Sequence independent lifting for $0-1$ knapsack problems with disjoint cardinality constraints

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Abstract

In this paper, we study the set of $0-1$ integer solutions to a single knapsack constraint and a set of non-overlapping cardinality constraints (MCKP). This set is a generalization of the traditional $0-1$ knapsack polytope and the $0-1$ knapsack polytope with generalized upper bounds. We derive strong valid inequalities for the convex hull of its feasible solutions by lifting the generalized cover inequalities presented in [32]. For problems with a single cardinality constraint, we derive a set of multidimensional superadditive lifting functions and prove that they are maximal and non-dominated under some mild conditions. We then show that these functions can also be used to build strong valid inequalities for problems with multiple disjoint cardinality constraints.

1 Introduction

In 1983, Crowder et al. [8] proposed to use strong inequalities for $0-1$ knapsack sets as cuts for more general $0-1$ mixed integer programs (MIPs) with multiple constraints. They demonstrated that this approach yields significant computational improvements over pure branch-and-bound algorithms on a set of general $0-1$ MIPs. Gu et al. [12] later implemented the idea in MINTO and confirmed these results. As a consequence, cuts from knapsack relaxations of MIPs were progressively implemented in commercial software. They are now an integral part of major commercial MIP solvers, such as CPLEX and XPRESS.

In this paper, we consider an extension of the basic $0-1$ knapsack model (KP) in which the $0-1$ integer solutions to a single knapsack constraint are also required to satisfy a set of non-overlapping cardinality constraints (MCKP). Specifically, for $N = \{1, \ldots, n\}$, $\emptyset \neq N_i \subseteq N$ for $i = 1, \ldots, r$ with $N_i \cap N_k = \emptyset$ for $i \neq k$, we study

$$\hat{S}_r = \left\{ x \in \{0,1\}^n : \sum_{j \in N} \hat{a}_j x_j \leq \hat{b}, \ \sum_{j \in N_i} x_j \leq \hat{K}_i, \ i = 1, \ldots, r, \right\}, \tag{1}$$

where $\hat{a}_j \in \mathbb{R}$ for $j \in N$, $\hat{b} \in \mathbb{R}$ and $\hat{K}_i \in \mathbb{Z}_+$ for $i = 1, \ldots, r$. Without loss of generality, we assume that $\hat{K}_i \geq 1$ for $i = 1, \ldots, r$. We also define $P\hat{S}_r := conv(\hat{S}_r)$. Because $\hat{S}_r$ is a finite set, the following proposition can easily be verified.

Proposition 1. *$P\hat{S}_r$ is a polytope.* □

The study of $P\hat{S}_r$ is interesting both theoretically and practically. From a theoretical perspective, $P\hat{S}_r$ generalizes the traditional $0-1$ knapsack problem (if $|N_i| = 1$ for all i), the $0-1$ knapsack problem with generalized upper bounds (GUBKP) (if $\hat{K}_i = 1$ for all i), and the $0-1$ knapsack problem with disjoint precedence constraints (if $N_i = \{j_i, k_i\}$, $\hat{a}_{j_i} \cdot \hat{a}_{k_i} \leq 0$ and $\hat{K}_i = 1$ for $i = 1, \ldots, r$). Also, it is a first step in the polyhedral study of unstructured $0-1$ problems with multiple constraints. In fact, we can always relax an MIP into $P\hat{S}_r$ by selecting any one of its row to be the knapsack constraint of $P\hat{S}_r$ and by generating a set of r disjoint extended cover inequalities from the problem formulation to constitute the cardinality constraints; see Nemhauser and Wolsey [20] for a discussion of extended cover inequalities. Clearly, strong valid inequalities for $P\hat{S}_r$ can be used as cuts for the original problem. Furthermore, these cuts are generated from a relaxation that is tighter than the traditional knapsack relaxation $P\hat{S}_0$. Therefore, it is reasonable to assume that these cuts will be strong.

From a practical perspective, MCKP appears as a subproblem of more complicated $0-1$ integer programs. As an example, GUBKP appears as a relaxation of product pricing, capital budgeting, scheduling and planning, resource allocation, computer and circuit layout design problems; see Armstrong et al. [1], Ferreira et al. [9], Sherali and Lee [23], Sinha and Zoltners [24], and Wolsey [29]. The general cardinality-constrained knapsack model, although less studied, also appears as a subproblem of many practical applications, including parallel task allocation, capacity allocation and various combinatorial problems; see Hollermann et al. [16], Toktay and Uzsoy [25], and Bruglieri et al. [7].

To the best of our knowledge, there are few polyhedral studies of the $0-1$ knapsack problem with cardinality constraints. Exceptions include recent work by Glover and Sherali [10, 11]. In [10], the authors study sets defined by a knapsack constraint and one additional cardinality constraint that covers all variables. Their focus is on deriving all strong *second-order cover inequalities*. We note that the set they studied is a special case of PS_1 since we do not assume that all the variables belong to the cardinality constraint. In [11], the authors design a procedure to generate similar *high-order cover inequalities* for KP with a single two-sided bounding constraint over all variables and a set of disjoint two-sided bounding inequalities. Our research is different from Glover and Sherali's work in that we concentrate on developing and applying superadditive lifting theory to efficiently generate strong valid inequalities from facet-defining extensions of classical cover inequalities.

Although polyhedral studies of $P\hat{S}_r$ are scarce, special cases such as KP and GUBKP have been studied extensively in the past. In 1975, Balas [4], Hammer et al. [15], Wolsey [27] presented facet-defining inequalities for KP based on the notion of a cover. Balas and Zemel [5] gave bounds on the lifted coefficients of minimal cover inequalities and Zemel [30] proposed a general simultaneous lifting procedure to obtain facets. Balas and Zemel [6] also showed that all the facet-defining inequalities of KP can be obtained by lifting cover inequalities after possibly complementing variables. Padberg [21] introduced $(1, k)$-configurations inequalities for KP. Weismantel [26] presented a complete linear description of the convex hull of particular 0–1 knapsack polytopes and proposed a "reduction principle" that can be used to generate strong valid inequalities for KP.

The polyhedral structure of GUBKP has been less studied. Johnson and Padberg [17] described an equivalent transformation from instances of GUBKP with arbitrarily signed coefficients to instances of GUBKP where all the variable coefficients are nonnegative. This implies that when studying $P\hat{S}_r$ with $\hat{K}_i = 1$ for $i = 1, \ldots, r$, it is not restrictive to assume that $a_j \geq 0$ for $j \in N$. Wolsey [29] gave a set of valid inequalities for GUBKP and proved that they are strong by deriving conditions under which they describe the convex hull of the problem. Nemhauser and Vance [19] presented a method based on independent sets to lift cover inequalities and obtain facet-defining inequalities. Sherali and Lee [23] also generated inequalities from covers using lifting. Their lifting procedure is different because it simultaneously lifts all variables within a GUB, which yields a polynomial lifting scheme. Finally, Gu et al. [12] discussed some computational aspects of using lifted GUB cover inequalities in the solution of $0-1$ integer programs.

In this paper, we are interested in studying the $0-1$ knapsack with disjoint cardinality constraints. To simplify the presentation, we first complement the $0-1$ variables x_j of $\hat{S}_r$ whose coefficients $\hat{a}_j$ are negative. We obtain the equivalent set

$$
\begin{aligned}
S_r = \quad \{x \in \mathbb{Z}^n : \quad & \sum_{j\in N} a_j x_j \le b, && (2)\\
& \sum_{j\in N_i^+} x_j - \sum_{j\in N_i^-} x_j \le K_i, \ \forall i = 1,\ldots,r && (3)\\
& x_j \le 1 \quad \forall j \in N, && (4)\\
& x_j \ge 0 \quad \forall j \in N\} && (5)
\end{aligned}
$$

where $b \in \mathbb{R}$, $K_i \in \mathbb{Z}$, $a_j \in \mathbb{R}_+$ for $j \in N$, $N_i^+ = \{j \in N_i : \hat{a}_j \ge 0\}$ and $N_i^- = \{j \in N_i : \hat{a}_j < 0\}$. Clearly, from (1), we have $K_i = \hat{K}_i - |N_i^-|$ and $N_i = N_i^+ \cup N_i^- \subseteq N$ with $N_i^+ \cap N_i^- = \emptyset$. We define $N_0 := N \backslash \bigcup_{i=1}^{r} N_i$. We assume without loss of generality that if $j, k \in N_i^+$ (or N_i^-, or N_0) and $j < k$, then $a_j \ge a_k$. We also assume that $b \ge 0$ (since otherwise, $S_r = \emptyset$), $a_j \le b$ for $j \in N$ (since otherwise $x_j = 0$) and $|N_i^-| \ge |K_i|$ if $K_i < 0$ (since otherwise $S_r = \emptyset$). We denote the convex hull of S_r by PS_r. Furthermore, when $N_i^- = \emptyset$ for $i = 1,\ldots,r$, we denote the set S_r as S_r^+ and its convex hull as PS_r^+. We observe that PS_r^+ is an independence system.

In Section 2, we describe basic lifting results and discuss general conditions under which lifting is sequence independent. In particular, we show that for PS_r, conditions weaker than superadditivity already imply that lifting is sequence independent. In Section 3, we briefly review some results about *generalized cover inequalities*, a family of inequalities described in [32] that are strong for PS_r. In Section 4, we derive a set of superadditive lifting functions for PS_1^+ and prove that they are non-dominated and maximal. In Section 5, we present a set of superadditive lifting functions for PS_1 and show that the valid inequalities obtained using sequence independent lifting in PS_1 are either coefficient-wise stronger than those obtained from superadditive lifting in PS_0, or cannot be obtained from PS_0. In Section 6, we extend the results of Section 4 and present a $r+1$-dimensional maximal and non-dominated superadditive lifting function for generalized cover inequalities in PS_r^+.

2 Lifting for PS_r

In this section, we give a brief review of how lifting can be used to generate valid inequalities in integer programming. We also generalize sequence independence results and describe how we will use these results to generate inequalities for PS_r. In Section 2.1, we give basic definitions and results about lifting. In Section 2.2, we describe general conditions for which lifting is sequence independent. These superadditive conditions were introduced by Wolsey [28] in the case of $0-1$ integer programs. Because of the specific nature of our problem, we show that it is sufficient to consider a subset of the superadditive conditions to guarantee sequence independent lifting in PS_r. This result is important since it permits an easier derivation of stronger lifting coefficients for MCKP in Sections 4, 5 and 6. In Section 2.3, we discuss the use of superadditive functions to obtain strong valid inequalities.

2.1 Basic Lifting Results

We consider G to be the set of $0-1$ integer solutions to a finite set of linear inequalities, i.e.

$$G = \{x \in \{0,1\}^n : \sum_{j\in N} A_j x_j \le b\} \tag{6}$$

where $A_j \in \mathbb{R}^m$ for $j = 1,\ldots,n$, $b \in \mathbb{R}^m$ and $N = \{1,\ldots,n\}$. We denote the convex hull of G as PG. We consider the restriction of PG obtained by fixing some of the variables at 0, i.e. we define $PG(N') := conv\{G \cap \{x_j = 0 : j \in N'\}\}$ where $N' = \{1,\ldots,n'\} \subseteq N$. Assuming that all the variables are fixed at 0 is without loss of generality since $0-1$ variables can be complemented. Assume that the *seed inequality*

$$\sum_{j\in N\backslash N'} \alpha_j x_j \leq \alpha_0 \tag{7}$$

is valid for $PG(N')$. *Sequential lifting* is the process by which (7) is converted into a valid inequality for PG of the form

$$\sum_{j\in N} \alpha_j x_j \leq \alpha_0 \tag{8}$$

by reintroducing the variables $x_1, \ldots, x_{n'}$ in (7) one at a time. Since we can always reorder variables, we assume that x_i is the i^{th} variable to be lifted. For sequential lifting, the coefficients α_i for $i \in N'$ can be obtained by solving the lifting problems

$$\begin{aligned} f_i(\vec{Z}) = \min\ & \alpha_0 - \sum_{j\in N\backslash N'} \alpha_j x_j - \sum_{1\leq j<i} \alpha_j x_j \\ \text{s.t.}\ & \sum_{j\in N\backslash N'} A_j x_j + \sum_{1\leq j<i} A_j x_j \leq b - \vec{Z} \\ & x_j \in \{0,1\}\ \forall j \in N\backslash N' \cup \{1,\ldots,i-1\} \end{aligned} \tag{9}$$

where $\vec{Z} \in \mathbb{R}^m$ and where we define $f_i(\vec{Z}) = +\infty$ when $f_i(\vec{Z})$ is infeasible. When $f_i(A_i) < +\infty$ for $i \in N'$, it is clear that setting $\alpha_i = f_i(A_i)$ in (8) yields a valid inequality for PG. In particular, it can easily be proven that if (7) is facet-defining for $PG(N')$ and if $PG(N')$ is full-dimensional, then setting $\alpha_i = f_i(A_i)$ for $i \in N'$ yields a facet-defining inequality of PG. The case where $f_i(A_i) = +\infty$ for some $i \in N'$ corresponds to situations where the variable x_i to be lifted cannot take any value other than the one it is fixed at in the current set. This typically means that x_i should be lifted later in the sequence.

In the remainder of this paper, we will use f to denote f_1 and refer to it as the *exact lifting function* of (7). In the following propositions, we give some properties of f_i. These properties can easily be derived from the fact that (7) is valid for PG and from the fact that the problem defining f_i is a relaxation of that defining f_{i-1}.

Proposition 2. *Let $i \in N'$. Then, $f_i(\vec{0}) \geq 0$.* □

Proposition 3. *Let $i \leq j \leq n'$ and $\vec{Y} \leq \vec{Z} \in \mathbb{R}^m$ be such that $f_i(\vec{Y}) < +\infty$ and $f_i(\vec{Z}) < +\infty$. Then $f_i(\vec{Y}) \leq f_i(\vec{Z})$ if $f_i(\vec{Y}) \geq f_j(\vec{Y})$.* □

2.2 Sequence Independent Lifting

The inequalities generated by computing the lifting coefficients α_i exactly through the computation of $f_i(\vec{A_i})$ are strong. However, the amount of computation needed to obtain them is often prohibitive as a different optimization problem must be solved for every variable that is lifted. Fortunately this computational burden can be significantly reduced if the exact lifting function f is well-structured.

Definition 1. *A function $g : \mathbb{D} \mapsto \mathbb{R}$ is superadditive if $g(x) + g(y) \leq g(x+y)$ for all $x, y, x+y \in \mathbb{D}$.*

In 1977, Wolsey [28] showed that for 0−1 knapsack problems, lifting coefficients can be directly from f when f is superadditive. Gu et al. [14] extended Wolsey's results to $0-1$ mixed-integer programs. Atamtürk [3] later generalized the results to general mixed integer programs. Because of its computational advantages, the superadditive lifting theory has been used in different applications to derive strong inequalities for MIPs. As an example, Marchand and Wolsey [18] used superadditive lifting to derive two families of closed form facet-defining inequalities for $0-1$ knapsack sets with a single continuous variable.

The condition that f be superadditive over $\mathbb{D} = \mathbb{R}^n$ is sufficient for sequence independent to hold. However, there are weaker conditions that still imply sequence independent lifting. We give such conditions next.

Theorem 4. *Assume that $A_i \le b$ for $i = 1, \ldots, n'$. Assume also that $f(A_i) + f(\sum_{j\in R} A_j) \le f(\sum_{j \in R\cup\{i\}} A_j)$ for all $R \subseteq N'\backslash\{i\}$ such that $\sum_{j\in R\cup\{i\}} A_j \le b$. Then $f_i(\vec{x}) = f(\vec{x})$ for $i = 1, \ldots, n'$ and for $\vec{x} \in \{A_i, \ldots, A_{n'}\}$.*

Proof. The proof is by induction. The proof is obvious when $i = 1$. So, we assume that we have already established that for $i \le k$ where $2 \le k \le n' - 1$, $f_i(\vec{x}) = f(\vec{x})$ for $\vec{x} \in \{A_i, \ldots, A_{n'}\}$. It remains to prove that the result holds for $i = k + 1$.

It follows from Proposition 3, that $f_i(A_t) \le f(A_t)$ for $i = 1, \ldots, n'$. Since $x_j \in \{0, 1\}$ for $1 \le j \le i - 1$, we observe from (9) that $f_i(A_t) = \min_{R\subseteq\{1,\ldots,i-1\}}\{f(\sum_{j\in R} A_j + A_t) - \sum_{j\in R} f_j(A_j)\}$. From the inductive hypothesis, we have $\sum_{j\in R} f_j(A_j) = \sum_{j\in R} f(A_j)$ since $R \subseteq \{1, \ldots, i-1\}$. From the theorem assumption on f, it can easily be verified that $\sum_{j\in R} f(A_j) \le f(\sum_{j\in R} A_j)$ for $R \subseteq \{1, \ldots, i-1\}$. So, $f_i(A_t) \ge \min_{R\subseteq\{1,\ldots,i-1\}}\{f(\sum_{j\in R} A_j + A_t) - f(\sum_{j\in R} A_j)\}$ for $t = i, \ldots, n'$. Again, from the theorem assumption on f, we observe that $f_i(A_t) \ge f(A_t)$. We conclude that $f_i(A_t) = f(A_t)$ for $t = i, \ldots, n'$. □ □

Theorem 4 shows that it is sufficient to prove that f is superadditivity over the set consisting of all possible values of $\sum_{j\in R} A_j$ for $R \subseteq N'$ to ensure sequence independent lifting. In many cases, the number of such conditions increases exponentially with the number of variables to be lifted. As a result, it becomes more convenient to verify that f is superadditive over $\mathbb{R}$ or $\mathbb{R}_+$. However, the number of conditions to verify for a cardinality constraint does not increase exponentially. This is because all the coefficients of the variables are either $-1, 0$ or 1. By combining this observation with the result of Theorem 4, we derive next simpler conditions for sequence independent lifting in PS_r. To simplify the presentation we define for $N^0, N^1 \subseteq N$ satisfying $N^0 \cap N^1 = \emptyset$,

$$PS_r(N^0, N^1) := conv\{x \in S_r : x_j = 0,\ \forall j \in N^0, x_j = 1,\ \forall j \in N^1\}.$$

We define $PS_r^+(N^0, N^1)$ similarly.

Proposition 5. *Let f be the lifting function of a valid inequality for $PS_r(N', \emptyset)$. Then, the lifting of variables in N' is sequence independent if*

$$f\begin{pmatrix} y \\ \vec{I} \end{pmatrix} + f\begin{pmatrix} z \\ \vec{h} \end{pmatrix} \le f\begin{pmatrix} y + z \\ \vec{I} + \vec{h} \end{pmatrix} \tag{10}$$

for $(y, \vec{I}) \in [0, b] \times \{\vec{0}, \pm\vec{e}_1, \ldots, \pm\vec{e}_r\}$ and $(z, \vec{h}) \in [0, b] \times \mathbb{Z}^r$ in PS_r or for $(y, \vec{I}) \in [0, b] \times \{\vec{0}, \vec{e}_1, \ldots, \vec{e}_r\}$ and $(z, \vec{h}) \in [0, b] \times \mathbb{Z}^r_+$ in PS_r^+ where $\vec{e}_1, \ldots, \vec{e}_r$ are the unit vectors in $\mathbb{R}^r$. □

Note that a lifting function satisfying the conditions of Proposition 5 is not necessarily superadditive over $[0, b] \times \mathbb{Z}^r$. However, in the remainder of this paper, we will refer to functions satisfying (10) as superadditive. Verifying that a function is superadditive is often cumbersome, even for one-dimensional functions. Therefore, reducing the set of conditions to verify is a very crucial step in developing strong sequence independent lifting schemes for problems with multiple constraints. For this reason, Proposition 5 will be used extensively in later sections.

2.3 Approximate Superadditive Lifting

Very often, the exact lifting function of a seed inequality of interest is not superadditive and so the result of Theorem 4 cannot be used directly to derive valid inequalities. In such as situation, a superadditive lower approximation of the exact lifting function can be used to obtain strong valid inequalities without having to solving lifting problems repeatedly. This idea was first used in Gu et al. [13, 14] to derive efficient lifting procedures for $0-1$ knapsack and single node flow problems. An application of this procedure is also given by Atamtürk [2] for general mixed-integer knapsack sets and by Shebalov and Klabjan [22] for mixed-integer program with variable upper bounds.

For any given lifting function, Gu et al. [14] give a constructive proof of the existence of a superadditive lower approximation. In practice, there are usually many such approximations. Therefore, evaluating the quality of a proposed approximation is important. To measure the strength of a superadditive approximation, Gu et al. [13, 14] propose two criteria: non-dominance and maximality. Here, we summarize these concepts and present them for higher dimensions. We also introduce a new criterion to describe superadditive approximations of lifting functions defined for multiple constraints that yield coefficients stronger than those obtained from the individual constraints. Thereafter, we refer to the exact lifting function as f and to its superadditive approximation as g. We denote the domain of f as $\mathbb{X} \subseteq \mathbb{R}^m$.

Definition 2. *We say that a valid superadditive approximation $v(x)$ of $u(x)$ is non-dominated over $\mathbb{X}$ if there is no valid superadditive approximation $v'(x)$ of $u(x)$ such that $v(x) \leq v'(x)$ for all $x \in \mathbb{X}$ and $v(x') < v'(x')$ for some $x' \in \mathbb{X}$.*

Note that, for lifting, it is not crucial that the superadditive approximation be non-dominated over the domain of f. In fact, it is only important that it is non-dominated in a range $\underline{\mathbb{X}}$ that contains all the coefficients of the variables to be lifted. In PS_r for example, we must verify that the exact lifting function is superadditive over $[0, b] \times \mathbb{Z}^r$ so as to guarantee sequence independent lifting. However, we only care about the strength of the superadditive approximation over $\underline{\mathbb{X}} = [0, b] \times \{\vec{0}, \pm\vec{e}_1, \ldots, \pm\vec{e}_r\}$ since all the lifting coefficients will be obtained by evaluating the lifting function in this range. This observation motivates the following definitions.

Definition 3. *Let $\underline{\mathbb{X}} \subseteq \mathbb{X}$ be a set that contains all the coefficients of the variables to be lifted. Given that $f(x)$ is the exact lifting function over $\mathbb{X}$, we say $g(x)$ is a non-dominated superadditive approximation of $f(x)$ if $g(x)$ is a superadditive approximation of $f(x)$ over $\mathbb{X}$ and is non-dominated over $\underline{\mathbb{X}}$.*

Next, we describe the concept of a maximal approximation. We begin with a definition of maximal set.

Definition 4. *Let n' be the number of variables to be lifted. Let $\underline{\mathbb{X}}$ be a set that contains all the coefficients of the variables to be lifted, i.e. $A_i \in \underline{\mathbb{X}}$ for $i = 1, \ldots, n'$. We say that $E = \{x \in \underline{\mathbb{X}} : f_i(x) = f(x)$ for all choices of $A_i \in \underline{\mathbb{X}},\ i = 1, \ldots, n'$ and for all lifting orders$\}$ is the maximal set of the lifting function $f(x)$.*

Definition 5. *Given a valid superadditive approximation $g(x)$ of $f(x)$, we say that $g(x)$ is maximal over $\underline{\mathbb{X}}$ if $g(x) = f(x)$ for $x \in E$.*

Clearly, the best possible superadditive approximations of f are maximal and non-dominated. However, non-dominance and maximality are sometimes difficult to achieve and to verify for multiple-constraint problems. In these situations, it is important to guarantee that the approximation obtained for multiple constraints is at least as good as that obtained from a single row relaxation. It is clear that a function ϕ that is superadditive over $\mathbb{X}_1 \subseteq \mathbb{R}^m$ can be extended into a superadditive function over $\mathbb{X}_2 \subseteq \mathbb{X}_1 \times \mathbb{R}^s$ by setting $\psi\binom{x}{y} = \phi(x)$ for $(x, y) \in \mathbb{X}_1 \times \mathbb{R}^s$. When applied to lifting, this observation simply means that a valid superadditive lifting function for an integer program with m constraints can be extended into a valid lifting function for a program with $m+s$ constraints if the former model is the relaxation of the latter model obtained by dropping the last s constraints. For this reason, we introduce the following definition where we use $E[g(x)]$ to denote the trivial extension of a superadditive lifting function $g(x)$ to a higher dimensional space.

Definition 6. *Let g and g' be two valid lifting functions that are superadditive over $\mathbb{X}_1$ and $\mathbb{X}_2$ respectively. Assume also that $\mathbb{X}_1 \subseteq \mathbb{X}_2 \times \mathbb{R}^s$. We say that g is superior to g' on $\mathbb{X}_1$ if $g\binom{x}{y}$ dominates $E[g'(x)]$.*

3 Deriving Strong Valid Inequalities from Generalized Cover Inequalities

In this section, we describe the procedure we will use to generate strong valid inequalities for PS_r. A traditional approach to generate strong valid inequalities for KP is to lift minimal cover inequalities that are facet-defining for restrictions of the set; see [5, 6, 30]. However, for MCKP, there may not exist any minimal cover C that yields a facet-defining inequality for $PS_r(N\backslash C, \emptyset)$ even when PS_r is non-trivial. We illustrate this observation in Example 1.

Example 1. *Consider* $S_1 = \{x \in \{0,1\}^5 : 6x_1 + 5x_2 + 4x_3 + 4x_4 + 2x_5 \leq 17, x_1 + x_2 + x_3 - x_4 \leq 2\}$. *Observe first that* $S_1 \neq S'$ *where* $S' = \{x \in \{0,1\}^5 : x_1+x_2+x_3-x_4 \leq 2\}$ *since the point* $(1,1,1,1,0) \in S'$ *but* $(1,1,1,1,0) \notin S_1$. *Clearly,* $C = (1,2,3,4)$ *is the only minimal cover that can be obtained from the knapsack constraint. It is easily seen that* $PS_1(N\backslash C, \emptyset)$ *is full-dimensional. Finally, observe that* $x_1 + x_2 + x_3 + x_4 \leq 3$ *is not facet-defining for* $PS_1(N\backslash C, \emptyset)$. *This illustrates the fact that* PS_1 *might be non-trivial and might not have minimal cover inequalities that can be lifted into facets of* PS_1. □

Example 1 illustrates that minimal covers from the knapsack constraint do not always produce the best seed inequalities for lifting when studying PS_r. Therefore, to derive strong inequalities for PS_r, we propose to first fix variables x_j with $j \in \hat{N} \subseteq \bigcup_{i=1}^{r} N_i^-$ to 1 so as to loosen the cardinality constraints and then to identify a minimal cover C that is strong. The minimal cover inequality is then lifted with respect to these variables to obtain a facet-defining inequality for $PS_r(N\backslash\{\hat{N} \cup C\}, \emptyset)$.

Next we describe necessary and sufficient conditions for PS_r to be full-dimensional. Throughout this subsection, we define $N_i^+ := \{p_i, \ldots, q_i\}$ and $N_i^- := \{q_i + 1, \ldots, s_i\}$ for $i = 1, \ldots, r$, and $N_0 = \{s_r + 1, \ldots, n\}$ where $p_1 = 1$ and $p_i = s_{i-1} + 1$ for $i \geq 2$. We define $a_k(W) := \max\{\sum_{j \in V} a_j : V \subseteq W, |V| = k\}$, and $\underline{a}_k(W) := \min\{\sum_{j\in V} a_j : V \subseteq W, |V| = k\}$. We also define $a_k(W) := -\infty$ if $|W| < k$ and $\underline{a}_k(W) := \infty$ if $|W| < k$.

Proposition 6. *([32]) Let* $\underline{b} = max\{0, \sum_{i:K_i \leq -1} \underline{a}_{|K_i|}(N_i^-)\}$. PS_r *is full-dimensional iff* $a_1(N_0) + \underline{b} \leq b$ *and for each* $i = 1, \ldots, r$, *one of the following conditions holds*

(i) $K_i \geq 1$ *and* $a_1(N_i) + \underline{b} \leq b$,

(ii) $K_i = 0$, $a_1(N_i^+) + \underline{a}_1(N_i^-) + \underline{b} \leq b$ *and* $a_1(N_i^-) + \underline{b} \leq b$,

(iii) $K_i \leq -1$, $|N_i^-| \geq |K_i| + 1$, $a_1(N_i^-) + \underline{b} - a_{s_i - |K_i| + 1} \leq b$, *and* $\max\{a_1(N_i^+), 0\} + \underline{b} + a_{s_i - |K_i|} \leq b$. □

In Proposition 7, we present necessary and sufficient conditions under which a minimal cover C of PS_r yields an inequality

$$\sum_{j \in C} x_j \leq |C| - 1 \tag{11}$$

that is facet-defining for $PS_r(N\backslash C, \emptyset)$. Let $C_i = C \cap N_i$, $C_i^+ = C \cap N_i^+$ and $C_i^- = C \cap N_i^-$ for $i = 1, \ldots, r$. We define $\eta_i := K_i - |C_i^+| + |C_i^-|$. Note that η_i is an integer that represents the amount of slack there is in the i^{th} cardinality constraint when all the members of the cover are set to 1.

Proposition 7. *([32]) Let* C *be a minimal cover for* $PS_r(N\backslash C, \emptyset)$ *and assume that* $PS_r(N\backslash C, \emptyset)$ *is full-dimensional. Then, the minimal cover inequality* (11) *is facet-defining for* $PS_r(N\backslash C, \emptyset)$ *if and only if for each* $i = 1, \ldots, r$, *one of the following conditions is satisfied*

(i) $C_i^- \neq \emptyset$ *and* $\eta_i \geq 1$;

(ii) $C_i^+ \neq \emptyset$, $C_i^- = \emptyset$:
a) $C \neq C_i^+$ *and* $\eta_i \geq 0$;
b) $C = C_i^+$ *and* $\eta_i \geq -1$;

(iii) $C_i^+ = C_i^- = \emptyset$ *and* $\eta_i \geq 0$. □

For PS_r^+, the conditions of Proposition 7 become simpler.

Corollary 8. *Let C be a minimal cover. The cover inequality* (11) *is facet-defining for $PS_r^+(N\backslash C, \emptyset)$ if and only if one of the following conditions is satisfied*

(i) $C_i = \emptyset$ *for* $i = 1, \ldots, r$*, i.e.* $C \subseteq N_0$*;*

(ii) $C = C_i$ *for some* $i \in \{1, \ldots, r\}$*, and* $\eta_i \geq -1$*;*

(iii) $C \neq C_i$ *and* $\eta_i \geq 0$ *for* $i = 1, \ldots, r$*.* □

We prove in [32] that if PS_r is not completely determined by the cardinality constraints, there always exist $\hat{N} \subseteq N$ and $C \subseteq N\backslash\hat{N}$ such that (11) is facet-defining for $PS_r(N\backslash(C \cup \hat{N}), \hat{N})$. For PS_r^+, a minimal cover from the knapsack constraint that satisfies conditions of Corollary 8 can always be obtained if we choose $\hat{N} = \emptyset$. Observe also that in the case of GUBKP, condition (iii) corresponds to the notion of minimal GUB cover; see Nemhauser and Vance [19], Sherali and Lee [23], and Wolsey [29].

Therefore, it is possible to obtain a facet-defining inequality for $PS_r(N\backslash(C \cup \hat{N}), \emptyset)$ by lifting a minimal cover inequality from 1. It is shown in Zeng and Richard [32] that, except for the case $C = C_i^+$ with $\eta_i = -1$ and the case $C = C_i^-$ with $\eta_i = 1$ for some $i \in \{1, \ldots, r\}$, all the lifted inequalities are of the form

$$\sum_{j \in C} x_j + \sum_{j \in \overline{EZ}} x_j \leq |C| - 1 + |\overline{EZ}| \tag{12}$$

where $\overline{EZ} \subseteq \hat{N}$ is the set of exact lifting coefficients that are equal to 1 and where we denote $EZ = \hat{N}\backslash\overline{EZ}$. Note that although (16) has the form of a cover inequality, $C \cup \overline{EZ}$ may not be a cover for the knapsack constraint. We illustrate this observation in the following example.

Example 2. *Consider*

$$PS_1 = conv\{x \in \{0,1\}^6 : 7x_1 + 7x_2 + x_3 + x_4 + x_5 + x_6 \leq 15, \tag{13}$$

$$x_1 - x_2 - x_3 - x_4 - x_5 - x_6 \leq -2\}. \tag{14}$$

Note that $C = \{1, 2\}$ is a minimal cover for the full-dimensional polyhedron $PS_1(\emptyset, \hat{N})$ where $\hat{N} = \{3, 4, 5, 6\}$. The minimal cover inequality derived from C is

$$x_1 + x_2 \leq 1. \tag{15}$$

It follows from Proposition 7 that (15) *is facet-defining for $PS_1(N\backslash(C \cup \hat{N}), \hat{N})$. The lifting coefficients for x_3, x_4, x_5, x_6 obtained through exact lifting are zeros. Therefore,* (15) *is a facet-defining inequality for PS_1. It is easily seen that C is not a cover for the knapsack constraint* (13)*.* □

To streamline the notation, we will denote in the remainder of this paper the set $C \cup \overline{EZ}$ as C and the set $C^- \cup \overline{EZ}$ as C^-. Using this new definition of C, (12) becomes

$$\sum_{j \in C} x_j \leq |C| - 1. \tag{16}$$

with

$$|C_i^+| - |C_i^-| - |EZ \cap N_i| + \eta_i = K_i \tag{17}$$

for $i \in \{1, \ldots, r\}$. Because of the similarity and difference to the traditional cover inequality, we refer to (12) as a *generalized cover inequality*; see Zeng and Richard [32]. Note that, for PS_r^+, a generalized cover is always a minimal cover for the knapsack constraint with $C^- = EZ = \emptyset$.

The following corollary presents some properties of generalized cover.

Corollary 9. *([32]) For a generalized cover C, we have $a_j > 0$ for $j \in C$. Furthermore, if C is a cover for the knapsack constraint, then it is also a minimal cover for the knapsack constraint.* □

In this paper, we show how to lift generalized cover inequalities with respect to the variables x_j from 0 for $j \in N\backslash(C\cup\hat{N})$ from 0 to obtain facet-defining inequality for PS_r. Since lifting is difficult to perform exactly, we compute strong approximate lifting coefficients using a superadditive lower approximation of the exact lifting function. We note that the lifting function used here is multidimensional since we need to consider the coefficients of the variables in the knapsack constraint and in the different cardinality constraints.

To the best of our knowledge, this is the first time superadditive lower approximations of multi-dimensional lifting functions are proposed and proven to be strong. In fact, research on approximate superadditive lifting has been limited to one-dimensional lifting functions even when the sets studied have multiple constraints; see [13, 14, 22].

4 Superadditive Approximation of Lifting Functions in PS_1^+

In this section, we study how to lift generalized cover inequalities that satisfy conditions of Corollary 8 for PS_1^+. Because the lifting function of the generalized cover inequality is not superadditive, we approximate it from below to obtain a valid superadditive lifting function. We first construct a superadditive lifting function for the case where $K_1 = 1$. Then, for the case where $K_1 \geq 2$, we introduce a composition method that can be used to create a 2-dimensional superadditive approximation from a single-dimensional one. Such a composition method is very useful since it is difficult in general to build strong multidimensional superadditive lifting functions. For both the cases where $K_1 = 1$ and $K_1 \geq 2$, we prove that the proposed approximations are strong by showing that they are non-dominated and maximal. In the remainder of this paper, we refer to the exact lifting function of the generalized cover inequality in PS_1^+ as Θ and to its superadditive approximation as θ.

4.1 Exact Lifting and Superadditive Lifting function

Because PS_1^+ has a single cardinality constraint with only nonnegative coefficients, we use the notation N^+, C^+ and K to represent $N_1^+ = N_1, C_1^+$ and K_1 respectively. We assume without loss of generality that the variables x_j for $j \in C$ are sorted in non-decreasing order of their coefficients, i.e. $a_1 \geq a_2 \geq \cdots \geq a_{|C|}$. We also denote $A_i = \sum_{1\leq j\leq i} a_j$ for $i = 1, \ldots, |C|$, $A_0 = 0$ and $\lambda = A_{|C|} - b$. Because the generalized cover C is also a minimal cover for the knapsack constraint, we have $\lambda > 0$. The i^{th} lifting problem associated with a generalized cover inequality of PS_1^+ is given by

$$\begin{aligned}\Theta_i\binom{z}{h} = \min\ & |C| - 1 - \sum_{j\in C} x_j - \sum_{1\leq j\leq i-1} \alpha_j x_j \\ \text{s.t.}\ & \sum_{j\in C} a_j x_j + \sum_{1\leq j\leq i-1} a_j x_j \leq b - z \\ & \sum_{j\in C^+} x_j + \sum_{1\leq j\leq i-1} I_j x_j \leq K - h \\ & x_j \in \{0,1\} \text{ for } j \in C \cup \{1,\ldots,i-1\}\end{aligned} \tag{18}$$

where $(z,h) \in [0,b] \times \mathbb{Z}_+$ and I_j is the coefficient of x_j in the only cardinality constraint. Similar to Proposition 3, we present the following result.

Corollary 10. *The function $\Theta_i\binom{z}{h}$ is non-decreasing over $(z,h) \in [0,b] \times \mathbb{Z}_+$.* □

Instead of computing exact solutions to a set of different lifting problems, we derive a superadditive lower approximation of Θ to generate strong valid inequalities for PS_1^+. From Corollary 8, we observe that there are different types of generalized cover inequalities that lead to different lifting functions. We will show next in Sections 4.2 and 4.3 that all the valid superadditive approximations of generalized cover inequalities we propose can be constructed as in Theorem 11.

Theorem 11. *For the lifting function $\Theta\binom{z}{h}$ of the generalized cover inequality, there exists a non-dominated and maximal superadditive approximation of $\Theta\binom{z}{h}$ over $[0,b]\times\{0,\ldots,K\}$ that is of the form*

$$\theta\begin{pmatrix} z \\ h \end{pmatrix} = \begin{cases} \theta_0(z) & \text{if } h=0 \\ \max\{\theta_0(z-a^*)+1, \theta_0(z)\} & \text{if } h=1 \\ \sup\limits_{\{z=\sum_{j=1}^h z_j, z_j\geq 0, j=1,\ldots,h\}} \sum_{j=1}^{h} \theta\begin{pmatrix} z_j \\ 1 \end{pmatrix} & \text{if } h=2,\ldots,K \end{cases} \tag{19}$$

where $\theta_0(z)$ is a superadditive approximation of $\Theta\binom{z}{0}$ and a^* depends on C and N^+. □

We now describe how to obtain $\theta_0(z)$ and a^*. The case $K=1$ is described in Section 4.2 and the case $K\geq 2$ is described in Section 4.3.

4.2 Building a Superadditive Approximation of Θ when $K=1$

First we use Theorem 4 and Proposition 5 to derive sufficient conditions for sequence independent lifting.

Corollary 12. *Define $d^+ = \max\{|S| : S \subseteq (N^+ \backslash C^+), \sum_{j\in S} a_j \leq b\}$. For PS_1^+ with $K=1$, lifting is sequence independent over $[0,b]\times\{0,1\}$ if*

$$\Theta\begin{pmatrix} y \\ 0 \end{pmatrix} + \Theta\begin{pmatrix} z \\ 0 \end{pmatrix} \leq \Theta\begin{pmatrix} y+z \\ 0 \end{pmatrix} \tag{20}$$

$$\Theta\begin{pmatrix} y \\ 0 \end{pmatrix} + \Theta\begin{pmatrix} z \\ h \end{pmatrix} \leq \Theta\begin{pmatrix} y+z \\ h \end{pmatrix} \tag{21}$$

where $h=\min\{1,d^+\}$ and $y,z,y+z\in[0,b]$. □

Note that when $d^+=0$, i.e. $N^+=C^+$, (20) and (21) reduce to the traditional superadditive conditions for sequence independent lifting in the $0-1$ knapsack polytope. In fact, the traditional knapsack set can be viewed as a special case of PS_1^+ where a cardinality constraint of the form $x_j\leq 1$ is imposed for some $j\in N$. There are two cases. On the first hand, if $j\in C^+$, $N^+\backslash C^+=\emptyset$ then $h=d^+=0$, and therefore (21) reduces to (20). On the other hand, if $j\notin C^+$, $\Theta\binom{z}{0}=\Theta\binom{z}{1}$ and therefore (21) reduces again to (20). We conclude that in both cases, the conditions of Corollary 12 reduces to those given by Gu et al. [14].

The nontrivial case is when $d^+=1$, i.e. there exists a variable x_j such that $j\in N^+\backslash C^+$. From Corollary 8 we see that, in order to yield a facet-defining inequality, the generalized cover inequality must satisfy $|C^+|\leq 2$. Therefore, we discuss next the following 3 cases: (i) $|C^+|=1$, (ii) $|C^+|=0$ and (iii) $|C|^+=2$. Because the case $|C^+|=1$ is the most interesting, we consider it first.

4.2.1 Case i: $|C^+|=1$

Assume that $C^+=\{l\}$. Note that since any single variable of PS_1^+ cannot form a cover, we must have $C\neq C^+$. We first derive the exact lifting function.

Theorem 13. *The exact lifting function $\Theta^*\binom{z}{I}$ of the generalized cover inequality is*

$$\Theta^*\binom{z}{I} = \begin{cases} \begin{cases} 0 & \text{if } 0 \leq z \leq A_1-\lambda \\ i & \text{if } A_i-\lambda < z \leq A_{i+1}-\lambda \quad i=1,\ldots,|C|-1 \end{cases} & \text{if } I=0 \\ \begin{cases} 0 & \text{if } 0 \leq z \leq a_l-\lambda \\ i & \text{if } A_{i-1}+a_l-\lambda < z \leq A_i+a_l-\lambda \quad i=1,\ldots,l-1 \\ i & \text{if } A_i-\lambda < z \leq A_{i+1}-\lambda \quad i=l,\ldots,|C|-1. \end{cases} & \text{if } I=1 \end{cases} \tag{22}$$

□

Note that we use Θ^* to denote this lifting function rather than Θ since we will use this function in this section and Section 6. We observe that $\Theta^*\binom{z}{1} = \Theta^*\binom{z}{0}$ for $z \geq A_l - \lambda$. Note also that $\Theta^*\binom{z}{1} = \Theta^*\binom{z}{0}$ for all z if $a_l = a_1$. In the following corollary, we derive additional relations between $\Theta^*\binom{z}{1}$ and $\Theta^*\binom{z}{0}$.

Corollary 14. *$\Theta^*\binom{z}{1} \geq \Theta^*\binom{z}{0}$ $\forall z \in [0, b]$. Furthermore,*

$$\Theta^*\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{cases} 0 & \text{if} \quad 0 \leq z \leq a_l - \lambda \\ 1 & \text{if} \quad a_l - \lambda < z < a_l \\ \Theta^*\binom{z-a_l}{0} + 1 & \text{if} \quad a_l \leq z \leq A_l - \lambda \\ \Theta^*\binom{z}{0}, & \text{if} \quad A_l - \lambda < z \leq b. \end{cases} \tag{23}$$

□

Next, we derive a valid superadditive approximation θ^* for the lifting function Θ^*. To derive this approximation, we first observe that $\Theta^*\binom{z}{0}$ is the lifting function of a cover inequality. Therefore it can be approximated with the superadditive function proposed by Gu et al. [14]. We then build $\theta^*\binom{z}{1}$ from $\theta^*\binom{z}{0}$ using the relations presented in Corollary 14. The result of this construction is presented in Theorem 15.

Theorem 15. *Let $\rho_i = \max\{0, a_{i+1} - (a_1 - \lambda)\}$ for $i = 0, \ldots, |C| - 1$. The function*

$$\theta^*\begin{pmatrix} z \\ I \end{pmatrix} = \begin{cases} \begin{cases} 0 & \text{if} \quad z = 0 \\ i & \text{if} \quad A_i - \lambda + \rho_i < z \leq A_{i+1} - \lambda \\ i - (A_i - \lambda + \rho_i - z)/\rho_1 & \text{if} \quad A_i - \lambda < z \leq A_i - \lambda + \rho_i \end{cases} & \text{if } I = 0 \\ \\ \begin{cases} \theta^*\binom{z}{0} & \text{if} \quad 0 \leq z < a_l \\ \max\{\theta^*\binom{z-a_l}{0} + 1, \theta^*\binom{z}{0}\} & \text{if} \quad a_l \leq z \leq b \end{cases} & \text{if } I = 1 \end{cases} \tag{24}$$

for $i = 1, \ldots, |C| - 1$ is a valid superadditive approximation of $\Theta^*\binom{x}{I}$.

Proof. The proof is in two steps. We first show that $\theta^*\binom{z}{I} \leq \Theta^*\binom{z}{I}$ for $(z, I) \in [0, b] \times \{0, 1\}$ and then prove that the lifting function $\theta^*\binom{z}{I}$ satisfies conditions (20) and (21) of Corollary 12 with $d^+ = 1$.

Since $\theta^*\binom{z}{0}$ is the valid superadditive approximation of $\Theta^*\binom{z}{0}$ proposed by Gu et al. [14], $\theta^*\binom{z}{0} \leq \Theta^*\binom{z}{0}$ for $z \in [0, b]$. Now, for $z \in [0, a_l)$, we deduce from Corollary 14 that $\Theta^*\binom{z}{1} \geq \Theta^*\binom{z}{0} \geq \theta^*\binom{z}{0} = \theta^*\binom{z}{1}$. Observe now that because $a_i \geq a_l$ for $i \in C$ such that $i < l$, $\Theta^*\binom{z-a_l}{0} + 1 \geq \Theta^*\binom{z}{0}$ for $a_l \leq z \leq A_l - \lambda$. Similarly, because $a_j \leq a_l$ for $j \in C$ such that $j > l$, we have $\Theta^*\binom{z-a_l}{0} + 1 \leq \Theta^*\binom{z}{0}$ for $A_l - \lambda < z \leq b$. Therefore, we conclude from Corollary 14 that for $z \in [a_l, b]$

$$\begin{aligned} \Theta^*\begin{pmatrix} z \\ 1 \end{pmatrix} &= \max\{\Theta^*\begin{pmatrix} z - a_l \\ 0 \end{pmatrix} + 1, \Theta^*\begin{pmatrix} z \\ 0 \end{pmatrix}\} \\ &\geq \max\{\theta^*\begin{pmatrix} z - a_l \\ 0 \end{pmatrix} + 1, \theta^*\begin{pmatrix} z \\ 0 \end{pmatrix}\} = \theta^*\begin{pmatrix} z \\ 1 \end{pmatrix}. \end{aligned}$$

We now verify that θ^* satisfies the conditions of Corollary 12. The fact that condition (20) is satisfied follows from Gu et al. [14]. To verify condition (21), we must show that

$$\theta^*\begin{pmatrix} y \\ 0 \end{pmatrix} + \theta^*\begin{pmatrix} z \\ 1 \end{pmatrix} \leq \theta^*\begin{pmatrix} y + z \\ 1 \end{pmatrix}$$

for $y, z, y + z \in [0, b]$. For $y \in [0, b]$, $z \in [0, a_l)$ and $y + z \in [0, b]$, we obtain that

$$\theta^*\begin{pmatrix} y \\ 0 \end{pmatrix} + \theta^*\begin{pmatrix} z \\ 1 \end{pmatrix} = \theta^*\begin{pmatrix} y \\ 0 \end{pmatrix} + \theta^*\begin{pmatrix} z \\ 0 \end{pmatrix} \leq \theta^*\begin{pmatrix} y + z \\ 0 \end{pmatrix} \leq \theta^*\begin{pmatrix} y + z \\ 1 \end{pmatrix}$$

since it is easily verified that $\theta^*\binom{u}{0} \leq \theta^*\binom{u}{1}$ for $u \in [0,b]$. For $y \in [0,b]$, $z \in [a_l, b]$ and $y+z \in [0,b]$, we have

$$\begin{aligned}\theta^*\binom{y}{0} + \theta^*\binom{z}{1} &= \max\{\theta^*\binom{y}{0} + \theta^*\binom{z-a_l}{0} + 1, \theta^*\binom{y}{0} + \theta^*\binom{z}{0}\} \\ &\leq \max\{\theta^*\binom{y+z-a_l}{0} + 1, \theta^*\binom{y+z}{0}\} = \theta^*\binom{y+z}{1}\end{aligned}$$

since $y + z \geq a_l$. □ □

We next prove in Theorem 17 that the approximation θ^* we propose is non-dominated and maximal. In the proof, we make use of the following result of Gu et al. [14].

Theorem 16. *Function* $\theta^*\binom{z}{0}$ *is a non-dominated approximation of* $\Theta^*\binom{z}{0}$ *that satisfies* $\theta^*\binom{y}{0} + \theta^*\binom{z}{0} \leq \theta^*\binom{y+z}{0}$ *for* $y, z, y+z \in [0,b]$*. Furthermore, it is maximal over* $[0,b] \times \{0\}$*.* □

Theorem 17. *The function* $\theta^*\binom{z}{I}$ *is a superadditive approximation of* $\Theta^*\binom{z}{I}$ *that is non-dominated and maximal over* $[0,b] \times \{0,1\}$*.*

Proof. We first prove that $\theta^*\binom{z}{I}$ is non-dominated. When $a_l = a_1$, $\Theta^*\binom{z}{1} = \Theta^*\binom{z}{0}$ and $\theta^*\binom{z}{1} = \theta^*\binom{z}{0}$. It follows from Theorem 16 that θ^* is non-dominated. It is therefore sufficient to consider the case where $a_l < a_1$. Assume by contradiction that θ^* is dominated, i.e. there exists $\theta' : [0,b] \times \{0,1\} \longmapsto \mathbb{R}$ such that $\theta'\binom{z}{I} \geq \theta^*\binom{z}{I}$ for $(z, I) \in [0,b] \times \{0,1\}$ and $\theta'\binom{z'}{I'} > \theta^*\binom{z'}{I'}$ for some $(z', I') \in [0,b] \times \{0,1\}$. It follows from Theorem 16 that $I' = 1$. It is easily verified that

$$\theta^*\binom{z}{1} = \begin{cases} \theta^*\binom{z}{0} & \text{if} \quad 0 \leq z < a_l \\ \theta^*\binom{z-a_l}{0} + 1 & \text{if} \quad a_l \leq z \leq A_l - \lambda \\ \theta^*\binom{z}{0} & \text{if} \quad A_l - \lambda < z \leq b \end{cases} \tag{25}$$

using an argument similar to that of Theorem 15. We now consider three cases.

(1) If $z' \in [0, a_l)$, then $0 \leq \theta^*\binom{z'}{1} < \theta'\binom{z'}{1} \leq \Theta^*\binom{z'}{1}$. Because $\Theta^*\binom{z}{1} = 0$ when $z \in [0, a_l - \lambda]$, we conclude that $z' \in (a_l - \lambda, a_l)$ with $\Theta^*\binom{z'}{1} = 1$ and $\theta^*\binom{z'}{1} = \theta^*\binom{z'}{0} < 1$. Now consider $z = A_l - \lambda$. Clearly, $z - z' \in (A_{l-1} - \lambda, A_l - \lambda]$. It follows that $\theta^*\binom{z-z'}{0} \in (l-2, l-1]$. On the first hand, if $\theta^*\binom{z-z'}{0} = l - 1$, then $\theta'\binom{z-z'}{0} = \Theta^*\binom{z-z'}{l} = l - 1$. Furthermore,

$$l - 1 + \theta'\binom{z'}{1} = \theta'\binom{z-z'}{0} + \theta'\binom{z'}{1} \leq \theta'\binom{z}{1} \leq \Theta^*\binom{z}{1} = l - 1.$$

It follows that $\theta'\binom{z'}{1} = \theta^*\binom{z'}{1} = 0$, which is the desired contradiction.

On the other hand, if $l - 2 < \theta^*\binom{z-z'}{0} < l - 1$, then $A_{l-1} - \lambda < z - z' \leq A_{l-1} - \lambda + \rho_{l-1}$ with $\rho_{l-1} = a_l - (a_1 - \lambda) > 0$. It follows that $z' > a_1 - \lambda$. Furthermore,

$$\begin{aligned} l - 1 \quad &= \quad \Theta^*\binom{z}{1} \geq \theta'\binom{z}{1} \geq \theta'\binom{z'}{1} + \theta'\binom{z-z'}{0} > \theta^*\binom{z'}{1} + \theta^*\binom{z-z'}{0} \\ &= \quad \frac{z' - a_1 + \lambda}{\rho_1} + l - 1 - \frac{a_l - \rho_{l-1} + z'}{\rho_1} = l - 1 \end{aligned}$$

which is the desired contradiction.

(2) If $z' \in (a_l, A_l - \lambda]$, then $\theta^*\binom{z'}{1} = \theta^*\binom{z'-a_l}{0} + 1$ and $\Theta^*\binom{z'}{1} = \Theta^*\binom{z'-a_l}{0} + 1$. Also, because $\theta^*\binom{z'}{1} < \theta'\binom{z'}{1} \leq \Theta^*\binom{z'}{1}$, it follows that $z' - a_l \in (A_{j-1} - \lambda, A_{j-1} - \lambda + \rho_{j-1})$ for some $j \in \{1, \ldots, l-1\}$ such that $\rho_{j-1} > 0$. Define now $z = A_j + a_l - \lambda$. From Theorem 13 and Theorem 15, $\Theta^*\binom{z}{1} = \theta^*\binom{z}{1} = j$. Clearly, $z - z' \in (a_j - \rho_{j-1}, a_j) \subseteq (a_1 - \lambda, a_j)$. From (24), we obtain

$$j \geq \theta'\binom{z}{1} \geq \theta'\binom{z'}{1} + \theta'\binom{z-z'}{0} > \theta^*\binom{z'}{1} + \theta^*\binom{z-z'}{0} = \theta^*\binom{z'-a_l}{0} + 1 + \theta^*\binom{z-z'}{0} = j,$$

which is the desired contradiction.

(3) If $z' \in (A_l - \lambda, b]$, then $\theta^*\binom{z'}{1} = \theta^*\binom{z'}{0}$ and $\Theta^*\binom{z'}{1} = \Theta^*\binom{z'}{0}$. This is a contradiction to the fact that $\theta^*\binom{z}{0}$ is non-dominated over $[0, b]$.

Second, we prove that $\theta^*\binom{z}{I}$ is a maximal approximation of $\Theta^*\binom{z}{I}$ over $[0, b] \times \{0, 1\}$. Let $E \subseteq [0, b] \times \{0, 1\}$ be the maximal set of $\Theta^*\binom{z}{I}$. We show that if $\theta^*\binom{z'}{I'} < \Theta^*\binom{z'}{I'}$ for some $(z', I') \in [0, b] \times \{0, 1\}$, then $(z', I') \notin E$. When $I' = 0$, the proof reduces to that of Theorem 16 given in [14]. Assume therefore that $I' = 1$.

(1) In the case $z' \in [0, a_l)$, we must have $z' \in (a_l - \lambda, a_l)$ since $\theta^*\binom{u}{1} = \Theta^*\binom{u}{1}$ when $u \leq a_l - \lambda$. Let $z = A_l - \lambda$. Then $\Theta^*\binom{z}{1} = l - 1$. Note also that $z - z' > A_{l-1} - \lambda$ and so $\Theta^*\binom{z-z'}{0} = l - 1$. It follows that

$$\Theta_2^*\binom{z'}{1} = \min\{\Theta^*\binom{z'}{1}, \Theta^*\binom{z}{1} - \Theta^*\binom{z-z'}{0}\} = \min\{1, 0\} = 0.$$

We conclude that $(z', 1) \notin E$.

(2) If $z' \in [a_l, A_l - \lambda]$, we must have $z' - a_l \in (A_{j-1} - \lambda, A_{j-1} - \lambda + \rho_{j-1})$ for some $j \in \{1, \ldots, l-1\}$ with $\rho_{j-1} > 0$. Consider $z = A_j + a_l - \lambda$. Then, $\Theta^*\binom{z'}{1} = \Theta^*\binom{z}{1} = j$ and $z - z' > a_j - \rho_{j-1} = a_1 - \lambda$. We conclude that $\Theta^*\binom{z-z'}{0} \geq 1$. Furthermore, during the sequential lifting procedure, we have

$$\Theta_2^*\binom{z'}{1} = \min\{\Theta^*\binom{z'}{1}, \Theta^*\binom{z}{1} - \Theta^*\binom{z-z'}{0}\} = j - 1,$$

showing that $(z', 1) \notin E$.

(3) If $z' \in (A_l - \lambda, b]$, then $\theta^*\binom{z'}{1} = \theta^*\binom{z'}{0}$ and $\Theta^*\binom{z'}{1} = \Theta^*\binom{z'}{0}$. Therefore, the proof reduces to that that of Theorem 16. □

□

In the remainder of this paper, we define $\theta^*\binom{z}{0} := -M$ for $z < 0$ with $M \geq |C| - 1$. This is not restrictive since $a_j \geq 0$ for $j \in N$. Using this definition, we see that the result of Theorem 15 fits the mold presented in Theorem 11 when $a^* := a_l$.

4.2.2 Case *ii*: $|C^+| = 0$

In this case, it is easily seen from (18) that $\Theta\binom{z}{I} = \Theta^*\binom{z}{0}$ for $(z, I) \in [0, b] \times \{0, 1\}$. Therefore, it is sufficient to verify (20) to guarantee that lifting is sequence independent. Theorem 18 naturally follows from Theorem 16.

Theorem 18. *Function $\theta\binom{z}{I} := \theta^*\binom{z}{0}$ is a valid superadditive approximation of $\Theta\binom{z}{I}$ that is non-dominated and maximal over $(z, I) \in [0, b] \times \{0, 1\}$.* □

Here also, we observe that the approximation θ we propose can be obtained from Theorem 11 by setting $a^* := a_1$.

4.2.3 Case *iii*: $|C^+| = 2$

In this case, we see that $C = C^+$ because of Corollary 8. Let $C = \{1, 2\}$ with $a_1 \geq a_2$ and $b = a_1 + a_2 - \lambda$. We derive the exact lifting function of the corresponding generalized cover inequality next.

Theorem 19. *The lifting function of the generalized cover inequality is*

$$\Theta\binom{z}{I} = \begin{cases} \begin{cases} 0 & \text{if} \quad 0 \leq z \leq a_1 - \lambda \\ 1 & \text{if} \quad a_1 - \lambda \leq z \leq a_1 + a_2 - \lambda \end{cases} & \text{if } I = 0 \\ \\ 1 \quad \text{if} \quad 0 \leq z \leq b & \text{if } I = 1. \end{cases} \tag{26}$$

□

Observe that $\Theta\binom{z}{0}$ is identical to $\Theta^*\binom{z}{0}$ with $C = \{1, 2\}$. Because Θ is simple, a non-dominated and maximal valid superadditive approximation can easily be derived.

Theorem 20. *The function*

$$\theta\binom{z}{I} = \begin{cases} \theta^*\binom{z}{0} & \text{if } I = 0 \\ \\ \begin{cases} \theta^*\binom{z}{0} & \text{if} \quad 0 \leq z < a_2 \\ 1 & \text{if} \quad a_2 \leq z \leq a_1 + a_2 - \lambda \end{cases} & \text{if } I = 1 \end{cases} \tag{27}$$

is a valid superadditive approximation of Θ that is non-dominated and maximal over $(z, I) \in [0, b] \times \{0, 1\}$. □

It is easy to verify that $\theta\binom{z}{1} = \max\{\theta\binom{z-a_2}{0} + 1, \theta\binom{z}{0}\}$. Therefore, θ takes the form described in Theorem 11 with $a^* := a_2$.

4.3 Building a Superadditive Approximation of Θ when $K \geq 2$

Similar to the derivation of the superadditive approximation when $K = 1$, we present first conditions that follow from Theorem 4 and Proposition 5 that ensure sequence independent lifting. Again, we define $d^+ := \max\{|S| : S \subseteq (N^+ \backslash C^+), \sum_{j \in S} a_j \leq b\}$ and denote $K^+ := \min\{K, d^+\}$.

Corollary 21. *For PS_1^+ with $K \geq 2$, lifting is sequence independent over $[0, b] \times \{0, 1\}$ if*

$$\Theta\binom{y}{0} + \Theta\binom{z}{0} \leq \Theta\binom{y+z}{0} \tag{28}$$

$$\Theta\binom{y}{0} + \Theta\binom{z}{h} \leq \Theta\binom{y+z}{h} \quad \forall h \in \{1, \dots, K^+\} \tag{29}$$

$$\Theta\binom{y}{1} + \Theta\binom{z}{h} \leq \Theta\binom{y+z}{h+1} \quad \forall h \in \{1, \dots, K^+ - 1\} \tag{30}$$

where $y, z, y + z \in [0, b]$. □

Because the conditions of Corollary 21 are most stringent when $K^+ = K$, we will show that the approximation we propose satisfy conditions (28)-(30) when $K^+ = K$.

The derivation of an approximation is more difficult in this case since the function proposed must satisfy superadditive conditions over $[0, b] \times \{0, K^+\}$ rather than $[0, b] \times \{0, 1\}$. We observe however that because $\theta\binom{z}{h}$ for $h \geq 2$ is not used in the derivation of lifting coefficients, it is not necessary that the approximation over $[0, b] \times \{2, \dots, K\}$ be strong. We take advantage of this observation in developing θ. We first build a strong approximation of Θ over $[0, b] \times \{0, 1\}$. We then extend it into a valid approximation over $[0, b] \times \{2, \dots, K\}$. In this case, there are four cases: (*i*) $C \neq |C^+|$ and $|C^+| = K$, (*ii*) $C \neq |C^+|$ and $|C^+| \leq K - 1$, (*iii*) $C = C^+$ and $|C^+| = K + 1$, and (*iv*) $C = C^+$ and $|C^+| \leq K$.

4.3.1 Case i: $C \neq C^+$ and $|C^+| = K$

Let $C^+ = \{l_1, \dots, l_K\}$ with $l_1 < l_2 < \dots < l_K$. Because $a_i \geq a_j$ for $i, j \in C$ such that $i < j$, we have $a_{l_1} \geq \dots \geq a_{l_K}$. We denote $\hat{A}_i = \sum_{j=1}^{i} a_{l_j}$ and $\hat{A}_i^h = \sum_{j=h-i+1}^{h} a_{l_j}$ for $h = 1, \dots, K$ and for $i = 1, \dots, h$. When $i = 0$, we define $\hat{A}_0^h = 0$ for $h = 1, \dots, K$. We next give in Theorem 22 the exact lifting function of the generalized cover inequality (11). It has a form that is similar to that presented in Corollary 14.

Theorem 22. *The exact lifting function of the generalized cover inequality is*

$$\Theta\binom{z}{h} = \begin{cases} \Theta^*\binom{z}{0} & \text{if } \ h = 0 \\ \begin{cases} h-1 & \text{if } \quad 0 \leq z \leq \hat{A}_h - \lambda \\ h & \text{if } \ \hat{A}_h - \lambda < z < \hat{A}_h \\ \max_{i=0,\dots,h} \{\Theta^*\binom{z - \hat{A}_i^h}{0} + i\} & \text{if } \quad \hat{A}_h \leq z \leq b \end{cases} & \text{if } \ h = 1, \dots, K. \end{cases} \tag{31}$$

Proof. Since it is easy to verify the value of $\Theta\binom{z}{0}$ for $z \in [0, b]$ and $\Theta\binom{z}{h}$ for $z \in [0, \hat{A}_h)$ and $h = 1, \dots, K$, we only derive the value of $\Theta\binom{z}{h}$ for $z \in [\hat{A}_h, b]$ and $h = 1, \dots, K$. Fix $z \in [\hat{A}_h, b]$ and $h = 1, \dots, K$. Define $T := \{j \in C : j \neq l_i, \ \forall i = 1, \dots, h\}$ and assume that $T = \{k_1, \dots, k_{|C|-h}\}$ with $k_1 < \dots < k_{|C|-h}$. Let s be the only index such that $\hat{A}_h - \lambda + \sum_{j=1}^{s-1} a_{k_j} < z \leq \hat{A}_h - \lambda + \sum_{j=1}^{s} a_{k_j}$. It is easy to verify that the solution x^* defined as

$$x_j^* = \begin{cases} 0 & \text{if } \ j \in \{l_1, \dots, l_h\} \cup \{k_1, \dots, k_s\} \\ 1 & \text{if } \ j \in \{k_{s+1}, \dots, k_{|C|-h}\} \end{cases}$$

is optimal for $\Theta\binom{z}{h}$ and that $\Theta\binom{z}{h} = h + s - 1$.

First we prove that $\max_{i=0,\dots,h}\{\Theta^*\binom{z-\hat{A}_i^h}{0} + i\} \leq \Theta\binom{z}{h}$. For $i \in \{0, \dots, h\}$, we define $\tilde{x}^i$ as the solution obtained by setting the i largest elements of $\{l_1, \dots, l_h\}$ to 1, i.e.

$$\tilde{x}_j^i = \begin{cases} x_j^* & \text{if } \ j \in C \backslash \{l_{h-i+1}, \dots, l_h\} \\ 1 & \text{if } \ j \in \{l_{h-i+1}, \dots, l_h\}. \end{cases}$$

The solution $\tilde{x}^i$ satisfies

$$\sum_{j \in C} a_j \tilde{x}_j^i = \sum_{j \in C} a_j x_j^* + \sum_{t=h-i+1}^{h} a_{l_t} \leq b - z + \sum_{t=h-i+1}^{h} a_{l_t} = b - z + \hat{A}_i^h$$

and also

$$\sum_{j \in C} \tilde{x}_j^i = \sum_{j \in C} x_j^* + \sum_{t=h-i+1}^{h} 1 \leq K - h + i \leq K.$$

It follows that $\tilde{x}^i$ is a feasible solution for the problem $\Theta\binom{z-\hat{A}_i^h}{0}$ with objective value $\Theta\binom{z}{h} - i$, i.e. $\Theta\binom{z-\hat{A}_i^h}{0} \leq \Theta\binom{z}{h} - i$. We conclude that $\max_{i=0,\dots,h}\{\Theta^*\binom{z-\hat{A}_i^h}{0} + i\} \leq \Theta\binom{z}{h}$.

Second we prove that $\max_{i=0,\dots,h}\{\Theta^*\binom{z-\hat{A}_i^h}{0} + i\} \geq \Theta\binom{z}{h}$. Define $M_0 = \{j \in \{l_1, \dots, l_h\} : j \geq k_s + 1\}$. Consider the solution $\hat{x}$ defined as

$$\hat{x}_j = \begin{cases} x_j^* & \text{if } \ j \in C \backslash M_0 \\ 1 & \text{if } \ j \in M_0. \end{cases}$$

The solution $\hat{x}$ satisfies

$$\sum_{j \in C} a_j \hat{x}_j = \sum_{j \in C} a_j x_j^* + \sum_{j \in M_0} a_j \leq b - z + \hat{A}_{|M_0|}^h$$

and also

$$\sum_{j\in C}\hat{x}_j = \sum_{j\in C} x^*_j + \sum_{j\in M_0} 1 \le K - h + |M_0| \le K.$$

It follows that $\hat{x}$ is a feasible solution to the problem $\Theta\binom{z-\hat{A}^h_{|M_0|}}{0}$ with objective value $\Theta\binom{z}{h} - |M_0| = h - |M_0| + s - 1$. We now prove that $\hat{x}$ is an optimal solution to $\Theta\binom{z-\hat{A}^h_{|M_0|}}{0}$. Because $z \in (\hat{A}_h - \lambda + \sum_{j=1}^{s-1} a_{k_j}, \hat{A}_h - \lambda + \sum_{j=1}^{s} a_{k_j}]$, we have that

$$z - \hat{A}^h_{|M_0|} \le \hat{A}_h - \lambda + \sum_{j=1}^{s} a_{k_j} - \hat{A}^h_{|M_0|} = \sum_{j=1}^{h-|M_0|} a_{l_j} - \lambda + \sum_{j=1}^{s} a_{k_j} = A_{k_s} - \lambda.$$

The last equality holds because $\{l_1, \ldots, l_{h-|M_0|}\} \cup \{k_1, \ldots, k_s\} = \{1, \ldots, k_s\}$. Similarly, we can show that $z - \hat{A}^h_{|M_0|} > A_{k_s-1} - \lambda$. This follows from (22) that $\Theta\binom{z-\hat{A}^h_{|M_0|}}{0} = \Theta^*\binom{z-\hat{A}^h_{|M_0|}}{0} = k_s - 1 = h - |M_0| + s - 1$. It implies that $\hat{x}$ is an optimal solution for $\Theta^*\binom{z-\hat{A}^h_{|M_0|}}{0}$ and therefore $\Theta^*\binom{z-\hat{A}^h_{|M_0|}}{0} = \Theta\binom{z}{h} - |M_0|$. As a consequence, we have $\max_{i=0,\ldots,h}\{\Theta^*\binom{z-\hat{A}^h_i}{0} + i\} \ge \Theta\binom{z}{h}$. □ □

Note that a closed form expression for $\Theta\binom{z}{h}$ can easily be derived. However, expressing $\Theta\binom{z}{h}$ as in Theorem 22 has advantages when deriving a valid superadditive approximation; see proof of Proposition 23. Observe that the lifting function Θ on $[0, b] \times \{0, 1\}$ is identical to Θ^* after replacing a_l with a_{l_1}. This suggests that a good approximation of Θ can be obtained by setting $\theta\binom{z}{h}$ equal to $\theta^*\binom{z}{I}$ for $h = I \in \{0, 1\}$ after replacing a_l by a_{l_1}. The function $\theta\binom{z}{h}$ over $[0, b] \times \{2, \ldots, K\}$ is then built from the value of θ over $[0, b] \times \{0, 1\}$. The above construction yields a valid approximation of Θ that is presented next.

Proposition 23. *The function*

$$\theta\binom{z}{h} = \begin{cases} \theta^*\binom{z}{0} & \text{if} \quad h = 0 \\ \begin{cases} \theta^*\binom{z}{0} & \text{if} \quad 0 \le z < a_{l_1} \\ \max\{\theta^*\binom{z-a_{l_1}}{0} + 1, \theta^*\binom{z}{0}\} & \text{if} \quad a_{l_1} \le z \le b \end{cases} & \text{if} \quad h = 1 \\ \sup\limits_{\{z=\sum_{j=1}^h z_j : z_j \ge 0, j=1,\ldots,h\}} \sum_{j=1}^{h} \theta\binom{z_j}{1} & \text{if} \quad h = 2, \ldots, K \end{cases} \tag{32}$$

is a valid superadditive approximation of Θ *over* $(z, h) \in [0, b] \times \{0, 1, \ldots, K\}$.

Proof. The proof is in two steps. We first show that θ is valid and then show that θ satisfies the conditions of Corollary 21. The proof that θ is valid is by induction over h. For $h = 0$ and $h = 1$, the fact that $\theta\binom{z}{h} \le \Theta\binom{z}{h}$ for $z \in [0, b]$ was proven in Theorem 15.

Assume that we have already proven that $\theta\binom{z}{h} \le \Theta\binom{z}{h}$ for $z \in [0, b]$ and $h = 1, \ldots, t$. We want to show that this result still holds for $h = t + 1$. For $z \in [0, b]$, we define

$$R_1 := \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \ge 0, j=1,\ldots,t, z_{t+1}\in[0,a_{l_1})\}} \sum_{j=1}^{t+1} \theta\binom{z_j}{1} \tag{33}$$

and

$$R_2 := \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \ge a_{l_1}, j=1,\ldots,t+1\}} \sum_{j=1}^{t+1} \theta\binom{z_j}{1}. \tag{34}$$

Clearly, $\theta\binom{z}{t+1} \le \max\{R_1, R_2\}$. We now prove that $R_1 \le \Theta\binom{z}{t+1}$ and $R_2 \le \Theta\binom{z}{t+1}$. For R_1, we observe first that $\theta\binom{z_{t+1}}{1} = \theta\binom{z_{t+1}}{0}$ since $z_{t+1} \in [0, a_{l_1})$. We have

$$\begin{aligned}
R_1 \leq & \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \geq 0, j=1,\ldots,t, z_{t+1} \in [0, a_{l_1})\}} \{\sum_{j=1}^{t-1} \theta\binom{z_j}{1} + \theta\binom{z_t + z_{t+1}}{1}\} \\
\leq & \sup_{\{z=\sum_{j=1}^{t} \tilde{z}_j : \tilde{z}_j \geq 0, j=1,\ldots,t\}} \{\sum_{j=1}^{t} \theta\binom{\tilde{z}_j}{1}\} \\
= & \theta\binom{z}{t} \leq \Theta\binom{z}{t} \leq \Theta\binom{z}{t+1}
\end{aligned}$$

where the first inequality holds because of the superadditivity of $\theta\binom{z}{h}$ over $[0, b] \times \{0, 1\}$. For R_2, we have

$$\begin{aligned}
R_2 \leq & \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \geq a_{l_1}, j=1,\ldots,t+1\}} \{\sum_{j=1}^{t+1} (\theta\binom{z_j - a_{l_1}}{0} + 1)\} \\
\leq & \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \geq a_{l_1}, j=1,\ldots,t+1\}} \{\theta\binom{\sum_{j=1}^{t+1} z_j - (t+1)a_{l_1}}{0} + t + 1\} \\
= & \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \geq a_{l_1}, j=1,\ldots,t+1\}} \{\theta\binom{z - (t+1)a_{l_1}}{0} + t + 1\} \\
\leq & \theta\binom{z - \hat{A}_{t+1}^{t+1}}{0} + t + 1 \leq \Theta\binom{z - \hat{A}_{t+1}^{t+1}}{0} + t + 1 \leq \Theta\binom{z}{t+1}.
\end{aligned}$$

where the third inequality follows from the fact that θ is non-decreasing and $\hat{A}_{t+1}^{t+1} \leq (t+1)a_{l_1}$, and the last inequality follows from Theorem 22.

We now prove that θ satisfies the conditions of Corollary 21. Condition (28) has already been established in Theorem 15. Condition (30) is satisfied because of the way $\theta\binom{z}{h}$ is defined for $h \geq 2$. We therefore only need to prove condition (29). Let $y, z, y + z \in [0, b]$ and $h \in \{2, \ldots, K\}$. We have

$$\begin{aligned}
\theta\binom{y}{0} + \theta\binom{z}{h} = & \theta\binom{y}{0} + \sup_{\{z=\sum_{j=1}^{h} z_j : z_j \geq 0, j=1,\ldots,h\}} \{\sum_{j=1}^{h} \theta\binom{z_j}{1}\} \\
= & \sup_{\{z=\sum_{j=1}^{h} z_j : z_j \geq 0, j=1,\ldots,h\}} \{\sum_{j=1}^{h} \theta\binom{z_j}{1} + \theta\binom{y}{0}\} \\
\leq & \sup_{\{z=\sum_{j=1}^{h} z_j : z_j \geq 0, j=1,\ldots,h\}} \{\sum_{j=1}^{h-1} \theta\binom{z_j}{1} + \theta\binom{z_h + y}{1}\} \\
\leq & \sup_{\{y+z=\sum_{j=1}^{h} \tilde{z}_j : \tilde{z}_j \geq 0, j=1,\ldots,h\}} \{\sum_{j=1}^{h} \theta\binom{\tilde{z}_j}{1}\} = \theta\binom{y + \tilde{z}}{h}
\end{aligned}$$

where the first inequality holds because of the superadditivity of $\theta\binom{z}{h}$ for $(z, h) \in [0, b] \times \{0, 1\}$. □ □

Clearly, deriving the value of $\theta\binom{z}{h}$ for $h \geq 2$ from its value for $h = 0$ and $h = 1$ significantly reduces the amount of work to be done to derive a superadditive approximation. Furthermore, we show next in Theorem 24 that the approximation derived in this manner is strong. The proof of Theorem 24 follows that of Theorem 17. The only difference is that the coefficient a_l must be replaced with a_{l_1} in the definition of $\Theta\binom{z}{1}$ and $\theta\binom{z}{1}$.

Theorem 24. *The function θ is a non-dominated and maximal superadditive approximation of Θ over $[0, b] \times \{0, 1\}$.* □

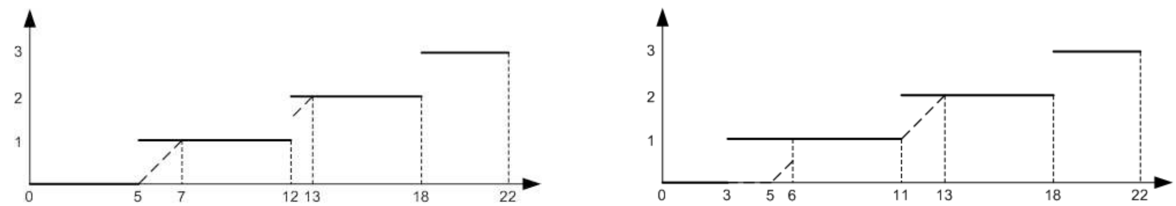
We observe again that $\theta\binom{z}{h}$ has the form described in Theorem 11 if we set $a^* := a_{l_1}$. We next illustrate the strength of lifted generalized cover inequalities. The example is obtained from Gu et al. [14] page 122 by adding an cardinality constraint.

Example 3. *Consider*

$$PS_1 = conv\{x \in \{0,1\}^5 : 8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 \leq 22,\ x_3 + x_4 + x_5 \leq 2\}.$$

Clearly, $C = \{1,2,3,4\}$ *is a generalized cover with* $\lambda = \sum_{j=1}^{4} a_j - 22 = 25 - 22 = 3$ *and* $a_{l_1} = 6$. *Figure 1 shows both the exact lifting function* Θ *and the superadditive approximation* θ *we propose.*

Assume that we reintroduce x_5 *through sequence independent lifting. Let* α_5 *be its lifting coefficient. Using the traditional superadditive lifting function from [14], we would obtain* $\alpha_5 = \theta\binom{6}{0} = \frac{1}{2}$. *Using our two-dimensional superadditive approximation, we obtain* $\alpha_5 = \theta\binom{6}{1} = 1$. *Furthermore, the resulting inequality* $x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$ *is facet-defining for* PS_1. □



(a) $\Theta\binom{z}{0}$ and $\theta\binom{z}{0}$ (b) $\Theta\binom{z}{1}$ and $\theta\binom{z}{1}$

Figure 1: Exact lifting function and superadditive lifting function of Example 3

4.3.2 Case *ii*: $C^+ \neq C$ and $|C^+| \leq K - 1$

Since $|C^+| \leq K - 1$, the lifting function $\Theta\binom{z}{h}$ is identical to $\Theta^*\binom{z}{0}$ for $(z,h) \in [0,b] \times \{0,1\}$. Therefore, we can easily build a superadditive approximation of Θ based on θ^*.

Theorem 25. *The function* $\theta\binom{z}{h} = \theta^*\binom{z}{0}$ *for* $(z,h) \in [0,b] \times \{0,\ldots,K\}$ *is a valid superadditive approximation of* $\Theta\binom{z}{h}$ *that is non-dominated and maximal over* $[0,b] \times \{0,1\}$.

Proof. From Corollary 10, we conclude that for $z \in [0,b]$, $\Theta\binom{z}{K} \geq \Theta\binom{z}{K-1} \geq \cdots \geq \Theta\binom{z}{1} \geq \Theta\binom{z}{0} = \Theta^*\binom{z}{0}$. It follows that $\Theta\binom{z}{h} \geq \Theta^*\binom{z}{0} \geq \theta^*\binom{z}{0} = \theta\binom{z}{h}$ for $z \in [0,b]$ and $h \in \{0,\ldots,K\}$. Furthermore, for $z_1, z_2 \in [0,b]$ and $h_1, h_2 \in \{0,\ldots,K\}$ such that $z_1 + z_2 \in [0,b]$ and $h_1 + h_2 \leq K$, we have

$$\theta\begin{pmatrix} z_1 \\ h_1 \end{pmatrix} + \theta\begin{pmatrix} z_2 \\ h_2 \end{pmatrix} = \theta^*\begin{pmatrix} z_1 \\ 0 \end{pmatrix} + \theta^*\begin{pmatrix} z_2 \\ 0 \end{pmatrix} \leq \theta^*\begin{pmatrix} z_1 + z_2 \\ 0 \end{pmatrix} = \theta\begin{pmatrix} z_1 + z_2 \\ h_1 + h_2 \end{pmatrix},$$

showing that the conditions of Corollary 21 are satisfied. The non-dominance and maximality of θ follows directly from the fact that $\Theta\binom{z}{1} = \Theta\binom{z}{0} = \Theta^*\binom{z}{0}$ and the fact that $\theta^*\binom{z}{0}$ is a non-dominated and maximal approximation of $\Theta^*\binom{z}{0}$; see Theorem 16. □ □

Again, $\theta\binom{z}{h}$ has the form described in Theorem 11 if we set $a^* := a_1$.

4.3.3 Case *iii*: $C = C^+$ and $|C| = K + 1$

In this case, $C = C^+ = \{1,\ldots,K+1\}$. Using the similar argument of Theorem 22, we derive an expression for the exact lifting function Θ in Corollary 26 that better describes the relations between $\Theta\binom{z}{h}$ for $h \geq 1$ and $\Theta\binom{z}{0}$. This result will be used to derive a superadditive approximation of $\Theta\binom{z}{h}$ in Theorem 27.

Corollary 26. *For $(z,h) \in [0,b] \times \{0,\ldots,K\}$,*

$$\Theta\binom{z}{h} = \begin{cases} \Theta^*\binom{z}{0} & \text{if} \quad h=0 \\ \begin{cases} h & \text{if} \quad 0 \le z \le A_{h+1}-\lambda \\ h+1 & \text{if} \quad A_{h+1}-\lambda < z < A_{h+1} \\ \max_{i=0,\ldots,h+1}\{\Theta^*\binom{z-\hat{A}_i^{h+1}}{0}+i\} & \text{if} \quad A_{h+1} \le z \le b \end{cases} & \text{if} \quad h=1,\ldots,K \end{cases} \tag{35}$$

where $\hat{A}_i^h = \sum_{j=h-i+1}^{h} a_j$ for $h=1,\ldots,K+1$ and $0 \le i \le h$. □

Theorem 27. *The function*

$$\theta\binom{z}{h} = \begin{cases} \theta^*\binom{z}{0} & \text{if } h=0 \\ \theta^*\binom{z}{1} \quad \text{replacing } a_l \text{ with } a_2 & \text{if } h=1 \\ \sup_{\{z=\sum_{j=1}^{h} z_j, z_j \ge 0, j=1,\ldots,h\}} \{\sum_{j=1}^{h} \theta\binom{z_j}{1}\} & \text{if } h \in \{2,\ldots,K\} \end{cases} \tag{36}$$

is a valid superadditive approximation of Θ over $[0,b] \times \{0,1,\ldots,K\}$.

Proof. The proof that $\theta\binom{z}{h}$ satisfies the conditions of Corollary 21 is identical to that of Theorem 23. We now prove that $\Theta\binom{z}{h} \ge \theta\binom{z}{h}$ for $(z,h) \in [0,b] \times \{0,\ldots,K\}$. We consider the following three cases.

(1) $h=0$. Clearly, $\Theta\binom{z}{0} = \Theta^*\binom{z}{0}$ for $z \in [0,b]$ and so $\theta\binom{z}{0} \le \Theta\binom{z}{0}$.

(2) $h=1$. For $z \in [a_1+a_2, b]$, it follows from the fact that $a_1 \ge \cdots \ge a_{K+1}$ that $\Theta\binom{z}{1} = \max\{\Theta^*\binom{z-a_1-a_2}{0}+2, \Theta^*\binom{z-a_2}{0}+1, \Theta^*\binom{z}{0}\} = \Theta^*\binom{z}{0}$. Furthermore, since for $z \in [a_1+a_2, b]$, it follows from (25), that $\theta\binom{z}{1} = \theta^*\binom{z}{0}$, we conclude that $\theta\binom{z}{1} \le \Theta\binom{z}{1}$. Therefore, it is sufficient to show that $\theta\binom{z}{1} \le \Theta\binom{z}{1}$ for $z \in [0, a_1+a_2)$. Because $\Theta\binom{z}{1}$ is a step function and because $\theta^*\binom{z}{1}$ is non-decreasing, it is sufficient to verify that $\theta\binom{a_1+a_2-\lambda}{1} \le \Theta\binom{a_1+a_2-\lambda}{1}$ and $\theta\binom{a_1+a_2-\epsilon}{1} \le \Theta\binom{a_1+a_2-\epsilon}{1}$ for all sufficiently small positive ϵ to prove that $\theta\binom{z}{1} \le \Theta\binom{z}{1}$ for $z \in [0, a_1+a_2)$. For $\tilde{z} = a_1+a_2-\lambda$, it follows from (15) and (35) that

$$\theta\binom{\tilde{z}}{1} = \max\{\theta^*\binom{\tilde{z}}{0}, \theta^*\binom{\tilde{z}-a_2}{0}+1\} = \max\{1,1\} = 1 = \Theta\binom{\tilde{z}}{1}.$$

For $\hat{z} = a_1+a_2-\epsilon$, we have that

$$\theta\binom{\hat{z}}{1} = \max\{\theta^*\binom{\hat{z}}{0}, \theta^*\binom{\hat{z}-a_2}{0}+1\} = \max\{2,2\} = 2 = \Theta\binom{\hat{z}}{1}$$

since $a_1-\lambda < \hat{z}-a_2 \le a_1+a_2-\lambda$.

(3) $h \ge 2$. The proof is by induction and is similar to that of Proposition 23. Assume that the result holds for $h=1,\ldots,t$. We define

$$R_1 := \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \ge 0, j=1,\ldots,t, z_{t+1} \in [0,a_2)\}} \sum_{j=1}^{t+1} \theta\binom{z_j}{1} \tag{37}$$

and

$$R_2 := \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \ge a_2, j=1,\ldots,t+1\}} \sum_{j=1}^{t+1} \theta\binom{z_j}{1} \tag{38}$$

and observe that $\theta\binom{z}{t+1} \leq \max\{R_1, R_2\}$. Similar to Proposition 23, it can be proven that $R_1 \leq \Theta\binom{z}{t}$. For R_2, we have

$$\begin{aligned} R_2 \leq & \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \geq a_2, j=1,\ldots,t+1\}} \{\sum_{j=1}^{t+1}(\theta\begin{pmatrix} z_j - a_2 \\ 0 \end{pmatrix} + 1)\} \\ \leq & \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \geq a_2, j=1,\ldots,t+1\}} \{\theta\begin{pmatrix} \sum_{j=1}^{t+1} z_j - (t+1)a_2 \\ 0 \end{pmatrix} + t + 1\} \\ = & \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \geq a_2, j=1,\ldots,t+1\}} \{\theta\begin{pmatrix} z - (t+1)a_2 \\ 0 \end{pmatrix} + t + 1\} \\ \leq & \theta\begin{pmatrix} z - \hat{A}_{t+1}^{t+2} \\ 0 \end{pmatrix} + t + 1 \leq \Theta\begin{pmatrix} z - \hat{A}_{t+1}^{t+2} \\ 0 \end{pmatrix} + t + 1 \leq \Theta\begin{pmatrix} z \\ t+1 \end{pmatrix}. \end{aligned}$$

where the third inequality follows from the fact that θ is non-decreasing and $\hat{A}_{t+1}^{t+2} \leq (t+1)a_2$ and the last inequality holds because of Corollary 26. □

□

The following theorem show that the proposed superadditive approximation is strong.

Theorem 28. *The function θ is a non-dominated and maximal superadditive approximation of Θ over $[0, b] \times \{0, 1\}$.*

Proof. The proof of non-dominance over $[0, b] \times \{0, 1\}$ is similar to that of Theorem 24. Next, we show θ is maximal. Let E to denote the maximal set of Θ. We show that if $\theta\binom{z'}{I} < \Theta\binom{z'}{I}$ for $(z', I) \in [0, b] \times \{0, 1\}$, then $(z', I) \notin E$. Since $\theta\binom{z}{0} = \theta^*\binom{z}{0}$ for $z \in [0, b]$ and $\theta^*\binom{z}{0}$ was shown to be maximal in Theorem 16, we may assume that $I = 1$.

(1) $a_2 \leq a_1 - \lambda$**.** From Theorem 13 and Theorem 15, we know that $\theta\binom{z}{0} = \theta^*\binom{z}{0} = \Theta^*\binom{z}{0} = \Theta\binom{z}{0}$ for $z \in [0, b]$. From (36), we derive that $\theta\binom{z}{1} = 0 \leq \Theta\binom{z}{1}$ for $z \in [0, a_2)$, $\theta\binom{z}{1} = \Theta\binom{z}{1}$ for $z \in [a_2, a_1 + a_2 - \lambda]$ and $\Theta\binom{z}{1} = \Theta^*\binom{z}{0}$ for $z \in (a_1 + a_2 - \lambda, b]$. Therefore, it is sufficient to consider $z' \in [0, a_2)$ such that $\Theta_2\binom{z'}{1} \neq 1$. Define now $z = a_1 + a_2 - \lambda$. Clearly, $z - z' > a_1 - \lambda$. Therefore, $\Theta\binom{z}{1} = \Theta\binom{z-z'}{0} = 1$. We conclude that

$$\Theta_2\begin{pmatrix} z' \\ 1 \end{pmatrix} = \min\{\Theta\begin{pmatrix} z' \\ 1 \end{pmatrix}, \Theta\begin{pmatrix} z \\ 1 \end{pmatrix} - \Theta\begin{pmatrix} z - z' \\ 0 \end{pmatrix}\} = \min\{1, 0\} = 0 < \Theta\begin{pmatrix} z' \\ 1 \end{pmatrix}.$$

This shows that $(z', 1) \notin E$.

(2) $a_2 > a_1 - \lambda$**.** From Theorem 15, Theorem 27 and the fact that $a_2 > a_1 - \lambda$, we know that $\theta\binom{z}{1} = \theta\binom{z}{0} = \theta^*\binom{z}{0}$ for $z \in [0, b]$. Because $\theta\binom{z'}{1} < \Theta\binom{z'}{1}$, we either have $z' \in [0, a_2)$ or $z' \in (A_j - \lambda, A_j - \lambda + \rho_j)$ where $j \geq 2$ and $\rho_j > 0$. If $z' \in [0, a_2)$, define $z = a_1 + a_2 - \lambda$. Then, $z - z' > a_1 - \lambda$ and $\Theta\binom{z}{1} = \Theta\binom{z-z'}{0} = 1$. We obtain

$$\Theta_2\begin{pmatrix} z' \\ 1 \end{pmatrix} = \min\{\Theta\begin{pmatrix} z' \\ 1 \end{pmatrix}, \Theta\begin{pmatrix} z \\ 1 \end{pmatrix} - \Theta\begin{pmatrix} z - z' \\ 0 \end{pmatrix}\} = 0 < \Theta\begin{pmatrix} z' \\ 1 \end{pmatrix}.$$

This shows that $(z', 1) \notin E$. If $z' \in (A_j - \lambda, A_j - \lambda + \rho_j)$, we define $z = A_{j+1} - \lambda$. Then, $z - z' > a_1 - \lambda$. Also, we have $\Theta\binom{z'}{1} = \Theta\binom{z}{1} = j$ and $\Theta\binom{z-z'}{0} = 1$. We conclude that

$$\Theta_2\begin{pmatrix} z' \\ 1 \end{pmatrix} = \min\{\Theta\begin{pmatrix} z' \\ 1 \end{pmatrix}, \Theta\begin{pmatrix} z \\ 1 \end{pmatrix} - \Theta\begin{pmatrix} z - z' \\ 0 \end{pmatrix}\} = j - 1 < \Theta\begin{pmatrix} z' \\ 1 \end{pmatrix},$$

which shows that $(z', 1) \notin E$. □

□

Again, we observe that $\theta\binom{z}{h}$ has the form presented in Theorem 11 if $a^* := a_2$.

4.3.4 Case iv: $C = C^+$ and $|C| \le K$

When $|C| \le K$, $\Theta\binom{z}{h} = \Theta^*\binom{z}{0}$ for $(z,h) \in [0,b] \times \{0,1\}$. Similar to the discussion in Section 4.3.2, $\theta\binom{z}{h} = \theta^*\binom{z}{0}$ for all $(z,h) \in [0,b] \times \{0,\dots,K\}$ is a strong valid superadditive approximation of Θ.

Theorem 29. *The function $\theta\binom{z}{h} = \theta^*\binom{z}{0}$ for $(z,h) \in [0,b] \times \{0,\dots,K\}$ is a valid superadditive approximation of Θ that is non-dominated and maximal over $[0,b] \times \{0,1\}$.* □

Again, $\theta\binom{z}{h}$ has the form described in Theorem 11 if we set $a^* := a_1$.

5 Superadditive Approximation of Lifting Functions in PS_1

In this section, we discuss how to derive strong lifted inequalities for PS_1 from generalized cover inequalities. We first describe in Section 5.1 the exact lifting function Π of generalized cover inequalities and present some properties that are helpful in describing its structure. Then, we derive in Section 5.2 a lower approximation π that can be used for sequence independent lifting.

5.1 Exact Lifting Function of the Generalized Cover Inequality

Again, because this section focuses on the case where $r = 1$, we use the notation N^+, N^-, C^+, C^-, η and K to represent $N_1^+, N_1^-, C_1^+, C_1^-, \eta_1$ and K_1 respectively. We also use EZ to denote $EZ \cap N_1$. Using this notation, the parameter η of a generalized cover is given by

$$\eta = |K| - |C^+| + |C^-| + |EZ|. \tag{39}$$

The lifting problems associated with generalized cover inequalities in PS_1 are given by

$$\begin{aligned} \Pi_i\begin{pmatrix} z \\ h \end{pmatrix} = \min |C| - 1 - \sum_{j \in C} x_j - \sum_{1 \le j \le i-1} \alpha_j x_j \\ \text{s.t.} \sum_{j \in C} a_j x_j + \sum_{j \in EZ} a_j x_j + \sum_{1 \le j \le i-1} a_j x_j \le b - z \\ \sum_{j \in C^+} x_j - \sum_{j \in C^-} x_j - \sum_{j \in EZ} x_j + \sum_{1 \le j \le i-1} I_j x_j \le K - h \end{aligned} \tag{40}$$

where $z \in [0,b], h \in \mathbb{Z}$ and I_j is the coefficient of x_j in the only cardinality constraint of PS_1.

Because PS_1 is not an independence system, the lifting function of the generalized cover inequality for PS_1 is more difficult to obtain. In fact, we observe that the shape of the exact lifting function $\Pi\binom{z}{h}$ is strongly influenced by both the knapsack constraint (2) and the cardinality constraint (3). This is unlike PS_1^+ where the shape of the exact lifting function was mostly dictated by the knapsack constraint.

To better identify which set C^+, C^-, C^0 and EZ that a variable x_j belongs to, we denote the coefficients a_j for $j \in C^+$ as $a_1^+, \dots, a_{|C^+|}^+$. We denote the coefficients a_j for $j \in C^-$, $j \in C^0$ and $j \in EZ$ similarly. We also assume without loss of generality that all the variables are sorted in non-increasing order of their knapsack constraint coefficients within each of these sets, i.e. $a_i^s \ge a_j^s$ for $s \in \{0, -, +, EZ\}$ and $i < j$. We define

$$\lambda := \sum_{j \in C} a_j + \sum_{j \in EZ} a_j - b. \tag{41}$$

From Corollary 9, it is easy to see that the lifting function of the generalized cover inequality is a step function with increments of height exactly equal to 1. Therefore, it is sufficient to know the points at which the function increases to completely characterize it. This motivates the introduction of the notation $b_h(i) - \lambda$ to represent the points where $\Pi\binom{z}{h}$ jumps from the value $i-1$ to the value i; see Figure 2 for an illustration. Let $A_k^s = \sum_{1 \le i \le k} a_i^s$ for $s \in \{0, +, -, EZ\}$. It can be verified that for a

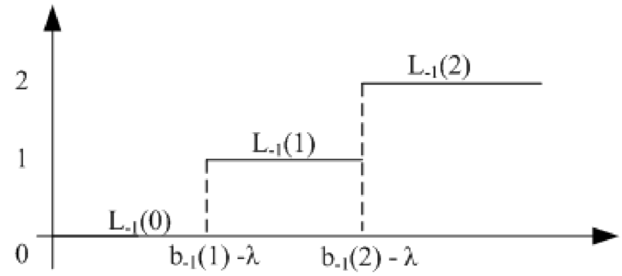


Figure 2: Lifting function $\Pi\binom{z}{h}$

given generalized cover inequality

$$b_h(i) := \max\{A_k^+ + A_m^0 + A_q^- + A_p^{EZ} : k + m + q = i, q + p - k \leq \eta - h\} \tag{42}$$

for $h \in \{-K^-, \dots, K^+\}$ and for $i \in \mathbb{Z}$. We set $b_h(i) = -\infty$ if (42) does not have a feasible solution. In (42), k represents the number of variables x_j with $j \in C^+$ to set to 0, m represents the number of variables x_j with $j \in C^0$ to set to 0, q represents the number of variables x_j with $j \in C^-$ to set to 0 and p represents the number of variables x_j with $j \in EZ$ to set to 0 in order to obtain the begining of the i^{th} step of $\Pi\binom{z}{h}$. Note that the number of parameters of $b_h(i)$ is polynomial in n, and the computation of $b_h(i)$ is polynomial in n for any given h. In fact, a dynamic programming algorithm is given in Zeng [31] that computes $b_h(i)$ in $O(n)$ for any given $h \in \{-K^-, \dots, K^+\}$ (assuming that the coefficients a_j are sorted).

Based on the parameter $b_h(i)$, we now define the function $\Pi\binom{z}{h}$. From (42), we see that in order for $\Pi\binom{z}{h} \geq i$ to hold for $i \in \mathbb{Z}_+$, we need to have $b - z < \sum_{j \in C \cup EZ} a_j - b_h(i) = b + \lambda - b_h(i)$, or equivalently $z > b_h(i) - \lambda$. Also, it is easily seen that $\Pi\binom{z}{h} \geq -1$ for all $(z, h) \in [0, b] \times \{-K^-, \dots, K^+\}$. For $h \geq 1$, the function $\Pi\binom{z}{h}$ might not have a step of height i for $z \geq 0$. We therefore define $i_0 - 1$ to be the lowest height of a step in $\Pi\binom{z}{h}$ for $z \geq 0$. More precisely, for a given h, $i_0(h) = \arg \min_{i \in \{0, \dots, |C|\}} \{b_h(i) - \lambda \geq 0\}$. Combining these observations, we obtain the following result.

Theorem 30. *The exact lifting function of the generalized cover inequality is*

$$\Pi\binom{z}{h} = \begin{cases} i_0(h) - 1 & \text{if} \quad 0 \leq z \leq b_h(i_0(h)) - \lambda \\ i - 1 & \text{if} \quad b_h(i-1) - \lambda < z \leq b_h(i) - \lambda \end{cases} \tag{43}$$

for $h \in \{-K^-, \dots, K^+\}$ and for $i \in \{i_0(h) + 1, \dots, |C|\}$. Furthermore, $\Pi\binom{z}{h}$ is non-decreasing over $[0, b] \times \{-K^-, \dots, K^+\}$. □

Note that, when $b_h(0) - \lambda < 0$ for $h = -K^-$, Theorem 30 implies that $\Pi\binom{0}{h} \geq 0$ for $h = -K^-, \dots, K^+$. This situation occurs for example when the given generalized cover inequality is also a cover inequality for the knapsack constraint.

An important characteristic of the function Π to consider when deriving a superadditive approximation is the length of the intervals over which Π is constant. To characterize this length, we introduce the notation $L_h(i) = b_h(i+1) - \lambda - \max\{b_h(i) - \lambda, 0\}$ if $i \geq 0$, and $L_h(-1) = \max\{b_h(0) - \lambda, 0\}$; see Figure 2. Note that, except for the first step of $\Pi\binom{z}{h}$ with $z \geq 0$, the length of the step of height i is equal to $L_h(i)$. It can be verified that for cover inequalities arising from a knapsack constraint, the intervals of the lifting function are non-increasing (except for the first one). This property was used by Gu et al. [14] to derive a superadditive approximation. Unfortunately, this property does not hold for the exact lifting function $\Theta\binom{z}{h}$ of the generalized cover inequality if $C^- \neq \emptyset$ or $EZ \neq \emptyset$. A weaker property can nevertheless be established. It is given in Proposition 31 and proven in Zeng [31].

Proposition 31. *$L_h(i) \geq L_h(i+2)$ for all $i \in \mathbb{Z}_+$ and $h \in \{-K^-, \dots, K^+\}$ for which $b_h(i) - \lambda \geq 0$ and $b_h(i+3)$ is finite.* □

The exact lifting function $\Pi\binom{z}{h}$ is less structured than the lifting functions studied in Section 4. This is quite a disadvantage as existing superadditive approximations of lifting functions are typically derived from well-structured functions; see Atamtürk [2] and Gu et al. [14] for examples. To overcome these difficulties, we propose to derive the approximation in two steps. First we approximate the lifting function Π slightly to obtain a function that is easier to describe. Then, given this well-structured approximation, we derive a strong lower superadditive function that can be used for sequence independent lifting.

5.2 Sequence Independent Lifting in PS_1

In this section, we first give a set of sufficient conditions that guarantee sequence independent lifting. We then construct an approximation of $\Pi\binom{z}{h}$ that satisfy these conditions. Similar to the notation used for PS_1^+, we define $K^+ = \max\{|S^+| : S^+ \subseteq (N^+ \backslash C^+), S_1 \subseteq C^-, S_2 \subseteq EZ, \sum_{j\in S^+} a_j + \sum_{j\in S_1\cup S_2} a_j \le b, |S^+| - |S_1 \cup S_2| \le K\}$, and $K^- = \max\{|S^-| : S^- \subseteq N^- \backslash (C^- \cup EZ), \sum_{j\in S^-} a_j \le b\}$.

Corollary 32. *For PS_1, lifting is sequence independent if*

$$\Pi\binom{y}{0} + \Pi\binom{z}{h} \le \Pi\binom{y+z}{h} \quad \forall h \in \{-K^-, \ldots, K^+\} \tag{44}$$

$$\Pi\binom{y}{1} + \Pi\binom{z}{h} \le \Pi\binom{y+z}{h+1} \quad \forall h \in \{-K^-, \ldots, K^+ - 1\} \tag{45}$$

$$\Pi\binom{y}{-1} + \Pi\binom{z}{h} \le \Pi\binom{y+z}{h-1} \quad \forall h \in \{-K^- + 1, \ldots, K^+\} \tag{46}$$

where $y, z, y+z \in [0, b]$. □

Because the seed inequality has the form of a cover inequality, $\Pi\binom{0}{-K^-}$ is either equal to 0 or to -1. We use different strategies to derive superadditive approximations of Π for these two cases. For the case $\Pi\binom{0}{-K^-} = 0$, the proposed superadditive approximation generalizes that of the $0-1$ knapsack problem introduced by Gu et al. [14].

5.2.1 Building a Superadditive Approximation of Π when $\Pi\binom{0}{-K^-} = 0$

The case $\Pi\binom{0}{-K^-}$ includes all situations where the generalized cover inequality is also a cover inequality for the knapsack constraint. We first show that there are non-dominated valid superadditive approximations of Π that satisfy $\pi\binom{z}{h} = \pi\binom{z}{-K^-}$ for $(z, h) \in [0, b] \times \{-K^-, \ldots, K^+\}$. We then use this result to derive a superadditive approximation π.

Proposition 33. *Assume that $\Pi\binom{0}{-K^-} = 0$. Let $\phi(z)$ be a non-dominated approximation of $\Pi\binom{z}{-K^-}$ over $z \in [0, b]$ such that*

$$\phi(y) + \phi(z) \le \phi(y + z) \tag{47}$$

for $y, z, y + z \in [0, b]$ and $\phi(z) \ge 0$ for $z \in [0, b]$. Then the function

$$\pi\binom{z}{h} = \phi(z) \tag{48}$$

is a valid non-dominated superadditive approximation of Π for $(z, h) \in [0, b] \times \{-K^-, \ldots, K^+\}$.

Proof. The validity of $\pi\binom{z}{h}$ for $(z, h) \in [0, b] \times \{-K^-, K^+\}$ follows directly from the fact that $\pi\binom{z}{h} = \phi(z) \le \Pi\binom{z}{-K^-}$ and the fact that $\Pi\binom{z}{h}$ is non-decreasing over $[0, b] \times \{-K^-, \ldots, K^+\}$. The conditions in Corollary 32 can also be verified easily using (47). Next, we prove that $\pi\binom{z}{h}$ is non-dominated.

Assume by contradiction that there is another superadditive approximation $g'\binom{z}{h}$ for $\Pi\binom{z}{h}$ such that $g'\binom{z}{h} \geq \pi\binom{z}{h}$ for all $(z,h) \in [0,b] \times \{-K^-,\ldots,K^+\}$ and such that $g'\binom{z_0}{h_0} > \pi\binom{z_0}{h_0}$ for some (z_0,h_0). We consider two situations based on whether $g'\binom{z}{h}$ is non-decreasing over $[0,b] \times \{-K^-,\ldots,K^+\}$.

Assume first that $g'\binom{z}{h}$ is non-decreasing. Because the generalized cover inequality (12) is facet-defining, we have $\Pi\binom{0}{0} = 0$. Also, because $\Pi\binom{0}{-K^-} = 0$ and by the fact that $\Pi\binom{z}{h}$ is non-decreasing over h, we have $0 = \Pi\binom{0}{-K^-} = \cdots = \Pi\binom{0}{-1} = \Pi\binom{0}{0} = 0$. It follows that $0 = \phi(0) = \pi\binom{0}{-1} \leq g'\binom{0}{-1} \leq \Pi\binom{0}{-1} = 0$, i.e. $g'\binom{0}{-1} = 0$. Because $g'\binom{z}{h}$ satisfies the conditions of Corollary 32, this implies that $g'\binom{z}{h} \leq g'\binom{z}{-K^-} \leq \Pi\binom{z}{-K^-}$ for $(z,h) \in [0,b] \times \{-K^-,\ldots,K^+\}$. Because $g'\binom{z}{h}$ is non-decreasing over $[0,b] \times \{-K^-,\ldots,K^+\}$, we have $g'\binom{z}{h} = g'\binom{z}{-K^-}$. Because

$$g'\begin{pmatrix} y \\ -K^- \end{pmatrix} + g'\begin{pmatrix} z \\ -K^- \end{pmatrix} = g'\begin{pmatrix} y \\ 0 \end{pmatrix} + g'\begin{pmatrix} z \\ 0 \end{pmatrix} \leq g'\begin{pmatrix} y+z \\ 0 \end{pmatrix} = g'\begin{pmatrix} y+z \\ -K^- \end{pmatrix},$$

and $g'\binom{z_0}{-K^-} = g'\binom{z_0}{h_0} > \phi(z_0)$, we conclude that $g'\binom{z}{-K^-}$ is a valid approximation of $\Pi\binom{z}{-K^-}$ that satisfies (47) and dominates $\phi(z)$, which is the desired contradiction.

In the second situation, there exist two points (z_1,h_1) and $(z_1+\epsilon, h_1+\mu)$ such that $g'\binom{z_1}{h_1} > g'\binom{z_1+\epsilon}{h_1+\mu}$, $\epsilon \geq 0$, $\mu \in \{0,1\}$ and $\epsilon + \mu > 0$. Because $g'\binom{z}{h}$ is superadditive, we have

$$g'\begin{pmatrix} \epsilon \\ \mu \end{pmatrix} \leq g'\begin{pmatrix} z_1+\epsilon \\ h_1+\mu \end{pmatrix} - g'\begin{pmatrix} z_1 \\ h_1 \end{pmatrix} < 0 \leq \phi(\epsilon) = \pi\begin{pmatrix} \epsilon \\ \mu \end{pmatrix},$$

which is the desired contradiction. □ □

We observe that because (42) does not yield an explicit form for $b_h(i)$, we do not have an explicit form for $\Pi\binom{z}{-K^-}$. Furthermore, it is difficult to derive closed form expressions for superadditive lifting functions that are not sufficiently structured. For these reasons, we will first build a lower approximation of the exact lifting function Π that has a simple structure and then derive a superadditive approximation for it.

Next we propose a superadditive approximation scheme for step functions that does not requires the lengths of steps are non-increasing; see Zeng [31] for a proof.

Theorem 34. *Assume that*

(i) (a) $u_0 \geq u_1 \geq u_2 \geq u_i \geq 0$, (b) $v_1 \geq v_2 \geq v_i \geq 0$ for $3 \leq i \leq t$ and (c) $v_1 > 0$;

(ii) (a) $u_{k_0+2i} \geq u_{k_0+2i+1}$, (b) $v_{k_0+2i} \geq v_{k_0+2i+1}$ for some $k_0 \geq 2$, $i \in \mathbb{Z}_+$ and $k_0+2i+1 \leq t$.

Then

$$\Psi(z) = \begin{cases} 0 & \text{if} \quad 0 \leq z \leq u_0 \\ i - \frac{u_0+M_{i-1}+v_i-z}{v_1} & \text{if} \quad u_0+M_{i-1} < z \leq u_0+M_{i-1}+v_i \\ i & \text{if} \quad u_0+M_{i-1}+v_i < z \leq u_0+M_i \end{cases} \tag{49}$$

is superadditive over $[0, u_0+M_t]$ where $m_i = v_i + u_i$ and $M_i = \sum_{j=1}^{i} m_k$ for $1 \leq i \leq t$. □

Clearly, if both u_i and v_i are non-increasing with i, Ψ reduces to the general superadditive function described in Gu et al. [13, 14]. However, the family of functions presented in Theorem 34 contains functions that are not derived in Gu et al. [13, 14]; see Figure 3 for an example. The exact lifting function $\Pi\binom{z}{-K^-}$ we obtained in Theorem 30 does not always satisfy the conditions of Theorem 34. Therefore, we propose to first generate an approximation $\check{\Pi}(z)$ of $\Pi\binom{z}{-K^-}$ that almost satisfies these conditions using Algorithm 1 and then to use Ψ to generate a strong superadditive approximation of $\check{\Pi}(z)$.

To simplify the notation, we refer to $L_{-K^-}(i)$ as $L(i)$ and to $b_{-K^-}(i)$ as $b(i)$ in this section. We also define $S_v^* = \{a_1^*,\ldots,a_v^*\}$ to be the set containing the v elements of the cover that have the largest

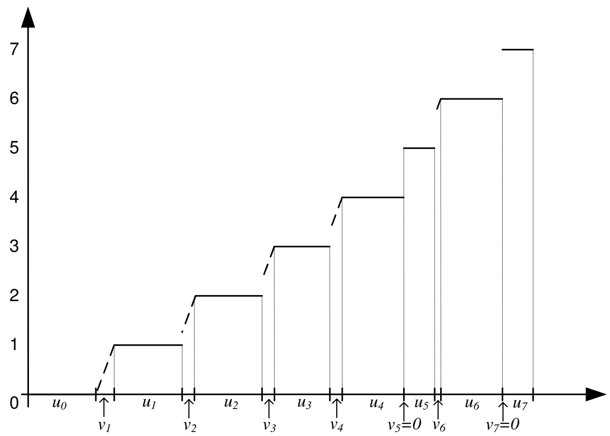


Figure 3: An example of superadditive function Ψ

coefficients. In Part 1 of Algorithm 1, we relax the cardinality constraint of the problem defining $\Pi\binom{z}{h}$, i.e. we replace $\sum_{j\in C^+} x_j - \sum_{j\in C^-\cup EZ} x_j \le K$ with $\sum_{j\in C^+} x_j - \sum_{j\in C^-\cup EZ\setminus S} x_j \le K + |S|$ where S is a small subset of the variables in the cover. More precisely, we define $S = S_v^* \setminus (S_v^* \cap (C^+ \cup C^0))$ where $v = 3$. Note that this relaxation is typically very modest. The lifting function obtained after this relaxation is therefore a lower approximation of $\Pi\binom{z}{h}$. Furthermore, it is close to satisfy condition (i) of Theorem 34. After Part 1 of Algorithm 1 has been executed, we define i_0 to be the smallest value of i for which $L(i) < L(i+1)$. Part 2 of Algorithm 1 is then carried out to derive an approximation that almost satisfies condition (ii) of Theorem 34.

Algorithm 1.

Part 1. *If $L(1)$ and $L(2)$ are not the largest and second largest intervals among $L(i)$ for $i \ge 1$, let $v = 3$ and let $\overline{S}_v^* = S_v^* \setminus (S^* \cap (C^+ \cup C^0))$. Then, define $C^0 \leftarrow C^0 \cup \overline{S}_v^*$, $C^- \leftarrow C^- \setminus \overline{S}_v^*$ and $K \leftarrow K + |\overline{S}_v^*|$. For these updated sets, recompute $\Pi\binom{z}{-K^-}$ and denote it by $\Pi'(z)$.*

Part 2. *For any $t \ge 1$, if $L(i_0+2t-1) < L(i_0+2t)$, update $b(i_0+2t) = b(i_0+2t) + (\frac{L(i_0+2t-1)+L(i_0+2t)}{2} - L_(i_0+2t-1))$ and $L(i_0+2t-1) = L(i_0+2t) = \frac{L(i_0+2t-1)+L(i_0+2t)}{2}$ to obtain $\check{\Pi}(z)$.* □

It is easily verified that $\check{\Pi}(z) \le \Pi\binom{z}{-K^-}$ since both parts of Algorithm 1 produce lower approximations of $\Pi\binom{z}{-K^-}$.

Corollary 35. *The function $\check{\Pi}(z)$ satisfies*

(1) *$L(1) \ge L(2) \ge L(i)$ for $i \in \mathbb{Z}_+$;*

(2) *$L(i_0+2t-1) \ge L(i_0+2t)$ for $t \in \mathbb{Z}_+$.* □

Observe that $\check{\Pi}(z)$ is a step function and each of its step L_i for $i \ge 1$ can easily be split into two parts corresponding to u_i and v_i in Theorem 34. Therefore, using Proposition 33 and Theorem 34, we obtain a superadditive approximation of $\Pi\binom{z}{h}$ for $(z,h) \in [0,b] \times \{-K^-, \ldots, K^+\}$.

Proposition 36. *Let $\rho_i = \max\{0, L(i) - L(0)\}$ for $i = 1, \ldots, |C|-1$. Then,*

$$\psi(z) = \begin{cases} 0 & \text{if} \quad 0 \le z \le b(1) - \lambda \\ i - \frac{b(i)-\lambda+\rho_i - z}{\rho_1} & \text{if} \quad b(i) - \lambda < z \le b(i) - \lambda + \rho_i \\ i & \text{if} \quad b(i) - \lambda + \rho_i < z \le b(i+1) - \lambda \end{cases} \tag{50}$$

for $i = 1, \ldots, |C|-1$ is a valid approximation of $\check{\Pi}$ that satisfies $\psi(y) + \psi(z) \le \psi(y+z)$ for $y, z, y+z \in [0,b]$. Moreover, function π defined as $\pi\binom{z}{h} := \psi(z)$ for $(z,h) \in [0,b] \times \{-K^-, \ldots, K^+\}$ is a valid superadditive approximation of Π over $(z,h) \in [0,b] \times \{-K^-, \ldots, K^+\}$. □

Next, we show that if Algorithm 1 does not change the lifting function $\Pi\binom{z}{-K^-}$, π is a provably strong approximation of Π.

Proposition 37. *Assume that* $\check{\Pi}(z) = \Pi\binom{z}{-K^-}$ *for* $z \in [0, b]$*, then* π *is a superadditive approximation of* Π *that is non-dominated over* $[0, b] \times \{-1, 0, 1\}$*.*

Proof. Because of Proposition 33, it is sufficient to show that $\psi(z)$ is a non-dominated nonnegative one-dimensional superadditive approximation of $\Pi\binom{z}{-K^-}$. Assume by contradiction that there exists a valid one-dimensional superadditive approximation $g(z)$ of $\Pi\binom{z}{-K^-}$ such that $g(z) \geq \psi(z)$ for $z \in [0, b]$ and such that $g(z_0) > \psi(z_0)$ for some $z_0 \in [0, b]$.

Clearly, from (43) and (50), $z_0 \in (b(i) - \lambda, b(i) - \lambda + \rho_i)$ for $i \in \{1, \ldots, |C| - 1\}$ with $\rho_i > 0$. Let $z = b(i + 1) - \lambda$. Then, $z - z_0 > b(i + 1) - b(i) - \rho_i = L(i) - \rho_i = L(0) = b(1) - \lambda$. Also, because $\Pi\binom{z}{-K^-}$ satisfies the properties of Corollary 35, $z - z_0 < b(i + 1) - b(i) = L(i) \leq L(1) = b(1) - \lambda + \rho_1$. Therefore, we have $\psi(z - z_0) = 1 - \frac{b(1)-\lambda+\rho_1-z+z_0}{\rho_1}$ and

$$\begin{aligned}
\psi(z_0) + \psi(z - z_0) &= i - \frac{b(i) - \lambda + \rho_i - z_0}{\rho_1} + 1 - \frac{b(1) - \lambda + \rho_1 - z + z_0}{\rho_1} \\
&= i + 1 - \frac{b(i + 1) - \lambda - z + L(1) - L(0)}{\rho_1} \\
&= i + 1 - \frac{\rho_1}{\rho_1} \\
&= i.
\end{aligned} \tag{51}$$

Because g dominates ψ at z_0, we conclude that $g(z_0)+g(z-z_0) > \psi(z_0)+\psi(z-z_0) = i = \Pi\binom{z}{-K^-} \geq g(z)$, which contradicts the superadditivity of g. □ □

In general, it is difficult to prove that π is non-dominated and maximal. However, we prove next that it is stronger than the traditional superadditive approximation of the lifting function of the cover when only the knapsack constraint is considered. First we introduce the notation $\Pi_0(z)$ to represent the exact lifting function obtained without considering the cardinality constraint, i.e. $\Pi_0(z) = \Pi\binom{z}{-K^-}$ for $z \in [0, b]$ with K^- is sufficiently large, i.e. $K^- \geq |C^-| + |EZ| - \eta$. In this case, it is easily seen that $L(i) \geq L(i+1)$ for all $i \geq 1$. This proves that $\Pi\binom{z}{-K^-}$ satisfies the properties of Corollary 35. We next show that the approximation $\check{\Pi}(z)$ of $\Pi\binom{z}{-K^-}$ is always at least as strong as Π_0.

Proposition 38. $\Pi\binom{z}{-K^-} \geq \check{\Pi} \geq \Pi_0(z)$ *for* $z \in [0, b]$*.*

Proof. Recall that Π' is the exact lifting function $\Pi\binom{z}{-K^-}$ obtained after Part 1 of Algorithm 1. It is easily proven that $\Pi_0(z) \leq \Pi'(z) \leq \Pi\binom{z}{-K^-}$ and $\check{\Pi}(z) \leq \Pi'(z) \leq \Pi\binom{z}{-K^-}$.

We now show that $\Pi_0(z) \leq \check{\Pi}(z)$. We denote the value of $b(i)$ for $\Pi_0, \check{\Pi}$ and Π' as $b_0(i), \check{b}(i)$ and $b'(i)$. It is sufficient to show that $\check{b}(i) \leq b_0(i)$ for all $0 < i \leq |C| - 1$. Because $\Pi_0(z) \leq \Pi'(z)$, we know that $b_0(i) \geq b'(i)$ for all i. Therefore, we only need to consider the points at which $b'(i)$ is converted into $\check{b}(i)$ in Part 2 of Algorithm 1. Let j_0 be such a point. Because Part 2 of Algorithm 1 only changes the value of b' for points of the form $j_0 = i_0 + 2t$ for some $t \in \mathbb{Z}_+$, we know that $b'(j_0 - 1) = \check{b}(j_0 - 1)$ and $b'(j_0 + 1) = \check{b}(j_0 + 1)$. Assume now for a contradiction that $\check{b}(j_0) > b_0(j_0)$. Then, we obtain

$$\begin{aligned}
&\check{b}(j_0) - b'(j_0 - 1) > b_0(j_0) - b'(j_0 - 1) \geq b_0(j_0) - b_0(j_0 - 1) \geq b_0(j_0 + 1) - b_0(j_0) \\
&\geq b'(j_0 + 1) - b_0(j_0) \geq b'(j_0 + 1) - \check{b}(j_0)
\end{aligned} \tag{52}$$

where the third inequality holds because the steps of the function Π_0 are non-increasing in length. Relation (52) contradicts the fact that $\check{b}(j_0)$ was obtained as the average of $b'(j_0-1)$ and $b'(j_0+1)$. □ □

For the case where C is a cover for the knapsack constraint, it is also a minimal cover for the knapsack constraint. Because of Corollary 9, Theorem 34, Proposition 36 and Proposition 38, we conclude next that the approximation π we propose is stronger than that presented in [14] that consider only the knapsack constraint.

Corollary 39. *If the generalized cover* (12) *is a cover inequality for the knapsack constraint, then the superadditive approximation* π *presented in Proposition 36 is superior to the non-dominated superadditive approximation of* Π_0 *presented in [14].* $\square$

We observe that the strength of our superadditive approximations depends on the value of K^-. When many variables with -1 coefficients in the cardinality constraint must be lifted, the improvement obtained by considering the cardinality constraint will not be important. In particular, we show next that if the generalized cover C is a cover for the knapsack constraint and if K^- is sufficiently large, the superadditive approximation presented in Proposition 36 is non-dominated and maximal over $[0, b] \times \{-1, 0, 1\}$.

Proposition 40. *If the generalized cover inequality is a cover inequality for the knapsack constraint and* $K^- \geq |C^-| + |EZ| - \eta + 1$*, then* $\pi\binom{z}{h}$ *is a valid non-dominated and maximal superadditive approximation of* $\Pi\binom{z}{h}$ *for* $(z, h) \in [0, b] \times \{-1, 0, 1\}$.

Proof. Note that when $K^- \geq |C^-| + |EZ| - \eta + 1$, the cardinality constraint in $\Pi\binom{z}{-K^-+1}$ is redundant and $\Pi\binom{z}{-K^-+1} = \Pi\binom{z}{-K^-} = \Pi_0(z)$. The non-dominance of π follows directly from Proposition 37.

Next, we prove that $\pi\binom{z}{h}$ is maximal. Let E be the maximal set of (40). It is sufficient to show that for $(z_0, I_0) \in [0, b] \times \{-1, 0, 1\}$ such that $\Pi\binom{z_0}{I_0} > \phi(z_0)$, $(z_0, I_0) \notin E$. Because the proof when $I_0 = -1$ is similar to that when $I_0 = 1$, we only prove the result for $I_0 \in \{0, 1\}$.

Using dynamic programming arguments, we can easily establish that

$$\Pi_j\binom{z_0}{I_0} = \min_{R \subseteq \{1,\ldots,j-1\}} \{\Pi\binom{\sum_{k \in R} z_k + z_0}{\sum_{k \in R} I_k + I_0} - \sum_{k \in R} \Pi_k\binom{z_k}{I_k}\}. \tag{53}$$

Define now $i := \Pi\binom{z_0}{-K^-}$. There are two cases. In the first case, $\Pi\binom{z_0}{I_0} \geq \Pi\binom{z_0}{-K^-} + 1 = i + 1$. In this case, define $(z_k, I_k) = (0, -1)$ for $k = 1, \ldots, j-1$. Using (53), we obtain that

$$\Pi_{K^-+1}\binom{z_0}{I_0} = \min_{R \subseteq \{1,\ldots,K^-\}} \{\Pi\binom{z_0}{I_0 - |R|} - \sum_{k \in R} \Pi_k\binom{0}{-1}\}. \tag{54}$$

It is easy to prove by induction that $\Pi_k\binom{0}{-1} = \Pi\binom{0}{-K^-} = 0$ for $k = 1, \ldots, K^-$ since lifting a variable with coefficient $(0, -1)$ basically corresponds to relaxing the cardinality constraint by 1 unit. By choosing $R = \{1, \ldots, K^-\}$ in (54), we therefore obtain that $\Pi_{K^-+1}\binom{z_0}{I_0} \leq \Pi\binom{z_0}{I_0 - K^-} = i$, which implies $(z_0, I_0) \notin E$.

In the second case, $\Pi\binom{z_0}{I_0} = \Pi\binom{z_0}{-K^-} = i$. Because $\psi(z_0)$ is a superadditive approximation of $\Pi\binom{z_0}{-K^-}$ that satisfies $\Pi\binom{z_0}{-K^-} - 1 < \psi(z_0) \leq \Pi\binom{z_0}{-K^-} = i$ and we assumed that $\psi(z_0) < \Pi\binom{z_0}{I_0}$, we conclude that $z_0 \in (b_{-K^-}(i) - \lambda, b_{-K^-}(i) - \lambda + \rho_i)$ with $\rho_i > 0$. In this case, we define $(z_k, I_k) = (0, -1)$ for $k = 1, \ldots, K^- - 1$ and $(z_{K^-}, I_{K^-}) = (y, -1)$ where $y = b_{-K^-}(i + 1) - \lambda - z_0 > b_{-K^-}(1) - \lambda$. From (53), we obtain that

$$\Pi_{K^-+1}\binom{z_0}{I_0} = \min_{R \subseteq \{1,\ldots,K^-\}} \{\Pi\binom{z_0 + c(R)y}{I_0 - |R|} - \sum_{k \in R} \Pi_k\binom{c(R)y}{-1}\}, \tag{55}$$

where $c(R) = 0$ if $K^- \notin R$ and $c(R) = 1$ otherwise. Similar to the first case, we can prove by induction that $\Pi_k\binom{0}{-1} = \Pi\binom{0}{-K^-}$ for $k = 1, \ldots, K^- - 1$ and $\Pi_k\binom{y}{-1} = \Pi\binom{y}{-K^-} \geq 1$. By choosing $R = \{1, \ldots, K^-\}$ in (55), we therefore obtain

$$\Pi_{K^-+1}\binom{z_0}{I_0} \leq \Pi\binom{z_0 + y}{I_0 - K^-} - 1 = i - 1,$$

which implies $(z_0, I_0) \notin E$. $\square$ $\square$

Next, we illustrate in Example 4 that inequalities obtained by superadditive lifting when considering cardinality constraints are stronger than those obtained without considering the cardinality constraint.

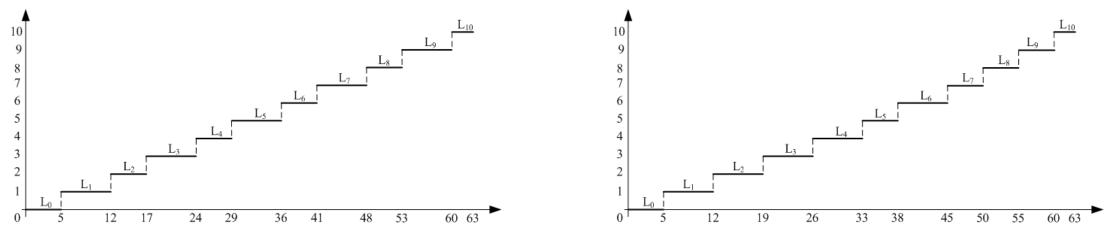
Example 4. *Consider*

$$PS_1 = conv\{x \in \{0,1\}^{12} : 7\sum_{j=1}^{6} x_j + 5\sum_{j=7}^{10} x_j + 3x_{11} + 40x_{12} \le 63, -\sum_{j=1}^{6} x_j + \sum_{j=7}^{11} x_j - x_{12} \le 0\}. \quad (56)$$

From Proposition 7, we observe that $C = \{1,\dots,11\}$ *is a generalized cover with* $C^0 = \emptyset$, $C^+ = \{7,\dots,11\}$, $C^- = \{1,\dots,6\}$ *and* $\lambda = 2$. *The generalized cover inequality*

$$\sum_{j=1}^{11} x_j \le 10$$

is facet-defining for $PS_1(\{12\},\emptyset)$. *The associated exact lifting function* $\Pi\binom{z}{-1}$ *is represented in Figure 4(a).*



(a) Lifting Function of $\Pi\binom{z}{-1}$ (b) Function $\check{\Pi}(z)$

Figure 4: Exact lifting function and its approximation after Algorithm 1

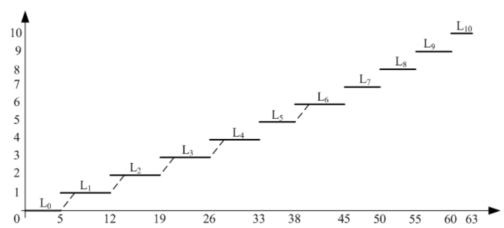


Figure 5: Superadditive Approximation of $\Pi\binom{z}{-1}$

The step lengths of $\Pi\binom{z}{-1}$ *are not monotonically decreasing and so* $\Pi\binom{z}{-1}$ *is different from the function studied by Gu et al. [14]. After applying Algorithm 1,* $\Pi\binom{z}{-1}$ *is converted into the function* $\check{\Pi}(z)$ *that is represented in Figure 4(b). The step lengths are still not monotonically decreasing. Finally, using Proposition 36, we derive the superadditive approximation* π *represented in Figure 5. Using this superadditive lifting function, we obtain that* $\pi\binom{40}{-1} = 6$. *Furthermore, the inequality*

$$\sum_{j=1}^{11} x_j + 6x_{12} \le 10,$$

is facet-defining for PS_1. *We note however that if we used the superadditive lifting function obtained without considering the cardinality constraint, the inequality obtained would have not been facet-defining for* PS_1. □

5.2.2 Building a Superadditive Approximation of Π when $\Pi\binom{0}{-K^-} = -1$

In this case, the set C is not a cover for the knapsack constraint. Unlike the case where $\Pi\binom{0}{-K^-} = 0$, strong superadditive approximations of $\Pi\binom{z}{h}$ are not independent of $h \in \{-K^-,\dots,K^+\}$. This is because the structure of $\Pi\binom{z}{h}$ itself depends on $h \in \{-K^-,\dots,K^+\}$.

Because Π satisfies the condition $\Pi\binom{z}{h-1} \leq \Pi\binom{z}{h}$ for $z \in [0,b]$ and for $h \in \{-K^-, \dots, K^+\}$, and because one of the conditions for sequence independence in Corollary 32 is $\pi\binom{y}{h} + \pi\binom{z}{-1} \leq \pi\binom{y+z}{h-1}$ for $y, z, y+z \in [0,b]$, we propose to first derive $\pi\binom{z}{-1}$ based on $\Pi\binom{z}{-K^-}$ and then to derive the approximation $\pi\binom{z}{0}$ and $\pi\binom{z}{1}$ based on $\pi\binom{z}{-1}$. Similar to the case where $\Pi\binom{0}{-K^-} = 0$, we apply first Algorithm 2 to $\Pi\binom{z}{-K^-}$ to obtain a function $\check{\Pi}(z)$ that has a simpler structure.

Algorithm 2.

Part 1. *If $L(0)$ and $L(1)$ are not the largest and second largest intervals among $L(i)$ for $i \geq 0$, let $v = 2$ and let $\overline{S}_v^* = S_v^* \backslash (S^* \cap (C^+ \cup C^0))$. Then, define $C^0 \leftarrow C^0 \cup \overline{S}_v^*$, $C^- \leftarrow C^- \backslash (\overline{S}_v^* \cap C^-)$ and $K \leftarrow K + |\overline{S}_v^*|$. For these updated sets, recompute $\Pi\binom{z}{-K^-}$ and denote it by $\Pi'(z)$.*

Part 2. *For any $t \geq 1$, if $L(i_0+2t-1) < L(i_0+2t)$, update $b(i_0+2t) = b(i_0+2t) + (\frac{L(i_0+2t-1)+L(i_0+2t)}{2} - L_(i_0 + 2t - 1))$ and $L(i_0 + 2t - 1) = L(i_0 + 2t) = \frac{L(i_0+2t-1)+L(i_0+2t)}{2}$ to obtain $\check{\Pi}(z)$.*

Similar to Corollary 35, we obtain the following result.

Corollary 41. *The function $\check{\Pi}(z)$ satisfies*

(1) *$L(0) \geq L(1) \geq L(i)$ for $i \in \mathbb{Z}_+$;*

(2) *$L(t_0 + 2i - 1) \geq L(t_0 + 2i)$ for $i \in \mathbb{Z}_+$.* □

Again, if $\Pi\binom{z}{-K^-}$ satisfies the property of Corollary 41, then $\check{\Pi}(z) = \Pi\binom{z}{-K^-}$. Clearly, $\check{\Pi}(z)$ is a non-decreasing step function.

Proposition 42. *The function $\check{\Pi}(z)$ satisfies*

$$\check{\Pi}(y) + \check{\Pi}(z) \leq \check{\Pi}(y+z) \tag{57}$$

for all $y, y+z \in [0,b]$.

Proof. Let $\psi'(z) = \max\{0, \check{\Pi}\binom{z}{-K^-}\}$. The function is of the form ψ given in Proposition 36 and is therefore superadditive. Consider now $y, y+z \in [0,b]$. To prove that (57) is satisfied, there are two cases. If $\check{\Pi}(y) < 0$ or $\check{\Pi}(z) < 0$, the condition (57) is clearly satisfied since $\check{\Pi}$ is non-decreasing. If $\check{\Pi}(y) \geq 0$ and $\check{\Pi}(z) \geq 0$, we have $\check{\Pi}(y) + \check{\Pi}(z) = \psi'(y) + \psi'(z) \leq \psi'(y+z) = \check{\Pi}(y+z)$. □ □

Next, we give in Theorem 43 a superadditive approximation for $\Pi\binom{z}{h}$ over $(z,h) \in [0,b] \times \{-K^-, \dots, K^+\}$ and show that it is valid and satisfies the conditions of Corollary 32.

Theorem 43. *The function $\pi\binom{z}{h}$*

$$\pi\binom{z}{h} = \begin{cases} \check{\Pi}\binom{z}{-K^-} & \text{if } h \in \{-K^-, \dots, -1\} \\ \begin{cases} \min\{0, \Pi\binom{z}{h}\} & \text{if } 0 \leq z \leq b_h(1) - \lambda \\ \max\{0, \check{\Pi}\binom{z}{-K^-}\} & \text{if } z > \max\{0, b_h(1) - \lambda\} \end{cases} & \text{if } h \in \{0, \dots, K^+\} \end{cases} \tag{58}$$

is a valid superadditive approximation for $\Pi\binom{z}{h}$ for $(z,h) \in [0,b] \times \{-K^-, \dots, K^+\}$.

Proof. The validity of $\pi\binom{z}{h}$ follows from the fact that $\check{\Pi}(z) \leq \Pi\binom{z}{h}$ for $z \in [0,b] \times \{-K^-, \dots, K^+\}$ and from the fact that $\Pi\binom{z}{h} \geq 0$ when $z > \max\{0, b_h(1) - \lambda\}$ for $h \in \{-K^-, \dots, K^+\}$.

We now prove that $\pi\binom{y}{h_1} + \pi\binom{z}{h_2} \leq \pi\binom{y+z}{h_1+h_2}$ for $y, z, y+z \in [0,b]$ and $h_1, h_2, h_1+h_2 \in \{-K^-, \dots, K^+\}$. If $h_1 < 0$ and $h_2 < 0$, the result follows from Proposition 42. If $h_1 \cdot h_2 < 0$, we assume without loss of generality that $h_1 < 0$ and $h_2 > 0$. If $z \leq b_{h_2}(1) - \lambda$, the result follows from the fact that $\Pi\binom{z}{h_2} < 0$ and π is non-decreasing. If $z > b_{h_2}(1) - \lambda$, the result follows from Proposition 42 and the fact that π is non-decreasing.

Finally, if $h_1 \geq 0$ and $h_2 \geq 0$. There are two subcases. If $y \leq b_{h_1}(1) - \lambda$ or $z \leq b_{h_2}(1) - \lambda$, the result follows from the fact that π is non-decreasing. If $y > b_{h_1}(1) - \lambda$ and $z > b_{h_2}(1) - \lambda$, the result follows from Proposition 42. □ □

We next illustrate Theorem 43 on an example.

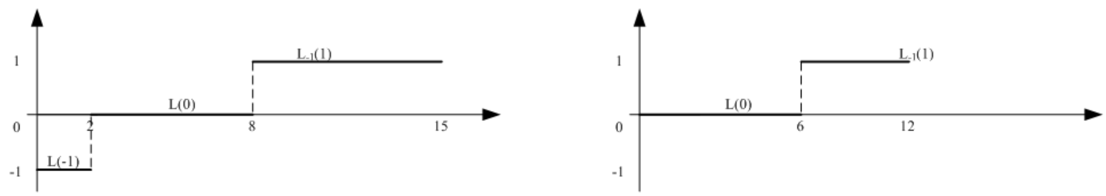
Example 5. *Consider*

$$PS_1 = conv\{x \in \{0,1\}^9 : 7x_1 + 7x_2 + x_3 + x_4 + x_5 + x_6 + 9x_7 + 10x_8 + 6x_9 \leq 15, \tag{59}$$
$$x_1 + x_9 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 \leq -2\}. \tag{60}$$

From Example 2, we know that

$$x_1 + x_2 \leq 1 \tag{61}$$

is facet-defining for $PS_1(N^0, \emptyset)$ *where* $N^0 = \{7,8,9\}$. *We now lift* x_7, x_8, x_9 *into* (61). *Note that* $K^- = 2$ *and* $K^+ = 1$. *Figure 6 shows the lifting function* $\Pi\binom{z}{h}$ *for* $h = -2$, $h = 1$. *We observe that*



(a) Lifting Function $\Pi\binom{z}{-2}$ (b) Lifting Function $\Pi\binom{z}{1}$

Figure 6: Exact lifting function

$\Pi\binom{z}{-2}$ *satisfies the properties of Corollary 41, which implies it satisfies* (57). *So, by Theorem 43, we have* $\pi\binom{9}{-1} = 1$, $\pi\binom{10}{-1} = 1$ *and* $\pi\binom{6}{1} = 0$. *As a result, we obtain that*

$$x_1 + x_2 + x_7 + x_8 \leq 1,$$

which is facet-defining for PS_1. *Note that this inequality cannot be obtained by only considering the knapsack constraint because it cuts off the integer solution* $\vec{e}_1 + \vec{e}_2$ *that is feasible for* (59). □

6 Sequence Independent Lifting in PS_r^+

In this section, we use the results obtained in Section 4 to study $0-1$ knapsack models with multiple disjoint cardinality constraints. Although the results obtained in Section 5 naturally extend to PS_r, we focus only on the extension of our results from PS_1^+ to PS_r^+. Since the discussion for PS_r is more technical and the improvements obtained over single-dimensional lifting are less pronounced for PS_r than for PS_r^+.

We first derive the exact lifting function of the generalized cover inequality and give sufficient conditions for lifting to be sequence independent. Then we derive a multidimensional superadditive approximation of the lifting function. We further prove that this approximation is maximal and non-dominated.

6.1 Exact lifting function of the generalized cover inequality

Let $\vec{I}_j = \{I_{j1}, \ldots, I_{jr}\}^T \in \mathbb{R}^r$ be the vector of the coefficients of x_j in the r cardinality constraints. Because we assume the cardinality constraints are disjoint, $\vec{I}_j$ can be either $\vec{0}$ or one of the unit vectors $\vec{e}_i$ of $\mathbb{R}^r$. Because for PS_r^+, generalized cover inequalities in PS_r^+ are cover inequalities of the knapsack constraint, the t^{th} lifting problem is given by

$$
\begin{aligned}
\Omega_t\binom{z}{\vec{v}} = \min\ & |C| - 1 - \sum_{j\in C} x_j - \sum_{1\le s<t} \alpha_s x_s \\
\text{s.t.}\ & \sum_{j\in C} a_j x_j + \sum_{1\le s<t} a_s x_s \le b - z \\
& \sum_{j\in N_i} x_j + \sum_{1\le s<t} I_{si} x_s \le K_i - v_i,\ i\in\{1,\dots,r\}
\end{aligned}
\tag{62}
$$

with $\alpha_s = \Omega_s\binom{a_s}{\vec{I}_s}$, $z\in[0,b]$ and $\vec{v} = \{v_1,\dots,v_r\}^T \in \mathbb{Z}^r_+$. As usual, we denote Ω_1 by Ω.

For the special case where $C \subseteq N_k$ for some $k\in\{1,\dots,r\}$, it is easy to see that the lifting function $\Omega\binom{z}{\vec{e}_i} = \Omega\binom{z}{\vec{0}} = \Theta^*\binom{z}{0}$ for $i\ne k$. Therefore, this case reduces to that of PS_1^+ where $C\subseteq N_1$. Furthermore, a superadditive approximation of $\Omega\binom{z}{\vec{e}_i}$ with $i\ne k$ can always be chosen equal to that of $\Omega\binom{z}{\vec{0}}$. A similar argument applies to the case where $C\subseteq N_0$. Therefore, unless we mention it otherwise, we assume in the remainder of this section that $C \not\subseteq N_i$ for $i=0,\dots,r$.

For a generalized cover C of PS_r^+, we have $C_i^+ = C_i$ and $|C_i| + \eta_i = K_i$ for $i=1,\dots,r$ with $\eta_i\ge 0$. Define $R^= := \{i\in\{1,\dots,r\} : \eta_i = 0\}$, and $\overline{R^=} := \{1,\dots,r\}\backslash R^=$. We denote $C_i = \{j_{i,1},\dots,j_{i,K_i-\eta_i}\}$ for $i=1,\dots,r$. We may assume that $a_{j_{i,1}} \ge \dots \ge a_{j_{i,K_i-\eta_i}}$. Similar to the discussion for PS_1^+, we define $A_{i,k} = \sum_{s=1}^k a_{j_{i,s}}$ and $\hat{A}_{i,k} = \sum_{s=K_i-\eta_i-k+1}^{K_i-\eta_i} a_{j_{i,s}}$ with $k=1,\dots,K_i-\eta_i$ for all i. We define $\mathbb{D} = \{0,\dots,K_1^+\}\times\dots\times\{0,\dots,K_r^+\}$, and $\mathbb{D}' = \{\vec{0},\vec{e}_1,\dots,\vec{e}_r\}$. In addition, for $\vec{v}\in\mathbb{D}$, we define $v_i^+ = \max\{v_i-\eta_i, 0\}$ for $i=1,\dots,r$ to represent the minimal number of elements of the cover that need to be removed from C_i in the optimal solution to $\Omega\binom{z}{\vec{v}}$. Finally, we let $\vec{v}^+ = \{v_1^+,\dots,v_r^+\}$.

From (62), it is easy to verify that $\Omega\binom{z}{\vec{0}} = \Theta^*\binom{z}{0}$. Using an argument similar to that of Theorem 22, we obtain the lifting function $\Omega\binom{z}{\vec{v}}$ described in Theorem 44.

Theorem 44. *The exact lifting function of the generalized cover inequality is*

$$
\Omega\binom{z}{\vec{v}} = \begin{cases} \Theta^*\binom{z}{0} & \text{if } \sum_{i=1}^r v_i^+ = 0 \\ \begin{cases} \sum_{i=1}^r v_i^+ - 1 & \text{if } 0 \le z \le \sum_{i=1}^r A_{i,v_i^+} - \lambda \\ \sum_{i=1}^r v_i^+ & \text{if } \sum_{i=1}^r A_{i,v_i^+} - \lambda < z < \sum_{i=1}^r A_{i,v_i^+} \\ \hat{\Omega}\binom{z}{\vec{v}} & \text{if } \sum_{i=1}^r A_{i,v_i^+} \le z \le b. \end{cases} & \text{if } \sum_{i=1}^r v_i^+ \ge 1 \end{cases}
\tag{63}
$$

for $(z,\vec{v})\in[0,b]\times\mathbb{D}$ *where* $\hat{\Omega}\binom{z}{\vec{v}} = \max_{k_i:0\le k_i\le v_i^+, i=1,\dots,r}\{\Theta^*\binom{z-\sum_{i=1}^r \hat{A}_{i,k_i}}{\vec{0}} + \sum_{i=1}^r k_i\}$. □

From (63), it is easily seen that $\Omega\binom{z}{\vec{e}_i} = \Theta^*\binom{z}{1}$ where a_l is replaced with $a_{j_{i,1}}$ if $i\in R^=$, and $\Omega\binom{z}{\vec{e}_i} = \Theta^*\binom{z}{0}$ if $i\in\overline{R^=}$.

6.2 Building a Superadditive Approximation of Ω

Let $d_i^+ = \max\{|S| : S\subseteq N_i\backslash(C_i), \sum_{k\in S} a_k \le b\}$ and $K_i^+ = \min\{d_i^+, K_i\}$ for $i\in\{1,\dots,r\}$. Note that when performing sequential lifting, we only need to solve lifting problems over $[0,b]\times\mathbb{D}'$. However, from Proposition 5, it is necessary to consider a larger range to ensure sequence independent lifting. We derive next in Corollary 45 sufficient conditions for sequence independent lifting. This corollary follows from Theorem 4 and Proposition 5.

Corollary 45. *For PS_r^+, lifting is sequence independent if*

$$\Omega\binom{y}{\vec{0}} + \Omega\binom{z}{\vec{0}} \leq \Omega\binom{y+z}{\vec{0}} \tag{64}$$

$$\Omega\binom{y}{\vec{0}} + \Omega\binom{z}{\vec{e}_i} \leq \Omega\binom{y+z}{\vec{e}_i} \tag{65}$$

$$\Omega\binom{y}{\vec{0}} + \Omega\binom{z}{\vec{v}} \leq \Omega\binom{y+z}{\vec{v}} \tag{66}$$

$$\Omega\binom{y}{\vec{e}_i} + \Omega\binom{z}{\vec{v}} \leq \Omega\binom{y+z}{\vec{e}_i+\vec{v}} \tag{67}$$

for all $y, z, y+z \in [0,b]$, $i = 1,\ldots,r$ and $\vec{v} \in \mathbb{D}$. □

Note that (64) and (65) are implied by (66) and (67). However, we list them separately because they are central to the derivation of the superadditive approximation. Now, by using a construction similar to that of Proposition 23, we give a valid approximation of Ω over $[0,b]\times\mathbb{D}$. The validity and superadditivity of this approximation can also easily be proven by an argument similar to that of Proposition 23.

Proposition 46. *The function*

$$\omega\binom{z}{\vec{v}} = \begin{cases} \theta^*\binom{z}{0}, & \text{if } \vec{v}^+ = \vec{0} \\ \max\{\theta^*\binom{z-a_{j_{i,1}}}{0} + 1, \theta^*\binom{z}{0}\}, & \text{if } \vec{v}^+ = \vec{e}_i \\ \sup\limits_{\{z=\sum_{i\in R^=,1\leq j\leq v_i^+} z_{i,j},\ z_{i,j}\geq 0\ \forall i,j\}} \{\sum\limits_{i\in R^=,1\leq j\leq v_i^+} \omega\binom{z_{i,j}}{\vec{e}_i}\} & \text{if } \vec{v}^+ \notin \mathbb{D}' \end{cases} \tag{68}$$

for $\vec{v}\in\mathbb{D}$ and $i\in\{1,\ldots,r\}$ is a valid superadditive approximation of $\Omega\binom{z}{\vec{v}}$ for $(z,\vec{v})\in[0,b]\times\mathbb{D}$. □

We show next that this superadditive approximation is strong by proving that it is non-dominated and maximal.

Theorem 47. *The function $\omega\binom{z}{\vec{v}}$ is a maximal and non-dominated superadditive approximation of $\Omega\binom{z}{\vec{v}}$ over $(z,\vec{v})\in[0,b]\times\mathbb{D}'$.*

Proof. Since the non-dominance is easy to verify, we only give the proof of maximality. Let $E_r \subseteq [0,b]\times\mathbb{D}'$ be the maximal set of (11) for PS_r^+. It is sufficient to show that $\omega\binom{z_0}{\vec{v}_0} < \Omega\binom{z_0}{\vec{v}_0}$ with $(z_0,\vec{v}_0)\in[0,b]\times\mathbb{D}'$ implies that $(z_0,\vec{v}_0)\notin E_r$.

Consider first the case where $v_0 = \vec{0}$. Because $\omega\binom{z_0}{0} = \theta^*\binom{z_0}{0}$ and $\Omega\binom{z_0}{0} = \Theta^*\binom{z_0}{0}$, the proof that $\Omega_2\binom{z_0}{\vec{0}} \leq \min\{\Omega\binom{z_0}{\vec{0}}, \Omega\binom{z_0+y}{\vec{0}} - \Omega\binom{y}{\vec{0}}\} < \Omega\binom{z_0}{\vec{0}}$ for some y such that $y, z_0+y\in[0,b]$ reduces to that of Theorem 16.

Consider now the case where $\vec{v}_0 = \vec{e}_i$ for some $i\in\{1,\ldots,r\}$. It is sufficient to consider $i\in R^=$ since $\Omega\binom{z}{\vec{e}_i} = \Omega\binom{z}{\vec{0}}$ when $i\in\overline{R^=}$. It follows from Proposition 46 and from the proof of Theorem 17 that there are two situations : (i) $\omega\binom{z_0}{\vec{e}_i} = \theta^*\binom{z_0}{0}$ and $\Omega\binom{z_0}{\vec{e}_i} = \Theta\binom{z_0}{0}$, or (ii) $\omega\binom{z_0}{\vec{e}_i} = \theta^*\binom{z_0-a_{j_{i,1}}}{0} + 1$ and $\Omega\binom{z_0}{e_i} = \Theta\binom{z_0-a_{j_{i,1}}}{0} + 1$. The proof of these cases reduces to those given in Theorem 17. Therefore, we conclude that $(z_0,\vec{e}_i)\notin E_r$. □ □

Next, we present an example from Nemhauser and Vance [19] to show the strength of our multidimensional superadditive approximation.

Example 6. *([19]) Consider*

$$PS_2 = conv\{x\in\{0,1\}^{10} : 37x_1 + 25x_2 + 23x_3 + 15x_4 + 14x_5 + 12x_6 + 11x_7 + 8x_8 + 7x_9 + 3x_{10} \leq 39,$$
$$x_6 + x_7 \leq 1, x_9 + x_{10} \leq 1\}.$$

Let $C = \{4, 5, 8, 10\}$. It is easy to verify that C is a generalized cover with $\lambda = 1$, $\eta_1 = 1$ and $\eta_2 = 0$. Therefore, from Theorem 44, we have

$$\Omega\binom{z}{\vec{0}} = \Omega\binom{z}{\vec{e}_1} = \begin{cases} 0 & \text{if} \quad 0 \le z \le 14 \\ 1 & \text{if} \quad 14 < z \le 28 \\ 2 & \text{if} \quad 28 < z \le 36 \\ 3 & \text{if} \quad 36 < z \le 39, \end{cases}$$

and

$$\Omega\binom{z}{\vec{e}_2} = \begin{cases} 0 & \text{if} \quad 0 \le z \le 2 \\ 1 & \text{if} \quad 2 < z \le 17 \\ 2 & \text{if} \quad 17 < z \le 31 \\ 3 & \text{if} \quad 31 < z \le 39. \end{cases}$$

From Theorem 15 and Proposition 46, it is easily seen that $\omega\binom{z}{\vec{0}} = \omega\binom{z}{\vec{e}_1} = \Omega\binom{z}{\vec{0}} = \Omega\binom{z}{\vec{e}_1}$ and $\omega\binom{z}{\vec{e}_2} = \max\{\omega\binom{z-3}{\vec{0}} + 1, \omega\binom{z}{\vec{0}}\}$. Using this superadditive approximation, we derive the following valid inequality

$$3x_1 + x_2 + x_3 + x_4 + x_5 + x_8 + x_9 + x_{10} \le 3. \tag{69}$$

This inequality is proven to be facet-defining for PS_2 in Nemhauser and Vance [19]. □

7 Conclusion

In this paper, we study the polyhedral structure of the set of 0−1 solutions to a knapsack problem with disjoint cardinality constraints. This model is a generalization of many classical models including KP and GUBKP. We derive a family of strong valid inequalities for these sets by lifting the generalized cover inequalities proposed by Zeng and Richard [32]. The lifting is performed using a set of multidimensional superadditive lifting functions that are proven to be non-dominated and maximal. The lifted generalized cover inequalities that we obtain either dominate those derived from the single knapsack constraint or cannot be obtained from the knapsack constraint directly. Therefore, our results strictly generalize classical results about lifted covers. This is, to the best of our knowledge, the first work in which multidimensional superadditive approximations of lifting functions are proven to be strong.

During the derivation of the superadditive lifting functions for PS_1^+ and PS_r^+, we observed that multidimensional superadditive lifting functions can typically be constructed from lower dimensional superadditive lifting functions. Using this observation, we were able to derive multidimensional superadditive lifting functions that are typically strong and can be computed efficiently if the lower dimensional functions can be computed efficiently. We believe that this is an important contribution as multi-constraint lifting is a computation tool for generating strong cuts that has been vastly under-investigated. Therefore, we believe that an interesting direction of research spawning from this work is to study how to systematically build multidimensional lifting functions from lower dimensional ones. Such a mechanism could yield inequalities stronger than those currently used and therefore produce improvements in general purpose MIP solvers.

As a final direction of future research, we mention that we are currently pursuing the empirical testing of the multidimensional lifting functions we propose in this paper.

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