Exchange Market Equilibria with Leontief's Utility: Freedom of Pricing Leads to Rationality *

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Abstract

This paper studies the equilibrium property and algorithmic complexity of the exchange market equilibrium problem with concave piece-wise linear functions, which include linear and Leontief's utility functions as special cases. We show that the Fisher model again reduces to the weighted analytic center problem, and the same linear programming complexity bound applies to computing its equilibrium. However, the story for the Arrow-Debreu model with Leontief's utility becomes quite different. We show that, for the first time, solving this class of Leontief exchange economies is equivalent to solving a linear complementarity problem whose algorithmic complexity is finite but not polynomially bounded.

1 Introduction

This paper studies the equilibrium property and algorithmic complexity of the Arrow-Debreu competitive equilibrium problem. In this problem, traders go to the market with

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initial endowments of commodities and utility functions. They sell and buy commodities to maximize their individual utilities under a market clearing price. Arrow and Debreu [1] have proved, under mild conditions, the existence of equilibrium prices when utility functions are concave and commodities are divisible. From then on, finding an efficient algorithm for computing a equilibrium price became an attractive research area; see [2, 4, 6, 7, 8, 9, 10, 15, 16, 17, 22, 23, 27].

Consider a special case of the Arrow-Debreu problem, the Fisher exchange market model, where traders are divided into two sets: producer and consumer. Consumers have money to buy goods and maximize their individual utility functions; producers sell their goods for money. A price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all the money is spent and all the goods are sold. Eisenberg and Gale [12, 13] gave a convex optimization setting to formulate Fisher's problem with linear and Leontief's utilities. They constructed an aggregated concave objective function that is maximized at the equilibrium. Thus, finding an equilibrium became solving a convex optimization problem, which can be done by using the Ellipsoid method or interior-point algorithms in polynomial time. Here, polynomial time means that one can compute an ϵ-approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$. On the other hand, Eaves [9] proved that the general Arrow-Debreu model with linear utilities could be solved as a *linear complementarity problem* (LCP) (introduced by Cottle at al. [5]) in finite time but not polynomially bounded.

It has turned out that the more general Arrow-Debreu problem with linear utilities is also equivalent to a convex optimization setting (see, e.g., Nenakhov and Primak [21] and Jain [15]). The best arithmetic operation complexity bound for solving both the Fisher and the Arrow-Debreu problems with linear utilities is $O(n^4 \log \frac{1}{\epsilon})$; see [27]. Moreover, if the input data are rational, then an exact solution can be obtained by solving a system of linear equations and inequalities when $\epsilon < 2^{-L}$, where L is the bit length of the input data. Thus, the arithmetic operation bound becomes $O(n^4 L)$, which is in line with the best complexity bound for linear programming of the same dimension and size.

In this paper we deal with a class of more general utility functions: concave piece-wise linear functions, which include Leontief's utility. We show that the Fisher model again reduces to the general analytic center model discussed in [27]. Thus, the same linear programming complexity bound applies to approximating the Fisher equilibrium with this utility. We also show that the solution to a (pairing) class of the Arrow-Debreu model with Leontief's utility can be decomposed into solutions of two systems of linear equalities and inequalities, and its price vector is the Perron-Frobenius eigen-vector of a scaled Leontief utility matrix. Consequently, if all input data are rational, then there always exists a rational Arrow-Debreu equilibrium, that is, the entries of the equilibrium vector are rational numbers. Additionally, the size (bit-length) of the equilibrium solution is bounded by the size of the input data. This result is interesting since rationality does not hold for Leontief's utility in general. Perhaps more importantly, it also implies, for the first time, that solving this class Arrow-Debreu model with Leontief's utility is also equivalent to solving a linear complementarity problem where its algorithmic complexity is finite but not polynomially bounded.

2 The Fisher equilibrium problem

Without loss of generality, assume that there is 1 unit of good from each producer $j \in P$ with $|P| = n$. Let consumer $i \in C$ (with $|C| = m$) have an initial money endowment w_i to spend and buy goods to maximize his or her concave piece-wise linear utility function:

$$u_i(x_i) = \min_k \{u_i^k(x_i)\}, \tag{1}$$

where $u_i^k(x_i)$ is a linear function in x_{ij}—the amount of good bought from producer j by consumer i, $j = 1, ..., n$, and k is the number of linear function pieces. More precisely,

$$u_i^k(x_i) = (u_i^k)^T x_i = \sum_{j \in P} u_{ij}^k x_{ij}.$$

In particular, the Leontief utility function is the one with

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \;\; k \in P \text{ and } a_{ik} > 0,$$

that is, vector u_i^k is an all zero vector except for the kth entry that equals $1/a_{ik}$.

We make the following assumptions temporarily: Every consumer's initial money endowment $w_i > 0$, at least one $u_{ij}^k > 0$ for every k and $i \in C$ and at least one $u_{ij}^k > 0$ for every k and $j \in P$. This is to say that every consumer in the market has money to spend and he or she likes at least one good; and every good is valued by at least one consumer. We will see that, with these assumptions, each consumer can have a positive utility value at equilibria. If a consumer has zero budget or his or her utility has zero value for every good, then buying nothing is an optimal solution for him or her so that he or she can be removed from the market; if a good has zero value to every consumer, then it is a "free" good with zero price in a price equilibrium and can be arbitrarily distributed among the consumers so that it can be removed from the market too.

For given prices p_j on good j, consumer i's maximization problem is

$$\begin{array}{ll} \text{maximize} & u_i(x_{i1}, ..., x_{in}) \\ \text{subject to} & \sum_{j \in P} p_j x_{ij} \leq w_i, \\ & x_{ij} \geq 0, \quad \forall j. \end{array} \tag{2}$$

Let x_i^* denote a maximal solution vector of (2). Then, vector p is called a Fisher price equilibrium if there exists an x_i^* for each consumer such that

$$\sum_{i \in C} x_i^* = e$$

where e is the vector of all ones representing available goods on the exchange market.

Problem (2) can be rewritten as a linear program, after introducing a scalar variable u_i, as

$$\begin{array}{ll} \text{maximize} & u_i \\ \text{subject to} & \sum_{j \in P} p_j x_{ij} \leq w_i, \\ & u_i - \sum_{j \in P} u_{ij}^k x_{ij} \leq 0, \ \forall k, \\ & u_i, \ x_{ij} \geq 0, \quad \forall j. \end{array} \tag{3}$$

Besides (u_i, x_i) being feasible, the optimality conditions of (3) are

$$\begin{array}{rcl} \lambda_i p_j - \sum_k \pi_i^k u_{ij}^k & \geq & 0, \ \forall j \in P \\ \sum_k \pi_i^k & = & 1 \\ \lambda_i w_i & = & u_i. \end{array} \tag{4}$$

for some $\lambda_i, \ \pi_i^k \geq 0$.

It has been shown by Eisenberg and Gale [12, 11, 13] (later independently by Codenotti et al. [3]) that a Fisher price equilibrium is an optimal Lagrange multiplier vector of an aggregated convex optimization problem:

$$\begin{array}{ll} \text{maximize} & \sum_{i \in C} w_i \log u_i \\ \text{subject to} & \sum_{i \in C} x_{ij} = 1, \quad \forall j \in P, \\ & u_i - \sum_{j \in P} u_{ij}^k x_{ij} \leq 0, \quad \forall k, i \in C, \\ & u_i, \ x_{ij} \geq 0, \quad \forall i, j. \end{array} \tag{5}$$

Conversely, an optimal Lagrange multiplier vector is also a Fisher price equilibrium, which can be seen from the optimality conditions of (5):

$$\begin{array}{rcl} p_j - \sum_k \pi_i^k u_{ij}^k & \geq & 0, \ \forall i, j \\ \pi_i^k (\sum_{j \in P} u_{ij}^k x_{ij} - u_i) & = & 0, \ \forall i, k \\ x_{ij} (p_j - \sum_k \pi_i^k u_{ij}^k) & = & 0, \ \forall i, j \\ u_i \sum_k \pi_i^k & = & w_i, \ \forall i. \end{array} \tag{6}$$

for some p_j, the Lagrange multiplier of equality constraint of $j \in P$, and some $\pi_i^k \geq 0$, the Lagrange multiplier of inequality constraint of $i \in C$ and k. Summing the second constraint over k we have

$$w_i = \sum_k \pi_i^k u_i = \sum_k \pi_i^k \sum_{j \in P} u_{ij}^k x_{ij} = \sum_{j \in P} \left(x_{ij} \sum_k \pi_i^k u_{ij}^k \right), \ \forall i;$$

then summing the third constraint over j we have

$$\sum_{j \in P} p_j x_{ij} = \sum_{j \in P} \left(x_{ij} \sum_k \pi_i^k u_{ij}^k \right) = w_i.$$

This implies that x_i from the aggregate problem is feasible for (3). Moreover, note that π_i^k in (6) equals π_i^k/λ_i in (4). Thus, finding a Fisher price equilibrium is equivalent to finding an optimal Lagrange multiplier of (5).

In particular, if each $u_i^k(x_i)$ has the Leontief utility form, i.e.,

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \ \forall k \in P \text{ and } a_{ik} > 0.$$

Then, upon using u_i to replace variable x_{ij}, the aggregated convex optimization problem can be simplified to

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log u_i \\ \text{subject to} & A^T u \leq e, \\ & u \geq 0; \end{array} \tag{7}$$

with the Leontief matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and variable vector} \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_m \end{pmatrix}. \tag{8}$$

Here, we assume that

Assumption 1. *A has no all-zero row, that is, every trader likes at least one good.*

With this assumption, we see that the feasible set of (7) is bounded so that the first order optimality or KKT conditions exist. Moreover, the feasible set has a non-empty (relative) interior, that is, it has an u such that all inequalities are held strictly.

3 The weighted analytic center problem

In [27] the Eisenberg-Gale aggregated problem was related to the (linear) analytic center problem studied in interior-point algorithms

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} w_j \log(x_j) \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \tag{9}$$

where the given A is an $m \times n$-dimensional matrix with full row rank, b is an m-dimensional vector, and w_j is the nonnegative weight on the jth variable. Any x who satisfies the constraints is called a primal feasible solution, while any optimal solution to the problem is called a weighted analytic center.

If the weighted analytic center problem has an optimal solution, the optimality conditions are

$$\begin{array}{rcl} Sx & = & w, \\ Ax & = & b,\ x \geq 0, \\ -A^T y + s & = & 0,\ s \geq 0, \end{array} \tag{10}$$

where y and s are the Lagrange or KKT multipliers or dual variable and slacks of the dual linear program:

$$\min\ b^T y \quad \text{subject to} \quad s = A^T y \geq 0,$$

and S is the diagonal matrix with slack vector s on its diagonals.

Let the feasible set of (9) be bounded and has a (relative) interior, i.e., has a strictly feasible point $x > 0$ with $Ax = b$ (clearly holds for problem (5) and (7)). Then, there is a strictly feasible dual solution $s > 0$ with $s = A^T y$ for some y. Furthermore, [27], based on the literature of interior-point algorithms (e.g., Megiddo and Kojima et al. [20, 19] and Güler [14]), has shown that

Theorem 1. *Let A, b be fixed and consider a solution $(x(w), y(w), s(w))$ of (10) as a mapping of $w \geq 0$ with $\sum_j w_j = 1$. Then,*

- *The mapping of* $S^n_{++} = \{x > 0 \in R^n : \ e^T x = 1\}$ *to* $F_{++} = \{(x > 0, y, s > 0) : \ Ax = b, \ s = A^T y\}$ *is one-to-one, continuously differentiable.*

- *The mapping of* $S^n_{+} = \{x \geq 0 \in R^n : \ e^T x = 1\}$ *to* $F_{+} = \{(x \geq 0, y, s \geq 0) : \ Ax = b, \ s = A^T y\}$ *is upper semi-continuous.*

- *The pair* $(x_j(w), s_j(w))$ *is unique for any* $j \in W = \{j : \ w_j > 0\}$*, and*

$$x'_j(w)s''_j(w) = x''_j(w)s'_j(w) = 0, \ \forall j \notin W$$

and for any two solutions $(x'(w), y'(w), s'(w))$ *and* $(x''(w), y''(w), s''(w))$ *of (10).*

From this theorem, we see that, in the Fisher equilibrium problem (5) or (7), $u_i(w)$, the utility value of each consumer, is unique; but the price vector $p(w)$ can be non-unique.

In addition, a modified primal-dual path-following algorithm was developed in [27], for computing an ϵ-solution for any $\epsilon > 0$:

$$\begin{aligned} \|Sx - w\| &\leq \epsilon, \\ Ax &= b, \ x \geq 0, \\ -A^T y + s &= 0, \ s \geq 0. \end{aligned} \tag{11}$$

Theorem 2. *The primal-dual interior-point algorithm solves the weight analytic center problem (9) in* $O(\sqrt{n}\log(n\max(w)/\epsilon))$ *iterations and each iteration solves a system of linear equations in* $O(nm^2 + m^3)$ *arithmetic operations. If Karmarkar's rank-one update technique ([18]) is used, the average arithmetic operations per iteration can be reduced to* $O(n^{1.5}m)$*.*

A rounding algorithm is also developed for certain types of problems possessing a rational solution, and the total iteration bound would be $O(\sqrt{n}L)$ and the average arithmetic operation bound would be $O(n^{1.5}m)$ per iteration, where L is the bit-length of the input data A, b, w. These results indicate that the complexity of the Fisher equilibrium problem with concave piece-wise linear utility functions is completely in line with linear programming of the same dimension and size.

4 The Arrow-Debreu equilibrium problem

The Arrow-Debreu exchange market equilibrium problem was first formulated by Leon Walras in 1874 [25]. In this problem everyone in a population of m traders has an initial endowment of a divisible good and a utility function for consuming all goods—their own and others. Every trader sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that, under mild conditions, such equilibrium would exist if the utility functions were concave.

We consider a special class of Arrow-Debreu's problems, where each of the $m = n$ traders have exactly one unit of a divisible good for trade (e.g., see [15, 27]), and let trader i, $i = 1, ..., m$, bring good $j = i$ and have the concave piece-wise linear utility function of (1). We call this class of problems the *pairing class*. The main difference between Fisher's and Arrow-Debreu' models is that, in the latter, each trader is both producer and consumer and the initial endowment w_i of trader i is *not* given and will be the price assigned to his or her good i. Nevertheless, we can still write a (parametric) convex optimization model

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log u_i \\ \text{subject to} & \sum_i x_{ij} = 1, \quad \forall j, \\ & u_i \le \sum_j u_{ij}^k x_{ij}, \quad \forall i, k, \\ & u_i, \ x_{ij} \ge 0, \quad \forall i, j, \end{array} \tag{12}$$

where we wish to select weights w_i's such that an optimal Lagrange multiplier vector p equals w. It is easily seen that any optimal Lagrange multiplier vector p satisfies

$$p \ge 0 \quad \text{and} \quad e^T p = e^T w.$$

For fixed u_{ij}^k, consider p be a map from w, say $p(w)$. Then, the mapping is from S_+^n to S_+^n, and it is upper semi-continuous from Theorem 1. Thus, from the Kakutani fixed-point theorem (see, e.g., [23, 24, 26]), we have

Theorem 3. *There exists a $w \in S^n_+$ such that an Lagrange multiplier vector $p(w) = w$ for problem (12) if its feasible region is feasible and bounded and has a non-empty (relative) interior.*

We now focus on the Arrow-Debreu equilibrium with the Leontief utility function:

$$u_i^k(x_i) = \frac{x_{ik}}{a_{ik}}, \ \forall k = 1, ..., m,$$

where we temporarily assume $a_{ik} > 0$ for all i and k so that there exists at least one Arrow-Debreu equilibrium [1]. Recall the parametric convex optimization model (7) where the Leontief matrix A of (8) is an $m \times m$ positive matrix. Let $p \in R^m$ be an optimal Lagrange multiplier vector of the constraints. Then, we have

$$\begin{array}{ccc} u_i \sum_j a_{ij} p_j = w_i \ \forall i, & \text{and} & p_j(1 - \sum_i a_{ij} u_i) = 0 \ \forall j, \\ \sum_i a_{ij} u_i \le 1 \ \forall j, & \text{and} & u_i, \ p_j \ge 0 \ \forall i, j; \end{array}$$

or in matrix form

$$\begin{array}{rcl} UAp &=& w, \\ P(e - A^T u) &=& 0, \\ A^T u &\le& e, \\ u, \ p &\ge& 0, \end{array}$$

where U and P are diagonal matrices whose diagonal entries are u and p, respectively.

Note that the Arrow-Debreu equilibrium $p \in R^m$, together with $u \in R^m$, must satisfy

$$\begin{array}{rcl} UAp &=& p, \\ P(e - A^T u) &=& 0, \\ A^T u &\le& e, \\ u, \ p &\ge& 0, \\ p &\ne& 0. \end{array} \tag{13}$$

In the general case where $0 \le A \not> 0$, using Model (7) and Theorem 3 we can prove:

Corollary 1. *System (13) always has a solution (u, p) under Assumption 1 (i.e., A has no all-zero row).*

We comment that a solution to System (13) may not be an Arrow-Debreu equilibrium, although every Arrow-Debreu equilibrium satisfies System (13).

5 Characterization of an Arrow-Debreu equilibrium

If $u_i > 0$ at a solution $(u, p \neq 0)$ of System (13), we must have $p_i > 0$, that is, trader i's good must be priced positively in order to have a positive utility value. On the other hand, $p_i > 0$ implies that $\sum_k^m a_{ki} u_k = 1$, that is, good i must be all consumed and gone. Conversely, if $p_i > 0$, we must have $u_i > 0$, that is, trader i's utility value must be positive. Thus, there is a partition of all traders (or goods) such that

$$B = \{i : \ p_i > 0\} \quad \text{and} \quad N = \{i : \ p_i = 0\}$$

where the union of B and N is $\{1, 2, ..., m\}$. Then, (u, p) satisfies

$$\begin{array}{rcl} (U_B A_{BB}) p_B &=& p_B, \\ A_{BB}^T u_B &=& e, \\ A_{BN}^T u_B &\leq& e, \\ u_B, \ p_B &>& 0. \end{array}$$

Here A_{BB} is the principal submatrix of A corresponding to the index set B, A_{BN} is the submatrix of A whose rows in B and columns in N. Similarly, u_B and p_B are subvectors of u and p with entries in B, respectively.

Since the scaled Leontief matrix $U_B A_{BB}$ is a (column) stochastic matrix (i.e., $e^T U_B A_{BB} = e^T$), p_B must be the (right) Perron-Frobenius eigen-vector of $U_B A_{BB}$. Moreover, A_{BB} is irreducible because $U_B A_{BB}$ is irreducible and $u_B > 0$, and $U_B A_{BB}$ is irreducible because $p_B > 0$. To summarize, we have

Theorem 4. *Let $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$, A_{BB} be irreducible, and u_B satisfy the linear system*

$$A_{BB}^T u_B = e, \quad A_{BN}^T u_B \leq e, \quad \text{and} \quad u_B > 0.$$

Then the (right) Perron-Frobenius eigen-vector p_B of $U_B A_{BB}$ together with $p_N = 0$ will be a solution to System (13). And the converse is also true. Moreover, there is always a rational solution for every such B, that is, the entries of price vector are rational numbers, if the entries of A are rational. Furthermore, the size (bit-length) of the solution is bounded by the size of A.

Proof. In the first statement we only need to prove $p_B > 0$. But this is the result of the Perron-Frobenius theorem on $U_B A_{BB}$ since it is irreducible. Conversely, if $(p_B > 0, p_N = 0)$ is a price vector to System (13), then $u_B > 0$ and $A_{BB}^T u_B = e$ from the complementarity, and $U_B A_{BB}$ is irreducible from $p_B > 0$. The last fact implies that A_{BB} is irreducible since positive diagonal matrix U_B does not alter the irreducibility of A_{BB}.

To prove the rationality, we see that there is a rational vector u_B to the linear system, so that matrix $U_B A_{BB}$ will be rational, so that there will be a rational price solution p_B to the linear system

$$(U_B A_{BB} - I)\, p_B = 0,\ e^T p_B = 1,\ p_B > 0.$$

The size result is due to that the sizes of these two linear systems are bounded by the size of A. ∎

Our theorem implies that the traders in block B can trade among themselves and keep others goods "free". In particular, if one trader likes his or her own good more than any other good, that is, $a_{ii} \geq a_{ij}$ for all j, then $u_i = 1/a_{ii}$, $p_i = 1$, and $u_j = p_j = 0$ for all $j \neq i$, that is, $B = \{i\}$, makes an Arrow-Debreu equilibrium. The theorem thus establishes, for the first time, a combinatorial algorithm to compute an Arrow-Debreu equilibrium with Leontief's utility by finding a right block $B \neq \emptyset$, which is actually a non-trivial complementarity solution $(u \neq 0)$ to an LCP problem

$$A^T u + v = e,\ u^T v = 0,\ 0 \neq u, v \geq 0. \tag{14}$$

If $A > 0$, then any complementarity solution $u \neq 0$ and $B = \{j :\ u_j > 0\}$ of (14) induce an Arrow-Debreu equilibrium that is the (right) Perron-Frobenius eigen-vector of $U_B A_{BB}$, and it can be computed in polynomial time by solving a linear equation. If A is not strictly positive, then any complementarity solution $u \neq 0$ and $B = \{j :\ u_j > 0\}$, as long as A_{BB} is irreducible, induces a solution to System (13). The equivalence between the pairing Arrow-Debreu model and the LCP also implies

Corollary 2. *LCP (14) always has a non-trivial complementarity solution, where A_{BB} is irreducible with $B = \{j :\ u_j > 0\}$, under Assumption 1 (i.e., A has no all-zero row).*

If Assumption 1 does not hold, the corollary may not be true; see example below:

$$A^T = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}.$$

We comment that the pairing class of Arrow-Debreu's problems is a rather restrictive class of problems. Consider a general supply matrix $0 \le G \in R^{m \times n}$ where row i of G represents the multiple goods brought to the market by trader i, $i = 1, ..., m$. Then, the pairing model represents the case that $G = I$, the identity matrix, or $G = P$ where P is any permutation matrix of $m \times m$.

What to do if one trader brings two different goods? One solution is to copy the same trader's utility function twice and treat the trader as two traders with an identical Leontief utility function, where each of them brings only one type of good. Then, the problem reduces to the pairing model. Thus, we have

Corollary 3. *If all goods are different from each other in the general Arrow-Debreu problem with Leontief's utility, i.e., each column of $G \in R^{m \times n}$ has exactly one positive entry, then there is always a rational equilibrium, that is, the entries of a price vector are rational numbers.*

Now what to do if two traders bring the same type of good? In our present pairing class, they will be treated as two different goods, and one can set the same utility coefficients to them so that they receive an identical appreciation from all the traders. Again, the problem reduces to the pairing class, which leads to rationality. The difference is that now these two "same" goods may receive two different prices; for example, one is priced higher and the other is at a discount level. I guess this could happen in the real world since two "same" goods may not be really the same and the market does have "freedom" to price them.

6 An illustrative example

The rationality result is interesting since the existence of a rational equilibrium is not true for Leontief's utility in Fisher's model with rational data, see the following example converted in Arrow-Debreu's setting, with three consumers each of whom has 1 unit money (the first good) and two other goods (the second and third) brought by a seller (the fourth trader) who is only interested in money, adapted from Codenotti et al. [3] and Eaves [9].

$$A = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{5} \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

There is a unique equilibrium for this example, where the utility values of the three consumers are $u_1^* = \frac{2}{3\sqrt{3}}$, $u_2^* = \frac{1}{3} + \frac{1}{3\sqrt{3}}$, $u_3^* = \frac{10}{3} - \frac{10}{3\sqrt{3}}$, and the utility value of the seller $u_4^* = 3$. The equilibrium price for good 1 (money) is $p_1^* = 1$, and for other two goods are $p_2^* = 3(\sqrt{3} - 1)$, and $p_3^* = 3(2 - \sqrt{3})$.

However, if we treat the money from each consumer differently, that is, let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{5} \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then, there are multiple rational equilibria and here are a few:

1. $B = \{1, 4\}$, with $u_1^* = u_4^* = 1$ and $p_1^* = p_4^* = 1$, and $u_2^* = u_3^* = p_2^* = p_3^* = p_5^* = 0$;

2. $B = \{2, 5\}$, with $u_2^* = u_4^* = 1$ and $p_2^* = p_5^* = 1$, and $u_1^* = u_3^* = p_1^* = p_3^* = p_4^* = 0$;

3. $B = \{3, 4\}$, with $u_3^* = 4$ and $u_4^* = 1$ and $p_3^* = p_4^* = 1$, and $u_1^* = u_2^* = p_1^* = p_2^* = p_5^* = 0$;

4. $B = \{1, 2, 3, 4, 5\}$, with an equilibrium $u_1^* = \frac{11}{30}$, $u_2^* = \frac{31}{60}$, $u_3^* = \frac{3}{2}$, $u_4^* = 1$, $p_1^* = \frac{66}{80}$, $p_2^* = \frac{93}{80}$, $p_3^* = \frac{81}{80}$ and $p_4^* = p_5^* = \frac{3}{2}$, .

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