},g^{\mu_0},y_0^{\mu_0})$.

We assume throughout the paper that the assumptions below hold:

(A0) The mappings ℓ^{μ_0}, ϕ^{μ_0}, f^{μ_0} and g^{μ_0} are of class C^2, with locally Lipschitz continuous second order derivatives, and the dynamics f^{μ_0} is Lipschitz continuous;

(A1) the initial condition satisfies $g^{\mu_0}(y_0^{\mu_0}) < 0$.

These assumptions will not be repeated in the various results of the paper.

A parametrization $(\ell^\mu,\phi^\mu,f^\mu,g^\mu,y_0^\mu)$, identified with problem $(\mathcal{P}^\mu)$, is an *stable extension* of $(\mathcal{P})$, if there exists a neighborhood M_0 of μ_0, such that (i) there exists C^2 mappings $\hat{\ell} : \mathbb{R}\times\mathbb{R}^n\times M_0 \to \mathbb{R}$; $\hat{\phi} : \mathbb{R}^n\times M_0 \to \mathbb{R}$; $\hat{f} : \mathbb{R}\times\mathbb{R}^n\times M_0 \to \mathbb{R}^n$; $\hat{g} : \mathbb{R}^n\times M_0 \to \mathbb{R}$ and $\hat{y}_0 : M_0 \to \mathbb{R}^n$, such that $\ell^\mu(u,y) = \hat{\ell}(u,y,\mu)$ for all $(u,y)\in\mathbb{R}\times\mathbb{R}^n$ and all $\mu\in M_0$ (and similarly for ϕ^μ, f^μ, g^μ, and y_0^μ); (ii) the mappings ℓ^μ, f^μ, ϕ^μ, g^μ have Lipschitz continuous second-order derivatives and f^μ is Lipschitz continuous, uniformly over $\mu \in M_0$.

In this paper, we always consider stable extensions $(\mathcal{P}^\mu)$.

1.1 Definitions and Notations

The space of row vectors is denoted by $\mathbb{R}^{n*}$, and the adjoint and transposition operator in $\mathbb{R}^n$ are denoted by a star *. Fréchet derivatives of f, ℓ, etc. w.r.t. arguments $u \in \mathbb{R}$, $y \in \mathbb{R}^n$, are denoted by a subscript, for instance $f_u(u,y) = D_u f(u,y)$. The space $\mathrm{L}^r(0,T)$, $r \in [1,\infty]$, is the Lebesgue space of functions such that $\|u\|_r := (\int_0^T |u(t)|^r)^{1/r} < \infty$ for $1 \le r < \infty$ and $\|u\|_\infty := \mathrm{supess}_{t\in[0,T]} |u(t)| < \infty$, and $\mathrm{W}^{1,r}(0,T)$ is the Sobolev space of functions in $\mathrm{L}^r(0,T)$ with a weak derivative in $\mathrm{L}^r(0,T)$. The space of continuous functions and its dual space, the space of bounded Borel measures, are denoted respectively by $\mathrm{C}[0,T]$ and $\mathcal{M}[0,T]$. The cone of nonnegative measures is denoted by $\mathcal{M}_+[0,T]$, and $\mathrm{BV}([0,T];\mathbb{R}^n)$ denotes the space of vector-valued functions of bounded variation over $[0,T]$. The elements of $\mathcal{M}[0,T]$ are identified with functions of bounded variation vanishing at zero.

Given $\mu \in M_0$, a *trajectory* of $(\mathcal{P}^\mu)$ is an element $(u,y) \in \mathcal{U}\times\mathcal{Y}$ satisfying the state equation (2).

The first-order time derivative of the state constraint is the function defined by $(g^\mu)^{(1)} : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, $(u,y) \mapsto g_y^\mu(y) f^\mu(u,y)$. In this paper, we consider state constraints of *first order*, that is, the function $(g^\mu)^{(1)}(u,y)$ depends explicitly on the control variable u in the neighborhood of the contact set of the constraint, see assumption (A3).

The classical (resp. augmented) *Hamiltonian* functions $H^\mu : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n*} \to \mathbb{R}$ (resp. $\tilde{H}^\mu : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n*}\times\mathbb{R} \to \mathbb{R}$) are defined by:

$$H^\mu(u,y,p) \;:=\; \ell^\mu(u,y) + p f^\mu(u,y) \tag{4}$$
$$\tilde{H}^\mu(u,y,p_1,\eta_1) \;:=\; H^\mu(u,y,p_1) + \eta_1 (g^\mu)^{(1)}(u,y). \tag{5}$$

For (u,y) an admissible trajectory of $(\mathcal{P}^\mu)$, i.e. satisfying the constraint (3), define the *contact set* by:

$$I(g^\mu(y)) \;:=\; \{t \in [0,T]\;;\; g^\mu(y(t)) = 0\}. \tag{6}$$

A *boundary arc* (resp. *interior arc*) is a maximal interval of positive measure $\mathcal{I}$ such that $g^\mu(y(t)) = 0$ (resp. $g^\mu(y(t)) < 0$), for all $t \in \mathcal{I}$. Left and right endpoints of a boundary arc $[\tau_{en},\tau_{ex}]$ are called *entry* and *exit* point, respectively. A *touch point* τ_{to} is an isolated contact point, satisfying $g^\mu(y(\tau_{to})) = 0$ and $g^\mu(y(t)) < 0$, for $t \ne \tau_{to}$ in the neighborhood of τ_{to}. Entry, exit and touch points are called *junction* points.

When a trajectory (u,y) of $(\mathcal{P}^\mu)$ has a *finite* set of junction times, the latter is denoted by

$$\mathcal{T} =: \mathcal{T}_{en} \cup \mathcal{T}_{ex} \cup \mathcal{T}_{to},$$

with $\mathcal{T}_{en}$, $\mathcal{T}_{ex}$ and $\mathcal{T}_{to}$ the *disjoint* (and possibly empty) subsets of respectively regular entry, exit and touch points. We denote by $\mathcal{I}_b$ the union of boundary arcs, i.e. $\mathcal{I}_b := \cup_{i=1}^{N_b}[\tau_i^{en}, \tau_i^{ex}]$ for $\mathcal{T}_{en} := \{\tau_1^{en} < \ldots < \tau_{N_b}^{en}\}$ and similar definition of $\mathcal{T}_{ex}$, and we have $I(g^\mu(y)) = \mathcal{T}_{to} \cup \mathcal{I}_b$. The *arc structure* of a trajectory is the (finite) number of boundary arcs and touch points, and the order in which they occur.

Given a finite subset $\mathcal{S}$ of $(0, T)$, we denote by $\mathrm{PC}_{\mathcal{S}}^k[0,T]$ the set of functions over $[0,T]$ that are of class C^k outside $\mathcal{S}$, and have, as well as their first k derivatives, a left and right limit over $\mathcal{S} \cup \{0, T\}$. For all $\varphi \in \mathrm{PC}_{\mathcal{S}}^k[0,T]$, we denote left-and right limits at $\tau \in \mathcal{S}$ by $\varphi(\tau^\pm) := \lim_{t \to \tau^\pm} \varphi(t)$, and jump by $[\varphi(\tau)] := \varphi(\tau^+) - \varphi(\tau^-)$. The subset of functions in $\mathrm{PC}_{\mathcal{S}}^k[0,T]$ having continuous derivatives on $[0,T]$ until order $r \geq 0$ is denoted by $\mathrm{PC}_{\mathcal{S}}^{k,r}[0,T] := \mathrm{PC}_{\mathcal{S}}^k[0,T] \cap \mathrm{C}^r[0,T]$. We also use the notation $\nu_{\mathcal{S}} := (\nu_\tau)_{\tau \in \mathcal{S}} \in \mathbb{R}^{\mathrm{Card}\,\mathcal{S}}$.

1.2 Alternative Formulation of Optimality Conditions.

Let us first recall the definition of Pontryagin extremals.

Definition 1.1. A trajectory (u, y) a *Pontryagin extremal* for $(\mathcal{P}^\mu)$, if there exists $\alpha \in \mathbb{R}_+$, $\eta \in \mathcal{M}[0,T]$ and $p \in \mathrm{BV}([0,T]; \mathbb{R}^{n*})$, $(\eta, p, \alpha) \neq 0$, such that:

$$\dot{y}(t) = f^\mu(u(t), y(t)) \quad \text{a.e. on } [0,T] \quad ; \quad y(0) = y_0^\mu \tag{7}$$

$$\mathrm{d}p(t) = \{\alpha \ell_y^\mu(u(t), y(t)) + p(t) f_y^\mu(u(t), y(t))\}\mathrm{d}t + g_y^\mu(y(t))\mathrm{d}\eta(t) \text{ on } [0,T] \tag{8}$$

$$p(T) = \alpha \phi^\mu(y(T)) \tag{9}$$

$$u(t) \in \mathrm{argmin}_{\hat{u} \in \mathbb{R}}\{\alpha \ell_u^\mu(\hat{u}, y(t)) + p(t) f_u^\mu(\hat{u}, y(t))\} \text{ a.e. on } [0,T] \tag{10}$$

$$0 \geq g^\mu(y(t)) \quad ; \quad \mathrm{d}\eta \geq 0 \quad ; \quad \int_0^T g^\mu(y(t))\mathrm{d}\eta(t) = 0. \tag{11}$$

When $\alpha > 0$, dividing p and η by α, we can take $\alpha = 1$ in the above equations, and in that case we say that (u, y) is a *normal* Pontryagin extremal.

It is well known that optimal solutions of $(\mathcal{P}^\mu)$ are Pontryagin extremals. In presence of pure state constraints, a reformulation of the optimality conditions is needed to apply shooting methods. Our results are based on the following alternative formulation of optimality conditions, see e.g. [5, 10, 9, 14, 3].

Definition 1.2. A trajectory (u, y) having finitely many junction times $\mathcal{T}$, is solution of the *alternative formulation*, if there exists $p_1 \in \mathrm{PC}_{\mathcal{T}}^1([0,T], \mathbb{R}^{n*})$, $\eta_1 \in \mathrm{PC}_{\mathcal{T}}^1[0,T]$, and alternative jump parameters $\nu_{\mathcal{T}_{en}}^1$ and $\nu_{\mathcal{T}_{to}}$, such that the following relations are satisfied (time dependence is omitted):

$$\dot{y} = f^\mu(u, y) \quad \text{on } [0,T] \quad ; \quad y(0) = y_0^\mu \tag{12}$$

$$-\dot{p}_1 = \tilde{H}_y^\mu(u, y, p_1, \eta_1) \quad \text{on } [0,T] \setminus \mathcal{T} \tag{13}$$

$$0 = \tilde{H}_u^\mu(u, y, p_1, \eta_1) \quad \text{on } [0,T] \setminus \mathcal{T} \tag{14}$$

$$
\begin{aligned}
(g^\mu)^{(1)}(u,y) &= 0 \qquad \text{on } \mathcal{I}_b & (15)\\
\eta_1(t) &= 0 \qquad \text{on } [0,T]\setminus\mathcal{I}_b & (16)\\
p_1(T) &= \phi^\mu_y(y(T)) & (17)\\
g^\mu(y(\tau_{en})) &= 0, \qquad \tau_{en}\in\mathcal{T}_{en} & (18)\\
g^\mu(y(\tau_{to})) &= 0, \qquad \tau_{to}\in\mathcal{T}_{to} & (19)\\
[p_1(\tau_{en})] &= -\nu^1_{\tau_{en}} g^\mu_y(y(\tau_{en})), \qquad \tau_{en}\in\mathcal{T}_{en} & (20)\\
[p_1(\tau_{ex})] &= 0, \qquad \tau_{ex}\in\mathcal{T}_{ex} & (21)\\
[p_1(\tau_{to})] &= -\nu_{\tau_{to}} g^\mu_y(y(\tau_{to})), \qquad \tau_{to}\in\mathcal{T}_{to}. & (22)
\end{aligned}
$$

A solution of the alternative formulation satisfies the *additional conditions*, if the conditions below hold:

$$
\begin{aligned}
g^\mu(y(t)) &< 0 \qquad \text{on } [0,T]\setminus(\mathcal{I}_b\cup\mathcal{T}_{to}) & (23)\\
\dot\eta_1(t) &\le 0 \qquad \text{on } \operatorname{Int}\mathcal{I}_b & (24)\\
\nu^1_{\tau_{en}} &= \eta_1(\tau^+_{en}), \quad \tau_{en}\in\mathcal{T}_{en}\ ; \qquad \eta_1(\tau^-_{ex}) = 0, \quad \tau_{ex}\in\mathcal{T}_{ex} & (25)\\
\nu_{\tau_{to}} &= 0 \qquad \tau\in\mathcal{T}_{to}. & (26)
\end{aligned}
$$

We assume that problem $(\mathcal{P})$ has a local optimal solution, denoted in the sequel by $(\bar u,\bar y)$, and that the latter satisfies, with $\bar p_1, \bar\eta_1$ and $\bar{\mathcal{T}}$ its associated multipliers in the alternative formulation and junction times, the following assumptions:

(A2) Uniform strong convexity of the Hamiltonian w.r.t. the control variable, i.e. there exists $\alpha>0$, such that

$$\tilde H^{\mu_0}_{uu}(\hat u,\bar y(t),\bar p_1(t^\pm),\bar\eta_1(t^\pm)) \ge \alpha, \ \text{ for all } \hat u\in\mathbb{R} \text{ and all } t\in[0,T]. \quad (27)$$

(A3) Uniform regularity of the state constraint near the contact set, i.e., there exists $\beta,\varepsilon>0$ such that

$$|(g^{\mu_0})^{(1)}_u(\hat u,\bar y(t))| \ge \beta, \ \text{ for a.a. } t, \ \operatorname{dist}(t; I(g^{\mu_0}(y))) \le \varepsilon \text{ and all } \hat u\in\mathbb{R}. \quad (28)$$

(A4) The trajectory $(\bar u,\bar y)$ has a *finite set of junction times* $\bar{\mathcal{T}}$, and we suppose that $g(\bar y(T))<0$.

Proposition 1.3 (See e.g. [10, 9, 13]). *Let $(\bar u,\bar y)$ be an optimal solution of $(\mathcal{P}^{\mu_0})$, satisfying (A2)-(A4). Then $(\bar u,\bar y)$ is solution of alternative formulation (12)-(22), and satisfies additional conditions (23)-(26).*

Remark 1.4. It can be shown (see [3, Prop. 2.10]) that under (A2)-(A4), relations (12)-(26) characterize Pontryagin extremals. When (A3) (and (A1)) hold, the extremal is normal ($\alpha=1$), and the (unique) *classical* multipliers

$\eta \in \mathcal{M}_+[0,T]$ and $p \in \mathrm{BV}([0,T];\mathbb{R}^{n*})$ of Def. 1.1, are given by (recall that we adopted the convention $\eta(0)=0$):

$$\eta(t) = \sum_{\tau \in \mathcal{T}_{en}} \nu^1_\tau \mathbf{1}_{t\geq\tau}(t) - \eta_1(t^+) \qquad ; \qquad p(t) = p_1(t) + \eta_1(t) g^\mu_y(y(t)), \tag{29}$$

with $\mathbf{1}_{t\geq\tau}(t) = 1$ if $t \geq \tau$ and zero otherwise. Equivalently, η is given by $\mathrm{d}\eta(t) = -\dot\eta_1(t)\mathrm{d}t$.

Classical multipliers (p,η) and alternative ones (p_1,η_1) can be recovered from each other by (29) and (25). By (20)-(22) and additional conditions (25)-(26), we have $(p,\eta) \in \mathrm{PC}^{1,0}_{\mathcal{T}}([0,T],\mathbb{R}^{n*}) \times \mathrm{PC}^{1,0}_{\mathcal{T}}[0,T]$. It is also easy to see that, when (29) holds,

$$\tilde H^\mu(\cdot, y, p_1, \eta_1) = H^\mu(\cdot, y, p),$$

and hence, (A2) is equivalent (with $\bar p$ the classical costate associated with $\bar u$) to:

$$H^{\mu_0}_{uu}(\hat u, \bar y(t), \bar p(t^\pm)) \geq \alpha, \quad \text{for all } \hat u \in \mathbb{R} \text{ and all } t \in [0,T].$$

Remark 1.5. By (24)-(25), the following necessary condition holds:

$$\nu^1_{\mathcal{T}_{en}} \ \geq \ 0. \tag{30}$$

Remark 1.6. If (u,y) satisfies the alternative formulation and (A2)-(A4), then (25) is equivalent to the condition below (see [13] and [3, Prop. 2.15]):

$$(g^\mu)^{(1)}(u(\tau_{en}^-), y(\tau_{en})) = 0, \ \tau_{en} \in \mathcal{T}_{en} \ ; \ (g^\mu)^{(1)}(u(\tau_{ex}^+), y(\tau_{ex})) = 0, \ \tau_{ex} \in \mathcal{T}_{ex}. \tag{31}$$

Also (25) or (31) is equivalent to the continuity of the control at entry/exit points. By (26), (14) and (A2), we see that the control is also continuous at touch points. It follows that $(u,y) \in \mathrm{PC}^{1,0}_{\mathcal{T}}[0,T] \times \mathrm{PC}^{1,1}_{\mathcal{T}}([0,T];\mathbb{R}^n)$.

Remark 1.7. At a touch point τ_{to}, the function $t \mapsto g^\mu(y(t))$ has a local isolated maximum, and a continuous derivative at τ_{to} (due to the continuity of u), hence the condition below is satisfied (compare to (31)):

$$(g^\mu)^{(1)}(u(\tau_{to}), y(\tau_{to})) = 0, \quad \tau \in \mathcal{T}_{to}. \tag{32}$$

2 Statement of the main result

We make in addition to (A2)-(A4) the following assumptions:

(A5) *Uniform strict complementarity* on *boundary arcs*:

$$\exists \beta > 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\bar\eta_1(t) \leq -\beta \qquad \text{for all } t \in \operatorname{Int} \bar{\mathcal{I}}_b; \tag{33}$$

(A6) *Non tangentiality at second order* at touch points: for all $\tau_{to} \in \bar{\mathcal{T}}_{to}$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\bar{y}(t))|_{t=\tau_{to}} < 0. \tag{34}$$

Note that (34) makes sense, since $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(y(t))$ is a function of $(y, u, \dot{u})$ and u and $\dot{u}$ are continuous at a touch point (since $\nu_{\tau_{to}} = 0$). This condition is similar to the reducibility hypothesis when the state constraint is of order $q \geq 2$. The proof of the next lemma can be found in [3, Lemma 3.1].

Lemma 2.1. *Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal for $(\mathcal{P})$ satisfying (A2)-(A4). Then assumption (A5) implies that the following non-tangentiality at second-order holds at entry/exit points:*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\bar{y}(t))|_{t=\tau_{en}^-} < 0,\ \tau_{en} \in \bar{\mathcal{T}}_{en}\ ; \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\bar{y}(t))|_{t=\tau_{ex}^+} < 0,\ \tau_{ex} \in \bar{\mathcal{T}}_{ex}. \tag{35}$$

Let the linearized control and state spaces be respectively $\mathcal{V} := \mathrm{L}^2(0, T)$ and $\mathcal{Z} := \mathrm{H}^1(0, T; \mathbb{R}^n)$, where $\mathrm{H}^1(0, T)$ is the Sobolev space of functions in $\mathrm{L}^2(0, T)$ with a weak derivative in $\mathrm{L}^2(0, T)$. Define the quadratic cost function over $\mathcal{V} \times \mathcal{Z}$

$$\begin{aligned} \mathcal{J}_1(v, z) \ &:= \int_0^T \tilde{H}_{(u,y),(u,y)}(\bar{u}, \bar{y}, \bar{p}_1, \bar{\eta}_1)((v, z), (v, z))\mathrm{d}t \\ &\quad + z(T)^* \phi_{yy}(\bar{y}(T))z(T) \ + \sum_{\tau \in \bar{\mathcal{T}}_{en}} \bar{\nu}_\tau^1 z(\tau)^* g_{yy}(\bar{y}(\tau))z(\tau), \end{aligned} \tag{36}$$

where $\tilde{H}$ is the augmented Hamiltonian (5), and the set of constraints:

$$\dot{z} = f_y(\bar{u}, \bar{y})z + f_u(\bar{u}, \bar{y})v \quad \text{on } [0, T] \quad ; \qquad z(0) = 0 \tag{37}$$

$$g_y(\bar{y}(\tau))z(\tau) = 0 \qquad \tau \in \bar{\mathcal{T}}_{en} \tag{38}$$

$$g^{(1)}_{(u,y)}(\bar{u}(t), \bar{y}(t))(v(t), z(t)) = 0 \qquad t \in \bar{\mathcal{I}}_b \tag{39}$$

$$g_y(\bar{y}(\tau))z(\tau) \leq 0 \qquad \tau \in \bar{\mathcal{T}}_{to}. \tag{40}$$

We consider the following second-order condition:

$$\mathcal{J}_1(v, z) > 0, \quad \text{for all } (v, z) \in \mathcal{V} \times \mathcal{Z},\ (v, z) \neq 0, \text{ satisfying (37)-(39)}. \tag{41}$$

Remark 2.2. (i) We know by [3, Lemma 3.6] that we can express the quadratic cost $\mathcal{J}_1$, using $(\bar{p}, \bar{\eta})$ defined by (29) instead of $(\bar{p}_1, \bar{\eta}_1)$, over the space of linearized trajectories (v, z) satisfying (37), by:

$$\begin{aligned} \mathcal{J}_1(v, z) \ &= \int_0^T H_{(u,y),(u,y)}(\bar{u}, \bar{y}, \bar{p})((v, z), (v, z))\mathrm{d}t \ + \ z(T)^* \phi_{yy}(\bar{y}(T))z(T) \\ &\quad + \int_0^T z(t)^* g_{yy}(\bar{y}(t))z(t)\mathrm{d}\bar{\eta}(t). \end{aligned}$$

(ii) Condition (41) is stronger than the no-gap second-order sufficient condition, characterization of the second-order growth condition (see [2]):

$$\mathcal{J}_1(v,z) > 0, \quad \text{for all } (v,z) \in \mathcal{V} \times \mathcal{Z},\ (v,z) \neq 0, \text{ satisfying (37)-(40).} \quad (42)$$

Remark 2.3. The second-order sufficient condition (41) used in the stability and sensitivity analysis is weaker than the one given in [13], where the entry point constraint (38) is omitted. The authors present a numerical method, based on Riccati equations, allowing to check the coercivity of the quadratic form $\mathcal{J}_1$ over the subspace defined by (37) and (39), which if of interest in applications, while the verification of (41) in practice remains open.

Given $(\mu, u) \in M_0 \times \mathcal{U}$, denote by y_u^μ the state solution in $\mathcal{Y}$ of:

$$\dot{y}_u^\mu(t) \ = \ f^\mu(u(t), y_u^\mu(t)) \qquad \text{for a.a. } t \in [0,T] \quad ; \quad y_u^\mu(0) = y_0^\mu, \quad (43)$$

and the cost function $J : \mathcal{U} \to \mathbb{R}$, $u \mapsto \int_0^T \ell^\mu(u(t), y_u^\mu(t))\mathrm{d}t + \phi^\mu(y_u^\mu(T))$.

Since strict complementarity does not hold at touch points, the arc structure of the trajectory (in the sense of number and order of boundary arcs and touch points) *is not stable* under a small perturbation. However, we will see that by (A5), boundary arcs are locally preserved, and that by (A6), the only three possibilities for a touch point is to become a boundary arc, remain a touch point or become inactive at local solutions of the perturbed problems. That is what we call having a *neighboring arc structure* of active constraints to that of $(\bar{u}, \bar{y})$, in a sense made precise in section 4.

Below is our main result, that will be proved later.

Theorem 2.4. *Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal of $(\mathcal{P})$ satisfying (A2)-(A6). Then the following assertions are equivalent:*
(i) *For all stable extensions $(\mathcal{P}^\mu)$ of $(\mathcal{P}^{\mu_0})$, there exists neighborhoods (V_u, V_μ) of $(\bar{u}, \mu_0)$, such that for all $\mu \in V_\mu$, there exists locally a unique local optimal solution (u^μ, y^μ) of $(\mathcal{P}^\mu)$ with $u^\mu \in V_u$ satisfying the uniform quadratic growth condition: there exists $c, r > 0$, such that*

$$J^\mu(u) \geq J^\mu(u^\mu) + c\|u - u^\mu\|_2^2, \ \ \forall u \ ; \ \|u - \bar{u}\|_\infty < r, \ g^\mu(y_u^\mu) \leq 0 \text{ on } [0,T]; \quad (44)$$

(ii) *The strong second-order sufficient condition (41) holds.*

In addition, if either point (i) or (ii) is satisfied, denote by (p^μ, η^μ) the (unique) classical multipliers associated with the local solution (u^μ, y^μ). Then (u^μ, y^μ) is the unique Pontryagin extremal of $(\mathcal{P}^\mu)$ with u^μ in the neighborhood of $\bar{u}$, and the mapping

$$\mu \mapsto (u^\mu, y^\mu, p^\mu, \eta^\mu) \in \mathcal{U} \times \mathcal{Y} \times \mathrm{L}^\infty(0,T;\mathbb{R}^{n*}) \times \mathrm{L}^\infty(0,T)$$

is Lipschitz continuous on a neighborhood of μ_0.

In fact there is another main result behind Th. 2.4. We prove in Th. 4.4 that under (A2)-(A6), *all Pontryagin extremals*, i.e. stationary points (u, y) of $(\mathcal{P}^\mu)$, with (u, μ) in a neighborhood of $(\bar{u}, \mu_0)$ have a neighboring structure of active constraints to that of $(\bar{u}, \bar{y})$ (and in particular, have finitely many junction points). This is clearly a stronger result than the existence of a unique local optimal solution of $(\mathcal{P}^\mu)$ as in e.g. Th. 8.3 in [13] (where only the implication "$(ii) \Rightarrow (i)$" was investigated), and than the uniqueness of Pontryagin extremals satisfying some restrictions on the arc structure, as in [3, Th. 4.2]. Uniqueness of stationary points, in a certain sense, is needed to obtain the implication $(i) \Rightarrow (ii)$ in the above theorem.

In section 5, we will provide the first-order expansion of the local optimal solution and associated multipliers of the perturbed problem.

3 Shooting Formulation

By (A2)-(A4), applying the implicit function theorem to (14)-(16), we may express the algebraic variables (u, η_1) on each arc as C^1 functions of the differential variables (y, p_1). Denote by $\mathcal{F}_b^\mu$ and $\mathcal{F}_i^\mu$ the flows on (y, p_1) obtained respectively on boundary and interior arcs, by eliminating the algebraic variables, and write $(y, p_1)(t) = (y(t), p_1(t))$. On each arc (t_1, t_2), we have that

$$(y, p_1)(t_2^-) = \mathcal{F}_a^\mu((y, p_1)(t_1^+), t_2 - t_1) \tag{45}$$

where $\mathcal{F}_a^\mu$ equals $\mathcal{F}_b^\mu$ for a boundary arc, and $\mathcal{F}_i^\mu$ for an interior arc. So we can (and this is precisely the idea of shooting methods) describe the alternative optimality system (12)-(22) as a sequence of applications of mappings $\mathcal{F}_b^\mu$ and $\mathcal{F}_i^\mu$, combined with junction conditions. Note that the mappings $(x, t_1, t_2) \to \mathcal{F}_a^\mu(x, t_2 - t_1)$, $a = i, b$, are (locally) C^1 w.r.t. all arguments, and allow in particular $t_2 - t_1$ to be nonpositive.

Now let us view a touch point as a boundary arc of zero length. This makes sense since, as we will see later, under a small perturbation, a touch point may switch into a boundary arc. So we have an entry point and an exit point, τ_{en} and τ_{ex}, whose common value is the one of the touch point. The jump $\nu^1_{\tau_{en}}$ at entry point τ_{en} equals $\nu_{\tau_{to}}$ (i.e., zero). There is a zero jump of p_1 at the entry (and exit) time τ_{en}.

Assume that we have N_{ba} boundary arcs and N_{to} touch points. Let $N := N_{ba} + N_{to}$. We have now N entry and N exit points. Denote by t^{en} (resp. t^{ex}) the N dimensional vector of entry (resp. exit) points, taken in the chronological order. We use the notation $t_0^{ex} := 0$ and $t_{N+1}^{en} := T$. We may rewrite the alternative formulation as follows, taking into account the continuity of state and costate at exit points:

$$(y, p_1)(0) = (y_0, p_0) \tag{46}$$

$$(y, p_1)(t_i^{en-}) = \mathcal{F}_i^\mu((y, p_1)(t_{i-1}^{ex}), t_i^{en} - t_{i-1}^{ex}), \quad i = 1, \ldots, N+1, \tag{47}$$

$$
\begin{aligned}
(y, p_1)(t_i^{ex}) &= \mathcal{F}_b^\mu((y, p_1)(t_i^{en+}), t_i^{ex} - t_i^{en}), \quad i = 1, \dots, N, &(48)\\
[p_1(t_i^{en})] &= -\nu_i^1 g_y^\mu(t_i^{en}), \quad i = 1, \dots, n, &(49)\\
p_1(T) &= \phi_y^\mu(y(T)) &(50)\\
g^\mu(y(t_i^{en})) &= 0, \quad i = 1, \dots, n, &(51)
\end{aligned}
$$

where $p_0 \in \mathbb{R}^{n*}$ denotes the initial value of the costate.

We come now to the definition of the shooting mapping. Let $\Theta := \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ be the space of shooting parameters, of dimension $\bar{N} := n + 3N$. A vector of shooting parameters is denoted by

$$
\theta = (p_0^*, \nu^1, t^{en}, t^{ex}) \in \Theta. \tag{52}
$$

The shooting mapping F is defined over a neighborhood $V_\theta \times V_\mu$ of (θ_0, μ_0) in $\mathbb{R}^{\bar{N}} \times M_0$ into $\mathbb{R}^{\bar{N}}$, by

$$
F(\theta, \mu) = \begin{pmatrix} p_1(T) - \phi_y^\mu(y(T)) \\ g^\mu(y(t^{en})) \\ (g^\mu)^{(1)}(u(t^{en-}), y(t^{en})) \\ (g^\mu)^{(1)}(u(t^{ex+}), y(t^{ex})) \end{pmatrix}, \tag{53}
$$

where the values of (y, p_1, u) at times $t_i^{en\pm}, t_i^{ex\pm}, T$ are given by (46)-(49), and where we used e.g. the notation

$$
(g^\mu)^{(1)}(u(t^{en-}), y(t^{en})) := \Big((g^\mu)^{(1)}(u(t_i^{en-}), y(t_i^{en})) \Big)_{1 \le i \le N} \in \mathbb{R}^N.
$$

Being a composition of C^1 mappings, the shooting mapping is itself of class C^1.

Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal for $(\mathcal{P})$, satisfying (A2)-(A4), with finite set of junction times $\bar{\mathcal{T}}$. Define I_{ba} and I_{to} as the (disjoint) sets of index in $\{1, \dots, N\}$ corresponding respectively to boundary arcs and touch points of the trajectory $(\bar{u}, \bar{y})$. Split F into two components:

$$
F(\theta, \mu) = (\Phi(\theta, \mu)^*, \Psi(\theta, \mu)^*)^*,
$$

where Ψ corresponds to the components $g^\mu(y(t_i^{en}))$ for $i \in I_{to}$, denoted by the vector $g^\mu(y(t_{to}^{en})) \in \mathbb{R}^{N_{to}}$. Consider the following complementarity problem, for μ close to μ_0:

$$
\text{Find} \quad \theta \in \Theta \quad \text{such that} \qquad \Phi(\theta, \mu) = 0 \quad \text{and} \quad \Psi(\theta, \mu) \in N(\theta), \tag{54}
$$

where

$$
N(\theta) := \begin{cases} \mathbb{R}_-^{N_{to}} \cap (\nu_{to}^1)^\perp & \text{if } \nu_{to}^1 \in \mathbb{R}_+^{N_{to}}, \\ \emptyset & \text{otherwise,} \end{cases} \tag{55}
$$

with ν^1_{to} the vector of components ν^1_i, for $i \in I_{to}$.

Note that by (50)-(51) and (31)-(32), $\theta_0 := (\bar{p}_1(0)^*, \bar{\nu}^1, \bar{t}^{en}, \bar{t}^{ex})$ is solution of (54) for $\mu = \mu_0$, with $\bar{t}^{en}$ and $\bar{t}^{ex}$ the vectors of times in $\bar{\mathcal{T}}_{en} \cup \bar{\mathcal{T}}_{to}$ and $\bar{\mathcal{T}}_{ex} \cup \bar{\mathcal{T}}_{to}$ respectively, in increasing order, $\bar{\nu}^1_i = \bar{\nu}^1_{\bar{t}^{en}_i}$ if $i \in I_{ba}$, and $\bar{\nu}^1_i = 0$ if $i \in I_{to}$.

It should be underlined that we allow, in formulation of problem (54), entry times variables to be greater than exit times ones. However, we will check in the next section, after having shown that (54) has a unique solution, that the constraint $\nu^1_{to} \geq 0$ in (54) (compare with (30)) is sufficient, with assumption (A6), to ensure locally for μ in the neighborhood of μ_0 that the solution of (54) is such that $t^{en}_i \leq t^{ex}_i$ for all $i \in I_{to}$, and that in addition, strict complementarity $\dot{\eta}_1 < 0$ holds on the boundary arc (t^{en}_i, t^{ex}_i) whenever $t^{en}_i < t^{ex}_i$.

As we will see, the formulation (54) is strongly related with the associated linear-quadratic tangent problem $\min_{(v,z)\in\mathcal{V}\times\mathcal{Z}} \mathcal{J}_1(v, z)$ subject to the equality constraints (37)-(39) *and the inequality constraint* (40).

Remark 3.1. When the state constraint is of higher order, under small perturbations, a nonessential touch point satisfying (34) either becomes an essential one (with a nonzero jump of the costate), remains a non essential one, or becomes inactive, excluding the case when a boundary arc appears, see [3]. Consequently, the duplication of *reducible* touch points (i.e., satisfying (34)) into entry and exit points is unnecessary for state constraints of order greater than one.

4 Stability Analysis

Our stability analysis is based on the notion of strong regularity, introduced by Robinson in [16], applied to the complementarity problem (54). The point θ_0 solution of (54) for $\mu = \mu_0$ is *strongly regular*, if there exists neighborhoods (V_θ, V_δ) in $\mathbb{R}^{\bar{N}} \times \mathbb{R}^{\bar{N}}$ of $(\theta_0, 0)$, such that, for all $\delta \in V_\delta$, $\delta = (\delta_1, \delta_2) \in \mathbb{R}^{\bar{N}-N_{to}} \times \mathbb{R}^{N_{to}}$, there exists a unique solution $\theta \in V_\theta$ of:

$$\begin{cases} D_\theta\Phi(\theta_0, \mu_0)(\theta - \theta_0) - \delta_1 = 0 \\ D_\theta\Psi(\theta_0, \mu_0)(\theta - \theta_0) - \delta_2 \in N(\theta) \end{cases} \tag{56}$$

and the mapping $\Xi : \delta \to \theta(\delta)$ is Lipschitz continuous over V_δ.

If θ_0 is strongly regular, then by [16], there exists neighborhoods (V'_θ, V_μ) of (θ_0, μ_0), such that for each $\mu \in V_\mu$, (54) has in V'_θ a unique solution θ^μ,

$$\theta^\mu = (p^{\mu*}_0, \nu^{\mu,1}, t^{\mu,en}, t^{\mu,ex}), \tag{57}$$

and there exists $\kappa > 0$ such that for all $\mu, \mu' \in V_\mu$,

$$|\theta^\mu - \theta^{\mu'}| \leq \kappa\|\mu - \mu'\|. \tag{58}$$

In addition, the following expansion of θ^μ holds:

$$\theta^\mu = \theta_0 + \Xi(-D_\mu F(\theta_0, \mu_0)(\mu - \mu_0)) + o(\|\mu - \mu_0\|). \tag{59}$$

Lemma 4.1. *Under assumptions of Th.* 2.4*, (41) implies that θ_0 is a strongly regular solution of (54) for $\mu = \mu_0$. More precisely, given $\delta = (\delta_1, \delta_2) \in \mathbb{R}^{\bar{N}-N_{to}} \times \mathbb{R}^{N_{to}}$, $\delta_1 = (a_T, b^1_{ba}, c^{en}, c^{ex}) \in \mathbb{R}^n \times \mathbb{R}^{N_{ba}} \times \mathbb{R}^N \times \mathbb{R}^N$, $\delta_2 = b^1_{to}$, there exists a unique $\omega \in \Theta$, $\omega = (\pi_0, \gamma^1, \sigma^{en}, \sigma^{ex})$, solution of the following relation, equivalent to (56) with $\omega = \theta - \theta_0$:*

$$\begin{cases} D_\theta \Phi(\theta_0, \mu_0)\omega - \delta_1 = 0 \\ D_\theta \Psi(\theta_0, \mu_0)\omega - \delta_2 \in N(\omega), \end{cases} \tag{60}$$

and ω is given as follows. Let $(v_\delta, z_\delta, \pi_\delta, \zeta_\delta, \lambda^1_\delta)$ be the unique solution and associated multipliers of the following linear-quadratic problem:

$$(\mathcal{P}^\delta) \qquad \min_{(v,z)\in\mathcal{V}\times\mathcal{Z}} \quad \tfrac{1}{2}\mathcal{J}_1(v,z) + a_T^* z(T) \tag{61}$$

$$\text{subject to} \quad (37),\ (39),$$

$$g_y(y(t_i^{en}))z(t_i^{en}) = b_i^1, \quad i \in I_{ba} \tag{62}$$

$$g_y(y(t_i^{en}))z(t_i^{en}) \le b_i^1, \quad i \in I_{to}, \tag{63}$$

where the multipliers π_δ, ζ_δ and λ^1_δ are associated, respectively, with the constraint (37), (39) and (62)-(63). Then ω is given by: $\pi_0 = \pi_\delta(0)$, $\gamma^1 = \lambda^1_\delta$, and

$$\sigma_i^{en} = \frac{c_i^{en} - g^{(1)}_{(u,y)}(u(t_i^{en}), y(t_i^{en}))(v_\delta(t_i^{en-}), z_\delta(t_i^{en}))}{\frac{\mathrm{d}}{\mathrm{d}t} g^{(1)}(u,y)|_{t=t_i^{en-}}}, i = 1, \dots, N, \tag{64}$$

$$\sigma_i^{ex} = \frac{c_i^{ex} - g^{(1)}_{(u,y)}(u(t_i^{ex}), y(t_i^{ex}))(v_\delta(t_i^{ex+}), z_\delta(t_i^{ex}))}{\frac{\mathrm{d}}{\mathrm{d}t} g^{(1)}(u,y)|_{t=t_i^{ex+}}}, \ i = 1, \dots, N. \tag{65}$$

Proof. We only recall the main ideas of the proof, the latter being similar to that of Lemma 4.2 in [3], see also [13] where the block-decoupling property of the Jacobian of the shooting mapping was established for state constraints of first-order. We have that a part of (56) matches the first-order optimality conditions of $(\mathcal{P}_\delta)$, and the other part gives the variations of junction times by (64)-(65). By (A2), the quadratic form $\mathcal{J}_1$ is a *Legendre form* over the space of linearized trajectories (v, z) satisfying (37), so that (41) implies that $\mathcal{J}_1$ is *coercive* over the linear space of $(v, z) \in \mathcal{V} \times \mathcal{Z}$ satisfying (37)-(39). It follows that the first-order optimality system of $(\mathcal{P}_\delta)$ has a unique solution and multipliers, that are Lipschitz continuous w.r.t. δ. □

Lemma 4.2. *Under the assumptions and point (ii) of Th.* 2.4*, there exists a neighborhood V_μ of μ_0, such that the solution θ^μ given by (57) satisfies:*

$$t_i^{\mu,ex} \ge t_i^{\mu,en}, \qquad \text{for all } i \in I_{to} \tag{66}$$

and

$$t_i^{\mu,ex} = t_i^{\mu,en} \;\Leftrightarrow\; \nu_i^{\mu,1} = 0, \qquad i \in I_{to}. \tag{67}$$

In particular, the solution $(u^\mu, y^\mu, p_1^\mu, \eta_1^\mu)$ of (46)-(49) with $\theta = \theta^\mu$ is well-defined over $[0,T]$, and there exists a constant $\gamma > 0$, such that for all $i \in I_{to}$:

$$\dot{\eta}_1^\mu(t) < -\gamma \;\text{ on }\; [t_i^{\mu,en}, t_i^{\mu,ex}] \quad \text{whenever } t_i^{\mu,ex} > t_i^{\mu,en}. \tag{68}$$

Proof. Let $i \in I_{to}$. By strong regularity (Lemma 4.1), we have that

$$t_i^{\mu,ex} - t_i^{\mu,en} = \mathcal{O}(\|\mu - \mu_0\|) \quad ; \quad \nu_i^{\mu,1} = \mathcal{O}(\|\mu - \mu_0\|). \tag{69}$$

Denote by (u, y, p_1, η_1) the solution of (46)-(49) for $\theta = \theta^\mu$. Note that this is well-defined *on each arc*, but not as function of time, since it may take several values for $t \in ((t_i^{\mu,en}, t_i^{\mu,ex}))$ if $t_i^{\mu,en} > t_i^{\mu,ex}$ (where $((a,b))$ stands for (a,b) if $a \le b$ and (b,a) otherwise). We will see that this last case cannot occur, i.e. (66) holds, (and clearly also holds by continuity with a strict inequality for $i \in I_{ba}$), and is satisfied with equality iff $\nu_i^{\mu,1} = 0$.

Suppose first that $t_i^{\mu,ex} = t_i^{\mu,en}$. Then (u, y, η_1, p_1) is defined as function of time without ambiguity in the neighborhood of $t_i^{\mu,en}$ (the algebraic variables are given by the dynamics on interior arcs). By (46)-(50), there is a jump of p_1 at entry time and no jump at exit time, thus $(y, p_1)(t_i^{\mu,en+}) = (y, p_1)(t_i^{\mu,ex-}) = (y, p_1)(t_i^{\mu,ex+})$. By definition of (54), we have

$$(g^\mu)^{(1)}(u(t_i^{\mu,en-}), y(t_i^{\mu,en})) = (g^\mu)^{(1)}(u(t_i^{\mu,ex+}), y(t_i^{\mu,ex})) = 0,$$

and hence, since $t_i^{\mu,ex} = t_i^{\mu,en}$, hypothesis (A3) implies that u is *continuous* at time $t_i^{\mu,en}$. We deduce that:

$$0 = [H_u^\mu(u(t_i^{\mu,en}), y(t_i^{\mu,en}), p_1(t_i^{\mu,en}))] = -\nu_i^{\mu,1}(g^\mu)_u^{(1)}(u(t_i^{\mu,en}), y(t_i^{\mu,en})).$$

Since $(g^\mu)_u^{(1)}(u(t_i^{\mu,en}), y(t_i^{\mu,en})) \neq 0$ for $\|\mu - \mu_0\|$ small enough, it follows that $\nu_i^{\mu,1} = 0$. This proves the "$\Rightarrow$" implication in (67).

Suppose now that $t_i^{\mu,ex} \neq t_i^{\mu,en}$. In order to avoid any confusion, denote the solution of (46)-(49) for $\theta = \theta^\mu$ by $(u^-, y^-, p_1^-, \eta_1^-)$ on the boundary arc $((t_i^{\mu,en}, t_i^{\mu,ex}))$, and by $(u^+, y^+, p_1^+, \eta_1^+)$ on the succeeding interior arc $(t_i^{\mu,ex}, t_{i+1}^{\mu,en})$. Note that the limits of these functions and of their time derivative at endpoints of the interval where they are defined do exist, and here the jump denotes for instance $[u(t_i^{\mu,ex})] := u^+(t_i^{\mu,ex}) - u^-(t_i^{\mu,ex})$.

Using the same local arguments as in Rem. 1.6, we can show that by (A2)-(A3), $(u^+, y^+, p_1^+, \eta_1^+)(t_i^{\mu,ex}) = (u^-, y^-, p_1^-, \eta_1^-)(t_i^{\mu,ex})$, and we denote this common value by $(u(t_i^{\mu,ex}), y(t_i^{\mu,ex}), p_1(t_i^{\mu,ex}), \eta_1(t_i^{\mu,ex}))$. By (A6), there exists by continuity a constant $c > 0$ such that, for μ close enough to μ_0,

$$\frac{\mathrm{d}}{\mathrm{d}t}(g^\mu)^{(1)}(u^+(t), y^+(t))|_{t \to t_i^{\mu,ex}} < -c. \tag{70}$$

On the other hand, we have on the boundary arc $(\!(t_i^{\mu,en}, t_i^{\mu,ex})\!)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(g^\mu)^{(1)}(u^-(t), y^-(t))|_{t\to t_i^{\mu,ex}} = 0. \tag{71}$$

Since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(g^\mu)^{(1)}(u^\pm(t), y^\pm(t)) = (g^\mu)^{(1)}_u(u^\pm, y^\pm)\dot{u} + (g^\mu)^{(1)}_y(u^\pm, y^\pm)f^\mu(u^\pm, y^\pm),$$

it follows that the jump of $\dot{u}$ at $t_i^{\mu,ex}$ satisfies, by (70)-(71):

$$[\frac{\mathrm{d}}{\mathrm{d}t}(g^\mu)^{(1)}(u(t), y(t))|_{t\to t_i^{\mu,ex}}] = (g^\mu)^{(1)}_u(u(t_i^{\mu,ex}), y(t_i^{\mu,ex}))[\dot{u}(t_i^{\mu,ex})] < -c, \tag{72}$$

and hence, $\dot{u}^-(t_i^{\mu,ex}) \neq \dot{u}^+(t_i^{\mu,ex})$. By time-derivation of (14) on the boundary arc $(\!(t_i^{\mu,en}, t_i^{\mu,ex})\!)$ of nonzero length and on the interior arc $(t_i^{\mu,ex}, t_{i+1}^{\mu,en})$, we obtain (omitting arguments $(u^\pm(t), y^\pm(t), p_1^\pm(t), \eta_1^\pm(t))$):

$$\tilde{H}^\mu_{uu}\dot{u}^\pm + \tilde{H}^\mu_{yu}f^\mu - \tilde{H}^\mu_y f^\mu_u + (g^\mu)^{(1)}_u\dot{\eta}_1^\pm = 0, \tag{73}$$

and hence, taking the jump at time $t_i^{\mu,ex}$ gives, since (u, y, p_1, η_1) is continuous at $t_i^{\mu,ex}$:

$$\tilde{H}^\mu_{uu}(u, y, p_1, \eta_1)(t_i^{\mu,ex})[\dot{u}(t_i^{\mu,ex})] + (g^\mu)^{(1)}_u(u, y)(t_i^{\mu,ex})[\dot{\eta}_1(t_i^{\mu,ex})] = 0.$$

Since $\dot{\eta}_1^+(t_i^{\mu,ex}) = 0$, by (72) and (A2)-(A3), we obtain by continuity that there exists a constant $C > 0$ such that, for $\|\mu - \mu_0\|$ small enough,

$$\begin{aligned}\dot{\eta}_1^-(t_i^{\mu,ex}) = -[\dot{\eta}_1^-(t_i^{\mu,ex})] &= \frac{\tilde{H}^\mu_{uu}(u, y, p_1, \eta_1)(t_i^{\mu,ex})}{((g^\mu)^{(1)}_u(u, y)(t_i^{\mu,ex}))^2}(g^\mu)^{(1)}_u(u, y)(t_i^{\mu,ex})[\dot{u}(t_i^{\mu,ex})] \\ &< -C. \end{aligned} \tag{74}$$

By (73) and time derivation of (15), we see that $\dot{\eta}_1^-(t)$ is given by a Lipschitz continuous function of time on $(\!(t_i^{\mu,en}, t_i^{\mu,ex})\!)$, uniformly w.r.t. μ, so there exists $m > 0$ independent of μ, such that

$$\dot{\eta}_1^-(t) \ \leq\ -C + m|t_i^{\mu,ex} - t_i^{\mu,en}|, \qquad t \in (\!(t_i^{\mu,en}, t_i^{\mu,ex})\!). \tag{75}$$

In view of (69), $\dot{\eta}_1^-$ is negative on $(\!(t_i^{\mu,en}, t_i^{\mu,ex})\!)$ for sufficiently small $\|\mu-\mu_0\|$, and consequently, $\eta_1^-(t_i^{\mu,en}) = \eta_1^-(t_i^{\mu,en}) - \eta_1^-(t_i^{\mu,ex})$ is nonzero and has the sign of $t_i^{\mu,ex} - t_i^{\mu,en}$. By similar arguments to Rem. 1.6, we can show that $\eta_1^-(t_i^{\mu,en}) = \nu_i^{\mu,1}$, and since $\nu_i^{\mu,1} \geq 0$ by definition of (54), it follows that $t_i^{\mu,ex} > t_i^{\mu,en}$ necessarily holds whenever $t_i^{\mu,en} \neq t_i^{\mu,ex}$, which proves (66). In addition, (75) implies that $\nu_i^{\mu,1} = \eta_1(t_i^{\mu,en+}) > 0$ for μ close enough to μ_0, which show by "contraposition" the "$\Leftarrow$" implication in (67). Finally, relation (68) follows from (74) and (69), which completes the proof. □

Lemma 4.3. *Under assumptions and condition (ii) of Th.* 2.4*, the solution* $(u^\mu, y^\mu, p_1^\mu, \eta_1^\mu)$ *of (46)-(49) for* $\theta = \theta^\mu$*, where* θ^μ *is solution of (54), is such that* (u^μ, y^μ) *is a Pontryagin extremal of* $(\mathcal{P}^\mu)$*, with classical multipliers* (p^μ, η^μ) *given by (29), and the mapping* $\mu \mapsto (u^\mu, y^\mu, p^\mu, \eta^\mu) \in \mathcal{U} \times \mathcal{Y} \times \mathrm{L}^\infty(0,T;\mathbb{R}^{n*}) \times \mathrm{L}^\infty(0,T)$ *is Lipschitz continuous on a neighborhood of* μ_0.

Proof. By Lemma 4.2, we see that $(u^\mu, y^\mu, p_1^\mu, \eta_1^\mu)$ is well-defined over $[0,T]$, and by definition of (54), satisfies alternative formulation (12)-(22), as well as additional condition (25) by Rem. 1.6 (and in particular, u^μ is continuous on $[0,T]$ and so are p^μ and η^μ given by (29)). In view of Rem. 1.4, it remains to show that the additional conditions (23), (24) and (26) are satisfied.

Let $i \in N_{to}$. If $\nu_i^{\mu,1} = 0$, then by Lemma 4.2, $t_i^{\mu,en} = t_i^{\mu,ex}$, u^μ and its time derivative are continuous at $t_i^{\mu,en}$, and $(g^\mu)^{(1)}(u^\mu(t_i^{\mu,en}), y^\mu(t_i^{\mu,en})) = 0$. By (A6) and standard continuity arguments, there exists $\varepsilon > 0$ such that $g^\mu(y^\mu(\cdot))$ attains its maximum over $(\bar{t}_i^{en} - \varepsilon, \bar{t}_i^{en} + \varepsilon)$ at the unique point $t_i^{\mu,en}$. Therefore if $g^\mu(y^\mu(t_i^{\mu,en})) < 0$, the state constraint is locally not active. If $g^\mu(y^\mu(t_i^{\mu,en})) = 0$, then $t_i^{\mu,en}$ is a touch point of the perturbed problem, and (26) holds by (67).

If $\nu_i^{\mu,1} > 0$, then by Lemma 4.2, $t_i^{\mu,en} < t_i^{\mu,ex}$ and we have a boundary arc. By (68), additional condition (24) holds on this boundary arc, and (23) holds near the entry/exit points by (A6). If $i \in I_{ba}$, then (24) holds by continuity on the boundary arc $[t_i^{\mu,en}, t_i^{\mu,ex}]$ by (A5), and (23) holds near the entry/exit points by (35). Finally, outside a small neighborhood of contact points, we obtain $g^\mu(y^\mu) < 0$ by a standard compactness argument. Hence (u^μ, y^μ) is a Pontryagin extremal, with classical multipliers (p^μ, η^μ) given by (29).

Lipschitz continuity of the mapping $\mu \mapsto (u^\mu, y^\mu, p^\mu, \eta^\mu)$ follows from Lipschitz continuity of the mapping $\mu \mapsto \theta^\mu$ by strong regularity (Lemma 4.1), Lipschitz continuity of $(\theta, \mu) \mapsto (u, y, p, \eta)|_k$, where $(u, y, p, \eta)|_k$ denotes the restriction of the solution of (46)-(49) and (29) to "arc" k (possibly a singleton), for all $k = 1, \dots, 2N+1$, and finally Lipschitz continuity of u^μ, $\dot{y}^\mu$, p^μ and η^μ as functions of time on $[0,T]$, uniformly w.r.t. μ. □

So far, we have proved that there exists a unique solution of (54), with which is associated a unique Pontryagin extremal (u^μ, y^μ) for $(\mathcal{P}^\mu)$. It remains to prove the uniqueness of (u^μ, y^μ) among all the Pontryagin extremals (u, y) of $(\mathcal{P}^\mu)$ with u in a L^∞-neighborhood of $\bar{u}$, without restrictions on the arc structure of (u, y).

More precisely, for $\delta > 0$, define

$$\Omega_i^\delta := (\bar{t}_i^{en} - \delta, \bar{t}_i^{ex} + \delta), \qquad i = 1, \dots, N. \tag{76}$$

In view of (A6) and (35), we may fix $\kappa, \bar{\delta} > 0$ satisfying the conditions below:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g^{\mu_0}(\bar{y}(t)) \leq -\kappa < 0 \quad \text{on } \Omega_i^{\bar{\delta}} \setminus [\bar{t}_i^{en}, \bar{t}_i^{ex}], \quad i = 1, \dots, N; \tag{77}$$

$$\text{The sets } \Omega_i^{\bar{\delta}} \text{ are pairwise disjoint and contained in } [0,T]. \tag{78}$$

The next theorem gives a direct result (i.e. without using a shooting formulation) that Pontryagin extremals of the perturbed problem have a neighboring structure to that of $(\bar{u},\bar{y})$, when assumptions (A2)-(A6) are satisfied. Its proof is given in the Appendix.

Theorem 4.4. *Let $(\bar{u},\bar{y})$ be a Pontryagin extremal for $(\mathcal{P}^{\mu_0})$ satisfying (A2)-(A6). Then for all $0<\delta<\bar{\delta}$ and all stable extensions $(\mathcal{P}^{\mu})$ of $(\mathcal{P}^{\mu_0})$, there exists a neighborhood $V_u \times V_\mu \times V_\theta$ of $(\bar{u},\mu_0,\theta_0)$ in $\mathcal{U}\times M \times \Theta$, such that all* Pontryagin extremals *(u,y) of $(\mathcal{P}^{\mu})$ with $(u,\mu)\in V_u \times V_\mu$ satisfies the following properties, with the contact set $I(g^\mu(y))$ defined in (6):*

(S1) *$I(g^\mu(y)) \subset \cup_{i=1}^N \Omega_i^\delta$;*

(S2) *for all $i\in I_{ba}$, $I(g^\mu(y))\cap\Omega_i^\delta$ is an interval of positive measure;*

(S3) *for all $i\in I_{to}$, $I(g^\mu(y))\cap\Omega_i^\delta$ is either empty, or a singleton, or an interval of positive measure,*

and there exists $\theta\in V_\theta$ such that θ is solution of (54) and (u,y) is the trajectory associated with θ.

An immediate consequence of Th. 4.4 is the next lemma.

Lemma 4.5. *Under assumptions and point (ii) of Th.* 2.4*, there exists a* L^∞ *neighborhood V_u of $\bar{u}$ and a neighborhood V_μ of μ_0, such that for all $\mu\in V^\mu$, (u^μ,y^μ) is locally the unique Pontryagin extremal of $(\mathcal{P}^\mu)$ with $u\in V_u$.*

Proof. Let (u,y) be a Pontryagin extremal of $(\mathcal{P}^\mu)$ with (u,μ) in the neighborhood of $(\bar{u},\mu_0)$. By Th. 4.4, there exists θ in the neighborhood of θ_0 solution of (54) and (u,y) is the (unique) trajectory associated with θ. By strong regularity (Lemma 4.1), (54) has in a neighborhood of μ_0 a unique solution θ^μ. Consequently, $\theta=\theta^\mu$ and $(u,y)=(u^\mu,y^\mu)$ is the unique Pontryagin extremal of $(\mathcal{P}^\mu)$ with (u,μ) in the neighborhood of $(\bar{u},\mu_0)$. □

Now we can prove the main result.

Proof of Theorem 2.4. The proof is somewhat similar to that of Th. 4.1 in [3]. By Lemmas 4.1-4.5, (ii) implies that there exists a unique Pontryagin extremal and associated multipliers $(u^\mu,y^\mu,p^\mu,\eta^\mu)$ with (u^μ,μ) in the neighborhood of $(\bar{u},\mu_0)$, and the mapping $\mu\mapsto(u^\mu,y^\mu,p^\mu,\eta^\mu)$ is Lipschitz continuous. To achieve the proof of $(ii)\Rightarrow(i)$, it remains to show that u^μ satisfies the uniform quadratic growth condition, which can be done by slight modifications in the proof of Lemmas 4.7 and 4.8 in [3], that we omit here.

To prove the converse implication, we construct a perturbation of the constraint function g^μ, so that (nonessential) touch points becomes inactive on the perturbed problem $(\mathcal{P}^\mu)$, and $(\bar{u}, \bar{y})$ is a Pontryagin extremal of $(\mathcal{P}^\mu)$ (see the proof of Th. 4.1 in [3]). By (i) and Lemma 4.5, $(\bar{u}, \bar{y})$ is then the local solution of $(\mathcal{P}^\mu)$ satisfying (44), so it follows from the characterization of quadratic growth in [2, Th. 5.1] that the strong second-order sufficient condition (41) holds. □

5 Sensitivity Analysis

By Lemma 4.1, strong regularity holds, and by (56) the mapping $\Xi : V_\delta \to V_\theta$, $\delta \mapsto \theta(\delta)$ solution of (56) is positively homogeneous of degree one. It follows then from (59) that the mapping $\mu \mapsto \theta^\mu$ is Fréchet directionally differentiable. The directional derivatives in direction d are obtained by substituting into (56) δ by $-D_\mu F(\theta_0, \mu_0)d$. Therefore,

$$\theta^{\mu_0+d} = \theta_0 + \omega_d + o(\|d\|), \tag{79}$$

where

$$\omega_d \;=\; (\pi^*_{d,0}, \gamma^1_d, \sigma^{en}_d, \sigma^{ex}_d) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \tag{80}$$

is as follows. Denote by

$$(v_d, z_d, \pi_{1,d}, \zeta_{1,d}, \lambda^1_d) \tag{81}$$

the (unique) optimal solution, costate and multipliers of the linear-quadratic problem below:

$$\begin{aligned}(\mathcal{P}_d)\quad \min_{(v,z)\in\mathcal{V}\times\mathcal{Z}} \ &\tfrac{1}{2}\int_0^T D^2_{(u,y,\mu),(u,y,\mu)}\tilde{H}(\bar{u},\bar{y},\bar{p}_1,\bar{\eta}_1)((v,z,d),(v,z,d))\mathrm{d}t \\ &+ \tfrac{1}{2}D^2\hat{\phi}(\bar{y}(T),\mu_0)((z(T),d),(z(T),d)) \\ &+ \tfrac{1}{2}\sum_{i\in I_{ba}} \bar{\nu}^1_i D^2\hat{g}(\bar{y}(\bar{t}^{en}_i),\mu_0)((z(\bar{t}^{en}_i),d),(z(\bar{t}^{en}_i),d))\end{aligned}$$

$$\begin{aligned}\text{subject to:}\quad &\dot{z}(t) = D\hat{f}(\bar{u},\bar{y},\mu_0)(v,z,d) \quad \text{on } [0,T],\quad z(0)=0 &(82)\\ &D\hat{g}^{(1)}(\bar{u},\bar{y},\mu_0)(v,z,d) = 0 \quad \text{on } \bar{\mathcal{I}}_b &(83)\\ &D\hat{g}(\bar{y}(\bar{t}^{en}_i),\mu_0)(z(\bar{t}^{en}_i),d) = 0, \quad i\in I_{ba} &(84)\\ &D\hat{g}(\bar{y}(\bar{t}^{en}_i),\mu_0)(z(\bar{t}^{en}_i),d) \le 0, \quad i\in I_{to}, &(85)\end{aligned}$$

with $\pi_{1,d}$ associated with the constraint (82), $\zeta_{1,d}$ with (83), and λ^1_d with (84)-(85). Then we have $\pi_{d,0} = \pi_{1,d}(0)$, and

$$\gamma^1_d = \lambda^1_d, \tag{86}$$

$$\sigma^{en}_{d,i} = -\frac{D\hat{g}^{(1)}(\bar{u}(\bar{t}^{en}_i),\bar{y}(\bar{t}^{en}_i),\mu_0)(v_d(\bar{t}^{en-}_i), z_d(\bar{t}^{en}_i), d)}{\frac{\mathrm{d}}{\mathrm{d}t}g^{(1)}(\bar{u},\bar{y})|_{t=\bar{t}^{en-}_i}},\ i=1,\dots,N, \tag{87}$$

$$\sigma^{ex}_{d,i} = -\frac{D\hat{g}^{(1)}(u(\bar{t}^{ex}_i),y(\bar{t}^{ex}_i),\mu_0)(v_d(\bar{t}^{en+}_i), z_d(\bar{t}^{ex}_i), d)}{\frac{\mathrm{d}}{\mathrm{d}t}g^{(1)}(\bar{u},\bar{y})|_{t=\bar{t}^{en+}_i}},\ i=1,\dots,N. \tag{88}$$

Since the mapping $\mu \mapsto \theta^\mu$ is Fréchet directionally differentiable, and the solution $(u^\mu, y^\mu, p_1^\mu, \eta_1^\mu)$ of (46)-(50) is, on each arc, a C^1 function of (θ^μ, μ), combining with the continuity of u^μ and of the classical multipliers p^μ and η^μ given by (29) (since by Lemma 4.3, (u^μ, y^μ) is a Pontryagin extremal of $(\mathcal{P}^\mu)$, see then Remarks 1.4 and 1.6), we obtain (as in [3, Lemma 4.1]) the following result.

Theorem 5.1. *Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal of $(\mathcal{P})$ satisfying (A2)-(A6). If either point (i) or (ii) of Th. 2.4 is satisfied, then there exists a neighborhood V_μ of μ, such that the mapping $\mu \mapsto (u^\mu, y^\mu, p^\mu, \eta^\mu)$ is Fréchet directionally differentiable in the space*

$$\mathrm{L}^r(0,T) \times \mathrm{W}^{1,r}(0,T;\mathbb{R}^n) \times \mathrm{L}^r(0,T;\mathbb{R}^{n*}) \times \mathrm{L}^r(0,T), \qquad \text{for all } 1 \leq r < +\infty,$$

and the derivatives of the state and control in direction d are the optimal solution (v_d, z_d) of linear-quadratic problem $(\mathcal{P}_d)$, while those of the costate p^μ and state constraint multiplier η^μ are obtained, respectively, a.e. by

$$\pi_d(t) = \pi_{1,d}(t) + \zeta_{1,d}(t) g_y^{\mu_0}(\bar{y}(t)) + \bar{\eta}_1(t) D\hat{g}_y(\bar{y}(t), \mu_0)(z_d(t), d) \tag{89}$$

$$\zeta_d(t) = \sum_{i=1}^{N} \gamma_{d,i}^1 \mathbf{1}_{t \geq \bar{t}_i^{en}}(t) - \zeta_{1,d}(t). \tag{90}$$

In addition, all jumps parameters and junction times are Fréchet directionally differentiable w.r.t. μ, and their directional derivative in direction d are given by (86)-(88).

Remark 5.2. We can show that an equivalent formulation of $(\mathcal{P}_d)$ is (see Rem. 2.2(i)) to minimize

$$\begin{aligned} \int_0^T D^2_{(u,y,\mu),(u,y,\mu)} H(\bar{u}, \bar{y}, \bar{p}, \mu_0)((v,z,d),(v,z,d)) \mathrm{d}t \\ + D^2\hat{\phi}(\bar{y}(T), \mu_0)((z(T), d), (z(T), d)) \\ + \int_0^T D^2 g(\bar{y}(t), \mu_0)((z(t), d), (z(t), d) \mathrm{d}\bar{\eta}(t) \end{aligned} \tag{91}$$

for $(v, z) \in \mathcal{V} \times \mathcal{Z}$ subject to the constraints (82), (85) and

$$D\hat{g}(\bar{y}, \mu_0)(z, d) = 0 \qquad \text{on } \bar{\mathcal{I}}_b. \tag{92}$$

This last constraint is equivalent to (83)-(84) since $D\hat{g}^{(1)}(\bar{u}, \bar{y}, \mu_0)(v, z, d) = \frac{\mathrm{d}}{\mathrm{d}t} D\hat{g}(\bar{y}(t), \mu_0)(z(t), d)$. Then, using the relation (89), we can show that π_d, the directional derivative of p^μ w.r.t. μ, is the multiplier associated with (82) in formulation (91)-(92) of $(\mathcal{P}_d)$, and that the directional derivative of $\frac{\mathrm{d}\eta^\mu}{\mathrm{d}t}$ w.r.t. μ, equal by (90) to $\dot{\zeta}_d = -\dot{\zeta}_{1,d}$, is also equal to the multiplier associated with the constraint (92). In addition, it is not difficult to check that $\zeta_{1,d}(\bar{t}_i^{en+}) = \gamma_{d,i}^1$ for all $i \in I_{ba}$, so that discontinuities in π_d and ζ_d may occur only at points $\bar{t}_i^{en}$ for $i \in I_{to}$ when the constraint (85) is active and the associated multiplier is positive.

Let us conclude this section by the following remark. For $i \in I_{to}$, since $\bar{t}_i^{en} = \bar{t}_i^{ex}$, the optimality system of $(\mathcal{P}_d)$, that it easy to derive and omitted here, yields that $H_{uu}v_d + H_{uy}z_d + \pi_{1,d}f_u = 0$ at $\bar{t}_i^{en\pm}$, and that the jump of $\pi_{1,d}$ is given by $[\pi_{1,d}(\bar{t}_i^{en})] = -\gamma_{d,i}^1 g_y(\bar{y}(\bar{t}_i^{en}))$. Hence, the jump of v_d is given by

$$\begin{aligned}[v_d(\bar{t}_i^{en})] &= \gamma_{d,i}^1 H_{uu}^{-1}(\bar{u},\bar{y},\bar{p})(\bar{t}_i^{en}) g_y(\bar{y}(\bar{t}_i^{en})) f_u(\bar{u},\bar{y})(\bar{t}_i^{en}) \\ &= \gamma_{d,i}^1 H_{uu}^{-1}(\bar{u},\bar{y},\bar{p})(\bar{t}_i^{en}) g_u^{(1)}(\bar{u},\bar{y})(\bar{t}_i^{en}),\end{aligned}$$

and we obtain from (87)-(88)

$$\sigma_{d,i}^{ex} - \sigma_{d,i}^{en} = -\frac{g_u^{(1)}(\bar{u},\bar{y})(\bar{t}_i^{en})[v_d(\bar{t}_i^{en})]}{\frac{\mathrm{d}}{\mathrm{d}t} g^{(1)}(\bar{u},\bar{y})|_{t=\bar{t}_i^{en}}} = C_i \gamma_{d,i}^1, \tag{93}$$

with

$$C_i := \frac{H_{uu}^{-1}(\bar{u},\bar{y},\bar{p})(\bar{t}_i^{en})(g_u^{(1)}(\bar{u},\bar{y})(\bar{t}_i^{en}))^2}{-\frac{\mathrm{d}}{\mathrm{d}t} g^{(1)}(\bar{u},\bar{y})|_{t=\bar{t}_i^{en}}} > 0.$$

Since $\gamma_{d,i}^1 \geq 0$ for $i \in I_{to}$, we see that $\sigma_{d,i}^{ex} - \sigma_{d,i}^{en} \geq 0$, with equality iff $\gamma_{d,i}^1 = 0$. It follows that, for $\mu - \mu_0 = d$, the length of the boundary arc and the jump parameter are related, at first order, by

$$t_i^{\mu,ex} - t_i^{\mu,en} = C_i \nu_i^{\mu,1} + o(\|\mu - \mu_0\|). \tag{94}$$

6 Example

We illustrate the results of this paper on a very basic example. We consider the problem of an elastic line of positive mass, fixed at its endpoints and submitted to a vertical uniform force (g). The problem is to find the equilibrium position, i.e. minimize the potential energy. This can be written as the optimal control problem (with t replaced by $x \in [0,1]$):

$$\min \int_0^1 \left(\frac{u(x)^2}{2} + gy(x) \right) \mathrm{d}x \quad ; \quad \dot{y}(x) = u(x) \quad ; \quad y(0) = 0 = y(1).$$

We add a first-order state constraint, e.g. the level of the floor

$$y(x) \geq -h. \tag{95}$$

Here g and h denotes positive constants. Note that the case of final constraint on the state (here, $y(T) = y_T$ given in $\mathbb{R}^n$) are dealt with similarly, replacing in the shooting equations the final condition $p(T) - \phi_y(y(T)) = 0$ by $y(T) - y_T = 0$. The unconstrained optimal trajectory when $h/g \geq 1/8$ is given by:

$$y(x) = \tfrac{1}{2}gx^2 - \tfrac{1}{2}gx \qquad ; \qquad u(x) = gx - \tfrac{1}{2}g. \tag{96}$$

The resolution of the constrained problem when $h/g \leq 1/8$ is as follows. The trajectory is:

$$u(x) = \begin{cases} g(x - x_{en}) & \text{on } [0, x_{en}] \\ 0 & \text{on } [x_{en}, x_{ex}] \\ g((x-1) - (x_{ex} - 1)) & \text{on } [x_{ex}, 1] \end{cases}$$

$$y(x) = \begin{cases} g(x^2/2 - x_{en}x) & \text{on } [0, x_{en}] \\ -h & \text{on } [x_{en}, x_{ex}] \\ g((x-1)^2/2 - (x_{ex} - 1)(x-1)) & \text{on } [x_{ex}, 1] \end{cases}$$

with junction conditions:

$$g(x_{en}^2/2 - x_{en}^2) = -h \qquad ; \quad g((x_{ex}-1)^2/2 - (x_{ex}-1)^2) = -h.$$

Entry and exit positions x_{en} and x_{ex} are given by:

$$x_{en} = \sqrt{2h/g} \qquad ; \qquad x_{ex} = 1 - \sqrt{2h/g}. \tag{97}$$

The alternative state constraint multiplier on $[x_{en}, x_{ex}]$ is given by:

$$\eta_1(x) = p_1(x) = -g(x - x_{ex}) \geq 0, \qquad \dot{\eta}_1(x) = -g < 0,$$

and hence, the jump parameter at entry time is:

$$\nu_{en}^1 = \eta_1(x_{en}) = g(x_{ex} - x_{en}) = g\left(1 - 2\sqrt{2h/g}\right) \geq 0. \tag{98}$$

We consider perturbations w.r.t. nominal values of parameters $g = g_0 = 1$ and $h = h_0 = 1/8$, for which there is a touch point at $x = 1/2$. The no-gap strong sufficient second-order condition clearly holds, since the linear-quadratic problem:

$$\min \int_0^1 \frac{v^2(x)}{2} \mathrm{d}x \quad ; \quad \dot{z}(x) = v(x) \quad ; \quad z(0) = 0 = z(1)$$

having a strongly convex cost function, has $(v, z) = 0$ for unique solution. Let us then study the perturbed quadratic problem at (g_0, h_0) in direction $d := (\gamma, \eta)$:

$$\min \int_0^1 \left(\frac{v(x)^2}{2} - \gamma z(x) \right) \mathrm{d}x \quad ; \quad \dot{z}(x) = v(x) \quad ; \quad z(0) = 0 = z(1),$$

subject to the interior point inequality constraint:

$$z(1/2) \geq -\eta. \tag{99}$$

The unconstrained trajectory is:

$$z_d(x) = \gamma \left(\frac{x^2}{2} - \frac{x}{2} \right) \qquad ; \qquad v_d(x) = \gamma \left(x - \frac{1}{2} \right). \tag{100}$$

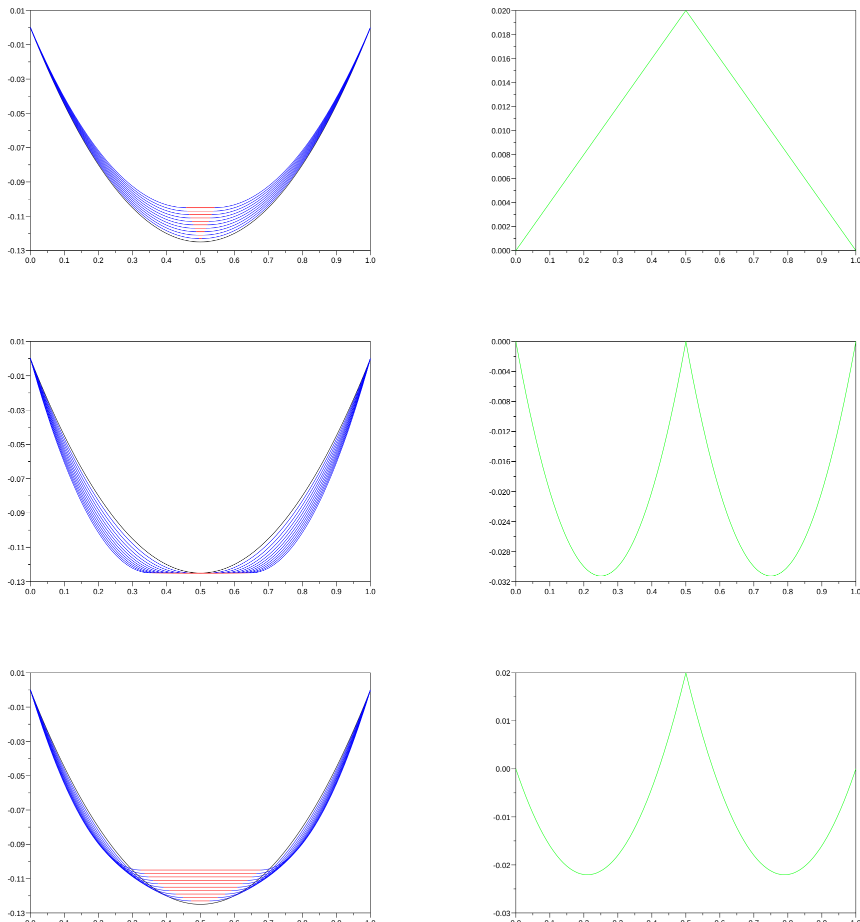

Figure 1: Perturbation of the state (left) and directional derivatives (right) in case (a) to (c) (from top to bottom)

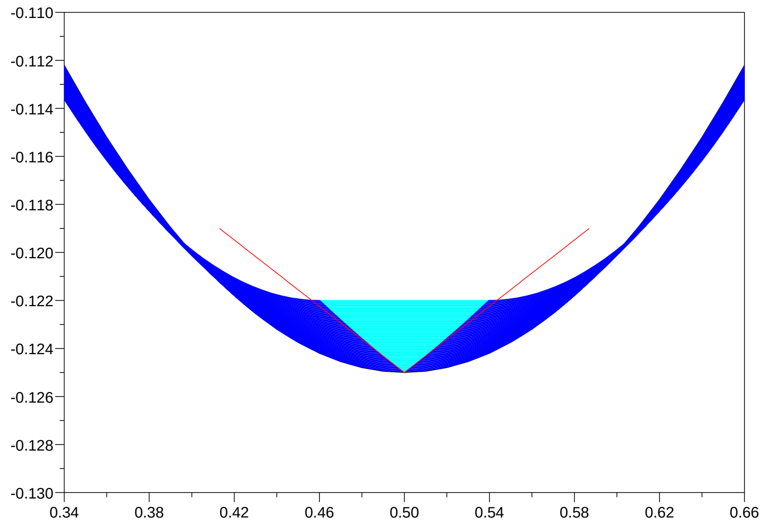

Figure 2: Variation of the lenght of the boundary arc in case (c).

Therefore, the constraint is active, iff $\eta \le \gamma/8$. If $\eta \ge \gamma/8$, (100) corresponds to the directional derivative of the unconstrained trajectory (96). When $\eta \le \gamma/8$, the constraint (99) is active, i.e. $z_d(1/2) = 0$, and therefore, the solution of the linear-quadratic problem is as follows:

$$v_d(x) = \begin{cases} \gamma x - (2\eta + \gamma/4) & \text{on } [0, 1/2] \\ \gamma(x-1) + (2\eta + \gamma/4) & \text{on } [1/2, 1]. \end{cases}$$

$$z_d(x) = \begin{cases} \gamma x^2/2 - (2\eta + \gamma/4)x & \text{on } [0, 1/2] \\ \gamma(x-1)^2/2 + (2\eta + \gamma/4)(x-1) & \text{on } [1/2, 1]. \end{cases}$$

The multiplier λ_d associated with the constraint (99) is, by (93):

$$\lambda_d = [\pi_d(1/2)] = -[v_d(1/2)] = -2(2\eta - \gamma/4) \ge 0, \tag{101}$$

and, by (87)-(88), the variations of entry and exit points $\sigma_{d,en}$ and $\sigma_{d,ex}$ are given by:

$$\sigma_{d,en} = -\frac{v(1/2^-)}{g_0} = -\gamma/4 + 2\eta \quad ; \quad \sigma_{d,ex} = -\frac{v(1/2^+)}{g_0} = \gamma/4 - 2\eta. \tag{102}$$

By (98) and (97), we check that the above formula corresponds to the first-order variations, with $g = g_0 + \gamma$ and $h = h_0 + \eta$, $|\gamma|, |\eta|$ small, of:

$$\nu_{en}^1 = 1 - 2\sqrt{\frac{1/4 + 2\eta}{1+\gamma}} \quad ; \quad x_{en} = \sqrt{\frac{1/4 + 2\eta}{1+\gamma}} \quad ; \quad x_{ex} = 1 - \sqrt{\frac{1/4 + 2\eta}{1+\gamma}}.$$

We consider perturbations in three directions $d = (\gamma, \eta)$:

Case (a) $(\gamma, \eta) = (0, -0.02)$

Case (b) $(\gamma, \eta) = (1, 0)$

Case (c) $(\gamma, \eta) = (1, -0.02)$.

Case (a) corresponds to an elevation of the ground level, case (b) corresponds to an increasing of the "gravitational" force g, both of them leading to the apparition of a boundary arc, and case (c) combines elevation of the ground and increasing of g. The perturbed trajectories and directional derivatives of the state in $\mathrm{W}^{1,r}$, $1 \le r < +\infty$, are presented for each case in Fig. 1. The unconstrained trajectory for (g_0, h_0) is a parabola. In Fig. 2, we focus on the apparition of the boundary arc in case (c), check that its lenght is of order the perturbation and compare with the directional derivatives of variation of junction times (102).

A Proof of Theorem 4.4

The proof of Theorem 4.4 will use two lemmas. Note that by continuity of the mapping $(u, \mu) \mapsto g^\mu(y_u^\mu)$, it is immediate that all Pontryagin extremals of a stable extension $(\mathcal{P}^\mu)$ with (u, μ) in the neighborhood of $(\bar{u}, \mu_0)$ satisfy (S1). Let us first define new alternative multipliers needed in lemma A.2 below (see also [13, 12, 9, 6, 7] where these multipliers are used)

$$\eta^1(t) \;:=\; \int_t^T \mathrm{d}\eta(s) \;=\; \eta(T) - \eta(t^+) \tag{103}$$

$$p^1(t) \;:=\; p(t) - \eta^1(t) g_y^\mu(y(t)). \tag{104}$$

Remark A.1. On $[0, T] \setminus \mathcal{T}$, η^1 and p^1 are related to p_1 and η_1 by the following relations:

$$\eta^1(t) = \sum_{\tau \in \mathcal{T}_{en}} \nu_\tau^1 \mathbf{1}_{t<\tau}(t) + \eta_1(t) \quad ; \quad p^1(t) = p_1(t) - \sum_{\tau \in \mathcal{T}_{en}} \nu_\tau^1 \mathbf{1}_{t<\tau}(t) g_y^\mu(y(t)), \tag{105}$$

with $\mathbf{1}_{t<\tau}(t) = 1$ if $t < \tau$ and zero otherwise.

With this definition, and without any assumptions on the arc structure of the trajectory (i.e. without assuming a finite number of junction points), we have that

$$-\mathrm{d}p^1 = (H_y^\mu(u, y, p^1) + (g^\mu)^{(1)}(y)\eta^1)\mathrm{d}t,$$

and hence, the new alternative costate p^1 is absolutely continuous. Consequently, an equivalent form of (8)-(10) (when $\alpha = 1$) is, a.e. on $[0, T]$:

$$\left\{ \begin{array}{rcl} -\dot{p}^1(t) &=& H_y^\mu(u(t), y(t), p^1(t)) + (g^\mu)_y^{(1)}(u(t), y(t))\eta^1(t) \\ p^1(T) &=& \phi_y^\mu(y(T)) \end{array} \right. \tag{106}$$

$$0 \;=\; H_u^\mu(u(t), y(t), p^1(t)) + (g^\mu)_u^{(1)}(u(t), y(t))\eta^1(t). \tag{107}$$

In addition, (11) implies the following (weaker) relations, since η^1 is constant on interior arcs:

$$
\begin{aligned}
0 &= g^{(1)}(u(t), y(t)) \qquad \text{on boundary arcs} && (108) \\
0 &= \dot{\eta}^1(t) \qquad \text{on interiors arcs.} && (109)
\end{aligned}
$$

It follows that when relations (27)-(28) are satisfied, η^1 is absolutely continuous. We shall need the following result (convergence of multipliers).

Lemma A.2. *Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal of $(\mathcal{P}^{\mu_0})$ satisfying (A2)-(A3). Consider a stable extension $(\mathcal{P}^{\mu})$, and let $(u_n, y_n = y_{u_n}^{\mu_n})$ be a Pontryagin extremal for $(\mathcal{P}^{\mu_n})$, such that $u_n \to \bar{u}$ in L^∞ and $\mu_n \to \mu_0$. Denote by p_n, η_n the (unique) multipliers associated with (u_n, y_n), and let p_n^1, η_n^1 be given by (103)-(104). Then:*

1. *The sequence (η_n) is bounded in $\mathcal{M}[0,T]$;*
2. *$\|\eta_n - \bar{\eta}\|_{1,\infty*} \to 0$, where $\|\cdot\|_{1,\infty*}$ denote the norm of the dual of $\mathrm{W}^{1,\infty}$;*
3. *$p_n^1 \to \bar{p}^1$ uniformly over $[0,T]$;*
4. *$\eta_n^1 \to \bar{\eta}^1$ uniformly over $[0,T]$.*

Remark A.3. Note that under the assumptions of Lemma A.2, by (104) we deduce that $\|p_n - \bar{p}\|_\infty \to 0$, and by (103), $\|\eta_n - \bar{\eta}\|_\infty \to 0$.

By (A3) (and (A1)), it is not difficult to see that Robinson's constraint qualification holds at point $\bar{u}$ for $(\mathcal{P}^{\mu_0})$, i.e. there exists $\gamma > 0$, such that

$$
\gamma B_{\mathrm{C}[0,T]} \subset G^{\mu_0}(\bar{u}) + DG^{\mu_0}(\bar{u})\mathcal{U} - K, \tag{110}
$$

with $B_{\mathrm{C}[0,T]}$ the unit (open) ball of space of continuous functions, $K = \mathrm{C}_-[0,T]$ the cone of nonpositive continuous functions, and G^μ the mapping $\mathcal{U} \to \mathrm{C}[0,T]$, $u \mapsto g^\mu(y_u^\mu)$.

Now let (u, y) be a Pontryagin extremal of a stable extension $(\mathcal{P}^\mu)$, and let p, η, α be the associated multipliers. For (u, μ) close enough to $(\bar{u}, \mu_0)$, Robinson's constraint qualification (110) still holds, so it is easy to see that (u, y) is necessarily a *normal* Pontryagin extremal (i.e. $\alpha = 1$), and the associated multipliers (p, η) are unique. Note that equation (11) can be equivalently rewritten as:

$$
g^\mu(y) \in K \quad ; \quad \eta \in \mathcal{M}_+[0,T] \quad ; \quad \operatorname{supp}(\mathrm{d}\eta) \subset I(g^\mu(y)), \tag{111}
$$

where $\operatorname{supp}(\mathrm{d}\eta)$ denotes the support of the measure $\mathrm{d}\eta$.

Proof of Lemma A.2. Let $\delta > 0$. By continuity of the mapping $(u, \mu) \mapsto g^\mu(y_u^\mu)$, there exists $\delta > 0$, such that for n large enough (this is precisely assertion (S1)),

$$
I(g^{\mu_n}(y_n)) \subset \Omega^\delta := \cup_{i=1}^N \Omega_i^\delta. \tag{112}
$$

The first assertion of the lemma is a classical consequence of Robinson's constraint qualification (110) (see e.g. [4, Prop. 4.43]). By (A3), reducing δ if necessary, the linear mapping

$$\mathcal{U} \to \mathrm{W}^{1,\infty}(\Omega^{\delta}) \quad ; \quad v \mapsto (DG^{\mu_0}(\bar{u})v)|_{\Omega^{\delta}},$$

where $|_{\Omega^{\delta}}$ denotes the restriction to the set Ω^{δ}, is onto (e.g. Lemma 2.3 in [2]). Since $\mathrm{supp}(\mathrm{d}\eta_n) \subset I(g^{\mu_n}(y_n)) \subset \Omega^{\delta}$ by (112), the second assertion follows from [4, Prop. 4.44 and Rem. 4.45(i)].

Set $\bar{p}^1_n := p^1_n - \bar{p}^1$, $\bar{\eta}_n := \eta_n - \bar{\eta}$ and $\bar{\eta}^1_n := \eta^1_n - \bar{\eta}^1$. By (106), $\bar{p}^1_n$ is solution of:

$$-\dot{\bar{p}}^1_n(t) \;=\; \bar{p}^1_n A(t) + c(t)\bar{\eta}^1_n + r_n(t),$$

with $A(t) := f^{\mu_0}_y(\bar{u}(t), \bar{y}(t))$, $c(t) := (g^{\mu_0})^{(1)}_y(\bar{u}(t), \bar{y}(t))$ and $\|r_n\|_\infty \to 0$ when $n \to +\infty$. Denote by $R^{t_1}_{t_2}$ the flow of the linear system

$$-\dot{x}(t) = x(t)A(t) \tag{113}$$

i.e. $R^{t_1}_{t_2}x_2 = x(t_1)$, where x is solution of (113) on $[t_1, t_2]$ with initial condition $x(t_2) = x_2$. Then we have, for all $0 \le t \le t_2 \le T$:

$$\bar{p}^1_n(t) = \bar{p}^1_n(t_2)\mathcal{R}^t_{t_2} + \int_t^{t_2} c(s)\mathcal{R}^t_s\bar{\eta}^1_n(s)\mathrm{d}s + o_\infty(1)(t), \tag{114}$$

where $o_\infty(1)$ denotes a function that goes to zero in L^∞ when $n \to +\infty$.

Let us show that $\bar{\eta}^1_n \to 0$ in $\mathrm{L}^1(0,T)$. Indeed, we have:

$$\int_0^T |\bar{\eta}^1_n(t)|\mathrm{d}t \;=\; \int_0^T |\int_t^T \mathrm{d}\bar{\eta}_n(s))|\mathrm{d}t \;\le\; \int_0^T \int_t^T |\mathrm{d}\bar{\eta}_n(s)|\mathrm{d}t.$$

By Fubini's Theorem, it follows that

$$\begin{aligned}\int_0^T |\bar{\eta}^1_n(t)|\mathrm{d}t \;&\le\; \int_0^T (\int_0^s \mathrm{d}t)|\mathrm{d}\bar{\eta}_n|(s)\\ &=\; \int_0^T s|\mathrm{d}\bar{\eta}_n|(s) = \langle |\mathrm{d}\bar{\eta}_n|, I_d\rangle \;\le\; C\|\mathrm{d}\bar{\eta}_n\|_{1,\infty*} \;\to\; 0,\end{aligned}$$

with I_d the identity function in $[0,T]$, $\langle\cdot,\cdot\rangle$ the duality product over $\mathrm{W}^{1,\infty*} \times \mathrm{W}^{1,\infty}$ and $|\cdot|$ the total variation. By (114) and final condition of costate, we deduce that

$$\begin{aligned}|\bar{p}^1_n(t)| \;&\le\; |\phi^{\mu_n}_y(y_n(T)) - \phi^{\mu_0}_y(\bar{y}(T))|\|\mathcal{R}^{\cdot}_T\|_\infty + \int_t^T |c(s)\mathcal{R}^t_s||\bar{\eta}^1_n(s)|\mathrm{d}s + o_\infty(1)\\ &\le\; C|\phi^{\mu_n}_y(y_n(T)) - \phi^{\mu_0}_y(\bar{y}(T))| + \|c\|_\infty\|\mathcal{R}^{\cdot}_T\|_\infty\|\int_0^T |\bar{\eta}^1_n(s)|\mathrm{d}s + o_\infty(1)\\ &=\; o_\infty(1),\end{aligned}$$

which shows the third assertion. Finally, by (107), we have near the contact set on Ω^δ:

$$\eta^1 \;=\; -\frac{H_u^{\mu_n}(u_n, y_n, p_n^1)}{(g^{\mu_n})_u^{(1)}(u_n, y_n)} \;\rightarrow\; -\frac{H_u^{\mu_0}(\bar{u}, \bar{y}, \bar{p}^1)}{(g^{\mu_0})_u^{(1)}(\bar{u}, \bar{y})} \;=\; \bar{\eta}^1 \quad \text{uniformly on } \Omega^\delta,$$

and η^1 is piecewise constant on $[0,T]\backslash\Omega^\delta$, which shows the last assertion. □

Lemma A.4. *Let $(\bar{u}, \bar{y})$ be a Pontryagin extremal for $(\mathcal{P}^{\mu_0})$ satisfying (A2)-(A6). Then for all $0 < \delta < \bar{\delta}$ and all stable extensions $(\mathcal{P}^\mu)$ of $(\mathcal{P}^{\mu_0})$, there exists neighborhoods $V_u \times V_\mu$ of $(\bar{u}, \mu_0)$ in $\mathcal{U} \times M$, such that if (u, y) is a* Pontryagin extremal *of $(\mathcal{P}^\mu)$ with $(u, \mu) \in V_u \times V_\mu$, then (u, y) has* no interior arc *contained in Ω_i^δ, for all $i = 1, \ldots, N$.*

We will denote in the proof of Lemma A.4 below the second-order time derivative of the state constraint by:

$$(g^\mu)^{(2)}(\dot{u}, u, y) := (g^\mu)_u^{(1)}(u, y)\dot{u} + (g^\mu)_y^{(1)}(u, y) f^\mu(u, y) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} g^\mu(y(t)). \quad (115)$$

Note that by (107), for any Pontryagin extremal (u, y) of $(\mathcal{P}^\mu)$ satisfying (27)-(28), $\dot{u}$ and $\dot{\eta}^1$ are solutions, on the interior of an interior and boundary arc, respectively, to the following systems (omitting time argument):

$$\left\{ \begin{array}{rcl} 0 &=& \tilde{H}_{uu}^\mu(u, y, p^1, \eta^1)\dot{u} + \tilde{H}_{uy}^\mu(u, y, p^1, \eta^1) f^\mu(u, y) \\ && \quad - \tilde{H}_y^\mu(u, y, p^1, \eta^1) f_u^\mu(u, y) + (g^\mu)_u^{(1)}(u, y)\dot{\eta}^1 \\ 0 &=& (g^\mu)_u^{(1)}(u, y)\dot{u} + g_y^{(1)}(u, y) f^\mu(u, y) = (g^\mu)^{(2)}(\dot{u}, u, y), \end{array} \right. \quad (116)$$

$$\left\{ \begin{array}{rcl} 0 &=& \tilde{H}_{uu}^\mu(u, y, p^1, \eta^1)\dot{u} + \tilde{H}_{uy}^\mu(u, y, p^1, \eta^1) f^\mu(u, y) \\ && \quad - \tilde{H}_y^\mu(u, y, p^1, \eta^1) f_u^\mu(u, y) \\ 0 &=& \dot{\eta}^1, \end{array} \right. \quad (117)$$

and therefore $(\dot{u}, \dot{\eta}^1)$ can be expressed, on the interior of each arc, as a C^1 function of (u, y, p^1, η^1) and μ.

Proof of Lemma A.4. We argue by contradiction. Suppose that the statement of the lemma were false. Then there exists a stable-extension $(\mathcal{P}^\mu)$, $\delta_0 \in (0, \bar{\delta}]$, and (u_n, μ_n) such that $\|u_n - \bar{u}\|_\infty, \|\mu_n - \mu_0\| \rightarrow 0$, $(u_n, y_n = y_{u_n}^{\mu_n})$ is a Pontryagin extremal for $(\mathcal{P}^{\mu_n})$, and there exists $i_n \in \{1, \ldots, N\}$ and $(t_n^1, t_n^2) \in \Omega_{i_n}^{\delta_0}$, such that (t_n^1, t_n^2) is an interior arc of $g^{\mu_n}(y_n)$. Taking if necessary a subsequence, assume that $i_n = i \in \{1, \ldots, N\}$ is constant. Denote by p_n, η_n the classical multipliers associated with (u_n, y_n) and p_n^1, η_n^1 given by (103)-(104). By Lemma A.2, (27) holds for sufficiently large n, implying the continuity of u_n over $[0, T]$ and also that of η_n^1 by (A3), so $\dot{u}_n$ and $\dot{\eta}_n^1$ are given by (116)-(117) on interior of arcs.

Since $g^{\mu_n}(y_n(t_n^1)) = 0 = g^{\mu_n}(y_n(t_n^2))$, and $g^{\mu_n}(y_n)$ is negative on (t_n^1, t_n^2), $g^{\mu_n}(y_n)$ attains its minimum over $[t_n^1, t_n^2]$ at a time $t_n^{min} \in (t_n^1, t_n^2)$. Since the second-order derivative $(g^{\mu_n})^{(2)}(\dot u_n, u_n, y_n)$ is continuous on the interior arc (t_n^1, t_n^2), we deduce that

$$(g^{\mu_n})^{(2)}(\dot u_n(t_n^{min}), u_n(t_n^{min}), y_n(t_n^{min})) \geq 0. \tag{118}$$

Assume that $i \in I_{to}$. Since the touch point $\bar t_i^{en} \in \Omega_i^{\delta_0}$ is non essential, $\dot u_n$ and $\dot u$ satisfy (117) on (t_n^1, t_n^2), and therefore can be expressed as C^1 functions of $(\mu_n, u_n, y_n, p_n^1, \eta_n^1)$ and $(\mu_0, \bar u, \bar y, \bar p^1, \bar\eta^1)$ respectively. Due to the uniform convergence of $(\mu_n, u_n, y_n, p_n^1, \eta_n^1)$ to $(\mu_0, \bar u, \bar y, \bar p^1, \bar\eta^1)$, we have that $\dot u_n \to \dot{\bar u}$ uniformly on (t_n^1, t_n^2) and hence, $(g^{\mu_n})^{(2)}(\dot u_n, u_n, y_n) \to (g^{\mu_0})^{(2)}(\dot{\bar u}, \bar u, \bar y)$ uniformly on (t_n^1, t_n^2). Consequently, by (77), for n large enough, we deduce that

$$(g^{\mu_n})^{(2)}(\dot u_n, u_n, y_n) \leq -\kappa/2 \qquad \text{on } (t_1^n, t_2^n), \tag{119}$$

which contradicts (118).

Assume now that $i \in I_{ba}$. Since we have (by definition of an interior arc) $(g^{\mu_n})^{(2)}(\dot u_n(t), u_n(t), y_n(t)) \leq 0$ when $t \to t_n^{1+}$ and $t \to t_n^{2-}$, it follows from (118) and the continuity of $(g^{\mu_n})^{(2)}(\dot u_n, u_n, y_n)$ on (t_n^1, t_n^2), that there exists at least one time $t_n^0 \in (t_n^1, t_n^2)$, such that

$$(g^{\mu_n})^{(2)}(\dot u_n(t_n^0), u_n(t_n^0), y_n(t_n^0)) = 0. \tag{120}$$

By the same arguments as above, we have that $(g^{\mu_n})^{(2)}(\dot u_n, u_n, y_n)$ converges uniformly to $(g^{\mu_0})^{(2)}(\dot{\bar u}, \bar u, \bar y)$ on $(t_n^1, t_n^2) \cap (\bar t_i^{en} - \delta_0, \bar t_i^{en})$ and on $(t_n^1, t_n^2) \cap (\bar t_i^{ex}, \bar t_i^{ex} + \delta_0)$, whenever these sets are nonempty. Using (77) again, we deduce that for sufficiently large n, (119) holds on $(t_n^1, t_n^2) \cap (\bar t_i^{en} - \delta_0, \bar t_i^{en})$ and $(t_n^1, t_n^2) \cap (\bar t_i^{ex}, \bar t_i^{ex} + \delta_0)$. Consequently, we necessarily have

$$t_n^0 \in (\bar t_i^{en}, \bar t_i^{ex}).$$

By (120), it follows that $(\mu_n, u_n, \dot u_n, y_n, p_n^1, \eta_n^1, \dot\eta_n^1)$ satisfies (116) punctually at time t_n^0 (with $\dot\eta_n^1(t_n^0) = 0$), and $(\mu_0, \bar u, \dot{\bar u}, \bar y, \bar p^1, \bar\eta^1, \dot{\bar\eta}^1)$ also satisfies (116) since we are on a boundary arc. By (A2)-(A3), it follows that the (punctual) values of $(\dot u_n(t_n^0), \dot\eta_n^1(t_n^0))$ and $(\dot{\bar u}(t_n^0), \dot{\bar\eta}^1(t_n^0))$ are given as C^1 functions of $(\mu_n, u_n, y_n, p_n^1, \eta_n^1)(t_n^0)$ and $(\mu_0, \bar u, \bar y, \bar p^1, \bar\eta^1)(t_n^0)$ respectively. Hence, by uniform convergence of $(\mu_n, u_n, y_n, p_n^1, \eta_n^1)$ to $(\mu_0, \bar u, \bar y, \bar p^1, \bar\eta^1)$, for all $\varepsilon > 0$, taking sufficiently large n, we obtain

$$|\dot u_n(t_n^0) - \dot{\bar u}(t_n^0)| + |\dot\eta_n^1(t_n^0) - \dot{\bar\eta}^1(t_n^0)| \leq \varepsilon. \tag{121}$$

But by (A5), we have $\dot{\bar\eta}^1(t_n^0) < -\beta$, whereas $\dot\eta_n^1(t_n^0) = 0$ since we are on an interior arc for (u_n, y_n), contradicting (121). Consequently, there is no interior arc of (u_n, y_n) included in Ω_i^δ, for all $i = 1, \ldots, N$, which achieves the proof of the lemma. □

Now we are ready to give the proof of Th. 4.4.

Proof of Th. 4.4. Assertion (S1) is immediate, and (S3) follows easily from Lemma A.4 since there is no interior arc of (u, y) in Ω_i^δ. In view of Lemma A.4, to complete the proof of (S2), it remains to show that $\Omega_i^\delta \cap I(g^\mu(y))$ is an interval of positive measure, i.e. a boundary arc. Suppose this is false. Then there exists a stable extension $(\mathcal{P}^\mu)$, sequences $u_n \to \bar{u}$ in L^∞, $\mu_n \to \mu_0$, and (u_n, y_n) a Pontryagin extremal of $(\mathcal{P}^{\mu_n})$, such that $\Omega_i^\delta \cap I(g^{\mu_n}(y_n))$ is either empty or a singleton. In both cases, this implies that there exists an interval of positive measure $[\bar{t}_1, \bar{t}_2]$, such that $[\bar{t}_1, \bar{t}_2] \subset [\bar{t}_i^{en}, \bar{t}_i^{ex}]$ and $[\bar{t}_1, \bar{t}_2] \cap I(g^{\mu_n}(y_n)) = \emptyset$ for all n. This entails in particular that $[\bar{t}_1, \bar{t}_2] \cap \mathrm{supp}(\mathrm{d}\eta_n) = \emptyset$. Let φ be a C^∞ function with support in $[\bar{t}_1, \bar{t}_2]$ which is positive on $(\bar{t}_1, \bar{t}_2)$, then we have $\int_0^T \varphi(t)\mathrm{d}\eta_n(t) = 0$, for all n. But by (A5), $\bar{\eta}$ has a positive density over $(\bar{t}_1, \bar{t}_2)$, and hence, $\int_0^T \varphi(t)\mathrm{d}\bar{\eta}(t) > 0$, which contradicts the second assertion in Lemma A.2. This achieves the proof of assertion (S2). Note that using the same argument, we can show that for any sequence of Pontryagin extremals (u_n, y_n) of $(\mathcal{P}^{\mu_n})$ with $u_n \to \bar{u}$ in L^∞ and $\mu_n \to \mu_0$, setting $\Omega_i^\delta \cap I(g^{\mu_n}(y_n)) := [\tau_n^1, \tau_n^2]$, it is necessary that

$$\bar{t}_i^{en} \geq \limsup_{n+\infty} \tau_n^1 \qquad ; \qquad \bar{t}_i^{ex} \leq \liminf_{n\to+\infty} \tau_n^2. \tag{122}$$

Note that whenever a Pontryagin extremal (u, y) for $(\mathcal{P}^\mu)$ satisfies (S1)-(S3), it has finitely many junction times, so it make sense to speak of the finite-dimensional vector of "shooting parameters" (initial costate, jump parameters at entry times, and junction times) such that (u, y) is solution of the alternative formulation. Now construct θ as follows. For all $i \in I_{to}$, if the constraint is not active on Ω_i^δ, add to the set of shooting parameters the (unique) time in Ω_i^δ where $g^\mu(y)$ attains its maximum over Ω_i^δ, duplicate all such times as well as touch points, add a zero jump parameter for each of them, and obtain then a $\theta \in \Theta$ such that θ is solution of (54), and (u, y) is the trajectory associated with θ.

The convergence of θ to θ_0 when $\mu \to \mu_0$ is easily obtained. More precisely, the convergence of initial costate follows from Rem. A.3. The convergence of jump parameters follows from assertion (4) in Lemma A.2 (recall that η^1 and η_1 are linked by (105) and $\eta_1 = 0$ on interior arcs). For $i \in I_{ba}$, knowing that $\Omega_i^\delta \cap I(g^\mu(y))$ is an interval (t_1^μ, t_2^μ), by letting $\delta \to 0$, we obtain $\bar{t}_i^{en} \leq \liminf_{\mu\to\mu_0} t_1^\mu$ and $\bar{t}_i^{ex} \geq \limsup_{\mu\to\mu_0} t_2^\mu$ (the converse inequality follows from (122)), which shows the convergence of entry/exit points for $i \in I_{ba}$. Similarly, letting $\delta \to 0$, we obtain the convergence of touch points and entry/exit points of boundary arcs to the common value $\bar{t}_i^{en}$, for $i \in I_{to}$. Finally, the convergence of nonactive local isolated maxima of $g^\mu(y)$ in Ω_i^δ when $i \in I_{to}$, is obtained by classical arguments, since (34) holds and locally on Ω_i^δ, u and $\dot{u}$ being continuous on interiors arcs, $g^\mu(y)$ belongs to a $\mathrm{W}^{2,\infty}$ (in fact C^2) neighborhood of $g^{\mu_0}(\bar{y})$. □

References

[1] P. Berkmann and H.J. Pesch. Abort landing in windshear: optimal control problem with third-order state constraint and varied switching structure. *J. of Optimization Theory and Applications*, 85, 1995.

[2] J.F. Bonnans and A. Hermant. No gap second order optimality conditions for optimal control problems with a single state constraint and control. INRIA Research Report 5837, to appear in Mathematical Programming.

[3] J.F. Bonnans and A. Hermant. Well-posedness of the shooting algorithm for state constrained optimal control problems with a single constraint and control. INRIA Research Report 5889, 2006.

[4] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems*. Springer-Verlag, New York, 2000.

[5] A.E. Bryson, W.F. Denham, and S.E. Dreyfus. Optimal programming problems with inequality constraints I: necessary conditions for extremal solutions. *AIAA Journal*, 1:2544–2550, 1963.

[6] A.L. Dontchev and W.W. Hager. Lipschitzian stability for state constrained nonlinear optimal control. *SIAM J. on Control and Optimization*, 36(2):698–718 (electronic), 1998.

[7] W.W. Hager. Lipschitz continuity for constrained processes. *SIAM J. Control Optim.*, 17(3):321–338, 1979.

[8] A. Haraux. How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. *Journal Mathematical Society of Japan*, 29:615–631, 1977.

[9] R.F. Hartl, S.P. Sethi, and R.G. Vickson. A survey of the maximum principles for optimal control problems with state constraints. *SIAM Review*, 37:181–218, 1995.

[10] D.H. Jacobson, M.M. Lele, and J.L. Speyer. New necessary conditions of optimality for control problems with state-variable inequality contraints. *J. of Mathematical Analysis and Applications*, 35:255–284, 1971.

[11] K. Malanowski. Two-norm approach in stability and sensitivity analysis of optimization and optimal control problems. *Advances in Mathematical Sciences and Applications*, 2:397–443, 1993.

[12] K. Malanowski. Stability and sensitivity of solutions to nonlinear optimal control problems. *Journal of Applied Mathematics & Optimization*, 32:111–141, 1995.

[13] K. Malanowski and H. Maurer. Sensitivity analysis for state constrained optimal control problems. *Discrete and Continuous Dynamical Systems*, 4:241–272, 1998.

[14] H. Maurer. On the minimum principle for optimal control problems with state constraints. Schriftenreihe des Rechenzentrum 41, Universität Münster, 1979.

[15] H. Maurer and H.J. Pesch. Solution differentiability for nonlinear parametric control problems. *SIAM Journal on Control and Optimization*, 32:1542–1554, 1994.

[16] S.M. Robinson. Strongly regular generalized equations. *Mathematics of Operations Research*, 5:43–62, 1980.

[17] J. Sokolowski. Sensitivity analysis of control constrained optimal control problems for distributed parameter systems. *SIAM Journal on Control and Optimization*, 25:1542–1556, 1987.

[18] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis*. Springer-Verlag, New-York, 1993.