Local versus Global Profit Maximization: The Case of Discrete Concave Production Functions

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Abstract

In this paper we show that for discrete concave functions, a local maximum need not be a global maximum. We also provide examples of discrete concave functions where this coincidence holds. As a direct consequence of this, we can establish the equivalence of local and global profit maximizers for an equivalent well-behaved production function that is generated by a (not necessarily discrete concave) production function.

1. Introduction: For a concave function defined on a convex subset of a Euclidean space, it is well known that a local maximum of a concave function is a global maximum. This result is valid even if the function is not differentiable. In fact, if a concave function is differentiable in the interior of its domain, then zero first derivative at an interior point is not only a necessary condition for an interior global maximum, but a sufficient condition for it as well.

In economics, a profit maximizing competitive firm is represented by a concave production function. Since at given prices the profit function is a concave function of input vectors, local profit maximization implies global profit maximization. For a thorough discussion of concavity as applied in economic theory one may refer to Madden (1986).

Our concern in this paper stems from the fact that many (if not most) inputs that are used in production, are available in integer units only. Thus for instance a firm employs an integer number of workers of each type, and hence given an eight-hour working day per worker, employs labor time of each type in integer units only. Even if an input is infinitely divisible, like for instance gasoline, transactions are in amounts that can be expressed up to at most two or three decimal places. Thus instead of liters, centiliters or milliliters may be required to express the amount of gasoline employed in integer units.

Suppose a firm employs a finite number of inputs that are available in integer units. If the number of inputs are L, then the production function of the firm is assumed to be non-decreasing and discrete concave, in the sense that it is the restriction of a non-decreasing and concave function defined on all non-negative L-vectors, to the set of all non-negative integer valued L-vectors. In such a context, the usual definition of neighborhood in the Euclidean metric, ceases to be effective and hence defining the concept of a local maximum as in the case of concave functions, seems even more problematic.

Bearing in mind that a neighborhood around a point in Euclidean spaces, allows some movement from that point in every feasible direction, we consider the following for integer valued vectors. Given an integer valued vector, its "neighborhood" would include

an integer valued vector, if and only if no coordinate of this vector exceeds the corresponding coordinate of the given vector by one.
The question that we seek an answer is the following: If an integer valued input vector maximizes profits in its neighborhood (i.e. is a local profit maximizer) then is it a global profit maximizer? Theorem 1 provides an example and answers the question in the negative. This clearly means that mere discrete concavity is not enough for a production function to inherit all or any of the properties of classical concave optimization. However, there is a certain well-behaved class of discrete concave functions, sufficiently rich in its own way, where local profit maximization implies global profit maximization. This class includes all additively separable production functions, and production functions where employing more than one unit of any input does not lead to any increase in output than what one unit of that input might yield. As a direct consequence of this result, we establish the equivalence of local and global profit maximizers for an equivalent well-behaved production function that is generated by a (not necessarily discrete concave) production function. However, this in a way defeats the purpose of computing local profit maximizers instead of, or as an alternative to a global search, since the number of computations required to find local maximizers become enormous, for even very simple profit maximization problems.

2. Discrete Concave Functions: Let $Z = \aleph \cup \{0\}$, where $\aleph$ denotes the set of natural numbers. A function f: $Z^L \rightarrow \Re_+$(: the set of non-negative real numbers) is said to be discrete concave if there exists a continuous concave function g: $\Re_+^L \rightarrow \Re$ such that the restriction of g to Z^L coincides with f.
Given functions f: $Z^L \rightarrow \Re$ and g: $\Re_+^L \rightarrow \Re$, let hypograph(f) $\equiv \{(x, \alpha) \in Z^L \times \Re / \alpha \leq f(x)\}$ and hypograph(g) $\equiv \{(x, \alpha) \in \Re_+^L \times \Re / \alpha \leq g(x)\}$.
Given a function f: $Z^L \rightarrow \Re_+$ its **<u>Least Concave Extension</u>** (LCE) is the function g^f: $\Re_+^L \rightarrow \Re$ such that the hypograph(g^f) = convex hull of hypograph(f). Clearly g^f is continuous and concave. Further, if g: $\Re_+^L \rightarrow \Re$ is a concave function whose restriction to Z^L coincides with f, then for all $x \in \Re_+^L$, $g(x) \geq g^f(x)$.
If f is discrete concave, then the restriction of its LCE g^f to Z^L coincides with f.
Let e denote the vector in $\Re^L$ all whose coordinates are equal to one and for j = 1,…,L, let e^j denote the vector in $\Re^L$ whose j^{th} coordinate is equal to one and all other coordinates are equal to zero.

For $x \in Z^L$, let $C(x) = \{z \in Z^L / z \leq x\}$ and $C^*(x)$ = Convex hull of C(x).

A discrete concave function is said to attain a local maximum at $z^* \in Z^L$ if $f(z^*) \geq f(x)$ for all $z \in C(z^*+e)$.
A discrete concave function is said to attain a maximum at $z^* \in Z^L$ if $f(z^*) \geq f(z)$ for all $z \in Z^L$.
Given $x \in Z^L$, a discrete concave function f: $Z^L \rightarrow \Re_+$ is said to be **well behaved up to x** if for all $y \in C^*(x)$: $g^f(y) = \text{Max} \{ \sum_{z \in C(x)} t(z) f(z) / t(z) \geq 0$ for all

$z \in C(x)$, $\sum_{z \in C(x)} t(z)z = y$ and $\sum_{z \in C(x)} t(z) = 1\}$.

A stronger definition of well-behavedness that is *not required for our purposes* in this paper could read as follows:

Given $z^* \in Z^L$, a discrete concave function f: $Z^L \rightarrow \Re_+$ is said to be **strongly well behaved up to z*** if for all $x \in C(z^*)$ and $y \in C^*(x)$: $g^f(y) = \text{Max}\ \{\sum_{z \in C(x)} t(z) f(z)$ / $t(z) \geq 0$ for all $z \in C(x)$, $\sum_{z \in C(x)} t(z)z = y$ and $\sum_{z \in C(x)} t(z) = 1\}$.

A discrete concave function is said to be **well-behaved** if it is well-behaved up to all $x \in Z^L$.

3. Discrete Concave Production Functions: Suppose there are L+1 > 1 commodities. A firm uses the first L commodities as inputs to produce the $L+1^{th}$ commodity, which is an infinitely divisible numeraire good.

A production function for the firm is a function f: $Z^L \rightarrow \Re$ which is both **discrete concave** ***and*** **non-decreasing** (i.e. for all x, $y \in Z^L$: $[x - y \in Z^L]$ implies $[f(x) \geq f(y)]$) such that for each input vector x, f(x) is the maximum amount of the numeraire consumption good which can be produced from x.

A price vector p is an element of $\Re^L_+ \backslash \{0\}$, where for j = 1,…,L, p_j denotes the price of input j.

Given a production function f and a price vector p, the objective of the firm is to maximize profits:

Maximize $[f(x) – p^T x]$

Subject to $x \in Z^L$.

$x^* \in Z^L$ is a local profit maximizer for the production function f at price vector p if $f(x^*) – p^T x^* \geq f(x) – p^T x$ for all $x \in C(x^* + e)$.

$x^* \in Z^L$ is a local profit maximizer for the production function f at price vector p if $f(x^*) – p^T x^* \geq f(x) – p^T x$ for all $x \in Z^L$.

3. Non- coincidence of Local and Global Maximum: Unlike concave production functions defined on Euclidean spaces a local profit maximizer for a production function f at a price vector p, need not be a global profit maximizer for the same.

Theorem 1: For L = 2, there exists a production function f, a price vector p and $x^* \in Z^L$ such that x* is a local profit maximizer for f at p, though not a global profit maximizer for f at p.

Proof: Let f: $Z^2 \rightarrow \Re$ be defined thus:

f(a,b) = minimum{35a + 20b, 60}

f is the restriction to Z^2 of a concave function on $\Re^2_+$. Further, f is non-decreasing.

f(0,3) = 60 and f attains a global maximum at (0,3).

Note, 55 = f(1,1).

Further, $g^f(\frac{1}{2}, 2) \geq \frac{1}{2} f(0,3) + \frac{1}{2} f(1,1) = 30 + \frac{55}{2} > 55$.

Now $(\frac{1}{2}, 2)$ belongs to C*((2,2)), and $(\frac{1}{2}, 2)$ is a convex combination of elements in C((2,2)) implies $(\frac{1}{2}, 2) = \alpha(0, 2) + \beta (1, 2) + [1-\alpha-\beta] (2,2)$ with $1 \geq \alpha + \beta$ and $\alpha, \beta \geq 0$.

Thus, $2\alpha + \beta = \frac{3}{2}$. This combined with $1 \geq \beta \geq 0$ implies $2\alpha + 1 \geq \frac{3}{2}$ i.e. $\alpha \geq \frac{1}{2}$.

Hence, it turns out that f is not well behaved up to (2,2).

For otherwise $g^f(\frac{1}{2}, 2) = 40\alpha + 60 (1-\alpha) = 60 - 20\alpha \leq 50$, contradicting $g^f(\frac{1}{2}, 2) > 55$.

Let $p = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$.

Note that

$$f(a,b) = 35a + 20b \text{ for } 7a + 4b \leq 12$$
$$= 60 \qquad \text{for } 7a + 4b > 12.$$

Then $h(a,b) = f(a,b) - p^T\begin{bmatrix} a \\ b \end{bmatrix} = 24a + 15b$ for $7a + 4b \leq 12$,

$= 60 - 11a - 5b$ for $7a + 4b > 12$.

h(2,2) = 28, h(2,1) = 33, h(2,0) = 38, h(1,2) = 39, h(1,1) = 39, h(1,0) = 24, h(0,2) = 30, h(0,1) = 15, h(0,0) = 0.

Thus, $f(1,1) - p^T\begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq f(a,b) - p^T\begin{bmatrix} a \\ b \end{bmatrix}$ for all $(a,b) \in C(2,2)$.

Thus (1,1) is a local profit maximizer for f at p.

However, $f(0,3) - p^T\begin{bmatrix} 0 \\ 3 \end{bmatrix} = 60 - 15 = 45 > 39 = f(1,1) - p^T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, (1,1) is not a global profit maximizer for f at p. Q.E.D.

Note: I am grateful to L. Ramprasath for suggesting the following alternative and simpler configuration for a proof of Theorem 1: f(a,b) = minimum{3a + 2b, 6} for all $(a,b) \in Z^2$ and $p^T = (2,1)$. Once again, (1,1) is a local profit maximizer of f at p, though (0,3) is a global profit maximizer of f at p.

However if a production function f is well-behaved up to x*+e and x* is a local profit maximizer for f at p, then it is a global profit maximizer for f at p as well.

Theorem 2: Let f be production function, $x^* \in Z^L$ and p a price vector. If f is well-behaved up to x* + e and x* is a local profit maximizer for f at p, then x* is a global profit maximizer for f at p as well.

Proof: Let $x^* \in Z^L$ and p a price vector. Suppose f is well-behaved up to x* + e and $f(x^*) - p^Tx^* \geq f(x) - p^Tx$ for all $x \in C(x^*+e)$. Towards a contradiction suppose there is a vector $u \in Z^L$ such that $f(u) - p^Tu > f(x^*) - p^Tx^*$.
Thus, there exist $t \in (0,1)$, such that $x^* + t(u-x) \in C^*(x^* + e)$.
Let $y = x^* + t(u-x^*)$.
Let g^f be the LCE of f.
By the concavity of g^f: $g^f(y) - p^Ty = g^f(x^* + t(u-x)) - p^T(x^* + t(u-x)) \geq g^f(x^*) - p^Tx^* + t\,[\{g^f(x^*) - p^Tx^*\} - \{g^f(u) - p^Tu\}] = f(x^*) - p^Tx^* + t\,[\{f(u) - p^Tu\} - \{f(x^*) - p^Tx^*\}] > f(x^*) - p^Tx^*$, since $f(u) - p^Tu > f(x^*) - p^Tx^*$.
Since, $y \in C^*(x^* + e)$ and f is well-behaved up to x* + e, $g^f(y) = \text{Max}\ \{\sum_{z \in C(x^*+e)} t(z) f(z) /$ $t(z) \geq 0$ for all $z \in C^*(x^*+e)$, $\sum_{z \in C(x^*+e)} t(z) f(z) = y$ and $\sum_{z \in C(x^*+e)} t(z) = 1\}$.
Since $f(x^*) - p^Tx^* \geq f(z) - p^Tz$ for all $z \in C(x^* + e)$, we get $f(x^*) - p^Tx^* \geq g^f(y) - p^Ty$ contradicting what we obtained earlier.
Thus, x* is a global maximizer of f at p. Q.E.D.

4. Types of Well- Behaved Production Functions: From Theorem 2 it immediately follows that well-behaved production functions have desirable computational properties for a profit-maximizing firm.
If L = 1, then any discrete concave function is well behaved: If $f: Z^2 \rightarrow \Re$ is discrete concave then for any $x \in Z^2$ and $y \in [x, x+1]$; $g^f(y) = f(x) + (y-x)[f(x+1)-f(x)]$.
For L = 2, the function $f: Z^2 \rightarrow \Re$ defined by $f(a,b) = a^{\frac{1}{2}} b^{\frac{1}{2}}$ for all $(a,b) \in Z^2$ is *not* well-behaved up to (2,2) and hence not well-behaved.
Since $(\frac{3}{2},2) = \frac{1}{2}(1,3) + \frac{1}{2}(2,1)$, $g^f(\frac{3}{2},2) \geq \frac{1}{2}(\sqrt{3} + \sqrt{2})$.
However, if f was well-behaved then since $(\frac{3}{2},2) \in C^*(2,2)$, we would have $g^f(\frac{3}{2},2) = \frac{1}{2}f(1,2) + \frac{1}{2}f(2,1) = \sqrt{2} < \frac{1}{2}(\sqrt{3} + \sqrt{2})$.
Thus, f is not well-behaved.

Yet, the following theorem guarantees that local profit maximizers are global profit maximizers for production functions of the above type at any price vector.

Proposition 1: Let f be a production function that is *homogeneous of degree one* (for all $x \in Z^L$ and $t \geq 0$: $f(tx) = tf(x)$).
If a global profit maximizer exists for f at a price vector p, then any local profit maximizer of f at p is a global profit maximizer of f at p as well.

Proof: Since f is homogeneous of degree one, at any global profit maximizer x of f at p, $f(x) - p^Tx = 0$.
Since $f(0) - p^T0 = 0$, clearly any local profit maximizer x of f at p must yield zero profits at p, and be a global profit maximizer of f at p as well. Q.E.D.

A function f: $Z^L \to \Re$ is said to be additively separable if there exists functions f^i: $Z \to \Re$, i = 1,…,L, such that for all x = $(x_1,\ldots,x_L) \in Z^L$: f(x) = $\sum_{i=1}^{L} f^i(x_i)$.
Let g denote the LCE of f and for each i, let g^i denote the LCE of f^i.
Then for y = $(y_1,\ldots,y_L) \in \Re^L_+$: g(y) = $\sum_{i=1}^{L} g^i(y_i)$.
Thus f is well-behaved.
However the class of well-behaved production functions is much larger than the class of separable production functions.
For $x \in \Re^L_+$, let e(x) be the vector in $\Re^L_+$ whose i^{th} coordinate is min$\{x_i, 1\}$. Thus, e(x)∈C*(e) for all $x \in \Re^L_+$ and e(x)∈C(e) for all $x \in Z^L$.
Let f be any production function such that for all $x \in Z^L$: f(x) = f(e(x) ≥ 0.
Note that if for a positive integer K, $x^k \in C(e)$ and $t^k \in (0,1)$ for k = 1,…,K, then $\sum_{k=1}^{K} t^k = 1$ and $\sum_{k=1}^{K} t^k x^k$ = x ∈C(e) implies x^k = x for k = 1,…,K. If for j∈{1,…,L}, $x^k(j)$ denotes the j^{th} coordinate of x^k and x(j) denotes the j^{th} coordinate of x, then: (a) 0 = $\sum_{k=1}^{K} t^k x^k(j)$ if and only if $x^k(j) = 0$ for k = 1,…,K; (b) 1 = $\sum_{k=1}^{K} t^k x^k(j)$ if and only if $x^k(j) = 1$ for k = 1,…,K.
Thus, $x^k(j) = x(j)$ for k = 1,…,K.
Let g: $\Re^L_+ \to \Re$ be defined as follows: (a) For y∈C*(e): g(y) = Max{ $\sum_{x \in C(e)} \alpha(x) f(x)$ / α(x) ≥ 0 for all x∈C(e), $\sum_{x \in C(e)} \alpha(x) = 1$, $\sum_{x \in C(e)} \alpha(x) x$ = y}; (b) For $y \in \Re^L_+ \setminus$ C*(e): g(y) = g(e(y)).
Thus the restriction of g to C*(e) is concave.
Let x, $y \in \Re^L_+$ with x ≥ y. Then e(x) ≥ e(y). Thus, g(x) ≥ g(y). Thus g is non-decreasing.
Let x, $y \in \Re^L_+$ and α∈(0,1). Let x* = αx + (1-α)y. Clearly x* ≥ e(x*) ≥ αe(x) + (1-α)e(y) and so g(x*) ≥ g(e(x*)) ≥ g(αe(x) + (1-α)e(y)) ≥ αg(e(x)) + (1-α)g(e(y)) = αg(x) + (1-α)g(y).
Thus g is concave and is the LCE of f.
Since for $x \in \Re^L_+$, g(x) = g(e(x)), f is well-behaved.

A consequence of this observation is noteworthy.
Given x,y $\in Z^L$, let m(x,y) be the L-vector whose i^{th} coordinate is min $\{x_i,y_i\}$.
Let f: $Z^L \to \Re_+$ be a non-decreasing function and let $z^* \in Z^L$ with z* >> 0. f is said to be a production function eventually satiable at z* (PFES at z*) if for all $x \in Z^L$: f(x) = f(m(x,z*)).
Note that we do not require f to satisfy discrete concavity.

Given a PFES at z* f, let M = $\sum_{i=1}^{L} z_i^*$, and F: $Z^M \rightarrow \Re_+$ such that for all $X \in Z^M$: F(X) = f(S(X)), where S(X) is the L-vector whose first coordinate is the sum of the first z_1^* coordinates of X, second coordinate is the sum of the first z_2^* coordinates of X,…, L^{th} coordinate is the sum of the last z_L^* coordinates of X.
From our observation above it follows that F is well-behaved.
Let E be the vector in Z^M all whose coordinates are equal to 1. Given $x \in Z^L$ with x ≤ z*, let Y(x) = {$X \in Z^M$/ X ≤ E and S(X) = x}. Clearly, F(X) = f(x) if X∈Y(x).
Given a price vector p, let P(p) be the M-vector whose first z_1^* coordinates are all equal to p_1, the next z_2^* are all equal to p_2, …, the last z_L^* coordinates are all equal to p_L.
Since for all $x \in Z^L$, $f(x) - p^Tx = f(m(x,z^*)) - p^Tx \leq f(m(x,z^*)) - p^Tm(x,z^*)$, there always exists a global profit maximizer x* such that m(x*,z) = x* (i.e. x* ≤ z*).
Since for all $x \in Z^L$ with x ≤ z*, $f(x) - p^Tx = F(X) - P(p)^TX$ for all X∈Y(x), $x^* \in Z^L$ with x* ≤ z* is a global profit maximizer of f at p if and only if all X∈Y(x*) is a local (and hence global) profit maximizer of F at P(p).
It is worth noting that there are 2^M integer valued vectors in an M-dimensional unit cube.

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References

1. Madden, P. (1986): "Concavity and Optimization in Microeconomics", Blackwell Publishers.