Polytopes and Arrangements: Diameter and Curvature

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Abstract

By analogy with the conjecture of Hirsch, we conjecture that the order of the largest total curvature of the central path associated to a polytope is the number of inequalities defining the polytope. By analogy with a result of Dedieu, Malajovich and Shub, we conjecture that the average diameter of a bounded cell of an arrangement is less than the dimension. We prove continuous analogues of two results of Holt-Klee and Klee-Walkup: we construct a family of polytopes which attain the conjectured order of the largest total curvature, and we prove that the special case where the number of inequalities is twice the dimension is equivalent to the general case. We substantiate these conjectures in low dimensions and highlight additional links.

1 Continuous Analogue of the Conjecture of Hirsch

Let P be a full dimensional convex polyhedron defined by m inequalities in dimension n. The diameter $\delta(P)$ is the smallest number such that any two vertices of the polyhedron P can be connected by a path with at most $\delta(P)$ edges. The conjecture of Hirsch, formulated in 1957 and reported in [2], states that the diameter of a polyhedron defined by m inequalities in dimension n is not greater than $m-n$. The conjecture does not hold for unbounded polyhedra. A polytope is a bounded polyhedron. No polynomial bound is known for the diameter of a polytope.

Conjecture 1.1. (Conjecture of Hirsch for polytopes)
The diameter of a polytope defined by m inequalities in dimension n is not greater than $m-n$.

Intuitively, the total curvature [15] is a measure of how far off a certain curve is from being a straight line. Let $\psi : [\alpha,\beta] \to \mathbb{R}^n$ be a $C^2((\alpha-\varepsilon,\beta+\varepsilon))$ map for some $\varepsilon > 0$ with a non-zero derivative in $[\alpha,\beta]$. Denote its arc length by $l(t) = \int_\alpha^t \|\dot{\psi}(\tau)\| d\tau$, its parametrization by the arc length by $\psi_{\text{arc}} = \psi \circ l^{-1} : [0, l(\beta)] \to \mathbb{R}^n$, and its curvature at the point t by $\kappa(t) = \ddot{\psi}_{\text{arc}}(t)$. The total curvature is defined as $\int_0^{l(\beta)} \|\kappa(t)\| dt$. The requirement $\dot{\psi} \neq 0$ insures that any given segment of the curve is traversed only once and allows to define a curvature at any point on the curve.

We present one useful proposition. Roughly speaking, it states that two similar curves might not differ greatly in their total curvatures either. This fact is used in Section 3 in proving the analogue of the d-step conjecture for the total curvature of the central path.

Proposition 1.1. *Let ψ be as above and $\{\phi^j\}_{j=0,1,\ldots}$ be a sequence of $C^2((\alpha-\varepsilon,\beta+\varepsilon))$ functions with non-zero derivatives in $[\alpha,\beta]$ that converge to ψ point-wise as $j \to \infty$, i.e., $\phi^j(t) \to \psi(t)$ for all $t \in [\alpha,\beta]$. Then the total curvature of ψ is bounded from above by the infimum limit of the total curvature of ϕ^j over all j.*

For compactness we only sketch an elementary proof of this proposition in Section 4. For a detailed exposition of a very similar argument see [5].

From now on we consider only polytopes, i.e., bounded polyhedra, and denote those by P.

For a polytope $P = \{x : Ax \geq b\}$ with $A \in \mathbb{R}^{m \times n}$, denote $\lambda(P)$ the largest total curvature of the primal central path corresponding to the standard logarithmic barrier function, $-\sum_{i=1}^{m} \ln(A_i x - b_i)$, of the linear programming problem $\min\{c^T x : x \in P\}$ over all possible c. Following the analogy with the diameter, let $\Lambda(m,n)$ be the largest total curvature $\lambda(P)$ of the primal central path over all polytopes P defined by m inequalities in dimension n.

Conjecture 1.2. (Continuous analogue of the conjecture of Hirsch)
The order of the largest total curvature of the primal central path over all polytopes defined by m inequalities in dimension n is the number of inequalities defining the polytopes, i.e., $\Lambda(m,n) = \mathcal{O}(m)$.

Remark 1.1. *In [5] the authors showed that a redundant Klee-Minty n-cube $\mathcal{C}$ satisfies $\lambda(\mathcal{C}) \geq (\frac{3}{2})^n$, providing a counterexample to the conjecture of Dedieu and Shub [4] that $\Lambda(m,n) = \mathcal{O}(n)$.*

For polytopes and arrangements, respectively central path and linear programming, we refer to the books of Grünbaum [9] and Ziegler [16], respectively Renegar [12] and Roos et al [13], and the references therein.

2 Discrete Analogue of the result of Dedieu, Malajovich and Shub

Let $\mathcal{A}$ be a simple arrangement formed by m hyperplanes in dimension n. We recall that an arrangement is called simple if $m \geq n+1$ and any n hyperplanes intersect at a unique distinct point. Since $\mathcal{A}$ is simple, the number of bounded cells (bounded connected component of the complement to the hyperplanes) of $\mathcal{A}$ is $I = \binom{m-1}{n}$. Let $\lambda^c(P)$ denote the total curvature of the primal central path corresponding to $\min\{c^T x : x \in P\}$. Following the approach of Dedieu, Malajovich and Shub [3], let $\lambda^c(\mathcal{A})$ denote the average value of $\lambda^c(P_i)$ over the bounded cells P_i of $\mathcal{A}$; that is,

$$\lambda^c(\mathcal{A}) = \frac{\sum_{i=1}^{i=I} \lambda^c(P_i)}{I}.$$

Note that each bounded cell P_i is defined by the same number m of inequalities, some being potentially redundant. Given an arrangement $\mathcal{A}$, the average total curvature of a bounded cell $\lambda(\mathcal{A})$ is the largest value of $\lambda^c(\mathcal{A})$ over all possible c. Similarly, $\Lambda_{\mathcal{A}}(m,n)$ is the largest possible average total curvature of a bounded cell of a simple arrangement defined by m inequalities in dimension n.

Proposition 2.1. (Dedieu, Malajovich and Shub [3])
The average total curvature of a bounded cell of a simple arrangement defined by m inequalities in dimension n is not greater than $2\pi n$.

By analogy, let $\delta(\mathcal{A})$ denote the average diameter of a bounded cell of $\mathcal{A}$; that is,

$$\delta(\mathcal{A}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I}.$$

Similarly, let $\Delta_{\mathcal{A}}(m,n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by m inequalities in dimension n.

Conjecture 2.1. (Discrete analogue of the result of Dedieu, Malajovich and Shub)
The average diameter of a bounded cell of a simple arrangement defined by m inequalities in dimension n is not greater than n.

3 Additional Links and Low Dimensions

3.1 Additional Links

Proposition 3.1. *If the conjecture of Hirsch holds, then* $\Delta_{\mathcal{A}}(m,n) \leq n + \frac{2n}{m-1}$.

Proof. Let m_i denote the number of hyperplanes of $\mathcal{A}$ which are non-redundant for the description of a bounded cell P_i. If the conjecture of Hirsch holds, we have $\delta(P_i) \leq m_i - n$. It implies:

$$\delta(\mathcal{A}) \leq \frac{\sum_{i=1}^{i=I}(m_i - n)}{I} = \frac{\sum_{i=1}^{i=I} m_i}{I} - n.$$

Since a facet belongs to at most 2 cells, the sum of m_i for $i = 1, \ldots, I$ is less than twice the number of bounded facets of $\mathcal{A}$. As a bounded facet induced by a hyperplane H of $\mathcal{A}$ corresponds to a bounded cell of the $(n-1)$-dimensional simple arrangement $\mathcal{A} \cap H$, the sum of m_i is less than $2m\binom{m-2}{n-1}$. Therefore, we have, for any simple arrangement $\mathcal{A}$,

$$\delta(\mathcal{A}) \leq \frac{2m\binom{m-2}{n-1}}{\binom{m-1}{n}} - n = \frac{2mn}{m-1} - n = \frac{n(m+1)}{m-1}.$$

□

Remark 3.1. *In the proof of Proposition 3.1, we overestimate the sum of* m_i *for* $i = 1, \ldots, I$ *as some bounded facets belong to exactly 1 bounded cell. Let call such bounded facets external. We hypothesize that any simple arrangement has at least* $n\binom{m-2}{n-1}$ *external facets, in turn, this would strengthen Proposition 3.1 to:* If the conjecture of Hirsch holds, then $\Delta_{\mathcal{A}}(m,n) \leq \frac{n(m-n+1)}{m-1}$.

Similarly to Proposition 3.1, the results of Kalai and Kleitman [10] and Barnette [1] which bounds the diameter of a polytope by, respectively, $2\, m^{\log(n)+1}$ and $\frac{2^{n-2}}{3}(m - n + \frac{5}{2})$, directly yield:

Proposition 3.2. $\Delta_{\mathcal{A}}(m,n) \leq \frac{4mn\left(2m\binom{m-2}{n-1}\right)^{\log n}}{m-1}$ *and* $\Delta_{\mathcal{A}}(m,n) \leq n(\frac{m+1}{m-1} + \frac{5}{2n})\frac{2^{n-2}}{3}$.

The special case of $m = 2n$ of the conjecture of Hirsch is known as the d-step conjecture (as the dimension is often denoted by d in polyhedral theory). In particular, it has been shown by Klee and Walkup [11] that the special case $m = 2n$ for all n is equivalent to the conjecture of Hirsch.

Proposition 3.3. (Continuous analogue of the result of Klee and Walkup)
If $\Lambda(2n,n) = \mathcal{O}(n)$ *for all* n*, then* $\Lambda(m,n) = \mathcal{O}(m)$.

Proof. Suppose $\Lambda(2n,n) \leq 2Kn$, $n \geq 2$. Consider $\min\{c^T x : x \in P\}$ where $P = \{x : Ax \geq b\}$ with $A \in \mathbb{R}^{m \times n}$ and $\mathcal{P}$ and χ respectively denotes the associated central path and the analytic center. The two cases $n < m < 2n$ and $m > 2n$ are considered separately. Denote $\mathbf{0}$ and $\mathbf{1}$ the vector of all zeros and all ones respectively, $\mathrm{int}P$ – the interior of P. We may assume P is full-dimensional, for if not, we may reduce the problem dimension to satisfy the assumption. Note A is full-rank since P is bounded.

Case $n < m < 2n$: without loss of generality assume $c = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and denote x_1^* the optimal value of $\min\{c^T x : x \in P\}$. Consider $\min\{c^T x : x \in \widetilde{P}\}$ where $\widetilde{P} = \{x : \widetilde{A}x \geq \widetilde{b}\}$ with $\widetilde{A} \in \mathbb{R}^{2n,n}$ and $\widetilde{b} \in \mathbb{R}^{2n}$ are given by:

$$\widetilde{A}_{i,j} = \begin{cases} A_{i,j} & \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n, \\ 1 & \text{for } i = m+1, \ldots, 2n \text{ and } j = 1, \\ 0 & \text{for } i = m+1, \ldots, 2n \text{ and } j = 2, \ldots, n, \end{cases} \qquad \widetilde{b}_i = \begin{cases} b_i & \text{for } i = 1, \ldots, m, \\ x_1^* - 1 & \text{for } i = m+1, \ldots, 2n, \end{cases}$$

and $\widetilde{\mathcal{P}}$ denotes the associated central path, see Figure 1(a). Since the central path is the collection of the analytic centers of the level sets between the optimal solution and χ, we have $\mathcal{P} \subseteq \widetilde{\mathcal{P}}$ and, therefore, $\lambda^c(P) \leq \lambda^c(\widetilde{P})$. As $\widetilde{P}$ is defined by $2n$ inequalities in dimension n, we have $\lambda^c(\widetilde{P}) \leq 2Kn$, and thus, for $n < m < 2n$, $\lambda^c(P) \leq 2Kn$, that is, $\Lambda(m,n) = \mathcal{O}(m)$.

Case $m > 2n$: without loss of generality assume $\chi = \mathbf{0}$. Consider $\min\{(c,-\theta)^T(x,y) : (x,y) \in \widetilde{P}\}$ where $\widetilde{P} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : Ax - by \geq 0,\, y \leq 1\}$ with the associated central path $\widetilde{\mathcal{P}}$. In particular, if the definition of P is non-redundant, $\widetilde{P}$ is the $(n+1)$-dimensional flipped pyramid with base $P \times \{1\}$ and the apex $(\chi, 0) = \mathbf{0}$. We show that for large enough value of $\theta \gg \|c\|$ the central path $\mathcal{P}$ of the original optimization problem may be well approximated by a segment of the central path $\widetilde{\mathcal{P}}$ so that the total curvature of $\mathcal{P}$ is bounded from above by the total curvature of $\widetilde{\mathcal{P}}$. Intuitively, by choosing θ large enough we should be able to force $\widetilde{\mathcal{P}}$ to first follow almost a straight line from the analytic center of $\widetilde{P}$ to the face containing $P \times \{1\}$, and once $\widetilde{\mathcal{P}}$ is forced almost onto this face the path should look just like the central path $\mathcal{P}$ for $\min\{c^Tx : x \in P\}$ in one-less dimension, see Figure 1(b). Consequently, we argue that the total curvature of $\mathcal{P}$ may not be less then that of $\widetilde{\mathcal{P}}$.

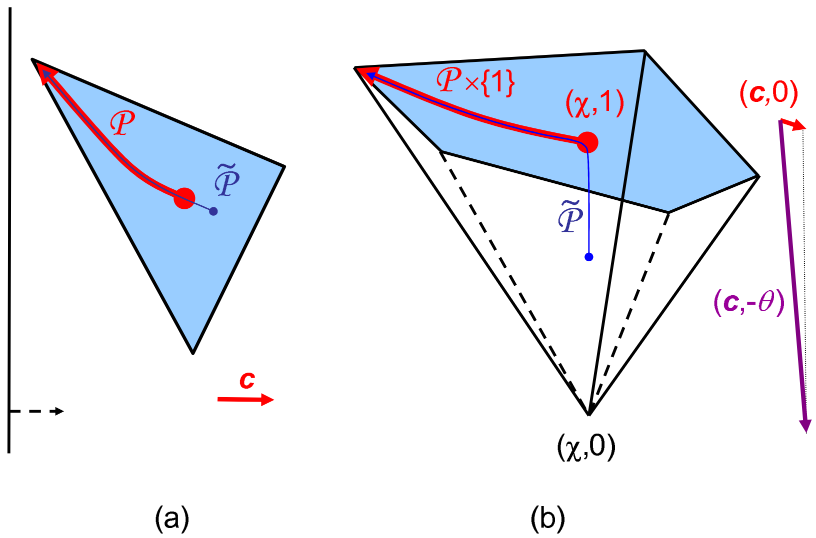

Figure 1: An illustration for the proof of Proposition 3.3 for $n = 2$: (a) $2n > m = 3$, (b) $2n < m = 5$

Denote $s = Ax - by$ and let S be the diagonal matrix with $S_{ii} = s_i$, $i = 1, \ldots, n$. Recall that the barrier functions $\widetilde{f}(x,y) : \mathrm{int}\widetilde{P} \to \mathbb{R}$ and $f(x) : \mathrm{int}P \to \mathbb{R}$ giving rise to the central paths $\widetilde{\mathcal{P}} = \{(\widetilde{x},\widetilde{y})(\nu) = \arg\min_{(x,y)} \nu(c,-\theta)^T(x,y) + \widetilde{f}(x,y),\, \nu \in [0,\infty)\}$ and $\mathcal{P} = \{x(\nu) = \arg\min_x \nu c^Tx + f(x),\, \nu \in [0,\infty)\}$ are $\widetilde{f}(x,y) = -\sum_{i=1}^m \ln s_i - \ln(1-y)$ and $f(x) = -\sum_{i=1}^m \ln s_i|_{y=1}$. The gradient of $\widetilde{f}$ at a point (x,y) satisfies

$$\widetilde{g} = \begin{pmatrix} -A^TS^{-1}\mathbf{1} \\ \frac{1}{1-y} + b^TS^{-1}\mathbf{1} \end{pmatrix}$$

and the Hessian of $\widetilde{f}$ satisfies

$$\widetilde{H} = \begin{pmatrix} A^TS^{-2}A & -A^TS^{-2}b \\ (-A^TS^{-2}b)^T & \frac{1}{(1-y)^2} + b^TS^{-2}b \end{pmatrix}$$

where its inverse may be computed using Schur complement and is equal to

$$\widetilde{H}^{-1} = \begin{pmatrix} (A^T S^{-2} A)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\frac{1}{(1-y)^2} - \gamma} \begin{pmatrix} (A^T S^{-2} A)^{-1} A^T S^{-2} b \\ 1 \end{pmatrix} \begin{pmatrix} (A^T S^{-2} A)^{-1} A^T S^{-2} b \\ 1 \end{pmatrix}^T$$

where

$$\gamma = b^T S^{-2} b - (A^T S^{-2} b)^T (A^T S^{-2} A)^{-1} A^T S^{-2} b.$$

In what follows we adapt the notation of [12]: we write $\|v\|_x$ for the local norm of v at x where the norm is induced by the intrinsic inner product at $x \in D_f$ arising from the strongly non-degenerate self-concordant function $f : D_f \to \mathbb{R}$, namely $\|v\| = v^T \nabla^2 f(x) v$. First we show that for θ large enough any segment of $\mathcal{P}$ corresponding to $\nu \in [\underline{\nu}, \overline{\nu}]$ may be well approximated by a suitably chosen segment of $\widetilde{\mathcal{P}}$; in doing so we will manufacture a good surrogate for a point $(\widetilde{x}, \widetilde{y})(\nu) \in \widetilde{\mathcal{P}}$ from $x\left(\frac{1}{2}(\nu + \sqrt{\nu^2 - \nu/\theta})\right) \in \mathcal{P}$.

Proposition 3.4. *Let $\nu \in [\underline{\nu}, \overline{\nu}]$ and M be such that $|b^T S^{-1} \mathbf{1}|, \sqrt{|\gamma|} \le M$ for all $(yx(\nu), y)$, $\frac{1}{2} \le y \le 1$, $\nu \in [\underline{\nu}, \overline{\nu}]$. If θ is chosen large enough to satisfy*

$$\theta \ge \frac{1}{\underline{\nu}} \max\{1, \sqrt{82} M - 1\}$$

then for $y = 1 - \frac{1}{1+\theta\nu}$ the point $(yx(\nu), y)$ approximates $(\widetilde{x}, \widetilde{y})(\nu/y)$, namely

$$\left\| \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} (\nu/y) - \begin{pmatrix} yx(\nu) \\ y \end{pmatrix} \right\|_{(yx(\nu),y)} \le \frac{\sqrt{82}}{3} \frac{M}{1+\theta\nu}.$$

Remark 3.2. *$M < \infty$ since the portion of the central path $\mathcal{P}$ corresponding to any finite interval $[\underline{\nu}, \overline{\nu}]$ lies in the interior of P.*

Remark 3.3. *With $y = 1 - \frac{1}{1+\theta\nu}$ the function $\nu \mapsto \frac{\nu}{y}$ is monotone increasing for $\nu \ge 0$ and hence is invertible with its inverse being $\nu \mapsto \frac{1}{2}(\nu + \sqrt{\nu^2 - \nu/\theta})$.*

Remark 3.4. *Recalling that $\widetilde{f}$ is strongly non-degenerate self-concordant and, therefore, an open unit ball in the local norm at $(x,y) \in \operatorname{int}\widetilde{P}$ satisfies $B_{(x,y)}((x,y),1) \subset \operatorname{int}\widetilde{P}$ where $\widetilde{P}$ is bounded by assumption, the proposition implies that $(\widetilde{x}, \widetilde{y})(\nu) \to (x(\nu), 1)$ as $\theta \to \infty$. Moreover, the convergence is uniform over any finite segment $[\underline{\nu}, \overline{\nu}]$ of $\mathcal{P}$.*

Proof. We rely on the fact that the Newton's method iterates exhibit local quadratic convergence to the central path for linear programming [13], which may be rephrased in a more general setting of strongly non-degenerate self-concordant functions [12]. For concreteness we use an intermediate statement in the proof of Proposition 2.2.8 of [12]: let $n(x) = -\nabla^2 f(x)^{-1} \nabla f(x)$ be the Newton step for strongly non-degenerate self-concordant function $f : D_f \to \mathbb{R}$, if $\|n(x)\|_x \le \frac{1}{9}$ then f has a minimizer z and $\|x - z\|_x \le 3\|n(x)\|_x$.

Consider the Newton step for minimizing $\nu'(c, -\theta)^T(x,y) + \widetilde{f}(x,y)$, and evaluate (the square of) its local norm at $(yx(\nu), y)$ where $y = 1 - \frac{1}{1+\theta\nu}$ and $\nu' = \frac{\nu}{y}$:

$$\left(\nu' \begin{pmatrix} c \\ -\theta \end{pmatrix} + \widetilde{g} \right)^T \widetilde{H}^{-1} \left(\nu' \begin{pmatrix} c \\ -\theta \end{pmatrix} + \widetilde{g} \right) \Bigg|_{(yx(\nu),y)}$$

$$= \begin{pmatrix} \frac{\nu}{y} c + \frac{1}{y} \nabla f \\ \frac{-\nu}{y}\theta + \frac{1}{1-y} + b^T S^{-1} \mathbf{1} \end{pmatrix}^T \widetilde{H}^{-1} \begin{pmatrix} \frac{\nu}{y} c + \frac{1}{y} \nabla f \\ \frac{-\nu}{y}\theta + \frac{1}{1-y} + b^T S^{-1} \mathbf{1} \end{pmatrix} \Bigg|_{(yx(\nu),y)}$$

and since $x(\nu) \in \mathcal{P}$

$$= \left(\begin{array}{c} 0 \\ \frac{-\nu}{y}\theta + \frac{1}{1-y} + b^T S^{-1}\mathbf{1} \end{array} \right)^T \widetilde{H}^{-1} \left(\begin{array}{c} 0 \\ \frac{-\nu}{y}\theta + \frac{1}{1-y} + b^T S^{-1}\mathbf{1} \end{array} \right) \Bigg|_{(yx(\nu),y)}$$

$$= \left(\frac{-\nu}{y}\theta + \frac{1}{1-y} + b^T S^{-1}\mathbf{1} \right)^2 \frac{1}{\frac{1}{(1-y)^2} + \gamma} \Bigg|_{(yx(\nu),y)}$$

and by the choice of y and θ

$$= \frac{(b^T S^{-1}\mathbf{1})^2}{\frac{1}{(1-y)^2+\gamma}} \Bigg|_{(yx(\nu),y)} \leq \frac{M^2}{(1+\theta\nu)^2 - M^2} \leq \frac{82}{81} \left(\frac{M}{1+\theta\nu} \right)^2 \leq \frac{1}{81}.$$

Now, since the size of the Newton step for minimizing $\nu'(c,-\theta)^T(x,y) + \widetilde{f}(x,y)$ measured with respect to the local norm at $(yx(\nu), y)$ is indeed $\leq \frac{1}{9}$, the statement of the proposition follows immediately. $\square$

Next we argue that the total curvature of any finite segment $[\underline{\nu}, \overline{\nu}]$ of the central path $\mathcal{P}$ may not be much less than the total curvature of $[\underline{\nu}, \overline{\nu}]$-segment of $\widetilde{\mathcal{P}}$. Indeed, this follows from Proposition 1.1 where the point-wise convergence of two paths follows from Remark 3.4. Note that the gradient to the central path does not vanish, since a point $x(\nu) \in \mathcal{P}$ is the minimizer for $\nu c^T x + f(x)$ and hence must satisfy $\nu c + \nabla f(x) = 0$, the derivative with respect to ν satisfies $\dot{x}(\nu) = -\nabla^2 f(x)^{-1} c \neq 0$, recalling that under our assumptions $\nabla^2 f(x)$ is non-singular. Using a similar argument one can show $\mathcal{P}$ is C^2 with respect to ν, although a particular parametrization of $\mathcal{P}$ is not important since it is already well-known that $\mathcal{P}$ is real-analytic [14]. Same considerations apply to $\widetilde{\mathcal{P}}$.

In turn, the total curvature of $\mathcal{P}$ may be arbitrary well approximated by the total curvature of a suitably chosen finite segment of the path, letting $\underline{\nu} \to 0$ and $\overline{\nu} \to \infty$. The later follows from the finiteness of the total curvature of $\mathcal{P}$ established in [3], or intuitively from the fact that $\mathcal{P}$ is asymptotically straight as $\nu \to 0$ and $\nu \to \infty$.

Finally, we may summarize our findings in the following Lemma.

Lemma 3.1. *With the construction above as $\theta \to \infty$ we have*

$$\liminf \lambda^{(c,-\theta)}(\widetilde{P}) > \lambda^c(P).$$

The proof that $\liminf \lambda^{(c,-\theta)}(\widetilde{P}) \geq \lambda^c(P)$ easily follows from the Remark 3.4 and Proposition 1.1 as already described. For strict inequality using the techniques in [5] (also similar to the technique used in Proposition 3.7) one can show that just before $\widetilde{\mathcal{P}}$ "starts to converge" to $\mathcal{P}$, $\widetilde{\mathcal{P}}$ is bound to make a first sharp turn which in the limit will contribute to $\pi/2$ additional total curvature for this path.

Now, inductively increasing the dimension and the number of inequalities by 1 and carefully using the limit argument in the above, the same construction gives a sequence of polytopes $\widetilde{P} = \widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_{m-2n}$ satisfying $\lambda^c(P) \leq \lambda^{(c,-\theta)}(\widetilde{P}_1) \leq \cdots \leq \lambda^{(c,-\theta,\ldots)}(\widetilde{P}_{m-2n})$. Since $\widetilde{P}_{m-2n}$ is defined by $2m-2n$ inequalities in dimension $m-n$, we have $\lambda^{(c,-\theta,\ldots)}(\widetilde{P}_{m-2n}) \leq 2K(m-n)$. This implies that, for $m > 2n$, $\lambda^c(P) \leq 2K(m-n)$, that is, $\Lambda(m,n) = \mathcal{O}(m)$. $\square$

Remark 3.5. *In contrast with Proposition 3.1, $\Lambda(m,n) = \mathcal{O}(m)$ does not imply that $\Lambda_{\mathcal{A}}(m,n) = \mathcal{O}(n)$ since all the m inequalities count for each $\lambda(P_i)$ while it is enough to consider the m_i non-redundant inequalities for each $\delta(P_i)$.*

3.2 Low Dimensions

The diameter of a polytope in dimension 2 and 3 satisfies, respectively, $\delta(P) \leq \lfloor \frac{m}{2} \rfloor$ and $\delta(P) \leq \lfloor \frac{2m}{3} \rfloor - 1$. It clearly implies:

Proposition 3.5. $\Delta_{\mathcal{A}}(m,2) \leq 2 + \frac{2}{m-1}$ *and* $\Delta_{\mathcal{A}}(m,3) \leq 3 + \frac{4}{m-1}$.

In dimension 2, let S_2 be the sphere of radius 1 centered on $(1,1)$ and consider the arrangement $\mathcal{A}^*_{m,2}$ made of the 2 lines forming the nonnegative orthant and an additional $m-2$ lines tangent to S_2 and separating the origin from the center of the sphere S_2. See Figure 2 for an illustration of $\mathcal{A}^*_{6,2}$. Besides $m-2$ triangles, the bounded cells of $\mathcal{A}^*_{m,2}$ are made of $\binom{m-2}{2}$ 4-gons. We have $\delta(\mathcal{A}^*_{m,2}) = \frac{2(m-2)}{m-1}$, and thus,

Proposition 3.6. $2 - \frac{2}{m-1} \leq \Delta_{\mathcal{A}}(m,2) \leq 2 + \frac{2}{m-1}$.

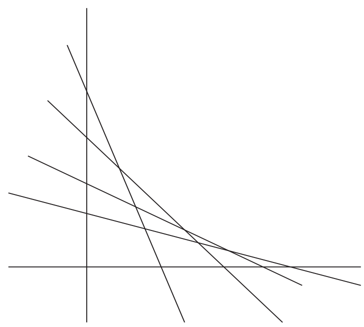

Figure 2: The arrangement $\mathcal{A}^*_{6,2}$

Remark 3.6. *The arrangement $\mathcal{A}^*_{m,2}$ was generalized in [6] to an arrangement with $\binom{m-n}{n}$ cubical cells yielding that the dimension n is an asymptotic lower bound for $\Delta_{\mathcal{A}}(m,n)$ for fixed n. A similar construction produces an order-n lower bound for $\Lambda_{\mathcal{A}}(m,n)$.*

In dimension 2, for $m \geq 4$, consider the polytope $P^*_{m,2}$ defined by the following m inequalities: $y \leq 1$, $x \leq \frac{y}{10} + \frac{1}{2}$, $-x \leq \frac{y}{3} + \frac{1}{3}$ and $(-1)^i x \leq \frac{10^{i-2}y}{11} + \frac{5}{11} - \frac{10^{-4}}{m}\frac{i}{m}$ for $i = 4, \ldots, m$. See Figure 3 for an illustration of $P^*_{6,2}$.

Proposition 3.7. *The total curvature of the central path of* $\min\{y : (x,y) \in P^*_{m,2}\}$ *satisfies*

$$\liminf_{m\to\infty} \frac{\lambda^{(0,1)}(P^*_{m,2})}{m} \geq \pi.$$

Proof. First we show that the central path $\mathcal{P}$ goes through a sequence of $m-2$ points $(x_j, \frac{10^{1-j}}{5})$ for $j = 1, \ldots, m-2$ with $x_j \geq 0$ for odd j and $x_j \leq \frac{-10^{-4}}{m}$ for even j. For $i = 2, \ldots, m$ and $j = 1, \ldots, m-2$, denote z^j_i the first coordinate of the intersection of the line $y = \frac{10^{1-j}}{5}$ and the facet of $P^*_{m,2}$ induced by the i^{th} inequality defining $P^*_{m,2}$, that is, $z^j_2 = \frac{10^{-j}}{5} + \frac{1}{2}$, $z^j_3 = -\frac{10^{-j+1}}{15} - \frac{1}{3}$, and $z^j_i = (-1)^i(\frac{10^{i-j-1}}{55} + \frac{5}{11} - \frac{10^{-4}}{m}\frac{i}{m})$ for $i = 4, \ldots, m$. As the central path may be characterized as the set of minimizers of the barrier function over appropriate level sets of the objective function, the point $(x_j, \frac{10^{1-j}}{5})$ of $\mathcal{P}$ satisfies

$$x_j = \arg\max_x \sum_{i=2}^{m} \ln(-1)^i(z^j_i - x).$$

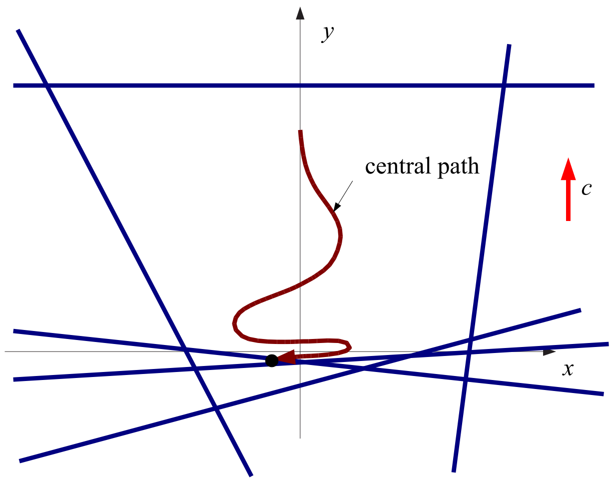

Figure 3: The polytope $P^*_{6,2}$ and its central path

Therefore, to show that $x_j \geq 0$ for odd j and that $x_j \leq \frac{-10^{-4}}{m}$ for even j, it is enough to prove that $g^j(0) > 0$ for odd j and $g^j(\frac{-10^{-4}}{m}) < 0$ for even j where

$$g^j(x) = \sum_{i=2}^{m} \frac{d}{dx} \ln(-1)^i (z_i^j - x).$$

For simplicity we assume that m is even. A similar argument applies for odd values of m. Since $(-1)^{k+1}\left(\frac{1}{x-z_k^j} + \frac{1}{x-z_{k+1}^j}\right) > 0$ for $k \geq j+4$ and $\frac{-10^{-4}}{m} \leq x \leq 0$, we have

$$\sum_{i=j+4}^{i=m} \frac{1}{x - z_i^j} \begin{cases} \geq 0, & j \text{ odd}, \quad x = 0, \\ \leq 0, & j \text{ even}, \quad x = \frac{-10^{-4}}{m}. \end{cases} \tag{1}$$

This yields

$$g^1(0) \geq \frac{-1}{\frac{1}{2} + \frac{1}{50}} + \frac{1}{\frac{1}{3} + \frac{1}{15}} - \frac{1}{\frac{100}{55} + \frac{5}{11} - 10^{-4}} = \frac{772}{5639} > 0.$$

For $j \geq 2$, rewrite

$$g^j(x) = \left(\frac{1}{x - z_2^j} + \frac{1}{x - z_3^j}\right) + \sum_{i=4}^{i<j+2} \frac{1}{x - z_i^j} + \sum_{i=j+2}^{i<j+4} \frac{1}{x - z_i^j} + \sum_{i=j+4}^{i=m} \frac{1}{x - z_i^j}.$$

Observe

$$\frac{1}{x - z_2^j} + \frac{1}{x - z_3^j} = \begin{cases} \frac{-1}{\frac{1}{2} + \frac{10^{-j}}{5}} + \frac{1}{\frac{1}{3} + \frac{10^{-j+1}}{15}} & \text{for } x = 0, \\ \frac{-1}{\frac{1}{2} + \frac{10^{-j}}{5} + \frac{10^{-4}}{m}} + \frac{1}{\frac{1}{3} + \frac{10^{-j+1}}{15} - \frac{10^{-4}}{m}} & \text{for } x = \frac{-10^{-4}}{m}, \end{cases} \tag{2}$$

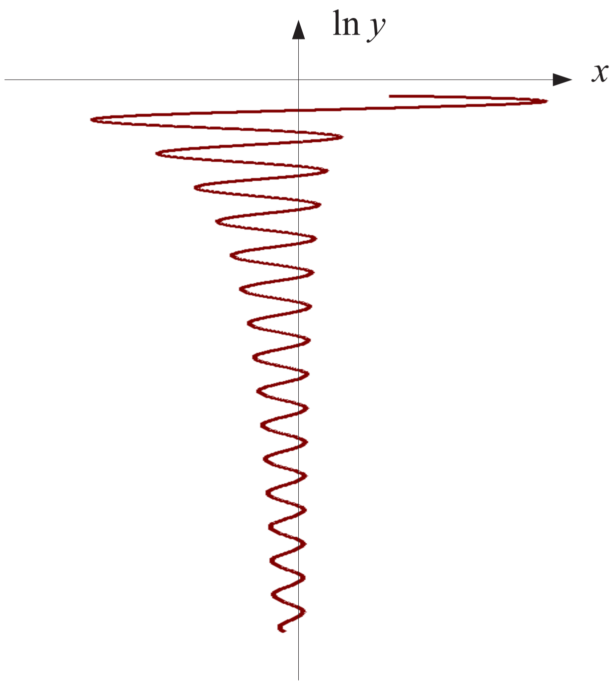

Figure 4: The central path for $P^*_{34,2}$

and

$$\sum_{i=j+2}^{i<j+4} \frac{1}{x - z_i^j} \begin{cases} \geq \frac{1}{\frac{10}{55}+\frac{5}{11}} - \frac{1}{\frac{100}{55}+\frac{5}{11}-\frac{10^{-4}}{m}}, & j \geq 3 \text{ odd}, \quad x = 0, \\ \leq \frac{-1}{\frac{10}{55}+\frac{5}{11}+\frac{10^{-4}}{m}} + \frac{1}{\frac{100}{55}+\frac{5}{11}-2\frac{10^{-4}}{m}}, & j \leq m-4 \text{ even}, \quad x = \frac{-10^{-4}}{m}, \\ \leq \frac{-1}{\frac{10}{55}+\frac{5}{11}+\frac{10^{-4}}{m}}, & j = m-2, \quad x = \frac{-10^{-4}}{m}. \end{cases} \tag{3}$$

For odd $j \geq 3$ and $x = 0$, we have

$$\sum_{i=4}^{i<j+2} \frac{1}{x - z_i^j} \geq -\frac{1}{\frac{10^{3-j}}{55} + \left(\frac{5}{11} - \frac{10^{-4}}{m}\right)} + \frac{1}{\frac{10^{4-j}}{55} + \frac{5}{11}} + \cdots - \frac{1}{\frac{1}{55} + \left(\frac{5}{11} - \frac{10^{-4}}{m}\right)}$$

$$= \frac{-1}{\frac{5}{11} - \frac{10^{-4}}{m}} \left(\frac{1}{1 + \frac{10^{3-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}} + \frac{1}{1 + \frac{10^{5-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}} + \cdots + \frac{1}{1 + \frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}} \right)$$

$$+ \frac{11}{5} \left(\frac{1}{1 + \frac{11 \cdot 10^{4-j}}{5 \cdot 55}} + \frac{1}{1 + \frac{11 \cdot 10^{6-j}}{5 \cdot 55}} + \cdots + \frac{1}{1 + \frac{11 \cdot 10^{-1}}{5 \cdot 55}} \right)$$

$$\geq \frac{-1}{\frac{5}{11}-\frac{10^{-4}}{m}}\left(1-\frac{10^{3-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}+\left(\frac{10^{3-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2\right.$$

$$\left.+1-\frac{10^{5-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}+\left(\frac{10^{5-j}}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2+\cdots+1-\frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}+\left(\frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2\right)$$

$$+\frac{11}{5}\left(1-\frac{11\cdot 10^{4-j}}{5\cdot 55}+\cdots+1-\frac{11\cdot 10^{-1}}{5\cdot 55}\right)$$

$$=\frac{-1}{\frac{5}{11}-\frac{10^{-4}}{m}}\left(\left\lfloor\frac{j}{2}\right\rfloor-\frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\cdot\frac{1-.01^{\lfloor\frac{j}{2}\rfloor}}{1-.01}+\left(\frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)}\right)^2\frac{1-.0001^{\lfloor\frac{j}{2}\rfloor}}{1-.0001}\right)$$

$$+\frac{11}{5}\left(\left\lfloor\frac{j}{2}\right\rfloor-1-\frac{11}{550}\cdot\frac{1-.01^{\lfloor\frac{j}{2}\rfloor-1}}{1-.01}\right)$$

$$\geq\frac{-\left\lfloor\frac{j}{2}\right\rfloor\frac{10^{-4}}{m}}{\left(\frac{5}{11}\right)^2-\frac{5}{11}\frac{10^{-4}}{m}}+\frac{1}{55\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)^2}-\frac{1}{55^2\left(\frac{5}{11}-\frac{10^{-4}}{m}\right)^3.9999}-\frac{11}{5}-\left(\frac{11}{5}\right)^2\frac{1}{550\cdot .9999},$$

where the second inequality is based on $1-v\leq\frac{1}{1+v}\leq 1-v+v^2$, $v\geq 0$ and the last equality is obtained by summing up the terms in three resulting geometric series. This, combined with observations (1), (2) and (3), gives, for odd $j\geq 3$,

$$g^j(0)\geq\left(-2+\frac{1}{\frac{1}{3}+\frac{1}{1500}}\right)+\left(\frac{1}{\frac{10}{55}+\frac{5}{11}}-\frac{1}{\frac{100}{55}+\frac{5}{11}-.0001}\right)$$

$$+\left(\frac{-.00005}{\left(\frac{5}{11}\right)^2-\frac{5}{11}\cdot .0001}+\frac{1}{55\left(\frac{5}{11}\right)^2}-\frac{1}{55^2\left(\frac{5}{11}-.0001\right)^3.9999}-\frac{11}{5}-\left(\frac{11}{5}\right)^2\frac{1}{550\cdot .9999}\right)=\frac{49}{63838}>0.$$

Similarly for even $j\geq 2$ and $x=\frac{-10^{-4}}{m}$ we have

$$\sum_{i=4}^{i<j+2}\frac{1}{x-z_i^j}\leq\frac{-1}{\frac{5}{11}+\frac{10^{-4}}{m}}\left(\left\lfloor\frac{j}{2}\right\rfloor-1-\frac{1}{550\left(\frac{5}{11}+\frac{10^{-4}}{m}\right)}\cdot\frac{1-.01^{\lfloor\frac{j}{2}\rfloor-1}}{1-.01}\right)$$

$$+\frac{1}{\frac{5}{11}-2\frac{10^{-4}}{m}}\left(\left\lfloor\frac{j}{2}\right\rfloor-1-\frac{1}{55\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)}\cdot\frac{1-.01^{\lfloor\frac{j}{2}\rfloor-1}}{1-.01}+\left(\frac{1}{55\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)}\right)^2\frac{1-.0001^{\lfloor\frac{j}{2}\rfloor-1}}{1-.0001}\right)$$

$$\leq\left(\left\lfloor\frac{j}{2}\right\rfloor-1\right)\frac{2\frac{10^{-4}}{m}+\frac{10^{-4}}{m}}{\left(\frac{5}{11}\right)^2-\left(\frac{-10^{-4}}{m}\right)^2-\frac{10^{-4}}{m}\left(\frac{5}{11}+\frac{10^{-4}}{m}\right)}+\frac{1}{550\left(\frac{5}{11}+\frac{10^{-4}}{m}\right)^2}\cdot\frac{1}{1-.01}$$

$$-\frac{1}{55\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)^2}+\frac{1}{55^2\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)^3}\cdot\frac{1}{1-.0001}.$$

Thus, for even $j\geq 2$.

$$g^j\left(\frac{-10^{-4}}{m}\right)\leq\left(\frac{-1}{\frac{1}{2}+\frac{1}{500}+.0001}+\frac{1}{\frac{1}{3}-.0001}\right)+\left(\frac{-1}{\frac{10}{55}+\frac{5}{11}+.0001}+\frac{1}{\frac{100}{55}+\frac{5}{11}-.0002}\right)$$

$$+\left(\frac{.00015}{\left(\frac{5}{11}\right)^2-.0001^2-.0001\left(\frac{5}{11}+.0001\right)}-\frac{89}{99}\cdot\frac{1}{55\left(\frac{5}{11}\right)^2}+\frac{1}{55^2\cdot .999}\cdot\frac{1}{\left(\frac{5}{11}-.0002\right)^3}\right)=\frac{-784}{3985}<0.$$

Therefore, the central path $\mathcal{P}$ goes through a sequence of $m-2$ points (x_j,y_j) with $y_j=\frac{10^{1-j}}{5}$ and $x_j\geq 0$ for odd j, $x_j\leq\frac{-10^{-4}}{m}$ for even j. One can easily check that $(x_j,y_j)\in\mathcal{P}$ for $j=1,\ldots,m-2$ by verifying that the analytic center χ is above the line $y=\frac{1}{5}$. We have

$$\chi=(\chi_1,\chi_2)=\arg\max_{(x,y)\in P^*_{m,2}}\left(\ln(1-y)+\ln\left(-x+\frac{y}{10}+\frac{1}{2}\right)+\ln\left(x+\frac{y}{3}+\frac{1}{3}\right)\right.$$
$$\left.+\sum_{i=4}^{m}\ln\left((-1)^{i+1}x+\frac{10^{i-2}y}{11}+\frac{5}{11}-\frac{10^{-4}}{m}\frac{i}{m}\right)\right).$$

Therefore, to show that $\chi_2>\frac{1}{5}$, it is enough to prove that the derivative with respect to y of the log-barrier function is negative for $(x,y)\in P^*_{m,2}$ and $y\leq\frac{1}{5}$, that is,

$$\frac{-1}{1-y}+\frac{1}{-10x+y+5}+\frac{1}{3x+y+1}+\sum_{i=4}^{m}\frac{10^{i-2}}{\left((-1)^{i+1}11x+10^{i-2}y+5-11\cdot\frac{10^{-4}}{m}\frac{i}{m}\right)}>0,$$

which is implied by

$$\frac{-1}{1-y}+\frac{100}{-11x+100y+5-11\cdot\frac{.0001}{m}\frac{4}{m}}>\frac{-5}{4}+\frac{100}{\frac{100}{5}+5+\frac{66}{15}}=\frac{1265}{588}>0.$$

To show that $\liminf_{m\to\infty}\frac{\lambda^{(0,1)^T}(P^*_{m,2})}{m}\geq\pi$, consider three consecutive points from this sequence, say $(x_{j-1},y_{j-1}),(x_j,y_j),(x_{j+1},y_{j+1})$, and observe that for any $\varepsilon>0$ we can choose m so that for all $\varepsilon m\leq j<m-2$ we have

$$\frac{|y_j-y_{j-1}|}{|x_j-x_{j-1}|}<\varepsilon,\ \frac{|y_{j+1}-y_j|}{|x_{j+1}-x_j|}<\varepsilon.$$

Let m be such a value and $j\geq\varepsilon m$. Without loss off generality j might be assumed odd and let $\tau_{j-1},\tau_j,\tau_{j+1}\in\mathbb{R}$ be such that $\mathcal{P}_{\mathrm{arc}}(\tau_k)=(x_k,y_k)$, $k=j-1,j,j+1$. We show by contradiction that there is a t_1 such that the first coordinate $\left(\dot{\mathcal{P}}_{\mathrm{arc}}(t_1)\right)_1>\sqrt{1-\varepsilon^2}$. Suppose that for all $t\in[\tau_{j-1},\tau_j]$ we have $\left(\dot{\mathcal{P}}_{\mathrm{arc}}(t)\right)_1\leq\sqrt{1-\varepsilon^2}$, then $\left(\dot{\mathcal{P}}_{\mathrm{arc}}(t)\right)_2\leq-\varepsilon$ since $\|\dot{\mathcal{P}}_{\mathrm{arc}}(t)\|=1$ and $(\mathcal{P}_{\mathrm{arc}}(t))_2$ is monotone-decreasing with respect to t. By the Mean-Value Theorem it follows that $\tau_j-\tau_{j-1}>x_j-x_{j-1}$, and thus, by the same theorem, we must have $(\mathcal{P}_{\mathrm{arc}}(\tau_j))_2-(\mathcal{P}_{\mathrm{arc}}(\tau_{j-1}))_2=y_j-y_{j-1}<-\varepsilon(x_j-x_{j-1})$, a contradiction. Similarly, there is a t_2 such that $\left(\dot{\mathcal{P}}_{\mathrm{arc}}(t_2)\right)_1<-\sqrt{1-\varepsilon^2}$. Since the total curvature K_j of the segment of $\mathcal{P}_{\mathrm{arc}}$ connecting the points $(x_{j-1},y_{j-1}),(x_j,y_j),(x_{j+1},y_{j+1})$ corresponds to the length of the curve $\dot{\mathcal{P}}_{\mathrm{arc}}$ connecting the corresponding derivative points on a unit 2-sphere, K_j may be bounded below by the length of the geodesic between the points $\dot{\mathcal{P}}_{\mathrm{arc}}(t_1)$ and $\dot{\mathcal{P}}_{\mathrm{arc}}(t_2)$, that is, bounded below by a constant arbitrarily close to π. Now simply add all K_j for all $\varepsilon m\leq j<m-2$. □

Holt and Klee [8] showed that, for $m>n\geq 13$, the conjecture of Hirsch is tight. Fritzsche and Holt [7] extended the result to $m>n\geq 8$. Since the polytope $P^*_{m,2}$ can be generalized to higher dimensions by adding the box constraints $0\leq x_i\leq 1$ for $i\geq 3$, we have:

Corollary 3.1. (Continuous analogue of the result of Holt and Klee)
$\liminf_{m\to\infty}\frac{\Lambda(m,n)}{m}\geq\pi$, *that is,* $\Lambda(m,n)$ *is bounded below by a constant times* m.

4 Sketch of the proof of Proposition 1.1

Observe that without loss of generality we may assume ψ is parameterized by the arc length: a short computation shows that if $\dot{\psi} \neq 0$ then the second derivative of the arc length parametrization of the curve is continuous since

$$\dot{\psi}_{\text{arc}}(t) = \psi(l^{-1}(t))'_t = \dot{\psi}(l^{-1}(t)) \cdot (l^{-1}(t))'_t = \frac{\dot{\psi}(l^{-1}(t))}{\|\dot{\psi}(l^{-1}(t))\|},$$

and consequently

$$\ddot{\psi}_{\text{arc}} \circ l = \frac{\ddot{\psi}\|\dot{\psi}\|^2 - \dot{\psi}\,(\ddot{\psi}^T\dot{\psi})}{\|\dot{\psi}\|^3}.$$

In what follows we remove the arc subscript from ψ to shorten the notation and write α for $l(\alpha)$ and β for $l(\beta)$. Note that the total curvature of ψ corresponds to the length of the gradient curve $\dot{\psi}$ between the points $\dot{\psi}(\alpha)$ and $\dot{\psi}(\beta)$ which in turn belong to the unit n-sphere.

First we argue that the total curvature of ψ may be arbitrarily well approximated by the sum of chordal distances between pairs $(\dot{\psi}(t_{i-1}), \dot{\psi}(t_i))$, $\sum_{i=1}^N \|\dot{\psi}(t_i) - \dot{\psi}(t_{i-1})\|$, where $t_0 = \alpha < t_1 < t_2 < \cdots < t_N = \beta$ as long as $\max_{i=1,\cdots,N}(t_i - t_{i-1}) \to 0$, see Figure 5(a). Trivially $\int_\alpha^\beta \|\ddot{\psi}(t)\|dt = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\ddot{\psi}(t)\|dt$ and observe

$$\left| \int_{t_{i-1}}^{t_i} \|\ddot{\psi}(t)\|dt - \int_{t_{i-1}}^{t_i} \frac{\|\dot{\psi}(t_i) - \dot{\psi}(t_{i-1})\|}{t_i - t_{i-1}} dt \right| \leq \int_{t_{i-1}}^{t_i} \left| \|\ddot{\psi}(t)\| - \frac{\|\dot{\psi}(t_i) - \dot{\psi}(t_{i-1})\|}{t_i - t_{i-1}} \right| dt$$

$$\leq \int_{t_{i-1}}^{t_i} \left\| \ddot{\psi}(t) - \frac{\dot{\psi}(t_i) - \dot{\psi}(t_{i-1})}{t_i - t_{i-1}} \right\| dt = \int_{t_{i-1}}^{t_i} \left\| \ddot{\psi}(t) - \begin{pmatrix} \ddot{\psi}_1(\bar{t}_1) \\ \ddot{\psi}_2(\bar{t}_2) \\ \vdots \\ \ddot{\psi}_n(\bar{t}_n) \end{pmatrix} \right\| dt \leq \int_{t_{i-1}}^{t_i} Mtdt = \frac{M(t_i - t_{i-1})^2}{2}$$

with $t_{i-1} \leq \bar{t}_i \leq t_i$ for all i, where the second inequality follows from the triangle inequality, the first equality follows from the Intermediate-Value Theorem, and the last inequality is implied by the Lipschitz continuity of the second derivative of ψ on the compact set where the Lipschitz constant is denoted by M. So, up to a quadratic error term the curvature contribution over the segment $[t_{i-1}, t_i]$ may be approximated by the length of the linear segment connecting the starting point $\dot{\psi}(t_{i-1})$ and the ending point $\dot{\psi}(t_i)$ with both points on the unit sphere, implying a linear error term for the total curvature approximation over $[\alpha, \beta]$. In particular, we may consider the partitioning $t_0 = \alpha < t_1 < t_2 < \cdots < t_N = \beta$ where each $\|\dot{\psi}(t_i) - \dot{\psi}(t_{i+1})\| = \gamma$ for some small fixed γ.

Note that any partitioning of $[\alpha, \beta]$ trivially gives a lower bound of $\sum_{i=1}^N \|\dot{\phi}^j(t_i) - \dot{\phi}^j(t_{i-1})\|$ on the total curvature of ϕ^j for all j. Also, note that since $\phi^j(t) \to \psi(t)$ as $j \to \infty$ on a compact set $[\alpha, \beta]$, the convergence is uniform.

Next we claim that for j large enough the changes in the first derivative of ϕ^j in the *neighboring segment* to $\psi([t_{i-1}, t_i])$ is at least almost as large as the change in the derivative of ψ itself over $[t_{i-1}, t_i]$, namely is bounded below by $\approx \|\dot{\psi}(t_i) - \dot{\psi}(t_{i+1})\| = \gamma$. If this is true, then since $\sum_{i=1}^N \|\dot{\phi}^j(t_i) - \dot{\phi}^j(t_{i-1})\|$ gives a lower bound on the total curvature of ϕ^j, the argument would be complete. To show that the former is indeed the case, consider $\delta < 1/2 \max_{i=1,\cdots,N}(t_i - t_{i-1})$ and small enough so that $\dot{\psi}(t_i) \approx \frac{\psi(t_i + \delta/2) - \psi(t_i - \delta/2))}{\delta}$ where the approximation error is quadratic in δ and uniform for all i (again, this may be achieved since ψ is C^2 over a compact). Since the convergence $\phi^j(t) \to \psi(t)$ is uniform, for any $r > 0$ there exists J so that for all $j \geq J$ we have $|\phi^j(t) - \psi(t)| < r$ for all $t \in [\alpha, \beta]$. Pick J so that $r \ll \delta \sin\gamma$. Consider two cylindrical tubes with rounded bases around segments $[\psi(t_{i-1} - \delta/2), \psi(t_{i-1} + \delta/2)]$ and $[\psi(t_i - \delta/2), \psi(t_i + \delta/2)]$ – each tube is a union of a cylinder of height $\approx \delta$ and radius r with base centers at $\psi(t_k - \delta/2)$ and $\psi(t_k + \delta/2)$, $k = i-1$ and i respectively, and two r-balls with the same centers. Align the coordinate system so that the first coordinate is collinear with the vector $(\psi(t_{i-1} - \delta/2), \psi(t_{i-1} + \delta/2))$ and the second coordinate chosen so that together with the first coordinate axis it spans a hyperplane parallel to $[\psi(t_i - \delta/2), \psi(t_i + \delta/2)]$, with the first two coordinates of $\psi(t_i)$ positive, see Figure 5(b). The following two inequalities should be interpreted as true up to higher order error terms. Note that by the Intermediate-Value Theorem there exists $\underline{\tau}$ such that $\left|\left(\dot{\phi}^j_{\text{arc}}(\underline{\tau})\right)_2\right| \leq \frac{2r}{\delta - 2r}$: considering the tube around $[\psi(t_{i-1}-\delta/2), \psi(t_{i-1}+\delta/2)]$ note that for ϕ^j_{arc} to traverse the tube will take at least $\delta - 2r$ change of the arc length parameter, while at the same time its second coordinate will change by at most $2r$. Similarly, considering the second tube we conclude that there exists $\overline{\tau}$ such that $\left|\left(\dot{\phi}^j_{\text{arc}}(\overline{\tau})\right)_2\right| \geq \frac{\delta \sin \gamma - 2r}{\delta + 2r}$. Clearly, with r small enough, or equivalently, j large enough, $\|\dot{\phi}^j_{\text{arc}}(t_i) - \dot{\phi}^j_{\text{arc}}(t_{i-1})\|$ is at least γ up to the higher order terms, since we consider the shortest distance between two points $\dot{\phi}^j_{\text{arc}}(\underline{\tau})$ and $\dot{\phi}^j_{\text{arc}}(\overline{\tau})$ on the n-unit sphere, see Figure 5(c).

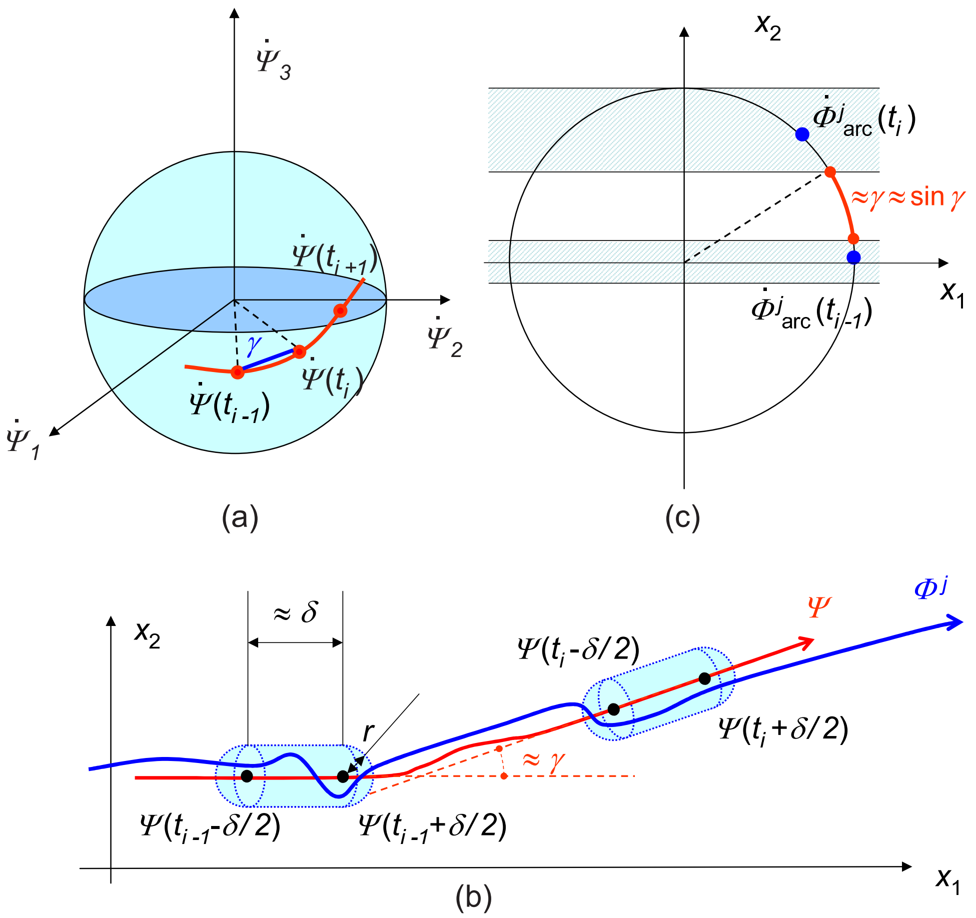

Figure 5: Illustration of the proof of Proposition 1.1

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References

[1] D. Barnette: An upper bound for the diameter of a polytope. *Discrete Mathematics* 10 (1974) 9–13.

[2] G. Dantzig: *Linear Programming and Extensions.* Princeton University Press (1963).

[3] J.-P. Dedieu, G. Malajovich and M. Shub: On the curvature of the central path of linear programming theory. *Foundations of Computational Mathematics* 5 (2005) 145–171.

[4] J.-P. Dedieu and M. Shub: Newton flow and interior point pethods in linear programming. *International Journal of Bifurcation and Chaos* 15 (2005) 827–839.

[5] A. Deza, T. Terlaky and Y. Zinchenko: Central path curvature and iteration-complexity for redundant Klee-Minty cubes. In: D. Gao and H. Sherali (eds.) *Advances in Mechanics and Mathematics* III (2007) 215–248.

[6] A. Deza and F. Xie: Hyperplane arrangements with large average diameter. *AdvOL-Report* 2007/5, McMaster University (2007).

[7] K. Fritzsche and F. Holt: More polytopes meeting the conjectured Hirsch bound. *Discrete Mathematics* 205 (1999) 77–84.

[8] F. Holt and V. Klee: Many polytopes meeting the conjectured Hirsch bound. *Discrete and Computational Geometry* 20 (1998) 1–17.

[9] B. Grünbaum: *Convex Polytopes.* Graduate Texts in Mathematics 221, Springer-Verlag (2003).

[10] G. Kalai and D. Kleitman: A quasi-polynomial bound for the diameter of graphs of polyhedra. *Bulletin of the American Mathematical Society* 26 (1992) 315–316.

[11] V. Klee and D. Walkup: The d-step conjecture for polyhedra of dimension $d < 6$. *Acta Mathematica* 133 (1967) 53–78.

[12] J. Renegar: *A Mathematical View of Interior-Point Methods in Convex Optimization.* SIAM (2001).

[13] C. Roos, T. Terlaky and J.-Ph. Vial: *Interior Point Methods for Linear Optimization.* Springer (2006).

[14] G. Sonnevend, J. Stoer and G. Zhao: On the complexity of following the central path of linear programs by linear extrapolation II. *Mathematical Programming* 52 (1-3) (1991) 527–553.

[15] M. Spivak: *Comprehensive Introduction to Differential Geometry.* Publish or Perish (1990).

[16] G. Ziegler: *Lectures on Polytopes.* Graduate Texts in Mathematics 152, Springer-Verlag (1995).

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