

# Alternating projections on manifolds

Adrian Lewis\*      Jérôme Malick†

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## Abstract

We prove that if two smooth manifolds intersect transversally, then the method of alternating projections converges locally at a linear rate. We bound the speed of convergence in terms of the angle between the manifolds, which in turn we relate to the modulus of metric regularity for the intersection problem, a natural measure of conditioning. We discuss a variety of problem classes where the projections are computationally tractable, and we illustrate the method numerically on a problem of finding a low-rank solution of a matrix equation.

**Key words:** alternating projections, nonconvex, linear convergence, subspace angle, metric regularity, low-rank approximation, spectral set

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## 1 Introduction

The method of alternating projections finds a point in the intersection of two closed convex sets by iteratively projecting a point onto one set and then the other. Popular because of its simplicity and intuitive appeal, the method has been rediscovered many times in the literature. The survey article [BB96] covers much of the history; a careful development of the method appears in [Deu01]. Many practitioners have experimented with the method and its enhancements, in a wide variety of applications: typical examples are signal processing [Com93], finance [Hig02], and the “perceptron algorithm” in machine learning (see for example [WW96]). The method extends in an obvious manner to find points in the intersection of several sets.

The attractive theory and extensive practice of alternating projections for convex feasibility makes it tempting to experiment with analogous heuristics for *nonconvex* feasibility problems. Two very important application areas well-suited to such techniques are low-order control design problems (see

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\*School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY 14853, U.S.A. [aslewis@orie.cornell.edu](mailto:aslewis@orie.cornell.edu) [legacy.orie.cornell.edu/~aslewis](http://legacy.orie.cornell.edu/~aslewis). Research supported in part by National Science Foundation Grant DMS-0504032.

†School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY 14853, U.S.A. [jerome.malick@inria.fr](mailto:jerome.malick@inria.fr).

for example [GB00], and [OHM05] for enhancements), and phase retrieval in image processing (see for example [BCL02]). Existing theory is sparse and much weaker than the convex case [CT90], and has not explained some substantial practical successes with such methods. Our aim in this work is to enhance theoretical understanding of nonconvex alternating projections. We consider the simplest case, that of alternating projections onto two smooth manifolds, intersecting transversally. Locally, the manifolds can be approximated by affine subspaces, and since in the case of subspaces, the method of alternating projections converges linearly, one might also hope (as expressed in [Ors06], supported by numerical evidence) for linear convergence in the manifold case. Our main result is a proof of local linear convergence.

Some of the appeal of the alternating projection method for convex feasibility problems is the ease of the projection subproblem. If a closed set in a Euclidean space is convex, then any point has a unique nearest point in the set (and indeed the converse is also true, by the Motzkin-Bunt theorem [BL05]). Furthermore, providing the set is reasonably described, computing the projection is tractable computationally: modern interior point methods provide one avenue [NN94].

By contrast, for nonconvex sets, the projection mapping can no longer be single-valued, and may be hard to compute. Furthermore, even if two closed nonconvex sets have a nonempty intersection, very simple examples show we cannot expect alternating projections to converge in general. On the other hand, smooth manifolds belong to a large class of interesting sets that admit unique projections *locally* (specifically, “prox-regular” sets [PRT00]). Furthermore, for some fundamental nonconvex sets, the projection problem is computationally cheap. An obvious example is projection onto the unit sphere, but, more generally, projection onto any set defined by a single quadratic equation or inequality is an easy problem numerically. By arguments analogous to the theory for the classical “trust region subproblem” [NW99], any optimal solution of the projection problem

$$\min_x \left\{ \|x - a\|^2 : x^\top Ax + b^\top x = c \right\},$$

associates with a (scalar) Lagrange multiplier solving the Lagrangian dual problem. This latter problem is a univariate maximization, solvable very efficiently by diagonalizing the symmetric matrix  $A$  and applying a specially-designed Newton-type method. The approach for an inequality constraint is very similar.

The singular value decomposition furnishes another efficient and well-known nonconvex projection technique [HJ89]. If the real matrix  $A$  has singular value decomposition  $UDV^\top$ , where the matrices  $U$  and  $V$  are orthogonal, and the matrix  $D$  is nonnegative on its main diagonal and zero off it, then by replacing by zero all the main diagonal entries of  $D$  except the  $r$  largest, we obtain a nearest matrix to  $A$  (with respect to the Frobenius

norm) from the set of matrices with rank no more than  $r$ .

The spectral decomposition for symmetric matrices gives access to a broad range of projection techniques onto nonconvex “spectral” sets: that is, sets of matrices defined via properties of their eigenvalues. For example, given any symmetric matrix, a nearest matrix, with respect to the Frobenius norm, with given eigenvalues (and multiplicities) is easy to compute. This observation was used recently in [Ors06] in an alternating projection method to solve nonnegative inverse eigenvalue problems. Equally easy to compute is a nearest matrix from the set of matrices having largest eigenvalue multiplicity at least  $k$ . The (locally identical) set of matrices having largest eigenvalue multiplicity exactly  $k$  is a manifold, and [Ous00] uses the corresponding projection as part of an eigenvalue optimization algorithm. We summarize general results about projections onto spectral sets of symmetric matrices in an appendix.

After outlining our notation in Section 2, we discuss the notion of angle between subspaces (and manifolds) in Section 3, a key idea both in the classical convergence theory for alternating projection on subspaces and for our extension to manifolds. We prove our main result—that the alternating projection method on transversal manifolds converges linearly locally—in Section 4. Just as in the classical theory for subspaces, the angle predicts the rate of linear convergence for the method. In Section 5, we relate this constant to a natural measure of the “conditioning” of the underlying feasibility problem. Finally in Section 6, we illustrate the theory with a numerical example, seeking a low rank solution of a linear matrix equation. Our aim in this work is not to develop efficient numerical schemes. Indeed, even for the classical alternating projection method on subspaces, many authors have observed the slow convergence of the raw method, and have experimented with enhancements. Our goal here is primarily to initiate a solid theoretical explanation for observed successes of heuristics based on nonconvex alternating projections.

## 2 Notation and basic results

We begin with elementary definitions. In this paper, we will consider a Euclidean space  $\mathbb{E}$  (in other words, a finite-dimensional real space with inner product denoted  $\langle \cdot, \cdot \rangle$ ). We denote by  $\mathbb{B}$  its unit ball and by  $\mathbb{S}$  its unit sphere. A sequence  $(x_k)_k$  in  $\mathbb{E}$  **converges linearly with rate**  $\kappa < 1$  to  $x$  if there is some constant  $\alpha$  such that

$$\|x_k - x\| \leq \alpha \kappa^k \quad \text{for all } k \geq 0.$$

More precisely, this property is “R-linear convergence” [DS83]: the infimum of all possible constants  $\kappa$ , namely

$$\limsup_{k \rightarrow \infty} \|x_k - x\|^{\frac{1}{k}}$$

is the “rate of R-linear convergence”.

**Manifolds.** A smooth manifold in  $\mathbb{E}$  is, loosely speaking, a set consisting locally of the solutions of some smooth equations. More precisely, we say that a set  $\mathcal{M} \subset \mathbb{E}$  is a  $C^k$ -**manifold (of codimension  $d$ ) around a point  $x \in \mathcal{M}$**  if there is an open set  $U \subset \mathbb{E}$  containing  $x$  such that

$$\mathcal{M} \cap U = \{x \in U : F(x) = 0\},$$

where  $F: U \rightarrow \mathbb{R}^d$  is a  $C^k$  function with surjective derivative throughout  $U$ . Note that  $k$ , the degree of smoothness of  $\mathcal{M}$ , will be omitted in statements if obvious or non-useful. Note also that the tangent space to  $\mathcal{M}$  at  $x \in \mathcal{M}$  is given by

$$\mathbb{T}_{\mathcal{M}}(x) = \ker \nabla F(x)$$

(which is actually independent of the choice of  $F$ ), and that the normal space at  $\mathcal{M}$  at  $x$  is then its orthogonal complement, namely

$$\mathbb{N}_{\mathcal{M}}(x) = \text{range } \nabla F(x)^*.$$

**Example 1** (Affine manifold). Particularly easy examples of smooth manifolds are affine subspaces. If  $\mathcal{M}$  is an affine subspace of  $\mathbb{E}$ , the equation  $F(x) = 0$  can be taken to be affine: that is, of the form  $\mathcal{A}(x) - b = 0$  with  $\mathcal{A}: \mathbb{E} \rightarrow \mathbb{R}^m$  a linear map and a vector  $b \in \mathbb{R}^d$ . The tangent space  $\ker \mathcal{A}$  is the same at any point in the affine subspace. ■

**Example 2** (Fixed rank matrices). Let  $\mathbb{E} = M_{n,m}(\mathbb{R})$  be the space of  $n \times m$ -matrices with the classical inner product  $\langle A, B \rangle = \text{trace}(A^T B)$ . Routine calculations show that the set of matrices with fixed rank  $r$ ,

$$\mathcal{R}_r = \{X \in M_{n,m}(\mathbb{R}) : \text{rank}(X) = r\},$$

is a smooth manifold around any matrix  $A \in \mathcal{R}_r$ . With the help of the singular value decomposition  $A = UDV^T$  (the two matrices  $U = [u_1, u_2, \dots, u_n]$  and  $V = [v_1, v_2, \dots, v_m]$  being orthogonal, and the diagonal entries in the diagonal matrix  $D$  being written in decreasing order) the tangent space at  $A$  to  $\mathcal{R}_r$  is

$$\mathbb{T}_{\mathcal{R}_r}(A) = \{H \in M_{n,m}(\mathbb{R}) : u_i^T H v_j = 0, \text{ for all } r < i \leq n, r < j \leq m\}. \quad \blacksquare$$

Let  $\mathcal{M}$  and  $\mathcal{N}$  be two  $C^k$ -manifolds around  $x \in \mathcal{M} \cap \mathcal{N}$ . The classical sufficient assumption to ensure that the intersection  $\mathcal{M} \cap \mathcal{N}$  is also a manifold around  $x$  is the following standard transversality assumption.

**Definition 3** (transversality). Suppose  $\mathcal{M}$  and  $\mathcal{N}$  are two  $C^k$ -manifolds around a point  $x \in \mathcal{M} \cap \mathcal{N}$ . We say that  $\mathcal{M}$  and  $\mathcal{N}$  are **transverse** at  $x$  if

$$\mathbb{T}_{\mathcal{M}}(x) + \mathbb{T}_{\mathcal{N}}(x) = \mathbb{E}.$$

In this case, the intersection  $\mathcal{M} \cap \mathcal{N}$  is a  $C^k$ -manifold around  $x$  and there holds  $\mathbb{T}_{\mathcal{M} \cap \mathcal{N}}(x) = \mathbb{T}_{\mathcal{M}}(x) \cap \mathbb{T}_{\mathcal{N}}(x)$ .

**Projections.** The projection of an element  $x \in \mathbb{E}$  onto a closed subset  $M \subset \mathbb{E}$  is defined by

$$\mathbb{P}_M(x) := \operatorname{argmin}\{\|x - y\| : y \in M\},$$

if it exists. In the case where  $M$  is convex,  $\mathbb{P}_M(x)$  exists and is unique for all  $x \in \mathbb{E}$ , then the projector operator  $\mathbb{P}_M: \mathbb{E} \rightarrow \mathbb{E}$  is well-defined. If furthermore the boundary of the closed convex  $M$  is a  $C^k$  manifold, the projection mapping  $\mathbb{P}_M$  is  $C^k$  [Hol73]. If  $M$  is no longer convex, versions of these results still hold locally. This is stated precisely in the following lemma, which will be a basic tool afterwards. We include a short proof.

**Lemma 4** (Projection onto a manifold). *Let  $\mathcal{M} \subset \mathbb{E}$  be a manifold of class  $C^k$  (with  $k \geq 2$ ) around a point  $\bar{x} \in \mathcal{M}$ . Then the projection  $\mathbb{P}_{\mathcal{M}}$  is well-defined around  $\bar{x}$ . Moreover  $\mathbb{P}_{\mathcal{M}}$  is of class  $C^{k-1}$  around  $\bar{x}$  and*

$$\nabla \mathbb{P}_{\mathcal{M}}(\bar{x}) = \mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x})}.$$

**Proof.** Let us introduce first the “normal bundle”

$$\mathbb{N}\mathcal{M} = \{(x, y) \in \mathbb{E}^2 : x \in \mathcal{M}, y \in \mathbb{N}_{\mathcal{M}}(x)\}.$$

It is well-known (and easy to check through local equations of  $\mathcal{M}$ ) that  $\mathbb{N}\mathcal{M}$  is a manifold of class  $C^{k-1}$  and of the same dimension as  $\mathbb{E}$ . Moreover there holds  $\mathbb{T}_{\mathbb{N}\mathcal{M}}(\bar{x}) = \mathbb{T}_{\mathcal{M}}(\bar{x}) \times \mathbb{N}_{\mathcal{M}}(\bar{x})$ . Let us now define

$$F: \begin{cases} \mathbb{N}\mathcal{M} & \longrightarrow \mathbb{E} \\ (x, v) & \longmapsto x + v, \end{cases}$$

which is also of class  $C^{k-1}$  with derivative at the point  $(\bar{x}, 0) \in \mathbb{N}\mathcal{M}$  given by

$$\nabla F(\bar{x}, 0): \begin{cases} \mathbb{T}_{\mathcal{M}}(\bar{x}) \times \mathbb{N}_{\mathcal{M}}(\bar{x}) & \longrightarrow \mathbb{E} \\ (u, v) & \longmapsto u + v. \end{cases}$$

Since this derivative is invertible, the local inverse theorem for manifolds yields that there are neighborhoods of  $(\bar{x}, 0)$  in  $\mathbb{N}\mathcal{M}$  and of  $F(\bar{x}, 0) = \bar{x}$  in  $\mathbb{E}$  such that  $F$  is a  $C^{k-1}$  onto diffeomorphism, and furthermore

$$\forall h \in \mathbb{E}, \quad \nabla F^{-1}(\bar{x})(h) = \nabla F(\bar{x}, 0)^{-1}(h) = (\mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x})}(h), \mathbb{P}_{\mathbb{N}_{\mathcal{M}}(\bar{x})}(h)).$$

Introducing now the  $C^{k-1}$  function

$$\pi: \begin{cases} \text{NM} & \longrightarrow \mathcal{M} \\ (x, v) & \longmapsto x, \end{cases}$$

we see that  $\text{P}_{\mathcal{M}} = \pi \circ F^{-1}$  is also  $C^{k-1}$  around  $\bar{x}$ , and

$$\nabla \text{P}_{\mathcal{M}}(\bar{x}) = \nabla \pi(\bar{x}, 0) \nabla F^{-1}(\bar{x}) = \text{P}_{\text{T}_{\mathcal{M}}(\bar{x})},$$

which completes the proof. ■

**Example 5** (Projection onto fixed rank matrices). In the notation of Example 2, consider a matrix  $X \in M_{n,m}(\mathbb{R})$  with singular value decomposition  $X = UDV^\top$ . Then the nearest matrix with rank no more than  $r$  is

$$\hat{X} = \sum_{i=1}^r \sigma_i u_i v_i^\top,$$

where the  $\sigma_i$  are the  $r$  first singular values (see [HJ89]). Consequently, if  $X$  is close to a matrix  $\bar{X} \in \mathcal{R}_r$ , the projection of  $X$  onto  $\mathcal{R}_r$  is also  $\hat{X}$ , since  $\hat{X}$  cannot have rank strictly less than  $r$ .

### 3 Angles between subspaces or manifolds

Let  $M$  and  $N$  be two subspaces of  $\mathbb{E}$ . Following [Fri37] and [Deu01], we define the **angle** between  $M$  and  $N$  as the angle between 0 and  $\pi/2$  whose cosine is

$$c(M, N) := \max\{ \langle x, y \rangle : \begin{array}{l} x \in \mathbb{S} \cap M \cap (M \cap N)^\perp \\ y \in \mathbb{S} \cap N \cap (M \cap N)^\perp \end{array} \}. \quad (1)$$

The quantity  $c(M, N)$  is well-defined unless one subspace is a subspace of the other, in which case we set  $c(M, N) = 0$ . Note that there holds (see [Deu01, 9.5])

$$\| \text{P}_M \text{P}_N - \text{P}_{M \cap N} \| = c(M, N), \quad (2)$$

and more generally [Deu01, Theorem 9.31],

$$\| (\text{P}_M \text{P}_N)^n - \text{P}_{M \cap N} \| = c(M, N)^{2n-1} \quad (3)$$

for  $n = 1, 2, \dots$

### 3.1 Properties of the angle between two subspaces

We begin by developing some basic properties of the angle, useful in our later discussion of metric regularity.

**Lemma 6.** *Let  $M$  and  $N$  be two subspaces of the space  $\mathbb{E}$ . Consider two vectors  $m \in \mathbb{S} \cap M \cap (M \cap N)^\perp$  and  $n \in \mathbb{S} \cap N \cap (M \cap N)^\perp$  such that  $c(M, N) = \langle m, n \rangle$ . Then*

$$P_M(n) = P_{M \cap (M \cap N)^\perp}(n) = c(M, N) m.$$

**Proof.** Consider first the decomposition

$$n = P_M(n) + P_{M^\perp}(n). \quad (4)$$

Observe from

$$n \in (M \cap N)^\perp \quad \text{and} \quad n - P_M(n) = P_{M^\perp}(n) \in M^\perp \subset (M \cap N)^\perp,$$

that we have  $P_M(n) \in M \cap (M \cap N)^\perp$ , and that we also have

$$P_{M^\perp}(n) \in M^\perp \subset (M \cap (M \cap N)^\perp)^\perp.$$

Thus equation (4) gives  $P_M(n) = P_{M \cap (M \cap N)^\perp}(n)$ . Now observe that the choice of  $n$  and  $m$  yields

$$m = \operatorname{argmax}_{x \in \mathbb{S} \cap M \cap (M \cap N)^\perp} \langle x, n \rangle,$$

which can be written

$$m = \operatorname{argmin}_{x \in \mathbb{S} \cap M \cap (M \cap N)^\perp} (\|n\|^2 + \|x\|^2 - 2\langle x, n \rangle) = \operatorname{argmin}_{x \in \mathbb{S} \cap M \cap (M \cap N)^\perp} \|n - x\|^2.$$

This finally gives us

$$m = P_{M \cap (M \cap N)^\perp}(n) / \|P_{M \cap (M \cap N)^\perp}(n)\| = P_M(n) / \|P_M(n)\|, \quad (5)$$

so

$$\|P_M(n)\| = \langle m, P_M(n) \rangle = \langle m, n \rangle.$$

The proof is complete. ■

This result permits us to prove that the angle between two subspaces is equal to the angle between their orthogonal complements, as stated in the next lemma.

**Lemma 7.** *Let  $M$  and  $N$  be two subspaces of  $\mathbb{E}$ . Then*

$$c(M, N) = c(M^\perp, N^\perp).$$

**Proof.** If one subspace is a subspace of the other, the result is immediate. Otherwise, denote  $c = c(M, N)$ , and consider vectors  $m \in \mathbb{S} \cap M \cap (M \cap N)^\perp$  and  $n \in \mathbb{S} \cap N \cap (M \cap N)^\perp$  such that  $\langle m, n \rangle = c$ . Then consider the two following vectors

$$\bar{m} = \alpha(n - cm) \quad \text{and} \quad \bar{n} = \alpha(cn - m), \quad \text{with} \quad \alpha = 1/\sqrt{1 - c^2}.$$

Let us check that  $\bar{m} \in \mathbb{S} \cap M^\perp \cap (M^\perp \cap N^\perp)^\perp$ . Firstly, by definition, we have  $\bar{m} \in N + M = (M^\perp \cap N^\perp)^\perp$ . Secondly, Lemma 6 shows  $\bar{m} = \alpha P_{M^\perp}(n) \in M^\perp$ . Thirdly, we obtain

$$\|\bar{m}\|^2 = \alpha^2(\|n\|^2 + c^2\|m\|^2 - 2c\langle m, n \rangle) = 1.$$

Similarly, we obtain  $\bar{n} \in \mathbb{S} \cap N^\perp \cap (M^\perp \cap N^\perp)^\perp$ . Finally we observe that

$$\langle \bar{m}, \bar{n} \rangle = \alpha^2 \langle n - cm, cn - m \rangle = c\alpha^2 \langle n - cm, n \rangle = c.$$

Thus by definition of the angle (1), we see that

$$c(M^\perp, N^\perp) \geq \langle \bar{m}, \bar{n} \rangle = c = c(M, N).$$

Changing the roles of  $M$  and  $N$  with their orthogonal complements, we get the reverse inequality  $c(M, N) \geq c(M^\perp, N^\perp)$  with the same argument. ■

We also state the following technical result that will be useful afterwards.

**Lemma 8.** *Let  $M$  and  $N$  be two subspaces of  $\mathbb{E}$  such that  $M \cap N = \{0\}$ . Then*

$$1 - c(M, N) = \min_{x \in \mathbb{S}, m \in M, n \in N} (\|x - m\|^2 + \|x - n\|^2).$$

**Proof.** Denote by  $R$  the right hand side of the equality to be proved. Developing the sum of squared norms, we get

$$\begin{aligned} R - 2 &= \min_{m \in M, n \in N} \left( \|n\|^2 + \|m\|^2 - 2 \max_{x \in \mathbb{S}} \langle x, m + n \rangle \right) \\ &= \min_{m \in M, n \in N} (\|n\|^2 + \|m\|^2 - 2\|m + n\|) \\ &= \min_{m \in M \cap \mathbb{S}, n \in N \cap \mathbb{S}} \min_{\alpha, \beta \in \mathbb{R}} (\alpha^2 + \beta^2 - 2\|\alpha m + \beta n\|^2) \end{aligned}$$

Observe now that the function  $f_{m,n}(\alpha, \beta) = \alpha^2 + \beta^2 - 2\|\alpha m + \beta n\|^2$  has compact lower level sets, and is smooth on  $\mathbb{R}^2 \setminus \{(0, 0)\}$  (since  $M \cap N = \{0\}$ ), and that it has a local maximum at  $(0, 0)$ . Hence  $f_{m,n}$  achieves its minimum at a critical point. Some algebra then gives

$$\min_{\alpha, \beta \in \mathbb{R}} f_{m,n}(\alpha, \beta) = -1 - \langle m, n \rangle$$



Minimizing now with respect to  $m$  and  $n$  we get

$$\min_{m \in M \cap \mathbb{S}, n \in N \cap \mathbb{S}} (-1 - \langle m, n \rangle) = -1 - \max_{m \in M \cap \mathbb{S}, n \in N \cap \mathbb{S}} \langle m, n \rangle = -1 - c(M, N),$$

the last equality holding again because of the assumption  $M \cap N = \{0\}$ . Finally we thus get  $R = 2 + (-1 - c(M, N)) = 1 - c(M, N)$  which is the targeted equality. ■

### 3.2 Angle between two manifolds

Let us now generalize the previous framework: in view of the definition of the angle between two subspaces, the following definition makes sense.

**Definition 9** (angle between two manifolds). Let  $\mathcal{M}$  and  $\mathcal{N}$  be two manifolds in  $\mathbb{E}$  around a point  $x \in \mathcal{M} \cap \mathcal{N}$ . The **angle** between  $\mathcal{M}$  and  $\mathcal{N}$  at  $x$  is the angle between the tangent spaces  $\mathbb{T}_{\mathcal{M}}(x)$  and  $\mathbb{T}_{\mathcal{N}}(x)$ . In other words, it is the angle between 0 and  $\pi/2$  with cosine

$$c(\mathcal{M}, \mathcal{N}, x) := c(\mathbb{T}_{\mathcal{M}}(x), \mathbb{T}_{\mathcal{N}}(x)).$$

If  $\mathcal{M}$  and  $\mathcal{N}$  are actually subspaces, it is clear that the angle between them does not depend on the point in their intersection and that the two definitions coincide. Let us add a lemma that formalizes an obvious smoothness property.

**Lemma 10** (Smoothness of the angle). *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two transverse  $C^k$ -manifolds in the space  $\mathbb{E}$  (with  $k \geq 2$ ) around the point  $\bar{x} \in \mathcal{M} \cap \mathcal{N}$ . Then the function*

$$c(\mathcal{M}, \mathcal{N}, \cdot): \begin{cases} \mathcal{M} \cap \mathcal{N} \longrightarrow [0, 1] \\ x \longmapsto c(\mathcal{M}, \mathcal{N}, x) \end{cases}$$

is of class  $C^{k-1}$  around  $\bar{x}$ .

**Proof.** From property (2) and Definition 9, we know for any point  $x \in \mathcal{M} \cap \mathcal{N}$ ,

$$c(\mathcal{M}, \mathcal{N}, x) = \|\mathbb{P}_{\mathbb{T}_{\mathcal{M}}(x)}\mathbb{P}_{\mathbb{T}_{\mathcal{N}}(x)} - \mathbb{P}_{\mathbb{T}_{\mathcal{M}}(x) \cap \mathbb{T}_{\mathcal{N}}(x)}\|. \quad (6)$$

Moreover, the projectors  $x \mapsto \mathbb{P}_{\mathbb{T}_{\mathcal{M}}(x)}$  are  $C^{k-1}$ : the columns of the derivative of a local  $C^k$  parametrization of the manifold form a basis for the tangent space that is a  $C^{k-1}$  function of the base point, so the projectors (expressed with this basis) are also  $C^{k-1}$ . Property (6) now proves the result. ■

## 4 Alternating projections onto manifolds

We are now ready to consider the alternating projection algorithm. We consider two manifolds  $\mathcal{M}$  and  $\mathcal{N}$  in the space  $\mathbb{E}$ , and study the **alternating projection sequence** defined iteratively as follows:

$$x_0 \in \mathbb{E} \text{ given, } \quad x_{k+1} = P_{\mathcal{M}}P_{\mathcal{N}}(x_k) \quad (7)$$

When  $\mathcal{M}$  and  $\mathcal{N}$  are actually affine subspaces, this algorithm is well-defined and its behaviour is well-understood, see [BB93]. In particular we have the following theorem (see [BB93, 4.11]).

**Theorem 11** (alternating projections for two affine subspaces). *Let  $M$  and  $N$  be two affine subspaces of the space  $\mathbb{E}$ . Then the alternating projection sequence (7) converges linearly with rate the cosine of the angle between the two subspaces,  $c(M, N)$ , independent of the starting point.*

When  $\mathcal{M}$  and  $\mathcal{N}$  are general smooth manifolds, we will see in Theorem 13 that the sequence (7) is also well-defined in a neighborhood of a point  $\bar{x}$  belonging to the intersection  $\mathcal{M} \cap \mathcal{N}$  (assuming transversality), and that the previous convergence result generalizes. The next result gives the main tool.

**Theorem 12** (Asymptotical improvement). *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two transverse  $C^2$ -manifolds around a point  $\bar{x} \in \mathcal{M} \cap \mathcal{N}$ . Then*

$$\limsup_{x \rightarrow \bar{x}, x \notin \mathcal{M} \cap \mathcal{N}} \frac{\|P_{\mathcal{M}}P_{\mathcal{N}}(x) - P_{\mathcal{M} \cap \mathcal{N}}(x)\|}{\|x - P_{\mathcal{M} \cap \mathcal{N}}(x)\|} \leq c(\mathcal{M}, \mathcal{N}, \bar{x}).$$

**Proof.** Lemma 4 implies that there exists  $\delta > 0$  such that the projection operators  $P_{\mathcal{M}}$ ,  $P_{\mathcal{N}}$  and  $P_{\mathcal{M} \cap \mathcal{N}}$  are well defined and of class  $C^1$ , on the ball  $B_{\delta}(\bar{x})$ . Restricting further to points  $x \in B_{\delta/2}(\bar{x})$ , we have

$$\|\bar{x} - P_{\mathcal{N}}(x)\| \leq \|\bar{x} - x\| + \|x - P_{\mathcal{N}}(x)\| \leq 2\|x - \bar{x}\| \leq \delta,$$

so  $P_{\mathcal{N}}(x) \in B_{\delta}(\bar{x})$ , and therefore  $P_{\mathcal{M}}P_{\mathcal{N}}$  is also well-defined and  $C^1$  on  $B_{\delta/2}(\bar{x})$ . We thus ensure that the fraction in the result makes sense.

Let  $(x_r)_r$  be an arbitrary sequence of points in  $B_{\delta/2}(\bar{x}) \setminus (\mathcal{M} \cap \mathcal{N})$  tending to  $\bar{x}$ . To simplify notation, we use  $\bar{x}_r = P_{\mathcal{M} \cap \mathcal{N}}(x_r)$ . Of course  $\bar{x}_r \in \mathcal{M} \cap \mathcal{N}$ , so

$$P_{\mathcal{M}}P_{\mathcal{N}}(x_r) - \bar{x}_r = P_{\mathcal{M}}P_{\mathcal{N}}(x_r) - P_{\mathcal{M}}P_{\mathcal{N}}(\bar{x}_r).$$

Observe also that the continuity of  $P_{\mathcal{M} \cap \mathcal{N}}$  yields that  $\bar{x}_r$  tends to  $\bar{x}$  too. So, the previous equation and continuous differentiability shows

$$P_{\mathcal{M}}P_{\mathcal{N}}(x_r) - \bar{x}_r = \nabla(P_{\mathcal{M}}P_{\mathcal{N}})(\bar{x}_r)(x_r - \bar{x}_r) + o(\|x_r - \bar{x}_r\|). \quad (8)$$

Using Lemma 4 and the chain rule, we get

$$\nabla(\mathbb{P}_{\mathcal{M}}\mathbb{P}_{\mathcal{N}})(\bar{x}_r) = \mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x}_r)}\mathbb{P}_{\mathbb{T}_{\mathcal{N}}(\bar{x}_r)}. \quad (9)$$

The transversality assumption now shows

$$\mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x}_r) \cap \mathbb{T}_{\mathcal{N}}(\bar{x}_r)}(x_r - \bar{x}_r) = \mathbb{P}_{\mathbb{T}_{\mathcal{M} \cap \mathcal{N}}(\bar{x}_r)}(x_r - \bar{x}_r) = 0,$$

since  $x_r - \bar{x}_r \in \mathbb{N}_{\mathcal{M} \cap \mathcal{N}}(\bar{x}_r) = \mathbb{T}_{\mathcal{M} \cap \mathcal{N}}(\bar{x}_r)^\perp$ . So we can write

$$\mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x}_r)}\mathbb{P}_{\mathbb{T}_{\mathcal{N}}(\bar{x}_r)}(x_r - \bar{x}_r) = (\mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x}_r)}\mathbb{P}_{\mathbb{T}_{\mathcal{N}}(\bar{x}_r)} - \mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x}_r) \cap \mathbb{T}_{\mathcal{N}}(\bar{x}_r)})(x_r - \bar{x}_r).$$

Combined with equations (8) and (9), this gives

$$\frac{\|\mathbb{P}_{\mathcal{M}}\mathbb{P}_{\mathcal{N}}(x_r) - \bar{x}_r\|}{\|x_r - \bar{x}_r\|} \leq \|\mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x}_r)}\mathbb{P}_{\mathbb{T}_{\mathcal{N}}(\bar{x}_r)} - \mathbb{P}_{\mathbb{T}_{\mathcal{M}}(\bar{x}_r) \cap \mathbb{T}_{\mathcal{N}}(\bar{x}_r)}\| + o(1),$$

that is

$$\frac{\|\mathbb{P}_{\mathcal{M}}\mathbb{P}_{\mathcal{N}}(x_r) - \bar{x}_r\|}{\|x_r - \bar{x}_r\|} \leq \cos(\mathcal{M}, \mathcal{N}, \bar{x}_r) + o(1),$$

by equation (2) and Definition 9. Taking the lim sup in this inequality, the result now follows by Lemma 10.  $\blacksquare$

A refinement of the above argument, using equation (3) in place of equation (2), shows the generalization

$$\limsup_{x \rightarrow \bar{x}, x \notin \mathcal{M} \cap \mathcal{N}} \frac{\|(\mathbb{P}_{\mathcal{M}}\mathbb{P}_{\mathcal{N}})^n(x) - \mathbb{P}_{\mathcal{M} \cap \mathcal{N}}(x)\|}{\|x - \mathbb{P}_{\mathcal{M} \cap \mathcal{N}}(x)\|} \leq c(\mathcal{M}, \mathcal{N}, \bar{x})^{2n-1} \quad (10)$$

for  $n = 1, 2, \dots$

Observe that, with the hypotheses of the above theorem, we have that, for all constants  $c > \cos(\mathcal{M}, \mathcal{N}, \bar{x})$ , there exists a radius  $\eta > 0$  such that

$$\forall x \in B_\eta(\bar{x}), \quad \|\mathbb{P}_{\mathcal{M}}\mathbb{P}_{\mathcal{N}}(x) - \mathbb{P}_{\mathcal{M} \cap \mathcal{N}}(x)\| \leq c\|x - \mathbb{P}_{\mathcal{M} \cap \mathcal{N}}(x)\|. \quad (11)$$

We can now prove our main result.

**Theorem 13** (Linear convergence). *In the space  $\mathbb{E}$ , let  $\mathcal{M}$  and  $\mathcal{N}$  be two transverse manifolds around a point  $\bar{x} \in \mathcal{M} \cap \mathcal{N}$ . If the initial point  $x_0 \in \mathbb{E}$  is close to  $\bar{x}$ , then the method of alternating projections*

$$x_{k+1} = \mathbb{P}_{\mathcal{M}}\mathbb{P}_{\mathcal{N}}(x_k) \quad (k = 0, 1, 2, \dots)$$

*is well-defined, and the distance  $d_{\mathcal{M} \cap \mathcal{N}}(x_k)$  from the iterate  $x_k$  to the intersection  $\mathcal{M} \cap \mathcal{N}$  decreases  $Q$ -linearly to zero. More precisely, given any constant  $c$  strictly larger than the cosine of the angle of intersection between the manifolds,  $\cos(\mathcal{M}, \mathcal{N}, \bar{x})$ , if  $x_0$  is close to  $\bar{x}$ , then the iterates satisfy*

$$d_{\mathcal{M} \cap \mathcal{N}}(x_{k+1}) \leq c \cdot d_{\mathcal{M} \cap \mathcal{N}}(x_k) \quad (k = 0, 1, 2, \dots), \quad (12)$$

Furthermore,  $x_k$  converges linearly to some point  $x^* \in \mathcal{M} \cap \mathcal{N}$ : for some constant  $\alpha > 0$ ,

$$\|x_k - x^*\| \leq \alpha c^k \quad (k = 0, 1, 2, \dots). \quad (13)$$

**Proof.** Choose  $c$  such that  $1 > c > \cos(\mathcal{M}, \mathcal{N}, \bar{x})$  and  $\eta > 0$  such that (11) is satisfied. Set  $\delta := (1 - c)\eta/4$  and choose any starting point  $x_0 \in B_\delta(\bar{x})$ .

*First step: properties of  $x_k$ .* Let us prove by induction that the sequence of points  $x_k$  is well-defined, and that both  $x_k$  and its projection  $\bar{x}_k = P_{\mathcal{M} \cap \mathcal{N}}(x_k)$  belong to the neighborhood  $B_\eta(\bar{x})$  and satisfy the properties

$$\|x_k - \bar{x}_{k-1}\| \leq \delta c^k \quad (\text{H1})$$

$$\|x_k - \bar{x}_k\| \leq \delta c^k \quad (\text{H2})$$

$$\|\bar{x}_k - \bar{x}_{k-1}\| \leq 2\delta c^k \quad (\text{H3})$$

$$\|\bar{x}_k - \bar{x}\| \leq 2\left(\sum_{i=0}^k c^i\right)\delta \quad (\text{H4})$$

$$\|x_k - \bar{x}\| \leq 2\left(\sum_{i=0}^k c^i\right)\delta. \quad (\text{H5})$$

Setting  $\bar{x}_{-1} = \bar{x}_0$  and using

$$\|x_0 - \bar{x}_0\| \leq \|x_0 - \bar{x}\| \leq \delta,$$

it is easy to see that these inequalities (H1)-(H5) hold for  $k = 0$ . Assume now that these inequalities hold for some  $k \geq 0$ : we prove they also hold with  $k$  replaced by  $k+1$ . Note that if  $x_k$  belongs to  $\mathcal{M} \cap \mathcal{N}$ , there is nothing to prove. Otherwise, since  $x_k$  belongs to  $B_\eta(\bar{x})$ , the next iterate  $x_{k+1}$  is well-defined and inequality (11) holds, so:

$$d_{\mathcal{M} \cap \mathcal{N}}(x_{k+1}) \leq \|x_{k+1} - \bar{x}_k\| \leq c\|x_k - \bar{x}_k\| = c \cdot d_{\mathcal{M} \cap \mathcal{N}}(x_k).$$

**(H1)** With the help of property (H2), the above inequality yields

$$\|x_{k+1} - \bar{x}_k\| \leq \delta c^{k+1}. \quad (14)$$

**(H2)** Note that  $\|x_{k+1} - \bar{x}_{k+1}\| \leq \|x_{k+1} - \bar{x}_k\|$  by definition of  $\bar{x}_{k+1}$ . With inequality (14), this implies

$$\|x_{k+1} - \bar{x}_{k+1}\| \leq \delta c^{k+1}. \quad (15)$$

**(H3)** We get property (H3) from inequalities (14) and (15), by observing

$$\|\bar{x}_{k+1} - \bar{x}_k\| \leq \|\bar{x}_{k+1} - x_{k+1}\| + \|x_{k+1} - \bar{x}_k\| \leq 2\delta c^{k+1}. \quad (16)$$

**(H4)** Finally, note

$$\|\bar{x}_{k+1} - \bar{x}\| \leq \|\bar{x}_{k+1} - \bar{x}_k\| + \|\bar{x}_k - \bar{x}\|$$

so inequality (16) and property (H4) enable us to write

$$\|\bar{x}_{k+1} - \bar{x}\| \leq 2\delta c^{k+1} + 2\delta \sum_{i=0}^k c^i \leq 2\delta \sum_{i=0}^{k+1} c^i. \quad (17)$$

(H5) Similarly,

$$\|x_{k+1} - \bar{x}\| \leq \|x_{k+1} - \bar{x}_k\| + \|\bar{x}_k - \bar{x}\|,$$

so we have from inequality (14) and property (H4)

$$\|x_{k+1} - \bar{x}\| \leq \delta c^{k+1} + 2\delta \sum_{i=0}^k c^i \leq 2\delta \sum_{i=0}^{k+1} c^i. \quad (18)$$

Observe now that inequality (17) yields

$$\|\bar{x}_{k+1} - \bar{x}\| \leq 2\delta/(1-c) \leq \eta/2$$

and inequality (18) yields

$$\|x_{k+1} - \bar{x}\| \leq \eta/2.$$

So  $\bar{x}_{k+1}$  and  $x_{k+1}$  belong to  $B_\eta(\bar{x})$  too. This ends the proof by induction.

*Second step: convergence.* We first prove the convergence of the sequence of projections  $(\bar{x}_k)_k$ : this sequence in  $\mathcal{M} \cap \mathcal{N} \cap B_\eta(\bar{x})$  is Cauchy. To see this, use property (H3) to write, for all indices  $k, p \geq 0$  with  $p \geq k$ ,

$$\|\bar{x}_p - \bar{x}_k\| \leq \sum_{i=k+1}^p \|\bar{x}_i - \bar{x}_{i-1}\| \leq 2\delta \sum_{i=k+1}^p c^i \leq \frac{2\delta}{1-c} c^{k+1}. \quad (19)$$

So  $(\bar{x}_k)_k$  converges to an element  $x^*$  in  $\mathcal{M} \cap \mathcal{N}$ . Passing to the limit in  $p$  in inequality (19), we obtain

$$\|\bar{x}_k - x^*\| \leq \frac{2\delta}{1-c} c^{k+1}.$$

With the help of property (H2), this implies

$$\|x_k - x^*\| \leq \|x_k - \bar{x}_k\| + \|\bar{x}_k - x^*\| \leq (1 + 2c/(1-c))\delta c^k,$$

which yields inequality (13) and completes the proof.  $\blacksquare$

**Remark 14** (Stronger bound). In fact, the distance  $d_k$  from the iterate  $x_k$  to the intersection of the two manifolds  $\mathcal{M} \cap \mathcal{N}$  decreases to zero with  $R$ -linear rate  $\cos^2(\mathcal{M}, \mathcal{N}, x^*)$ , a faster rate than predicted by inequality (12). To see this refinement, we argue as follows.

Fix any constant  $c$  in the interval  $(\cos(\mathcal{M}, \mathcal{N}, x^*), 1)$ , and any integer  $n > 0$ . We claim

$$\limsup_r d_r^{1/r} \leq c^{2-1/n}. \quad (20)$$

Our result then follows, by taking the infimum over  $c$  and  $n$ .

To verify the claim, note first that Theorem 13 and inequality (10) guarantee that there is an integer  $t_0$  such that  $d_{t+n} < c^{2n-1}d_t$  for all integers  $t > t_0$ , and hence by induction

$$d_{t+kn} < c^{k(2n-1)}d_t \quad \text{for all } t > t_0, k = 1, 2, 3, \dots \quad (21)$$

If inequality (20) fails, then there is a constant  $\epsilon > 0$  and a sequence of integers  $r_1 < r_2 < r_3 < \dots$ , all satisfying

$$\limsup_j d_{r_j}^{1/r_j} > c^{2-1/n} + \epsilon. \quad (22)$$

By considering the sequence  $(r_j)$  modulo  $n$ , and taking a further subsequence, we can suppose each  $r_j$  has the form  $a + b_j n$  for some fixed integer  $a$  and sequence of integers  $b_1 < b_2 < b_3 < \dots$ . Choose any integer  $b$  satisfying  $a + bn > t_0$ . Then we have

$$d_{r_j} = d_{a+b_j n} = d_{a+bn+(b_j-b)n} < c^{(b_j-b)(2n-1)}d_{a+bn},$$

using inequality (21). We deduce

$$d_{r_j} < c^{(n^{-1}[r_j-a]-b)(2n-1)}d_{a+bn}.$$

Raising both sides to the power  $1/r_j$  and letting  $j \rightarrow \infty$  now contradicts inequality (22). This completes the proof of our claim (20), and the result follows.  $\blacksquare$

Naturally, the convergence of Theorem 13 is only local, since the projections themselves are well-defined only locally in general. However by adding a convexity assumption, we can get a global convergence while preserving the local rate. A result of this kind is the following.

**Corollary 15.** *Let  $A$  and  $B$  two closed convex subsets of  $\mathbb{E}$  such that the boundaries  $\text{bd } A$  and  $\text{bd } B$  are smooth manifolds. If the intersection  $A \cap B$  is non-empty, then the alternating projection method*

$$x_0 \text{ given,} \quad x_{k+1} = P_A P_B(x_k)$$

*is well-defined and converges to a point  $x^* \in A \cap B$ . If furthermore  $\text{bd } A$  and  $\text{bd } B$  are transversal at  $x^*$ , the sequence  $(x_k)_k$  in fact converges linearly, with  $R$ -linear rate  $c(\text{bd } A, \text{bd } B, x^*)$ .*

**Proof.** Since  $C_1$  and  $C_2$  are closed and convex, the sequence  $(x_k)_k$  is well-defined for any starting point  $x_0$ . The classical theory of alternating projections (see for example [CG59]) implies global convergence to a point  $x^* \in A \cap B$ . Theorem 13 then gives the local linear convergence.  $\blacksquare$

## 5 Metric regularity and linear rate

The previous section shows that the rate of convergence of the method of alternating projections for two transverse manifolds is related to the angle between the manifolds. The speed of basic algorithms is often closely associated with Lipschitzian properties of the underlying generalized equations, “error bounds” for these equations (see [FP03], for example), and “metric regularity” (see [RW98]). Metric regularity in turn is related to the conditioning of a well-posed generalized equation, measured in terms of the size of allowable linear perturbations to the equation that preserve well-posedness (see the discussion in [DLR03]). In this section, we pursue this pattern in our context, by relating the angle between the manifolds to the metric regularity of a natural associated generalized equation. To accomplish this, we use a variety of tools from variational analysis: we refer the reader to [RW98] for terminology.

### 5.1 Regular intersection

We consider the metric regularity of the problem of finding a point in the intersection of two closed sets  $M$  and  $N$  in the space  $\mathbb{E}$ . To use variational tools for this analysis, we introduce the multifunction  $\phi: \mathbb{E}^2 \rightrightarrows \mathbb{E}$  defined by

$$\phi(x, y) = \begin{cases} \{x - y\} & \text{if } x \in M \text{ and } y \in N, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus we have

$$0 \in \phi(x, y) \iff x = y \in M \cap N.$$

Therefore we say  $M$  and  $N$  have **regular intersection at  $x$**  if  $\phi$  is metrically regular at  $(x, x)$  for 0. In that case we define the **regularity modulus** of the intersection at  $x$  via the regularity modulus of  $\phi$ :

$$\text{reg}_{M,N}(x) := \text{reg } \phi((x, x)|0). \quad (23)$$

**Lemma 16** (Coderivatives of  $\phi$ ). *Let  $M$  and  $N$  be two closed sets, and  $x$  be a point in the intersection  $M \cap N$ . Then the coderivative of the multifunction  $\phi$  at the point  $(x, x)$  is related to the normal cones to  $M$  and  $N$  at  $x$  by*

$$\forall z \in \mathbb{E}, \quad D^* \phi((x, x)|0)(z) = (z + N_M(x), -z + N_N(x)) \quad (24)$$

**Proof.** Let us write  $\phi$  as the sum

$$\phi(x, y) = x - y + \psi(x, y),$$

where the multifunction  $\psi: \mathbb{E}^2 \rightrightarrows \mathbb{E}$  is defined by

$$\psi(x, y) = \begin{cases} 0 & \text{if } x \in M \text{ and } y \in N, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since the function  $F: (x, y) \mapsto x - y$  is smooth, the calculus rule [RW98, 10.43] yields

$$D^*\phi((x, x)|0)(z) = \nabla F(x, x)^*(z) + D^*\psi((x, x)|0)(z),$$

so

$$D^*\phi((x, x)|0)(z) = (z, -z) + D^*\psi((x, x)|0)(z). \quad (25)$$

Thus we just have to compute  $D^*\psi((x, x)|0)(z)$ . Observe that

$$\text{graph } \psi = M \times N \times \{0\} \subset \mathbb{E}^3,$$

and, for  $x \in M \cap N$ , this yields

$$N_{\text{graph } \psi}(x, x, 0) = N_M(x) \times N_N(x) \times \mathbb{E}.$$

Returning to the definition of coderivatives [RW98, 8.33], we compute

$$\begin{aligned} (u, v) \in D^*\psi((x, x)|0)(z) &\iff (u, v, -z) \in N_{\text{graph } \psi}(x, x, 0) \\ &\iff (u, v) \in N_M(x) \times N_N(x). \end{aligned}$$

Thus equation (25) gives

$$D^*\phi((x, x)|0)(z) = (z, -z) + N_M(x) \times N_N(x),$$

which is exactly equation (24). ■

We can use this result to recognize regular intersections, as follows.

**Theorem 17** (Condition for regularity). *Two closed sets  $M$  and  $N$  have regular intersection at a point  $x \in M \cap N$  if and only if*

$$-N_M(x) \cap N_N(x) = \{0\}.$$

*In this case, we also have*

$$\frac{1}{\text{reg}_{M,N}(x)} = \min_{\|z\|=1} \sqrt{d(z, -N_M(x))^2 + d(z, N_N(x))^2}. \quad (26)$$

**Proof.** We apply [RW98, 9.43] : the metric regularity of  $\phi$  at  $(x, x)$  for 0 is equivalent to

$$(0, 0) \in D^*\phi((x, x)|0)(z) \implies z = 0.$$

In view of Lemma 16, this means

$$\left(0 \in z + N_M(x) \quad \text{and} \quad 0 \in -z + N_N(x)\right) \implies z = 0,$$

that is  $-N_M(x) \cap N_N(x) = \{0\}$ . Now combining [RW98, 9.43] and the Mordukhovich criterion [RW98], we obtain

$$\frac{1}{\text{reg } \phi((x, x)|0)} = \min_{\|z\|=1} d((0, 0), D^*\phi((x, x)|0)(z)),$$

and equation (26) follows using equations (23) and (24). ■



## 5.2 Regular intersection of two manifolds

The regularity of the intersection is easier to grasp when dealing manifolds. The following result proves that the nonsmooth regularity notion we introduced via metric regularity coincides with the regularity notion from smooth geometry, namely transversality.

**Theorem 18** (Regularity for two manifolds). *Consider two manifolds  $\mathcal{M}$  and  $\mathcal{N}$  around a point  $x \in \mathcal{M} \cap \mathcal{N}$ . Then their intersection is regular at  $x$  if and only if they are transverse at  $x$ . In this case, the intersection  $\mathcal{M} \cap \mathcal{N}$  is a smooth manifold around  $x$ , and the regularity modulus is related to the angle between them by*

$$\text{reg}_{\mathcal{M}, \mathcal{N}}(x) = \frac{1}{\sqrt{1 - c(\mathcal{M}, \mathcal{N}, x)}}. \quad (27)$$

**Proof.** The normal cone  $\mathbb{N}_{\mathcal{M}}(x)$  is linear in this case, so the condition for regularity of Theorem 17 becomes  $\{0\} = \mathbb{N}_{\mathcal{M}}(x) \cap \mathbb{N}_{\mathcal{N}}(x)$ . Taking orthogonal complements, the condition is then

$$\mathbb{E} = (\mathbb{N}_{\mathcal{M}}(x) \cap \mathbb{N}_{\mathcal{N}}(x))^{\perp} = \mathbb{N}_{\mathcal{M}}(x)^{\perp} + \mathbb{N}_{\mathcal{N}}(x)^{\perp} = \mathbb{T}_{\mathcal{M}}(x) + \mathbb{T}_{\mathcal{N}}(x),$$

which is exactly the transversality assumption (Definition 3). This property yields in particular that  $\mathcal{M} \cap \mathcal{N}$  is a smooth manifold around  $x$ . Let us prove now equation (27). From equation (26), we first get

$$(\text{reg}_{\mathcal{M}, \mathcal{N}}(x))^{-2} = \min_{z \in \mathbb{S}, m \in \mathbb{N}_{\mathcal{M}}(x), n \in \mathbb{N}_{\mathcal{N}}(x)} (\|z - m\|^2 + \|z - n\|^2),$$

Since we have  $\mathbb{N}_{\mathcal{M}}(x) \cap \mathbb{N}_{\mathcal{N}}(x) = \{0\}$ , Lemma 8 gives

$$(\text{reg}_{\mathcal{M}, \mathcal{N}}(x))^{-2} = 1 - c(\mathbb{N}_{\mathcal{M}}(x), \mathbb{N}_{\mathcal{N}}(x)).$$

So Lemma 7 and inequality (11) yield

$$(\text{reg}_{\mathcal{M}, \mathcal{N}}(x))^{-2} = 1 - c(\mathbb{T}_{\mathcal{M}}(x), \mathbb{T}_{\mathcal{N}}(x)) = 1 - c(\mathcal{M}, \mathcal{N}, x),$$

which completes the proof. ■

Having this connection between regularity modulus and the angle, the asymptotical rate of converge of the alternating projection method of Theorem 13 can be written

$$1 - (\text{reg}_{\mathcal{M}, \mathcal{N}}(\bar{x}))^{-2}.$$

## 6 A numerical illustration

In this section we give a numerical illustration showing the linear convergence of the alternating projection method. We focus on the following problem: using the notation of Example 2, we want to find an  $n \times m$ -matrix  $X$  of rank  $r$ , satisfying a linear system  $\mathcal{A}(X) = b$ . In other words, we seek a matrix in the intersection

$$\mathcal{R}_r \cap \{X \in M_{n,m}(\mathbb{R}) : \mathcal{A}(X) = b\},$$

for given linear map  $\mathcal{A}: M_{n,m}(\mathbb{R}) \rightarrow \mathbb{R}^d$  and vector  $b \in \mathbb{R}^d$ . This problem is a simple analogue of feasibility problems appearing in control, and treated by alternating projections in [GB00].

General features of alternating projection methods are that they can be implemented easily and that usually the amount of calculation in one iteration is very small. In our example, the projection onto  $\mathcal{R}_r$  is computed through a singular value decomposition (see Example 5). The projection onto the affine subspace  $\mathcal{A}$  of equation  $\mathcal{A}(X) = b$  is computed directly as

$$P_{\mathcal{A}}(X) = X - \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}(\mathcal{A}(X) - b),$$

with  $\mathcal{A}\mathcal{A}^*$  and its LU factorization computed only one time at the beginning of the algorithm. So the work of each iteration is dominated by the singular value decomposition.

Experiments with MATLAB on randomly generated problems (that is, the operator  $\mathcal{A}(X) = (\langle A_1, X \rangle, \dots, \langle A_d, X \rangle)$  being constructed with random matrices, and the vector  $b$  being chosen so that  $\mathcal{A}(X) = b$  has a rank  $r$  solution) always exhibit the linear convergence predicted by Theorem 13. For our experiments, we take in general matrix dimensions  $m \geq n$ , rank  $r$  rather small (lower than 10) and we pick the number of linear equations  $d$  satisfying

$$m r < d \leq r(m + n - r).$$

The left-hand inequality ensures we cannot solve the problem too easily, simply by setting all but  $r$  rows of the matrix  $X$  to zero, and the right-hand inequality, ensures (by counting dimensions) that a random problem typically has a solution and transversality holds. Starting at a random initial matrix  $X_0$ , we compute

$$X_{k+1} = P_{\mathcal{R}_r}(P_{\mathcal{A}}(X_k))$$

and we stop the algorithm when the absolute error satisfies

$$\|\mathcal{A}(X_k) - b\| \leq 10^{-7}.$$

We illustrate with one typical case.

**Example 19** (Linear convergence). We take  $n = 100$ ,  $m = 110$ ,  $r = 4$  and  $d = 450$ . The algorithm stops after 1869 iterations (with around 7 minutes of computing time on a standard PC). We give below a summary of the information printed at each iteration, that is:

- $\text{iter}$  is the number of the iteration,
- $\log|\text{AX-b}| = \log_{10}(\|\mathcal{A}(X_k) - b\|)$ ,
- $\log|\text{X-Xpre}| = \log_{10}(\|X_k - X_{k-1}\|)$ .

iter	$\log \text{AX-b} $	$\log \text{X-Xpre} $
1	-0.3010	0.4604
50	-1.3197	-2.5445
100	-1.6744	-3.0021
500	-3.1839	-4.6450
1000	-4.6511	-6.1343
1500	-6.0199	-7.5133
1850	-6.9516	-8.4491
1869	-7.0018	-8.4995

We plot on Figure 1 the value of  $\log_{10}(\|\mathcal{A}(X_k) - b\|)$  at each iteration  $k$ . We see that after 200 iterations the quantity decreases linearly as expected, illustrating the linear convergence.

A second, very simple, example illustrates the relationship between the angle of intersection and the convergence rate.

**Example 20** (Angle and rate of convergence). We repeat the same experiment but with  $d = 1$ , that is  $\mathcal{A}(X) = \langle A, X \rangle$ . In this case, we can simply compute the cosine of the angle at the intersection,  $c(\mathcal{R}_r, \mathcal{A}, X^*)$ . Indeed Lemmas 6 and 7 show

$$c(\mathcal{R}_r, \mathcal{A}, X^*) = \cos(\text{N}_{\mathcal{R}_r}(X^*), A) = \|\text{P}_{\text{N}_{\mathcal{R}_r}(X^*)}(A/\|A\|)\|. \quad (28)$$

In practice, this case is much easier and the algorithm stops after 178 iterations. Here is the information printed at some iterations:

iter	$\log \text{AX-b} $	improv	$\log \text{X-Xpre} $	c2
1	-1.1511	1.3350	0.4634	0.0004
50	-2.7735	0.9203	-5.1524	0.9197
100	-4.4291	0.9265	-6.8079	0.9265
150	-6.0846	0.9266	-8.4635	0.9266
178	-7.0117	0.9266	-9.3906	0.9266

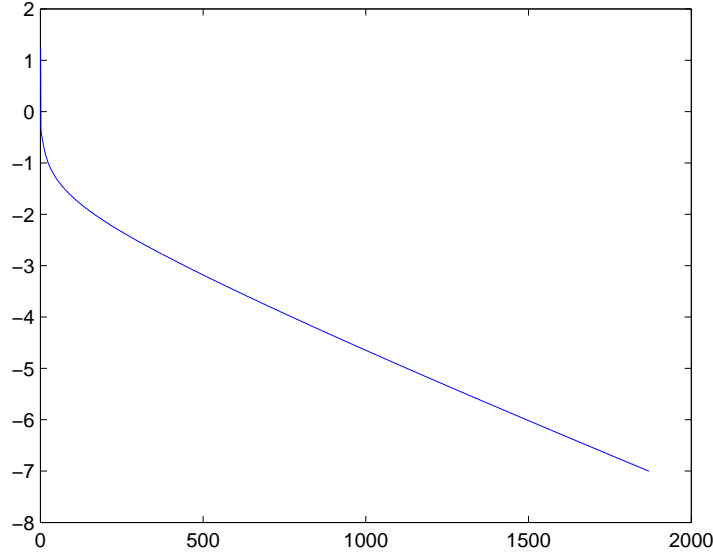


Figure 1: the plot of  $\log_{10}(\|\mathcal{A}(X_k) - b\|)$  for each iteration  $k$  (with parameters  $n = 100$ ,  $m = 110$ ,  $r = 4$  and  $d = 450$ )

At iteration  $k$ , the quantity

$$\left\langle \frac{x_k - x_{k+1}}{\|x_k - x_{k+1}\|}, \frac{\mathbb{P}_{\mathcal{A}}(x_k) - \mathbb{P}_{\mathcal{A}}(x_{k+1})}{\|\mathbb{P}_{\mathcal{A}}(x_k) - \mathbb{P}_{\mathcal{A}}(x_{k+1})\|} \right\rangle,$$

provides an approximation of  $c(\mathcal{R}_r, \mathcal{A}, X^*)$ . We print at each iteration `c2`, the square of this quantity. We also get `improv`, the improvement at each iteration, that is

$$\frac{\|\mathcal{A}(X_k) - b\|}{\|\mathcal{A}(X_{k-1}) - b\|}$$

We observe that the four quantities

- $10^s$  with  $s$  being the slope of the graph of  $\log_{10}(\|\mathcal{A}(X_k) - b\|)$ ,
- $c(\mathcal{R}_r, \mathcal{A})^2$  the square cosine of the angle (computed with (28)),
- the approximations `c2` (for the final iterations),
- the improvements `improv` (for the final iterations)

all coincide and the common value is here around 0.9266. This illustrates that the asymptotic convergence rate is the square cosine of the angle, as predicted by Remark 14.

## 7 Appendix: Projection onto spectral sets

In this appendix, we show that projection problems for “spectral” sets of symmetric matrices (that is, sets described solely by eigenvalue properties) are often easy. We begin with some basic ideas and notation, following [Lew96a, Lew96b] and the references therein.

The space  $\mathbf{S}^n$  of real symmetric  $n$ -by- $n$  matrices, equipped with the trace inner product, is a Euclidean space. A subset  $T$  is **spectral** if, for every matrix  $X \in T$  and every  $U$  in the group  $\mathbf{O}^n$  of orthogonal matrices, we have  $U^\top X U \in T$ . The eigenvalue map  $\lambda : \mathbf{S}^n \rightarrow \mathbb{R}^n$  maps any symmetric matrix  $X$  to its eigenvalues arranged in nonincreasing order,  $\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X)$ . It is easy to see that any spectral set can be written in the form  $\lambda^{-1}(K) = \{X : \lambda(X) \in K\}$ , for some set  $K \subset \mathbb{R}^n$ , and that we can further restrict  $K$  to be **permutation-invariant**: for every vector  $x \in K$  and every  $P$  in the group  $\mathbf{P}^n$  of permutation matrices, we have  $Px \in K$ .

Projecting a matrix  $Y \in \mathbf{S}^n$  onto a spectral set  $\lambda^{-1}(K)$  (where the set  $K \subset \mathbb{R}^n$  is permutation-invariant) is easy, providing we know how to project onto  $K$ . We proceed as follows: Calculate a spectral decomposition  $Y = U^\top \text{Diag}(y)U$ , where the matrix  $U$  is orthogonal and  $\text{Diag}(y)$  denotes the diagonal matrix with diagonal entries the components of the vector  $y \in \mathbb{R}^n$ ; next, find a nearest point  $x \in K$  to  $y$ ; now the matrix  $U^\top \text{Diag}(x)U$  is a nearest matrix to  $Y$  in  $\lambda^{-1}(K)$ .

This approach depends on the following classical result (see for example [Lew96a, Lemma 2.1 and Theorem 2.2]):

$$\sup_{V \in \mathbf{O}^n} \text{trace}(V^\top \text{Diag}(z)V \text{Diag}(y)) = \sup_{P \in \mathbf{P}^n} z^\top P y, \quad (29)$$

for any vectors  $y, z \in \mathbb{R}^n$ . We justify the projection procedure above in the following result.

**Theorem 21** (Spectral projection). *If the point  $x$  in the permutation-invariant set  $K \subset \mathbb{R}^n$  is a nearest point to the point  $y \in \mathbb{R}^n$ , then for any orthogonal matrix  $U$ , the matrix  $U^\top \text{Diag}(x)U$  is a nearest matrix in the spectral set  $\lambda^{-1}(K)$  to the matrix  $U^\top \text{Diag}(y)U$ .*

**Proof.** We can assume without loss of generality that the matrix  $U$  is the identity. Now using equation (29), the permutation-invariance of the set  $K$ ,

and the assumption on the point  $x$ , we have

$$\begin{aligned}
& \inf_{X \in \lambda^{-1}(K)} \|X - \text{Diag}(y)\|^2 \\
&= \inf_{V \in \mathbf{O}^n, z \in K} \|V^\top \text{Diag}(z)V - \text{Diag}(y)\|^2 \\
&= \inf_{V \in \mathbf{O}^n, z \in K} \left\{ \|z\|^2 + \|y\|^2 - 2\text{trace}(V^\top \text{Diag}(z)V \text{Diag}(y)) \right\} \\
&= \inf_{P \in \mathbf{P}^n, z \in K} \left\{ \|z\|^2 + \|y\|^2 - 2z^\top P y \right\} \\
&= \inf_{P \in \mathbf{P}^n, z \in K} \left\{ \|P^\top z\|^2 + \|y\|^2 - 2z^\top P y \right\} \\
&= \inf_{z \in K} \left\{ \|z\|^2 + \|y\|^2 - 2z^\top y \right\} \\
&= \inf_{z \in K} \|z - y\|^2 \\
&= \|x - y\|^2.
\end{aligned}$$

The first infimum is attained by  $X = \text{Diag}(x)$ , completing the proof.  $\blacksquare$

A useful tool for projecting onto permutation-invariant sets is the following easy result. We denote the vectors in  $\mathbb{R}^n$  with components in nonincreasing order by  $\mathbb{R}_{\geq}^n$ .

**Lemma 22.** *If the set  $K \subset \mathbb{R}^n$  is permutation-invariant, then for any vector  $y \in \mathbb{R}_{\geq}^n$  we have*

$$\inf_{x \in K} \|x - y\| = \inf_{x \in K \cap \mathbb{R}_{\geq}^n} \|x - y\|.$$

**Proof.** A classical inequality (see for example [Lew96a, Lemma 2.1]) shows that for any vector  $x \in \mathbb{R}_{\geq}^n$  we have  $\sup_{P \in \mathbf{P}^n} y^\top P x = y^\top x$ . The permutation-invariance of the set  $K$  now shows

$$\begin{aligned}
& \inf_{x \in K} \|x - y\|^2 \\
&= \inf_{x \in K \cap \mathbb{R}_{\geq}^n, P \in \mathbf{P}^n} \|P x - y\|^2 \\
&= \inf_{x \in K \cap \mathbb{R}_{\geq}^n, P \in \mathbf{P}^n} \left\{ \|x\|^2 + \|y\|^2 - 2y^\top P x \right\} \\
&= \inf_{x \in K \cap \mathbb{R}_{\geq}^n} \left\{ \|x\|^2 + \|y\|^2 - 2y^\top x \right\} \\
&= \inf_{x \in K \cap \mathbb{R}_{\geq}^n} \|x - y\|^2,
\end{aligned}$$

as desired.  $\blacksquare$

Our first example, showing how to project onto the “isospectral” set of all symmetric matrices with a given vector of eigenvalues, follows immediately. This result was observed in [Ors06].

**Example 23** (Isospectral projection). Suppose the matrix  $Y \in \mathbf{S}^n$  has spectral decomposition  $U^\top \text{Diag}(\lambda(Y))U$ . Then a nearest matrix to  $Y$  among all matrices with given eigenvalues  $x_1 \geq x_2 \geq \cdots \geq x_n$  is the matrix  $U^\top \text{Diag}(x)U$ . This follows by applying Lemma 22 and Theorem 21 to the set  $K = \mathbf{P}^n x$ . ■

Another interesting case concerns projection onto the set of matrices with maximum eigenvalue having a given multiplicity. The following example completes a partial result of [Ous00].

**Example 24** (Maximum eigenvalue multiplicity projection). Suppose the matrix  $Y \in \mathbf{S}^n$  has spectral decomposition  $U^\top \text{Diag}(\lambda(Y))U$ . Then a nearest matrix to  $Y$  among all matrices with maximum eigenvalue having multiplicity at least  $k$  is the matrix  $U^\top \text{Diag}(x)U$ , where

$$x_i = \begin{cases} k^{-1} \sum_{j=1}^k \lambda_j(Y) & (i \leq k) \\ \lambda_i(Y) & (i > k). \end{cases}$$

To see this result, we apply Lemma 22 and Theorem 21 to the set  $K \subset \mathbb{R}^n$  consisting of all vectors whose  $k$  largest components are equal. Suppose we wish to project a point  $y \in \mathbb{R}_{\geq}^n$  onto this set. By Lemma 22, we need to solve the problem

$$\inf_{x \in K \cap \mathbb{R}_{\geq}^n} \|x - y\|,$$

and it is not hard to check that a solution is given by

$$x_i = \begin{cases} k^{-1} \sum_{j=1}^k y_j & (i \leq k) \\ y_i & (i > k). \end{cases}$$

The result then follows. ■

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