Polyhedral combinatorics of a resource-constrained ordering problem part I: on the partial linear ordering polytope * †

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Abstract

This paper is the first of a series of two devoted to the polyhedral study of a strongly NP-hard resource-constrained scheduling problem, referred to as the *process move programming problem*. This problem arises in relation to the operability of certain high-availability real time distributed systems. After a brief introduction to the problem as well as a summary of previous results, we formulate it as an integer linear program using linear ordering variables. We then drop the capacity constraints and introduce the *partial linear ordering polytope*, defined as the convex hull of all incidence vectors of arc sets of linear orderings of a node subset of the complete digraph on n nodes (the nodes not in the subset being looped), study its basic properties as well as show several classes of inequalities to be facet-defining. The companion paper [6] is devoted to the study of the *process move program polytope* which is obtained when the capacity constraints are put back into the picture.

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1 Introduction

Let us consider a distributed system composed of a set U of *processors* and let R denote the set of *resources* they offer. For each processor $u \in U$ and each resource $r \in R$, $C_{u,r} \in \mathbb{N}$ denotes the amount of resource r offered by processor u. We are also given a set P of applications, hereafter referred to as *processes*, which consume the resources offered by the processors. The set P is sometimes referred to as the *payload* of the system. For each process $p \in P$ and each resource $r \in R$, $w_{p,r} \in \mathbb{N}$ denotes the amount of resource r which is consumed by process p. Note that neither $C_{u,r}$ nor $w_{p,r}$ vary with time. Also, when $|R| = 1$, $C_{u,r}$ and $w_{p,r}$ are respectively denoted C_u and w_p (this principle is applied to other quantities throughout this paper).

An *admissible state* for the system is defined as a mapping $f : P \longrightarrow U \cup \{u_\infty\}$, where u_∞ is a dummy processor having infinite capacity, such that for all $u \in U$ and all $r \in R$ we have

$$\sum_{p \in P(u;f)} w_{p,r} \leq C_{u,r}, \tag{1}$$

where $P(u; f) = \{p \in P : f(p) = u\}$. The processes in $\bar{P}(f) = P(u_\infty; f)$ are not instantiated and, when this set is non empty, the system is in *degraded mode*.

An instance of the *Process Move Programming* (PMP) problem is then specified by two arbitrary system states f_i and f_t and, roughly speaking, consists of, starting from state f_i, finding the least disruptive sequence of operations (e.g., process migrations) at the end of which the system is in state f_t. The two aforementioned system states are respectively referred to as the *initial system state* and the *final system state* or, for short, the *initial state* and the *final state*[1].

A process may be moved from one processor to another in two different ways: either it is *migrated*, in which case it consumes resources on both processors for the duration of the migration and this operation has virtually no impact on service, or it is *interrupted*, that is, removed from the first processor and later restarted on the other one. Of course, this latter operation

[1]Throughout the rest of this paper, it is assumed that $\bar{P}(f_i) = \bar{P}(f_t) = \emptyset$. When this is not the case the processes in $\bar{P}(f_t) \setminus \bar{P}(f_i)$ should be stopped before the reconfiguration, hence some resources are freed, the processes in $\bar{P}(f_i) \setminus \bar{P}(f_t)$ should be started after the reconfiguration and the processes in $\bar{P}(f_i) \cap \bar{P}(f_t)$ are irrelevant.

has an impact on service. Additionally, it is required that the capacity constraints (1) are always satisfied during the reconfiguration and that a process is moved (i.e., migrated or interrupted) at most once. This latest constraint is motivated by the fact that a process migration is far from being a lightweight operation (for reasons related to distributed data consistency which are out of the scope of this paper, see e.g. [5]) and, as a consequence, it is desirable to avoid processes hopping around processors.

Throughout this paper, when it is said that a process move is *interrupted*, it is meant that the process associated to the move is interrupted. This slightly abusive terminology significantly lightens our discourse. Additionally, it is now assumed that $|R| = 1$, unless otherwise stated.

For each processor u, a process p in $P(u; f_i) \setminus P(u; f_t)$ must be moved from u to $f_t(p)$. Let M denote the set of process moves. Then for each $m \in M$, w_m, s_m and t_m respectively denote the amount of resource consumed by the process moved by m, the processor from which the process is moved, that is, the *source* of the move and the processor to which the process is moved, that is, the *target* of the move. Lastly, $S(u) = \{m \in M : s_m = u\}$ and $T(u) = \{m \in M : t_m = u\}$.

A pair (I, σ), where $I \subseteq M$ and $\sigma : M \setminus I \longrightarrow \{1, \ldots, |M \setminus I|\}$ is a bijection, defines an admissible *process move program*, if provided that the moves in I are interrupted (for operational reasons, the interruptions are performed at the beginning) the other moves can be performed, through migrations, according to σ without inducing any violation of the capacity constraints (1). Formally, (I, σ) is an admissible program if for all $m \in M \setminus I$ we have

$$w_m \leq K_{t_m} + \sum_{\substack{m' \in I \\ s_{m'} = t_m}} w_{m'} + \sum_{\substack{m' \in S(t_m) \setminus I \\ \sigma(m') < \sigma(m)}} w_{m'} - \sum_{\substack{m' \in T(t_m) \setminus I \\ \sigma(m') < \sigma(m)}} w_{m'}, \tag{2}$$

where $K_u = C_u - \sum_{p \in P(u; f_i)} w_p$, thereby guaranteeing that the intermediate states are admissible.

Also note that because the final state is admissible, we have for each processor $u \in U$

$$K_u + \sum_{m \in S(u)} w_m - \sum_{m \in T(u)} w_m \geq 0. \tag{3}$$

Let c_m denote the cost of interrupting $m \in M$. The PMP problem then formally consists, given a set of moves, of finding a pair (I, σ) such that $c(I) = \sum_{m \in I} c_m$ is minimum.

Figure 1 provides an example of an instance of the PMP problem for a system with 10 processors, one resource and 46 processes. The capacity of each of the processors is equal to 30 and the sum of the consumptions of the processes is 281. The top and bottom figures respectively represent the initial and the final system states. For example, process number 23 must be moved from processor 2 to processor 6.

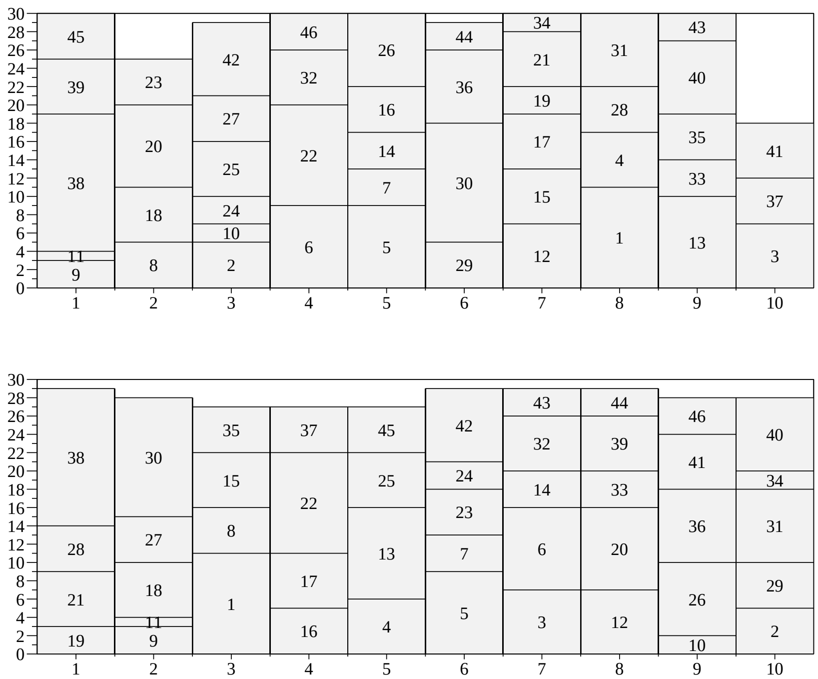

Figure 1: Example of an instance of the PMP problem.

In Sirdey et al. [9] we have shown that the PMP problem is strongly NP-hard, exhibited some polynomially solvable special cases (the most notable one being $|R| = 1$ and $w_m = w$ for all $m \in M$) as well as proposed a "combinatorial" branch-and-bound algorithm for the general case. Also, an extensive literature survey was provided in that paper. Additionally, approximate resolution algorithms have been proposed in Sirdey et al. [10] (simulated-annealing-based approach) as well as in Sirdey et al. [11] (Grasp-based approach). This paper and its companion focus on the study of the PMP problem from the point of view of polyhedral combinatorics. In Section 2 we formulate the problem as an integer linear program. In Section 3, we introduce the *partial linear ordering polytope*, denoted P^n_{PLO}, which is obtained when the capacity constraints of the PMP problem are relaxed, study its basic properties (dimension and simple facets) as well as demonstrate a useful trivial lifting lemma. In Section 4 we exhibit several classes of facet-defining inequalities for P^n_{PLO}, namely the k-clique, k-unicycle and k-l-bicycle inequalities, and discuss the associated separation problems.

2 An integer linear programming formulation

In this section, we formulate the PMP problem as an integer linear program. We first focus on obtaining a formulation for the decision problem which asks whether or not there exists an admissible process move program of the form $(\emptyset, \sigma)$, that is, an admissible program of zero cost. We subsequently refine the model to encompass the notion of interruption.

For each ordered pair of distinct moves of M, say m and m', we introduce the *linear ordering variables* [8]

$$\delta_{mm'} = \begin{cases} 1 & \text{if } m \text{ precedes } m', \\ 0 & \text{otherwise.} \end{cases}$$

In order for these variables to define a valid ordering, it is natural to ask for the following constraints to be satisfied

$$\begin{cases} \delta_{mm'} + \delta_{m'm} = 1 & \forall \{m, m'\} \subseteq M, & (4) \\ \delta_{mm'} + \delta_{m'm''} - \delta_{mm''} \leq 1 & m \neq m' \neq m'' \neq m \in M. & (5) \end{cases}$$

Constraints of type (4) simply express that either m precedes m' or m' precedes m. Constraints of type (5) are known as the *transitivity constraints* and simply state that if m precedes m' and if m' precedes m'' then m precedes m''. Along with the constraints

$$\delta_{mm'} \in \{0, 1\} \qquad m \neq m' \in M, \tag{6}$$

constraints of types (4) and (5) describe a *linear ordering polytope*, that is, the convex hull of the incidence vectors of the linear orderings of the moves in M (see for example Grötschel et al. [4] or Fishburn [2] for details).

Since interruptions are (so far) disallowed, constraints of type (2) can be expressed as follows

$$w_m \leq K_{t_m} + \sum_{m' \in S(t_m)} w_{m'} \delta_{m'm} - \sum_{m' \in T(t_m) \setminus \{m\}} w_{m'} \delta_{m'm} \qquad \forall m \in M. \quad (7)$$

It follows that any integral solution to the linear system of inequalities defined by the sets of constraints (4), (5), (6) and (7) (should such a solution exists) provides an admissible process move program of zero cost.

We now turn to the PMP problem and start by, for each move $m \in M$, introducing the variables

$$\delta_{mm} = \begin{cases} 1 & \text{if } m \text{ is interrupted,} \\ 0 & \text{otherwise.} \end{cases}$$

Constraints of type (2) can then be written as follows

$$(1 - \delta_{mm}) w_m \leq K_{t_m} + \sum_{m' \in S(t_m)} w_{m'} (\delta_{m'm'} + \delta_{m'm}) - \sum_{m' \in T(t_m) \setminus \{m\}} w_{m'} \delta_{m'm}, \quad (8)$$

for all $m \in M$. The transitivity constraints (5) remain unchanged and constraints of type (4) must be replaced by constraints

$$\delta_{mm'} + \delta_{m'm} = 1 - \max(\delta_{mm}, \delta_{m'm'}) \qquad \forall \{m, m'\} \subseteq M. \quad (9)$$

These constraints simply express that if either m or m' is interrupted then neither m precedes m' nor m' precedes m. (Recall that the interruptions are performed at the beginning.)

Proposition 1 *Constraints of type* (9) *are equivalent to the following set of constraints*

$$\begin{cases} \delta_{mm'} + \delta_{m'm} + \delta_{mm} + \delta_{m'm'} \geq 1 & \forall \{m, m'\} \subseteq M, \quad (10) \\ \delta_{mm'} + \delta_{m'm} + \delta_{mm} \leq 1 & m \neq m' \in M. \quad (11) \end{cases}$$

Proof. Left to the reader. □

For reasons which will soon become obvious, inequalities of types (10) and (11) are respectively referred to as the 2-*clique* and 1-*unicycle* inequalities.

The resulting integer linear program for the process move programming problem is given Figure 2. The polytope associated to this program is hereafter referred to as the *PMP polytope* and denoted P^M_{PMP}.

$$
\begin{cases}
\text{Minimize } \displaystyle\sum_{m \in M} c_m \delta_{mm} & \\
\text{s. t.} & \\
\delta_{mm'} + \delta_{m'm} + \delta_{mm} + \delta_{m'm'} \geq 1 & \forall \{m, m'\} \subseteq M, \\
\delta_{mm'} + \delta_{m'm} + \delta_{mm} \leq 1 & m \neq m' \in M, \\
\delta_{mm'} + \delta_{m'm''} - \delta_{mm''} \leq 1 & m \neq m' \neq m'' \neq m \in M, \\
(1 - \delta_{mm}) w_m \leq K_{t_m} + \displaystyle\sum_{m' \in S(t_m)} w_{m'} (\delta_{m'm'} + \delta_{m'm}) - \displaystyle\sum_{m' \in T(t_m) \setminus \{m\}} w_{m'} \delta_{m'm} & \forall m \in M, \\
\delta_{mm'} \in \{0, 1\} & m, m' \in M.
\end{cases}
$$

Figure 2: Formulation of the PMP problem as an integer linear program.

Another interesting approach to the problem, which is not studied in this paper, consists, interruptions not being allowed, of minimizing the maximum overflow occurring during the reconfiguration. That is, to minimize ε, subject to constraints (4), (5) and (6) along with constraints

$$w_m \leq K_{t_m} + \sum_{m' \in S(t_m)} w_{m'} \delta_{m'm} - \sum_{m' \in T(t_m) \setminus \{m\}} w_{m'} \delta_{m'm} + \varepsilon \qquad \forall m \in M.$$

3 The partial linear ordering polytope

3.1 Definition and basic properties

The partial linear ordering polytope, denoted P^n_{PLO}, is obtained from P^M_{PMP} by dropping the capacity constraints (8). Hence, it is defined as the convex hull of all incidence vectors of arc sets of linear orderings of a node subset of the complete digraph on n nodes, the nodes not in the subset being looped. Figure 3 provides an example of a point of P^6_{PLO} which orders nodes $\{1, 2, 3, 5\}$ in the order $(3, 5, 1, 2)$, nodes 4 and 6 being excluded and, hence, provided with a loop (i.e., $\delta_{44} = \delta_{66} = 1$).

Equivalently, P^n_{PLO} can be defined as the integral hull of the polytope defined by the following sets of inequalities (recall inequalities (10), (11) and (5))

$$\begin{cases} \delta_{ij} + \delta_{ji} + \delta_{ii} + \delta_{jj} \geq 1 & 1 \leq i < j \leq n, \qquad (12) \\ \delta_{ij} + \delta_{ji} + \delta_{ii} \leq 1 & i, j \in \{1, \dots, n\}, i \neq j, \qquad (13) \\ \delta_{ij} + \delta_{jk} - \delta_{ik} \leq 1 & i, j, k \in \{1, \dots, n\}, i \neq j \neq k \neq i, \qquad (14) \\ 0 \leq \delta_{ij} \leq 1 & i, j \in \{1, \dots, n\}, i \neq j. \end{cases}$$

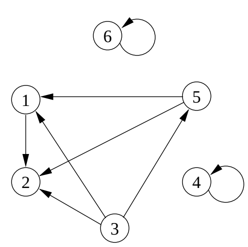

Figure 3: Example of a point of P^6_{PLO}.

P^n_{PLO} is of course interesting in its own right, it is moreover practically relevant in the sense that it can be expected that insights obtained from its study, in particular regarding its facet-defining inequalities, will be useful to gain insights regarding the structure of P^M_{PMP}.

Hereafter, in order to avoid any ambiguities, the terms *vertex* and *node* respectively refer to polytope vertices and graph vertices.

Proposition 2 *P^n_{PLO} is full-dimensional.*

Proof. P^n_{PLO} has n^2 variables, so it is required to exhibit n^2+1 affinely independent points. Let us consider the following points:

1. The unique point which orders no nodes (i.e., $\delta_{ii} = 1$ for all $i \in \{1,\dots,n\}$).
2. The points which only order node i (i.e., $\delta_{ii} = 0$ and $\delta_{jj} = 1$ for all $j \in \{1,\dots,n\} \setminus \{i\}$), for each $i \in \{1,\dots,n\}$. There are n such points.
3. The points which order only node i before node j (i.e., $\delta_{ii} = \delta_{jj} = 0$, $\delta_{kk} = 1$ for all $k \in \{1,\dots,n\} \setminus \{i,j\}$ and $\delta_{ij} = 1$), for each $i,j \in \{1,\dots,n\}$ with $i \neq j$. There are $n(n-1)$ such points.

It is obvious that the above n^2+1 points are linearly, hence affinely, independent. $\square$

3.2 Trivial lifting

In this section, we prove a trivial lifting lemma which is extensively used in the sequel to shorten facet-definition proofs. Note that this kind of results is quite frequent in the domain of ordering-related polytopes [4, 7, 1].

Lemma 1 *Let $a^T\delta \leq \alpha$ be a facet-defining inequality for P^n_{PLO}. Setting*

$$\overline{a}_{ij} = \begin{cases} a_{ij} & \text{for all } i,j \in \{1,\dots,n\}, \\ 0 & \text{if } i = n+1 \text{ or } j = n+1, \end{cases}$$

the inequality $\overline{a}^T\delta \leq \alpha$ then defines a facet of P^{n+1}_{PLO}.

Proof. It is obvious that $\overline{a}^T\delta \leq \alpha$ is valid for P^{n+1}_{PLO}. Let $\mathcal{F}$ be the face of P^{n+1}_{PLO} induced by $\overline{a}^T\delta \leq \alpha$. This face is clearly proper. We now need to exhibit $(n+1)^2$ affinely independent points of $\mathcal{F}$.

Since $a^T\delta \leq \alpha$ is facet-defining for P^n_{PLO}, there exist n^2 linearly independent points, say $\delta^1, \ldots, \delta^{n^2}$, so that $a^T\delta^k = \alpha$ for all $k \in \{1, \ldots, n^2\}$. Let D be the $n^2 \times n^2$ matrix having the points $\delta^1, \ldots, \delta^{n^2}$ as rows, and let $\overline{\delta}^k$ for $k = 1, \ldots, n^2$ so that

$$\overline{\delta}^k_{ij} = \begin{cases} \delta^k_{ij} & \text{for all } i, j \in \{1, \ldots, n\}, \\ 1 & \text{if } i = j = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the n columns of D corresponding to the variables δ_{ii} for $i = 1, \ldots, n$. Let $\overline{D}$ be the $n^2 \times n$ matrix composed of these n columns. Since D is a nonsingular matrix, these n columns are linearly independent and therefore, there exists a $n \times n$ submatrix N of $\overline{D}$ that is nonsingular. Let u^l, for $l = 1, \ldots, n$, be the row of D corresponding to the l^{th} row of N, and let U be the $n \times n^2$ submatrix of D composed of the points u^l, for $l = 1, \ldots, n$. Let v^l, for $l = 1, \ldots, n$, be the point so that

$$v^l_{ij} = \begin{cases} u^l_{ij} & \text{for all } i, j \in \{1, \ldots, n\}, \\ 0 & \text{if } i = j = n+1, \\ 1 - u^l_{jj} & \text{if } i = n+1 \text{ and } j \in \{1, \ldots, n\}, \\ 0 & \text{if } j = n+1 \text{ and } i \in \{1, \ldots, n\}. \end{cases}$$

and w^r, for $r = 1, \ldots, n$, be the point so that

$$w^r_{ij} = \begin{cases} u^r_{ij} & \text{for all } i, j \in \{1, \ldots, n\}, \\ 0 & \text{if } i = j = n+1, \\ 0 & \text{if } i = n+1 \text{ and } j \in \{1, \ldots, n\}, \\ 1 - u^r_{ii} & \text{if } j = n+1 \text{ and } i \in \{1, \ldots, n\}. \end{cases}$$

Without loss of generality, assume that the point δ^{n^2} is different to the points u^i for $i = 1, \ldots, n$. Let $\delta^{'}$ be the point so that

$$\delta^{'}_{ij} = \begin{cases} \delta^{n^2}_{ij} & \text{for all } i, j \in \{1, \ldots, n\}, \\ 0 & \text{if } i = j = n+1, \\ 1 - u^{n^2}_{jj} & \text{if } i = n+1 \text{ and } j \in \{1, \ldots, n\}, \\ 0 & \text{if } j = n+1 \text{ and } i \in \{1, \ldots, n\}. \end{cases}$$

The points $\overline{\delta}^k$ for $k = 1, \ldots, n^2$, v^l for $l = 1, \ldots, n$, w^r for $r = 1, \ldots, n$ and δ' all belong to $\mathcal{F}$. In fact, the inequality $\overline{a}^T \delta \leq \alpha$ is obviously tight for all these points. Moreover, the points v^l for $l = 1, \ldots, n$, w^r for $r = 1, \ldots, n$ and δ' are obtained by ordering the node $n+1$ either before or after the nodes of $\{1, \ldots, n\}$ that are already ordered. Each row of the following $(n+1)^2 \times (n+1)^2$ matrix A corresponds to one of these $(n+1)^2$ points: the n^2 first ones correspond to the points $\overline{\delta}^k$ for $k = 1, \ldots, n^2$, the n next ones to the points v^l for $l = 1, \ldots, n$, the n next ones to the points w^r for $r = 1, \ldots, n$ and the last row to the point δ'.

$$A = \begin{pmatrix} D & \mathbf{0} & \mathbf{0} & e^T \\ U & \mathbf{1} - N & \mathbf{0} & 0 \\ U & \mathbf{0} & \mathbf{1} - N & 0 \\ \delta^{n^2} & e - (\delta_{11}^{n^2}, \ldots, \delta_{nn}^{n^2}) & 0 & 0 \end{pmatrix}$$

Note that the last column of A corresponds to the variable $\delta_{(n+1)(n+1)}$ and that e is the point having all its entries equal to 1.

As mentioned above, the submatrix D is nonsingular. The matrix $\mathbf{1} - N$, where $\mathbf{1}$ is the $n \times n$ matrix having all its entries equal to 1, is nonsingular since N is so. Let μ be the point corresponding to the entries $\delta'_{(n+1)i}$ for $i = 1, \ldots, n$, that is, $\mu = e - (\delta_{11}^{n^2}, \ldots, \delta_{nn}^{n^2})$. By the definition of δ' and u^l for $l = 1, \ldots, n$, the point μ can be written as a linear combination of the n points $(v_{(n+1)1}^l, \ldots, v_{(n+1)n}^l)$, $l = 1, \ldots, n$, that is, $(\delta_{11}^{n^2}, \ldots, \delta_{nn}^{n^2}) = \sum_{l=1}^{n} \lambda_l (u_{11}^l, \ldots, u_{nn}^l)$. We then can replace the last row of the matrix A by

$$\begin{pmatrix} \delta^{n^2} - \sum_{l=1}^{n} \lambda_l u^l & \mathbf{0} & \mathbf{0} & 0 \end{pmatrix}.$$

We thus can deduce that the matrix A is nonsingular. We then have found $(n+1)^2$ linearly, hence affinely, independent points in $\mathcal{F}$. Therefore, $\mathcal{F}$ is a facet of P_{PLO}^{n+1}. $\square$

3.3 Simple facets

We now turn to the study of the simple facets of P_{PLO}^n.

Proposition 3 *Inequalities $\delta_{ij} \geq 0$ ($i \neq j$),* (12) *and* (13) *define facets of P_{PLO}^n.*

Proof. Inequalities (12) are a special case of the so-called k-clique inequalities studied in Section 4.1, hence the claim that inequalities (12) are facet-defining for P^n_{PLO} directly follows from Proposition 6.

Inequalities (13) are a special case of the so-called k-unicycle inequalities studied in Section 4.2, hence the claim that inequalities (13) are facet-defining for P^n_{PLO} directly follows from Proposition 8.

We now turn to inequality $\delta_{ij} \geq 0$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$.

First, we consider the following four points: the point which orders no nodes (i.e., $\delta_{kk} = 1$ for all $k \in \{1, \ldots, n\}$), orders only node i (i.e., $\delta_{ii} = 0$ and $\delta_{kk} = 1$ for all $k \in \{1, \ldots, n\} \setminus \{i\}$), orders only node j (i.e., $\delta_{jj} = 0$ and $\delta_{kk} = 1$ for all $k \in \{1, \ldots, n\} \setminus \{j\}$), orders only node j before node i (i.e., $\delta_{ii} = \delta_{jj} = 0$, $\delta_{kk} = 1$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$ and $\delta_{ji} = 1$). It is clear that $\delta_{ij} = 0$ for all these 4 points and that they are linearly, hence affinely, independent as for any of them there is a variable which is either 1 for the point and 0 for the others or vice-versa.

It follows that $\delta_{ij} \geq 0$ ($i \neq j$) is a facet of $P^{|\{i,j\}|}_{\mathrm{PLO}}$, hence a facet of P^n_{PLO} ($n \geq 2$) by Lemma 1. □

Proposition 4 *Inequalities $\delta_{ii} \geq 0$, $\delta_{ij} \leq 1$ and (14) do not define facets of P^n_{PLO}.*

Proof. We first consider inequality $\delta_{ii} \geq 0$ for $i \in \{1, \ldots, n\}$. First notice that if $\delta_{ii} = 0$ for some $i \in \{1, \ldots, n\}$ then, by (12) and (13), we have $\delta_{ij} + \delta_{ji} + \delta_{jj} = 1$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. Since P^n_{PLO} is full-dimensional we have

$$\{\delta \in P^n_{\mathrm{PLO}} : \delta_{ii} = 0\} \subsetneq \{\delta \in P^n_{\mathrm{PLO}} : \delta_{ij} + \delta_{ji} + \delta_{jj} = 1\} \subsetneq P^n_{\mathrm{PLO}},$$

for any $j \in \{1, \ldots, n\} \setminus \{i\}$. Hence $\delta_{ii} \geq 0$ is not facet-defining for P^n_{PLO}.

We now turn to inequality $\delta_{ij} \leq 1$ for $i, j \in \{1, \ldots, n\}$. If $i = j$ then it is easy to see that

$$\{\delta \in P^n_{\mathrm{PLO}} : \delta_{ii} = 1\} \subsetneq \{\delta \in P^n_{\mathrm{PLO}} : \delta_{ik} = 0\} \subsetneq P^n_{\mathrm{PLO}},$$

for any $k \in \{1, \ldots, n\} \setminus \{i\}$. Hence, $\delta_{ii} \leq 1$ is not facet-defining for P^n_{PLO}. If $i \neq j$, we then have

$$\{\delta \in P^n_{\mathrm{PLO}} : \delta_{ij} = 1\} \subsetneq \{\delta \in P^n_{\mathrm{PLO}} : \delta_{ii} = 0\} \subsetneq P^n_{\mathrm{PLO}},$$

for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, n\} \setminus \{i\}$. Therefore, $\delta_{ij} \leq 1$ does not define a facet of P^n_{PLO}.

Lastly, since $\delta_{ij} + \delta_{jk} - \delta_{ik} = 1$ $(i \neq j \neq k \neq i)$ implies that $\delta_{jj} = 0$, we have

$$\{\delta \in P^n_{\text{PLO}} : \delta_{ij} + \delta_{jk} - \delta_{ik} = 1\} \subsetneq \{\delta \in P^n_{\text{PLO}} : \delta_{jj} = 0\} \subsetneq P^n_{\text{PLO}}$$

and, hence, the transitivity inequalities are not facet-defining for P^n_{PLO}. $\square$

The transitivity inequalities can however be extended so as to obtain a class of facet-defining inequalities for P^n_{PLO}. This is the purpose of the following proposition.

Proposition 5 *Let $i, j, k \in \{1, \ldots, n\}$ with $i \neq j \neq k \neq i$. The extended transitivity inequality*

$$\delta_{ij} + \delta_{jk} - \delta_{ik} + \delta_{jj} \leq 1 \tag{15}$$

is a facet of P^n_{PLO}.

Proof. Validity is obvious since $\delta_{jj} = 1$ implies that $\delta_{ij} = \delta_{jk} = 0$ and since when $\delta_{jj} = 0$, inequality (15) is equivalent to a transitivity constraint (14).

We now prove that inequality (15) is indeed facet-defining for P^n_{PLO}. Let us consider the following points:

1. The point which orders no nodes (i.e., $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\}$), that is, such that $\delta_{ij} = \delta_{jk} = \delta_{ik} = 0$ and $\delta_{jj} = 1$.

2. The two points which order either only node i (i.e., $\delta_{ii} = 0$ and $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\} \setminus \{i\}$) or only node k (i.e., $\delta_{kk} = 0$ and $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\} \setminus \{k\}$), which are both such that $\delta_{ij} = \delta_{jk} = \delta_{ik} = 0$ and $\delta_{jj} = 1$.

3. The point which orders only node k before node i (i.e., $\delta_{ii} = \delta_{kk} = 0$, $\delta_{ll} \in \{1, \ldots, n\} \setminus \{i, k\}$ and $\delta_{ki} = 1$), that is, such that $\delta_{ij} = \delta_{jk} = \delta_{ik} = 0$ and $\delta_{jj} = 1$.

4. The two points which order either only node i before node j (i.e., $\delta_{ii} = \delta_{jj} = 0$, $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\} \setminus \{i, j\}$ and $\delta_{ij} = 1$) or only node j before node k (i.e., $\delta_{jj} = \delta_{kk} = 0$, $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\} \setminus \{j, k\}$ and $\delta_{jk} = 1$), which are respectively such that $\delta_{jk} = \delta_{ik} = \delta_{jj} = 0$ and $\delta_{ij} = 1$ and such that $\delta_{ij} = \delta_{ik} = \delta_{jj} = 0$ and $\delta_{jk} = 1$.

5. The point which orders only node i before node j, node i before node k and node j before node k (i.e., $\delta_{ii} = \delta_{jj} = \delta_{kk} = 0$, $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\} \setminus \{i, j, k\}$ and $\delta_{ij} = \delta_{ik} = \delta_{jk} = 1$), that is, such that $\delta_{jj} = 0$ and $\delta_{ij} = \delta_{ik} = \delta_{jk} = 1$.

6. The point which orders only node i before node j, node k before node i and node k before node j (i.e., $\delta_{ii} = \delta_{jj} = \delta_{kk} = 0$, $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\} \setminus \{i, j, k\}$ and $\delta_{ij} = \delta_{ki} = \delta_{kj} = 1$), that is, such that $\delta_{jj} = \delta_{ik} = \delta_{jk} = 0$ and $\delta_{ij} = 1$.

7. The point which orders only node j before node k, node k before node i and node j before node i (i.e., $\delta_{ii} = \delta_{jj} = \delta_{kk} = 0$, $\delta_{ll} = 1$ for all $l \in \{1, \ldots, n\} \setminus \{i, j, k\}$ and $\delta_{ji} = \delta_{jk} = \delta_{ki} = 1$), that is, such that $\delta_{jj} = \delta_{ij} = \delta_{ik} = 0$ and $\delta_{jk} = 1$.

It is clear that inequality (15) is tight for these 9 points of $P_{\mathrm{PLO}}^{|\{i,j,k\}|}$ and that they are linearly, hence affinely, independent. Hence, inequality (15) is facet-defining for $P_{\mathrm{PLO}}^{|\{i,j,k\}|}$ and, by Lemma 1, for P_{PLO}^{n}. □

4 Facets of the partial linear ordering polytope

Because optimizing over P_{PLO}^{n} allows to solve the linear ordering problem (simply set $c_{ii} = 0$ for all $i \in \{1, \ldots, n\}$), which is NP-hard, it is unlikely that an explicit linear description of it is ever to be obtained. Having said that, partial descriptions are known to be of the uttermost practical relevance, particularly when used within the framework of a branch-and-cut algorithm. Hence, this section is devoted to the study of several classes of facet-defining inequalities of P_{PLO}^{n}.

4.1 k-Clique inequalities

In this section we study a class of facet-defining inequalities for P_{PLO}^{n} which we call the k-clique inequalities.

Proposition 6 *Let $I \subseteq \{1, \ldots, n\}$ with $|I| = k$, the k-clique inequality*

$$\sum_{i \in I} \sum_{j \in I} \delta_{ij} \geq |I| - 1 \tag{16}$$

is a facet of P^n_{PLO}.

Proof. We first prove that inequality (16) is valid for P^n_{PLO}. Let D_I denote the complete digraph having I as node set. In order to order m nodes in I, it is required to select at least $m-1$ arcs of D_I. Hence a point of P^n_{PLO} which orders m nodes in I satisfies $\sum_{i \in I} \sum_{j \in I \setminus \{i\}} \delta_{ij} \geq m-1$ and $\sum_{i \in I} \delta_{ii} = |I|-m$. The claim of validity follows.

We now prove that inequality (16) is facet-defining for P^n_{PLO}. Let us consider the following points of $P^{|I|}_{\text{PLO}}$:

1. The points which only order node k, for each $k \in I$ (i.e., $\delta_{kk} = 0$ and $\delta_{ll} = 1$ for all $l \in I \setminus \{k\}$). There are $|I|$ such points and it is clear that inequality (16) is tight for them.

2. The points which only order node k_1 before node k_2, for each $k_1 \in I$ and $k_2 \in I$ with $k_1 \neq k_2$ (i.e., $\delta_{k_1k_1} = \delta_{k_2k_2} = 0$, $\delta_{ll} = 1$ for all $l \in I \setminus \{k_1, k_2\}$ and $\delta_{k_1k_2} = 1$). There are $|I|(|I|-1)$ such points and (again) it is clear that inequality (16) is tight for them.

Linear, hence affine, independence of the above $|I|^2$ points is straightforward. Hence, since they all belong to $P^{|I|}_{\text{PLO}}$, inequality (16) is facet-defining for $P^{|I|}_{\text{PLO}}$ and, as a consequence, is also facet-defining for P^n_{PLO} by Lemma 1. □

Figure 4 provides an example of a 4-clique inequality for P^6_{PLO} with $I = \{1, 2, 3, 5\}$.

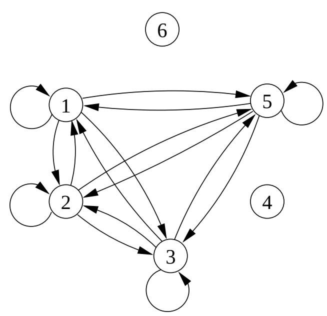

Figure 4: Example of a 4-clique on P^6_{PLO}.

As already stated, inequality (10) is nothing but a 2-clique inequality induced by $\{i, j\}$ and, hence, is facet-defining for P^n_{PLO}.

Also, note that for fixed k, the k-clique inequalities can be separated in polynomial time as there are only polynomially many (C_n^k) of them. This remark, however, is of little practical relevance.

Moreover, we have the following proposition.

Proposition 7 *The separation problem for the k-clique inequalities is NP-hard.*

Proof. It is easy to see that, given $\delta^\star \in \mathbb{R}^{n^2}$, the separation problem for the k-clique inequalities requires solving the mathematical program

$$\begin{cases} \text{Minimize } x^T \Delta x \\ \text{s. t. } x \in \{0,1\}^n, \end{cases}$$

where x is the incidence vector of set I and where Δ is a $n \times n$ matrix such that

$$\Delta_{ij} = \begin{cases} \delta^\star_{ij} & \text{if } i \neq j, \\ \delta^\star_{ii} - 1 & \text{otherwise.} \end{cases}$$

Then, consider an instance of the max-cut problem

$$\begin{cases} \text{Maximize } \sum_{i=1}^{n} \sum_{j=1}^{i-1} a_{ij} x_i (1 - x_j) \\ \text{s. t. } x \in \{0,1\}^n. \end{cases} \tag{17}$$

where x is the incidence vector of a node subset of the complete graph on n nodes and a_{ij} is the valuation of the edge incident to both i and j. Provided that (17) is equivalent to

$$\text{Minimize } \sum_{i=1}^{n} \sum_{j=1}^{i-1} a_{ij} x_i x_j - \sum_{i=1}^{n} x_i \sum_{j=1}^{i-1} a_{ij},$$

letting

$$\delta^\star_{ij} = \begin{cases} a_{ij} & \text{if } i > j, \\ 1 - \sum_{k=1}^{i-1} a_{ik} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

allows to use an algorithm solving the separation problem for the k-clique inequalities to solve the max-cut problem. Hence the claim follows. $\square$

Such a complexity result does not diminish the potential practical relevance of the k-clique inequalities, as, in general, separation problems do not need to be solved exactly when embedded within a branch-and-cut algorithm. Furthermore, there exists high quality heuristics for unconstrained binary quadratic problems (see e.g. Glover et al. [3]).

4.2 k-Unicycle inequalities

In this section we study a class of facet-defining inequalities for P^n_{PLO} which we call the k-unicycle inequalities.

Proposition 8 *Let $I \subset \{1, \dots, n\}$ with $|I| = k$ and $i_0 \in \{1, \dots, n\} \setminus I$. The k-unicycle inequality*

$$\delta_{i_0 i_0} + \sum_{i \in I} (\delta_{i i_0} + \delta_{i_0 i}) - \sum_{i \in I} \sum_{j \in I \setminus \{i\}} \delta_{ij} \leq 1 \tag{18}$$

is a facet of P^n_{PLO}.

Proof. We first show that inequality (18) is valid for P^n_{PLO} by considering the following cases:

1. A point which neither orders i_0 nor any nodes in I (i.e., $\delta_{ii} = 1$ for all $i \in I \cup \{i_0\}$) only selects the 1-valued loop of node i_0[2] and, hence, the left-hand side of (18) has value 1.

2. A point which orders i_0 and does not order any nodes in I (i.e., $\delta_{i_0 i_0} = 0$ and $\delta_{ii} = 1$ for all $i \in I$) does not select any arcs and, as a consequence, the left-hand side of (18) has value 0.

3. A point which does not order i_0 and orders nodes in $\emptyset \subset J \subseteq I$ (i.e., $\delta_{i_0 i_0} = 1$, $\delta_{ii} = 0$ for all $i \in J$ and $\delta_{ii} = 1$ for all $i \in I \setminus J$) selects the 1-valued loop of node i_0 as well as $\frac{|J|(|J|-1)}{2}$ (-1)-valued arcs (in-between the nodes in J), hence, the value of the left-hand side of (18) is less than or equal to 1.

4. A point which orders i_0 as well as the nodes in $\emptyset \subset J \subseteq I$ (i.e., $\delta_{i_0 i_0} = 0$, $\delta_{ii} = 0$ for all $i \in J$ and $\delta_{ii} = 1$ for all $i \in I \setminus J$) selects $|J|$ 1-valued

[2] Of the digraph induced by inequality (18).

arcs (between i_0 and the nodes in J) as well as $\frac{|J|(|J|-1)}{2}$ (-1)-valued arcs (in between the nodes in J), as a consequence, the value of the left-hand side of (18) is equal to $-|J|^2 < 0$.

The claim of validity then follows.

We now prove that inequality (18) is facet-defining for P^n_{PLO}. Let us consider the following points of $P^{|I\cup\{i_0\}|}_{\text{PLO}}$:

1. The point which orders no nodes (i.e., $\delta_{ii} = 1$ for all $i \in I \cup \{i_0\}$) and for which inequality (18) is obviously tight.

2. The points which only order node k, for each $k \in I$ (i.e., $\delta_{kk} = 0$, $\delta_{i_0 i_0} = 1$ and $\delta_{ii} = 1$ for all $i \in I \setminus \{k\}$). There are $|I|$ such points and inequality (18) is obviously tight for them (only the 1-valued loop of node i_0 is selected). Linear independence follows from the fact that a point in this set is the only point so far such that $\delta_{kk} = 0$.

3. The points which order either only node i_0 before node k or only node k before i_0, for each $k \in I$ (i.e., $\delta_{i_0 k} = 1$ or $\delta_{k i_0} = 1$ with $\delta_{ii} = 1$ for all $i \in I \setminus \{k\}$). There are $2|I|$ such points and inequality (18) is tight for them (only one 1-valued arc is selected, in-between i_0 and k). Linear independence follows from the fact that a point in this set is the only point so far such that either $\delta_{k i_0} = 1$ or $\delta_{i_0 k} = 1$.

4. The points which order only node i_0 before node k_1, node i_0 before node k_2 and either node k_1 before node k_2 or node k_2 before node k_1, for each $\{k_1, k_2\} \subseteq I$ (i.e., $\delta_{i_0 k_1} = \delta_{i_0 k_2} = 1$, $\delta_{k_1 k_2} = 1$ or $\delta_{k_2 k_1} = 1$, and $\delta_{ii} = 1$ for all $i \in I \setminus \{k_1, k_2\}$). There are $|I|(|I|-1)$ such points and inequality (18) is tight for them (two 1-valued arcs and one (-1)-valued arc are selected). Linear independence follows from the fact that a point in this set is the only point so far such that either $\delta_{k_1 k_2} = 1$ or $\delta_{k_2 k_1} = 1$.

Hence, we have exhibited a set of $(|I| + 1)^2$ linearly, hence affinely, independent points of $P^{|\{i_0\}\cup I|}_{\text{PLO}}$ for which inequality (18) is tight. It follows that inequality (18) is facet-defining for $P^{|\{i_0\}\cup I|}_{\text{PLO}}$ as well as for P^n_{PLO} by Lemma 1. □

Figure 5 provides an example of a 3-unicycle inequality for P^6_{PLO} with $i_0 = 1$ and $I = \{3, 4, 5\}$.

As already stated, inequality (11) is nothing but a 1-unicycle inequality with $i_0 = i$ and $I = \{j\}$. Hence, it is facet-defining for P^n_{PLO}.

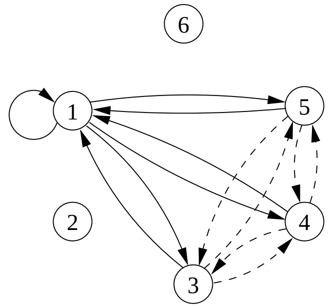

Figure 5: Example of a 3-unicycle on P_{PLO}^{6}. Dashed arcs are valued by -1.

Again, note that for fixed k, there are only polynomially many (nC_{n-1}^{k}) k-unicycle inequalities. Moreover, we have the following proposition.

Proposition 9 *The separation problem for the k-unicycle inequalities is NP-hard.*

Proof. Let $x_i = 1$, $i \in \{1, \ldots, n\}$, if and only if $i = i_0$ and let $y_i = 1$, $i \in \{1, \ldots, n\}$, if and only if $i \in I$. Given $\delta^\star \in \mathbb{R}^{n^2}$, the separation problem for the k-unicycle inequalities then requires solving the following mathematical program

$$\begin{cases} \text{Maximize } \displaystyle\sum_{i=1}^{n} \delta_{ii}^\star x_i + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\delta_{ij}^\star + \delta_{ji}^\star) x_i y_j - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \delta_{ij}^\star y_i y_j, \\ \text{s. t.} \\ \displaystyle\sum_{i=1}^{n} x_i = 1, & (19) \\ y_i \leq 1 - x_i \ \forall i \in \{1, \ldots, n\}, & (20) \\ x_i \in \{0, 1\} \ \forall i \in \{1, \ldots, n\}, \\ y_i \in \{0, 1\} \ \forall i \in \{1, \ldots, n\}, \end{cases}$$

where constraint (19) guarantees that an i_0 is chosen and where constraints (20) enforces that i_0 must not belong to I.

Then, consider an instance of the max-cut problem on the complete graph on n nodes (the notations are carried over from the proof of Proposition 7) as well as an instance of the separation problem for the k-unicycle inequalities in $\mathbb{R}^{(n+1)^2}$ where

$$\delta^\star_{ij} = \begin{cases} \delta^\star_{ij} = a_{ij} & \text{if } i < j, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$, where $\delta^\star_{i,n+1} = \sum_{i=1}^{j-1} a_{ij}$ and $\delta^\star_{n+1,i} = 0$ for $i \in \{1, \ldots, n\}$ as well as, finally, where $\delta^\star_{n+1,n+1}$ is set to a suitable huge value chosen in order to ensure that $i_0 = n+1$ (i.e., that $x_{n+1} = 1$) in all optimal solutions.

This allows to use an algorithm for solving the separation problem for the k-clique inequalities on P^{n+1}_{PLO} to solve the max-cut problem on the complete graph on n nodes. Hence the claim follows. $\square$

4.3 k-l-Bicycle inequalities

In this section we study a class of facet defining inequalities for P^n_{PLO} which we call the k-l-bicycle inequalities.

Proposition 10 *Let $i_0 \in \{1, \ldots, n\}$, $j_0 \in \{1, \ldots, n\} \setminus \{i_0\}$, $\emptyset \subset I \subset \{1, \ldots, n\} \setminus \{i_0, j_0\}$ and $\emptyset \subset J \subset \{1, \ldots, n\} \setminus \{i_0, j_0\}$ with $|I| = k$, $|J| = l$ and $I \cap J = \emptyset$. The k-l-bicycle inequality*

$$\begin{aligned} &\delta_{i_0 i_0} + \delta_{j_0 j_0} + \delta_{i_0 j_0} + \sum_{i \in I} (\delta_{i i_0} + \delta_{i_0 i} - \delta_{i j_0}) + \sum_{j \in J} (\delta_{j j_0} + \delta_{j_0 j} - \delta_{i_0 j}) \\ &- \sum_{i \in I} \sum_{i' \in I \setminus \{i\}} \delta_{i i'} - \sum_{j \in J} \sum_{j' \in J \setminus \{j\}} \delta_{j j'} - \sum_{i \in I} \sum_{j \in J} \delta_{j i} \leq 2 \end{aligned} \tag{21}$$

is a facet of P^n_{PLO}.

Proof. We first prove that inequality (21) is valid for P^n_{PLO}. Let us consider a point which orders the moves in $I' \subseteq I$ and the moves in $J' \subseteq J$. Note that such a point necessarily selects $\frac{|I'|(|I'|-1)}{2}$ and $\frac{|J'|(|J'|-1)}{2}$ (-1)-valued arcs (in-between the nodes in I' and J', respectively). Now we prove the validity claim by considering the following cases:

1. If the point orders neither node i_0 nor node j_0 (i.e., $\delta_{i_0 i_0} = \delta_{j_0 j_0} = 1$), then the left-hand side of (21) has value $2 - \frac{|I'|(|I'|-1)}{2} - \frac{|J'|(|J'|-1)}{2} \leq 2$.

2. If the point orders node i_0 and does not order node j_0 (i.e., $\delta_{i_0 i_0} = 0$ and $\delta_{j_0 j_0} = 1$), then the left-hand side of (21) has value at most $1 + |I'| - \frac{|I'|(|I'|-1)}{2} - \frac{|J'|(|J'|-1)}{2}$, that is, is less than or equal to 2 since $|I'| - \frac{|I'|(|I'|-1)}{2} \leq 1$.

3. If the point orders node j_0 and does not order node i_0 (i.e., $\delta_{j_0j_0} = 0$ and $\delta_{i_0i_0} = 1$), then the left-hand side of (21) has value at most $1 - \frac{|I'|(|I'|-1)}{2} + |J'| - \frac{|J'|(|J'|-1)}{2}$, that is, is less than or equal to 2 for the same reason as above.

4. If the point orders both node i_0 and node j_0 (i.e., $\delta_{i_0i_0} = \delta_{j_0j_0} = 0$), then there is at most $1+|I'|+|J'|$ selected 1-valued arcs ($\{i_0, j_0\}$ along with the arcs between the nodes in I' and i_0 and the arcs between the nodes in J' and j_0) along with the $\frac{|I'|(|I'|-1)}{2} + \frac{|J'|(|J'|-1)}{2}$ selected (-1)-valued arcs (in-between the nodes in I' and in-between the nodes in J'). We have $1 + |I'| - \frac{|I'|(|I'|-1)}{2} + |J'| - \frac{|J'|(|J'|-1)}{2} \leq 3$, equality occurring only when both I' and J' are of cardinality 1 or 2 and $\delta_{i_0j_0} = 1$. Let $i \in I'$ and $j \in J'$ and assume that $\delta_{i_0j_0} = 1$. If i precedes i_0 then since i_0 precedes j_0 the (-1)-valued arc from i_0 to j must be selected (by transitivity). If j_0 precedes j then since i_0 precedes j_0 the (-1)-valued arc from i_0 to j must be selected (by transitivity). So assume that i_0 precedes i and that j precedes j_0. We then have that either the (-1)-valued arcs between j and i is selected or both (-1)-valued arcs from i to j_0 and from i_0 to j must be selected so as to break the cycles in the digraph shown on Figure 6. Hence, at least one (-1)-valued arc is selected on top of the $\frac{|I'|(|I'|-1)}{2} + \frac{|J'|(|J'|-1)}{2}$ already identified ones. Therefore the left-hand side of (21) is less than or equal to 2.

The claim of validity then follows.

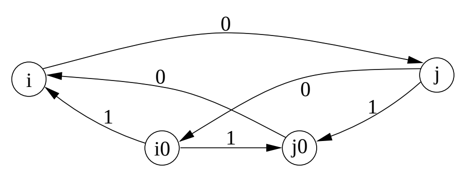

Figure 6: A non feasible ordering with $\delta_{i_0i} = \delta_{jj_0} = \delta_{i_0j_0} = 1$.

We now prove that inequality (21) is facet defining for P^n_{PLO}. Let us consider the following points of $P^{|I\cup J\cup\{i_0,j_0\}|}_{\mathrm{PLO}}$:

1. The point which orders no nodes (i.e., $\delta_{ii} = 1$ for all $i \in I \cup J \cup \{i_0, j_0\}$) for which inequality (21) is obviously tight.

2. The points which only order node k for each $k \in I \cup J$ (i.e., $\delta_{kk} = 0$, $\delta_{i_0 i_0} = \delta_{j_0 j_0} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k\}$). There are $|I| + |J|$ such points and it is clear that inequality (21) is tight for them.

3. The points which only order either node k before node i_0 or node i_0 before node k for each $k \in I$ (i.e., $\delta_{k i_0} = 1$ or $\delta_{i_0 k} = 1$, $\delta_{j_0 j_0} = 1$ and $\delta_{ii} = 1$ for each $i \in (I \cup J) \setminus \{k\}$). There are $2|I|$ such points and inequality (21) is tight for them.

4. The points which only order either node k before node j_0 or node j_0 before node k for each $k \in J$ (i.e., $\delta_{k j_0} = 1$ or $\delta_{j_0 k} = 1$, $\delta_{i_0 i_0} = 1$ and $\delta_{ii} = 1$ for each $i \in (I \cup J) \setminus \{k\}$). There are $2|J|$ such points and inequality (21) is tight for them.

5. The points which only order node k_1 before node k_2 for each $k_1 \in I$ and $k_2 \in J$ (i.e., $\delta_{i_0 i_0} = \delta_{j_0 j_0} = \delta_{k_1 k_2} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$)). There are $|I||J|$ such points and inequality (21) is tight for them.

6. The points which only order node i_0 before node j_0, node i_0 before node k and node j_0 before node k for each $k \in I$ (i.e., $\delta_{i_0 j_0} = \delta_{i_0 k} = \delta_{j_0 k} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k\}$). There are $|I|$ such points and inequality (21) is tight for them.

7. The points which only order node k before node i_0, node k before node j_0 and node i_0 before node j_0 for each $k \in J$ (i.e., $\delta_{k i_0} = \delta_{k j_0} = \delta_{i_0 j_0} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k\}$). There are $|J|$ such points and inequality (21) is tight for them.

8. Given any $k_1 \in I$ and any $k_2 \in J$, the point which only orders node j_0 before node k_1, node j_0 before node k_2, node j_0 before node i_0, node k_1 before node k_2, node k_1 before node i_0 and node k_2 before node i_0 (i.e., $\delta_{j_0 k_1} = \delta_{j_0 k_2} = \delta_{j_0 i_0} = \delta_{k_1 k_2} = \delta_{k_1 i_0} = \delta_{k_2 i_0} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). Inequality (21) is tight for this point.

9. The points which only order node i_0 before node k_1, node i_0 before node k_2 and node k_1 before node k_2 for each $k_1 \in I$ and $k_2 \in I \setminus \{k_1\}$ (i.e., $\delta_{i_0 k_1} = \delta_{i_0 k_2} = \delta_{k_1 k_2} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). There are $|I|(|I| - 1)$ such points and inequality (21) is tight for them. Note that variable $\delta_{k_1 k_2}$ has a -1 coefficient in inequality (21).

10. The points which only order node j_0 before node k_1, node j_0 before node k_2 and node k_1 before node k_2, for each $k_1 \in J$ and $k_2 \in J \setminus \{k_1\}$ (i.e., $\delta_{j_0k_1} = \delta_{j_0k_2} = \delta_{k_1k_2} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). There are $|J|(|J|-1)$ such points and inequality (21) is tight for them. Note that variable $\delta_{k_1k_2}$ has a -1 coefficient in inequality (21).

11. Given any $k_1 \in J$, the points which only order node k_2 before node k_1, node k_2 before node i_0, node k_2 before node j_0, node k_1 before node i_0, node k_1 before node j_0 and node i_0 before node j_0 for each $k_2 \in I$ (i.e., $\delta_{k_2k_1} = \delta_{k_2i_0} = \delta_{k_2j_0} = \delta_{k_1i_0} = \delta_{k_1j_0} = \delta_{i_0j_0} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). There are $|I|$ such points and inequality (21) is tight for them. Note that variable $\delta_{k_2j_0}$ has a -1 coefficient in inequality (21).

12. Given any $k_1 \in I$, the points which only order node i_0 before node j_0, node i_0 before node k_1, node i_0 before node k_2, node j_0 before node k_1, node j_0 before node k_2 and node k_1 before node k_2 for each $k_2 \in J$ (i.e., $\delta_{i_0j_0} = \delta_{i_0k_1} = \delta_{i_0k_2} = \delta_{j_0k_1} = \delta_{j_0k_2} = \delta_{k_1k_2} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). There are $|J|$ such points and inequality (21) is tight for them. Note that variable $\delta_{i_0k_2}$ has a -1 coefficient in inequality (21).

13. The points which only order node k_2 before node i_0, node k_2 before node j_0, node k_2 before node k_1, node i_0 before node j_0, node i_0 before node k_1 and node j_0 before node k_1 for each $k_1 \in I$ and each $k_2 \in J$ (i.e., $\delta_{k_2i_0} = \delta_{k_2j_0} = \delta_{k_2k_1} = \delta_{i_0j_0} = \delta_{i_0k_1} = \delta_{j_0k_1} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). There are $|I||J|$ such points and inequality (21) is tight for them. Note that variable $\delta_{k_2k_1}$ has a -1 coefficient in inequality (21).

14. Given any $k_1 \in I$ and any $k_2 \in J$, the point which only orders node k_1 before node k_2, node k_1 before node i_0, node k_2 before node i_0 (i.e., $\delta_{k_1k_2} = \delta_{k_1i_0} = \delta_{k_2i_0} = \delta_{j_0j_0} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). Inequality (21) is tight for this point.

15. Given any $k_1 \in I$ and any $k_2 \in J$, the point which only orders node j_0 before node k_1, node j_0 before node k_2, node k_1 before node k_2 (i.e., $\delta_{j_0k_1} = \delta_{j_0k_2} = \delta_{k_1k_2} = \delta_{i_0i_0} = 1$ and $\delta_{ii} = 1$ for all $i \in (I \cup J) \setminus \{k_1, k_2\}$). Inequality (21) is tight for this point.

Points in the union S of the sets 1 to 13 given above belong to $P_{\text{PLO}}^{|I \cup J \cup \{i_0, j_0\}|}$ and are linearly, hence affinely, independent. This is so because for each point in S there is a variable which takes value 1 (respectively 0) and which has value 0 (respectively 1) for all the preceding points.

Furthermore, it is easy to verify that point 14 is the only one satisfying

$$\delta_{j_0 j_0} + \delta_{k_1 i_0} + \delta_{k_2 i_0} = 3,$$

and that point 15 is the only one satisfying

$$\delta_{i_0 i_0} + \delta_{j_0 k_1} + \delta_{j_0 k_2} = 3.$$

Therefore, we have exhibited a set of $1 + (|I| + |J|) + 2|I| + 2|J| + |I||J| + |I| + |J| + 1 + |I|(|I| - 1) + |J|(|J| - 1) + |I| + |J| + |I||J| + 1 + 1 = (|I| + |J| + 2)^2$ affinely independent points of $P_{\text{PLO}}^{|I \cup J \cup \{i_0, j_0\}|}$. It follows that inequality (21) is facet-defining for $P_{\text{PLO}}^{|\{i_0, j_0\} \cup I \cup J|}$ and, by Lemma 1, also facet-defining for P_{PLO}^{n}. $\square$

Figure 7 provides an example of a 2-3-bicycle inequality on P_{PLO}^{9} with $i_0 = 7$, $j_0 = 1$, $I = \{8, 9\}$ and $J = \{3, 4, 5\}$.

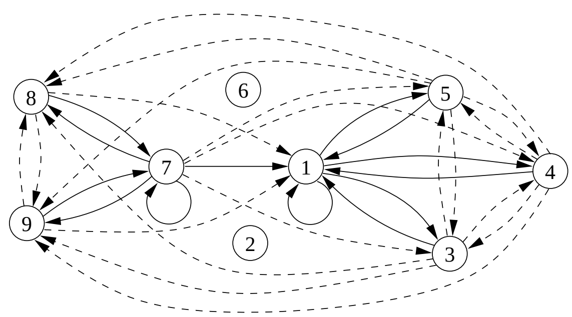

Figure 7: Example of a 2-3-bicycle on P_{PLO}^{9}. Dashed arcs are valued by -1.

Yet again, there are only polynomially many ($n^2 C_{n-2}^{k} C_{n-k-2}^{l}$) k-l-bicycle inequalities, for fixed k and l. Finally, we have the following proposition.

Proposition 11 *The separation problem for the k-l-unicycle inequalities is NP-hard.*

Proof. The proof is left to the reader, due to its obvious similarity with that of Proposition 9. $\square$

5 Conclusion

In this paper, we have introduced an integer linear program formulation, based on linear ordering variables, for the *process move programming problem*, a strongly NP-hard scheduling problem which consists, starting from an arbitrary initial process distribution on the processors of a distributed system, of finding the least disruptive sequence of operations (non-impacting process migrations or temporary process interruptions) at the end of which the system ends up in another predefined arbitrary state. The main constraint is that the capacity of the processors must not be exceeded during the reconfiguration. This problem has applications in the design of high availability real-time distributed switching systems such as the one discussed in Sirdey et al. [12].

Ignoring the capacity constraints, as a first step, lead us to define and study the *partial linear ordering polytope*. In particular, we have introduced several classes of facet-defining inequalities for this polytope and, as we shall demonstrate in the companion paper, it turns out that they all define facets of the *process move program polytope* under mildly restrictive assumptions.

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