Polyhedral combinatorics of a resource-constrained ordering problem part II: on the process move program polytope * †

Hervé Kerivin[a] and Renaud Sirdey[b] [c]

[a] UMR CNRS Limos (Université de Clermont-Ferrand II), Complexe scientifique des Cézeaux, 63177 Aubière Cedex, France.
[b] Service d'architecture BSC (PC 12A7), Nortel GSM Access R&D, Parc d'activités de Magny-Châteaufort, 78928 Yvelines Cedex 09, France.
[c] UMR CNRS Heudiasyc (Université de Technologie de Compiègne), Centre de recherches de Royallieu, BP 20529, 60205 Compiègne Cedex, France.

Corresponding author: R. Sirdey (*renauds@nortel.com*, +33 (0)1 69 55 41 18).

September 19, 2006

Abstract

This paper is the second of a series of two devoted to the polyhedral study of a strongly NP-hard resource-constrained scheduling problem, referred to as the *process move programming problem*. In the present paper, we put back into the picture the capacity constraints which were ignored in the first paper. In doing so, we introduce the *process move program polytope*, study its basic properties and show several classes of inequalities to be facet-defining. Some of the latter were proved to be facet-defining for the *partial linear ordering polytope* which was both introduced and studied in the companion paper.

1 Introduction

Recall that the *process move programming problem* (denoted *PMP problem*) consists, starting from an arbitrary initial distribution of processes on the processors of a distributed system, of finding the least disruptive sequence

*This research was supported in part by Association Nationale de la Recherche Technique grant CIFRE-121/2004.

†Technical report Nortel GSM Access R&D PE/BSC/INF/017913 V01/EN, submitted to Mathematical Programming.

of operations (i.e., non-impacting process migrations or temporary process interruptions) at the end of which the system ends up in another predefined arbitrary state. The main constraints are that the capacity of the processors must not be exceeded during the reconfiguration and that a process is moved (i.e., migrated or interrupted) exactly once.

More precisely, let U and M respectively denote the set of processors and the set of process moves. Then for each $m \in M$, c_m, w_m, s_m and t_m respectively denote the cost of interrupting the process moved by m (i.e., of *interrupting* m), the amount of resource consumed by the process moved by m (i.e., the *weight* of m), the processor from which the process is moved (i.e., the *source* of m) and the processor to which the process is moved (i.e., the *target* of m). Also, K_u denotes the initial remaining capacity of processor $u \in U$, $S(u) = \{m \in M : s_m = u\}$ and $T(u) = \{m \in M : t_m = u\}$. Lastly, a pair (I, σ), where $I \subseteq M$ and $\sigma : M \setminus I \longrightarrow \{1, \ldots, |M \setminus I|\}$ is a bijection, defines an admissible *process move program* if, provided that the moves in I are interrupted (for operational reasons, the interruptions are performed at the beginning of the reconfiguration), the other moves can be performed (i.e., migrated) according to σ without inducing any violation of the capacity constraints.

Let $\delta_{mm} = 1$ if and only if $m \in M$ is interrupted and $\delta_{mm'} = 1$ if and only if $m \in M$ is performed (without interruption) before $m' \in M \setminus \{m\}$ is performed (also without interruption). As shown in the companion paper [5], the process move programming problem can then be expressed as the integer linear program given in Figure 1. Constraints (1), (2), (3) and (4) are respectively referred to as the 2-*clique inequalities*, the 1-*unicycle inequalities*, the *extended transitivity inequalities* and the *capacity inequalities*.

This paper is devoted to the study of the polytope associated with the above program, called the *process move program polytope* and denoted P^M_{PMP}. In Section 2, we study the basic properties of P^M_{PMP}. In particular, we establish necessary and sufficient conditions for full dimensionality as well as investigate conditions under which the inequality classes $0 \leq \delta_{mm'}$ (for $m, m' \in M$), (1), (2), (3) and (4) are facet-defining for P^M_{PMP}. In Section 3, we build on results obtained in the companion paper [5] and investigate which of the classes of facet-defining inequalities indentified for the partial linear ordering polytope also define facets for P^M_{PMP} and under which conditions. We then subsequently introduce two classes of inequalities, the *source* and

$$
\mathcal{P}\begin{cases}
\text{Minimize} \displaystyle\sum_{m \in M} c_m \delta_{mm} & \\
\text{s. t.} & \\
\delta_{mm'} + \delta_{m'm} + \delta_{mm} + \delta_{m'm'} \geq 1 & \forall \{m, m'\} \subseteq M, \quad (1) \\
\delta_{mm'} + \delta_{m'm} + \delta_{mm} \leq 1 & m \neq m' \in M, \quad (2) \\
\delta_{mm'} + \delta_{m'm''} - \delta_{mm''} + \delta_{m'm'} \leq 1 & m \neq m' \neq m'' \neq m \in M, \quad (3) \\
(1 - \delta_{mm}) w_m - \displaystyle\sum_{m' \in S(t_m)} w_{m'} (\delta_{m'm'} + \delta_{m'm}) + \sum_{m' \in T(t_m) \setminus \{m\}} w_{m'} \delta_{m'm} \leq K_{t_m} & \forall m \in M, \quad (4) \\
\delta_{mm'} \in \{0, 1\} & m, m' \in M. \quad (5)
\end{cases}
$$

Figure 1: Formulation of the PMP problem as an integer linear program.

target cover inequalities, valid for P^M_{PMP}, and we provide both necessary and sufficient conditions for them to be facet-defining as well as pseudopolynomial separation algorithms.

2 The process move program polytope

The process move program polytope

$$P^M_{\text{PMP}} = \text{conv}\{\delta^{n^2} \in \mathbb{R}^{n^2} : \delta \text{ satisfies (1)-(5)}\}$$

where $n = |M|$, corresponds to the points of the partial linear ordering polytope P^n_{PLO} that satisfy the capacity constraints (4). Therefore we first study the necessary and sufficient conditions that make facet-defining inequalities of P^n_{PLO} remain facets of P^M_{PMP}. (Remark that any valid but not facet-defining inequality of P^n_{PLO} cannot obviously define a facet of P^M_{PMP}.) To do so, the following notations are introduced. Given two subsets of moves $\{m_1, \ldots, m_r\} \subseteq M$ and $X \subseteq M$ with $1 \leq r \leq n$ and $\{m_1, \ldots, m_r\} \cap X = \emptyset$ (X may be empty), a *incomplete process move program* is denoted by $[m_1, \ldots, m_r; X]$ if the only specified ordering is the one on $\{m_1, \ldots, m_r\}$, that is, the pair (I, σ) is chosen so that $I = M \setminus (\{m_1, \ldots, m_r\} \cup X)$ and $\sigma(m_i) < \sigma(m_j)$ for all $i,\ j \in \{1, \ldots, r\}$ with $i < j$. (Remark that a process move program corresponds to when $X = \emptyset$.) The incomplete process move program $[m_1, \ldots, m_r; X]$ is *admissible* if and only if there exists a point $\delta \in \mathbb{R}^{n^2}$ in P^M_{PMP} so that

$$\delta_{mm} = \begin{cases} 0 & \text{if } m \in \{m_1, \ldots, m_r\} \cup X, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\delta_{m_i m_j} = \begin{cases} 1 & \text{if } m_i,\ m_j \in \{m_1, \ldots, m_r\} \text{ with } i < j, \\ 0 & \text{if } m_i,\ m_j \in \{m_1, \ldots, m_r\} \text{ with } i \geq j. \end{cases}$$

Note that δ is not necesseraly unique because the imposed ordering is only on the process moves in $\{m_1, \ldots, m_r\}$. If $X = \emptyset$, the point δ is then unique and the process move program is simply denoted by $[m_1, \ldots, m_r]$.

2.1 Basic properties

Proposition 1 *Polytope P^M_{PMP} is full-dimensional if and only if $[m, m']$ is an admissible program for all $m, m' \in M$ with $m \neq m'$.*

Proof. If there exist $m, m' \in M$ with $m \neq m'$ and $[m, m']$ is not admissible, we then have $P^M_{\text{PMP}} \subseteq \{\delta \in \mathbb{R}^{n^2} : \delta_{mm'} = 0\}$. Hence $\dim P^M_{\text{PMP}} \leq n^2 - 1$.

Conversely, the admissibility of the program $[m, m']$ for all $m, m' \in M$ with $m \neq m'$ implies that the $n^2 + 1$ affinely independent vertices considered in the proof of Proposition 2 in [5] still belong to P^M_{PMP}. (Remark that with the aim of shortening some of the proofs throughout the paper, we will refer to points of P^n_{PLO} that we used in proofs in the companion paper [5] instead of introducing them again.) □

In the sequel, the full dimensional of P^M_{PMP} is assumed, unless otherwise stated. Given $N \subseteq M$, the process move programming problem restricted to N is obtained from the one associated with M by only considering the process moves in N and by increasing the initial remaining capacity of $u \in U$ by $\sum_{m \in S(u) \cap (M \setminus N)} w_m$, that is, by considering that the process moves in $M \setminus N$ are interrupted. Remark that the polytope P^N_{PMP} is full-dimensional as well.

Given two distinct admissible programs p and p', we say that p' is *dominated* by p if for any pair of distinct process moves $m_1, m_2 \in M$, m_1 is ordered before m_2 in p' implies m_1 is ordered before m_2 in p. This dominance relationship is denoted $p' \prec p$. Note that the set of interrupted process moves in p is included in the set of interrupted processes in p'.

Let $a^T\delta \leq \alpha$ induce a facet $\mathcal{F}$ of P^n_{PLO}. Given $m_1, m_2 \in M$ with $m_1 \neq m_2$, consider the restriction $\mathcal{F}_{m_1 m_2}$ of $\mathcal{F}$ onto the subspace of $\mathbb{R}^{n^2}$ defined by $\delta_{m_1 m_2} = 1$, that is, $\mathcal{F}_{m_1 m_2} = \mathcal{F} \cap \{\delta \in \mathbb{R}^{n^2} : \delta_{m_1 m_2} = 1\}$. Let $A_{m_1 m_2}$ be the set of programs associated with the points of $\mathcal{F}_{m_1 m_2}$ and so that for any program $p \in A_{m_1 m_2}$, there does not exist $p' \in A_{m_1 m_2} \setminus \{p\}$ with $p' \prec p$. The set $A_{m_1 m_2}$ contains nothing but the programs that order m_1 before m_2 and that involve the minimum number of process moves. Note that it does not mean that all the programs in $A_{m_1 m_2}$ involve the same number of moves. We can now give a necessary condition for facet-defining inequalities of P^n_{PLO} to be facet-defining for P^M_{PMP}.

Proposition 2 *Let $a^T\delta \leq \alpha$ define a nontrivial facet $\mathcal{F}$ of P^n_{PLO}. Then $a^T\delta \leq \alpha$ defines a facet of P^M_{PMP} only if for all $m_1, m_2 \in M$ with $m_1 \neq m_2$, there exists an admissible program in $A_{m_1 m_2}$.*

Proof. If there exist $m_1, m_2 \in M$ so that $m_1 \neq m_2$ and no programs in $A_{m_1 m_2}$ are admissible, the face of P^M_{PMP} induced by $a^T\delta \leq \alpha$ is then included in the

one induced by $\delta_{m_1 m_2} \geq 0$. Since $a^T\delta \leq \alpha$ is a nontrivial inequality and P^M_{PMP} is assumed to be full-dimensional, $a^T\delta \leq \alpha$ does not define a facet of P^M_{PMP}. $\square$

Unlike what we got for the partial linear ordering polytope (see Lemma 1 in [5]), we do not have a trivial lifting lemma for the process move program polytope that from a facet-defining inequality of P^M_{PMP}, gives a facet-defining inequality of $P^{M\cup\{m\}}_{\text{PMP}}$, $m \notin M$, by setting zero coefficients to the variables involving m. Nevertheless, we prove two restricted lifting lemmas for the process move program polytope that will be useful to prove some of the following results. In fact using these lifting results, we will mostly need to focus on the process move subset supporting the considered inequality to prove that the latter is facet-defining for P^M_{PMP}.
To make the statements of the lifting lemmas clearer (and their use as well), we introduce some additional notations. Given an inequality $a^T\delta \leq \alpha$ that defines a face $\mathcal{F}$ of P^M_{PMP}, let $M_0 = \{m_0 \in M : a_{m_0 m} = a_{m m_0} = 0 \text{ for all } m \in M\}$ and $M_s = M \setminus M_0$. (Remark that M_s is the process move subset that supports the inequality $a^T\delta \leq \alpha$, that is, for any $m \in M_s$, there exists a non-zero coefficient in $a^T\delta \leq \alpha$ that corresponds to a variable involving m.) Denote by $\mathcal{F}_s$ the face of $P^{M_s}_{\text{PMP}}$ induced by $a^T\delta \leq \alpha$. Let

$$F_i = \{m_i \in M_s : \max\{\sum_{m\in M_s} \delta_{mm} : \delta \in \mathcal{F}_s \text{ and } \delta_{m_i m_i} = 0\} = |M_s| - i\}$$

for $i = 1, \ldots, s$ where $s \leq n$ and $F_j = \emptyset$ for all $j \in \{s+1, \ldots, n_s\}$. The process move subsets F_i for all $\{1, \ldots, s\}$ are clearly pairwise disjoint and their union is nothing but M_s.

Proposition 3 *Let $a^T\delta \leq \alpha$ be a valid inequality of P^M_{PMP}. If $a^T\delta \leq \alpha$ is facet-defining for $P^{M_s}_{PMP}$ and the two following conditions hold*

i) the program $[\emptyset]$ induces a point of P^M_{PMP} satisfying $a^T\delta \leq \alpha$ with equality,

ii) for any $m_0 \in M_0$ and $m \in F_k$ with $k \in \{2, \ldots, s\}$, there exist $X_a \subseteq \bigcup_{h=1}^{k-1} F_h$ and $X_b \subseteq \bigcup_{h=1}^{k-1}$ so that the incomplete programs $[m_0, m; X_a]$ and $[m, m_0; X_b]$ induce points of P^M_{PMP} satisfying $a^T\delta \leq \alpha$ with equality,

then the inequality $a^T\delta \leq \alpha$ is also facet-defining for P^M_{PMP}.

Proof. Since $a^T\delta \leq \alpha$ defines a facet, say $\mathcal{F}_s$, of $P_{\text{PMP}}^{M_s}$, there exist n_s^2 affinely-independent points $\delta^1, \ldots, \delta^{n_s^2}$ of $\mathcal{F}_s$, where $n_s = |M_s|$. For $i = 1, \ldots, n_s^2$, let $\overline{\delta}^i \in \{0,1\}^{n^2}$ be so that

$$\overline{\delta}^i_{mm'} = \begin{cases} \delta^i_{mm'} & \text{for all } m,\ m' \in M_s, \\ 1 & \text{if } m = m' \in M_0, \\ 0 & \text{otherwise.} \end{cases}$$

We clearly have $\overline{\delta}^i \in \mathcal{F} = \{\delta \in P_{\text{PMP}}^M : a^T\delta = \alpha\}$ for $i = 1, \ldots, n_s^2$, and these n_s^2 points are affinely independent.
Let m_0 and m_0' be two distinct points of M_0. Because of Condition i) and the full-dimension assumption on P_{PMP}^M, the program $[m_0, m_0']$ clearly corresponds to a point of $\mathcal{F}$. Denote by S_0 the set of $n_0(n_0-1)$ points hence obtained, where $n_0 = |M_0|$. The sets of points $\{\overline{\delta}^i : i = 1, \ldots, n_s^2\}$ and S_0 form an affinely-independent set since any point in S_0 is the only one having a variable $\delta_{m_0 m_0'}$, $\{m_0, m_0'\} \subseteq M_0$, equal to 1.
Consider now a process move $m_k \in F_k$ with $k \in \{1, \ldots, s\}$. By Condition ii), for any $m_0 \in M_0$ there exist $X_a \subseteq \bigcup_{h=1}^{k-1} F_h$ and $X_b \subseteq \bigcup_{h=1}^{k-1} F_h$ so that the incomplete programs $[m_0, m; X_a]$ and $[m, m_0; X_b]$ induce points $\overline{\delta}^{m_k}$ and $\tilde{\delta}^{m_k}$, respectively, that belong to $\mathcal{F}$. (Remark that if $k = 1$, we have $X_a = X_b = \emptyset$ and from the full-dimension assumption on P_{PMP}^M those two points belong to $\mathcal{F}$.) It is obvious that the points $\overline{\delta}^{m_k}$, $\tilde{\delta}^{m_k}$ and $\overline{\delta}^1, \ldots, \overline{\delta}^{n_s^2}$ are affinely independent. Moreover because of $X_a \cap F_h = \emptyset$ and $X_b \cap F_h = \emptyset$ for $h = k, \ldots, s$, the $2n_0 \sum_{k=1}^{s} |F_k|$ points obtained as previously described are affinely independent. In fact, they can be ordered so that any of these points corresponds to the first one having a variable $\delta_{mm'}$ equal to 1 where $|\{m, m'\} \cap M_s| = 1$.
Finally for any process move m_0 of M_0, Condition i) implies that the program $[m_0]$ induces a point δ^{m_0} that belongs to $\mathcal{F}$. Point δ^{m_0} together with all the ones we have considered so far form an affinely-independent set since it is the only one satisfying

$$\delta_{m_0 m_0} + \sum_{m \in M_s} (\delta_{m_0 m} + \delta_{m m_0}) = 0.$$

Therefore, we have obtained $n_s^2 + n_0(n_0-1) + 2n_0 \sum_{k=1}^{s} |F_k| + n_0 = n_s^2 + n_0^2 + 2n_0 n_s = n^2$ affinely-independent points of $\mathcal{F}$. Our proof is then completed.

$\square$

Before stating our second restricted lifting result, we give a technical lemma that will be useful in some of the forthcoming proofs.

Lemma 1 *Let $m_1, m_2, m_3 \in M$ with $m_1 \neq m_2 \neq m_3 \neq m_1$. We then have $P^M_{PMP} \cap \{\delta \in \{0,1\}^{n^2} : \delta_{m_1 m_2} = 1$ and $\delta_{m_3 m_3} = 0\} \neq \emptyset$.*

Proof. Let $\overline{\delta} \in X = P^M_{\text{PMP}} \cap \{\delta \in \{0,1\}^{n^2} : \delta_{m_1 m_2} = 1 \text{ and } \delta_{m_3 m_3} = 0\}$. We first remark that if $\overline{\delta}_{mm} = 0$ for any $m \in M \setminus \{m_1, m_2, m_3\}$, then the point $\tilde{\delta}$ obtained from $\overline{\delta}$ by setting $\tilde{\delta}_{mm} = 1$, $\tilde{\delta}_{mm'} = \tilde{\delta}_{mm'} = 0$ for all $m' \in M \setminus \{m\}$ and keeping all the other components unchanged, also belongs to X. Therefore we can suppose without loss of generality that $\overline{\delta}_{mm} = 1$ for all $m \in M \setminus \{m_1, m_2, m_3\}$.
Assume $t_{m_1} \neq s_{m_2}$, that is, the target processor of move m_1 is different than the source processor of move m_2. Since full dimension is assumed, the process move program $[m_1, m_3, m_2]$ is admissible. Hence, the associated point belongs to X. Suppose now that $t_{m_1} = s_{m_2}$. If $s_{m_2} = s_{m_3}$, the program $[m_3, m_1, m_2]$ is admissible. If $s_{m_2} \neq s_{m_3}$, the program $[m_1, m_2, m_3]$ is admissible. In both cases, the point associated with the program belongs to X. Therefore, $X \neq \emptyset$.
$\square$

Proposition 4 *Let $a^T\delta \leq \alpha$ be a valid inequality of P^M_{PMP}. If $a^T\delta \leq \alpha$ is facet-defining for $P^{M_s}_{PMP}$ and the two following conditions hold*

i) $F_1 = M_s$,

ii) for any $m_0 \in M_0$, there exists $X \subseteq M_s$ so that $|X| \geq 2$ and the incomplete programs $[m_0; X]$ induces a point of P^M_{PMP} satisfying $a^T\delta \leq \alpha$ with equality,

then the inequality $a^T\delta \leq \alpha$ is also facet-defining for P^M_{PMP}.

Proof. The proof is quite similar to the one of Proposition 3. In fact denoting the points in the same way, Condition i) together with the full-dimension assumption and Lemma 1 is enough to get all the points of S_0, the ones of $\{\overline{\delta}^{m_1} : m_1 \in F_1\}$ and those of $\{\tilde{\delta}^{m_1} : m_1 \in F_1\}$. These points and those of $\{\overline{\delta}^i : i = 1, \ldots, n_s^2\}$ clearly form an affinely-independent set. The points of $\{\delta^{m_0} : m_0 \in M_0\}$ then exist because of Condition ii). The latter are affinely

independent with the first $n^2 - n_0$ ones considered so far in the proof since for any $m_0 \in M_0$, point δ^{m_0} is the only one that satisfies

$$\sum_{m \in M_s} (\delta_{m_0 m} + \delta_{m m_0}) \geq 2.$$

Therefore, the face of P^M_{PMP} induced by $a^T \delta \leq \alpha$ contains n^2 affinely-indenpendent points. □

2.2 Simple facets

We now investigate which of the inequality classes $\delta_{mm'} \geq 0$ (for $m, m' \in M$), (3) and (4) are facet-defining for P^M_{PMP}. We are postponing the study of constraints (1) and (2) to Section 3.1 because they are respectively special cases of the so-called k-clique and k-unicycle inequalities we will study later.

Proposition 5 *For $m_1, m_2 \in M$, the inequality $\delta_{m_1 m_2} \geq 0$ is facet-defining for P^M_{PMP} if and only if $m_1 \neq m_2$.*

Proof. From Proposition 4 in [5], $\delta_{mm} \geq 0$ cannot define a facet of P^M_{PMP} for all $m \in M$ since it is not facet-defining for P^n_{PLO}. Assume now that $m_1 \neq m_2$. From the full-dimensional assumption, any program $[m, m']$ is admissible where $m, m' \in M$ and $m \neq m'$. Consider then the point corresponding to the interruption of all the process moves and the $n^2 - 1$ points corresponding to all the admissible programs involving exactly two process moves but the program $[m_1, m_2]$. These n^2 points are clearly affinely independent and they belong to the face of P^M_{PMP} induced by $\delta_{m_1 m_2} \geq 0$. □

Proposition 6 *Let $m_1, m_2, m_3 \in M$ with $m_1 \neq m_2 \neq m_3 \neq m_1$. The extended transitivity inequality*

$$\delta_{m_1 m_2} + \delta_{m_2 m_3} - \delta_{m_1 m_3} + \delta_{m_2 m_2} \leq 1 \tag{6}$$

is facet-defining for P^M_{PMP} if and only if the following conditions hold

i) the programs $[m_1, m_2, m_3]$, $[m_2, m_3, m_1]$ and $[m_3, m_1, m_2]$ are admissible,

ii) for any $m \notin M \setminus \{m_1, m_2, m_3\}$,

ii.a) at least one of the following programs $[m_1, m, m_2]$, $[m, m_1, m_2]$ *or* $[m, m_2, m_3]$ *is admissible,*

ii.b) at least one of the following programs $[m_1, m_2, m]$, $[m_2, m_3, m]$ *or* $[m_2, m, m_3]$ *is admissible.*

Proof. To prove the necessary condition, let us express the sets $A_{mm'}$ for all $m, m' \in M$ with $m \neq m'$ and then, let us apply Proposition 2. For $m \in M \setminus \{m_1, m_2, m_3\}$, we have

$$A_{mm'} = \begin{cases} \{[m, m']\} & \text{if } m' \in M \setminus \{m, m_2\}, \\ \{[m_1, m, m_2], [m, m_1, m_2], [m, m_2, m_3]\} & \text{if } m' = m_2. \end{cases}$$

We also have

$$A_{m_1 m'} = \begin{cases} \{[m_1, m']\} & \text{if } m' \in M \setminus \{m_1, m_3\}, \\ \{[m_1, m_2, m_3]\} & \text{if } m' = m_3, \end{cases}$$

$$A_{m_2 m'} = \begin{cases} \{[m_2, m_3, m_1]\} & \text{if } m' = m_1, \\ \{[m_2, m_3]\} & \text{if } m' = m_3, \\ \{[m_1, m_2, m], [m_2, m_3, m], [m_2, m, m_3]\} & \text{if } m' \in M \setminus \{m_2, m_1, m_3\}, \end{cases}$$

$$A_{m_3 m'} = \begin{cases} \{[m_3, m']\} & \text{if } m' \in M \setminus \{m_3, m_2\}, \\ \{[m_3, m_1, m_2]\} & \text{if } m' = m_2. \end{cases}$$

From the full-dimension assumption on P^M_{PMP}, we can deduce the admissibility of all the programs in the sets $A_{mm'}$, $\{m, m'\} \subseteq M$ involving exactly two process moves. Condition i) (respectively ii)) is obviously implied by A_{m_1,m_3}, A_{m_2,m_1} and A_{m_3,m_2} (respectively A_{mm_2} and $A_{m_2 m}$ for all $m \in M \setminus \{m_1, m_2, m_3\}$) each having at least one admissible program. Therefore, inequality (6) is facet-defining for P^M_{PMP} only if conditions i) and ii) are fulfilled.

Suppose now these two conditions are satisfied. We have $M_s = \{m_1, m_2, m_3\}$. Let $\mathcal{F}_s$ be the face of $P^{M_s}_{\text{PMP}}$ induced by (6) and let $\overline{\delta}$ be one of the 9 points introduced in the proof of Proposition 5 in [5]. If $\overline{\delta}$ belongs to one of the first four sets, its associated program then orders at most two moves. Since the polytope $P^{M_s}_{\text{PMP}}$ is assumed to be full-dimensional, these 6 points belong to $\mathcal{F}_s$. The admissibility of $[m_1, m_2, m_3]$, $[m_2, m_3, m_1]$ and $[m_3, m_1, m_2]$ implies that the 3 points in sets 5 to 7 also belong to $\mathcal{F}_s$. Inequality (6) then is facet-defining for $P^{M_s}_{\text{PMP}}$.

Finally, it is straightforward to see that both conditions of Proposition 3 are

satisfied. We clearly have $F_1 = \{m_1, m_3\}$, $F_2 = \{m_2\}$ and the program interrupting all the process moves of M corresponds to a point of the face of P^M_{PMP} induced by (6). Moreover for any $m_0 \in M_0$, Condition ii.a) (respectively ii.b)) implies there exists $\emptyset \neq X_a \subsetneq \{m_1, m_3\}$ (respectively $\emptyset \neq X_b \subsetneq \{m_1, m_3\}$) so that A_{mm_2} (respectively A_{m_2m}) has an admissible program. We then deduce that inequality (6) defines a facet of P^M_{PMP}. □

Proposition 7 *Let $m \in M$. The capacity inequality*

$$(1-\delta_{mm})w_m + \sum_{m' \in T(t_m)\setminus\{m\}} w_{m'}\delta_{m'm} - \sum_{m' \in S(t_m)} w_{m'}(\delta_{m'm'} + \delta_{m'm}) \leq K_{t_m} \quad (7)$$

does not define a facet of P^M_{PMP}.

Proof. Given $m_0 \in M$, let $\mathcal{F}$ be the face of P^M_{PMP} induced by the capacity inequality (7) associated with m_0. Assume that $\mathcal{F}$ is a facet of P^M_{PMP}. From the assumption on the full dimension of polytope P^M_{PMP}, there exists $\delta^1 \in \mathcal{F}$ so that $\delta^1_{m_0m_0} = 1$. We then have $\delta^1_{m_0m} = \delta^1_{mm_0} = 0$ for all $m \in M \setminus \{m_0\}$, and (7) can be rewritten as

$$-\sum_{m \in S(t_{m_0})} w_m \delta^1_{mm} = K_{t_{m_0}}.$$

Since $w_m > 0$ for all $m \in M$ and $K_u \geq 0$ for all $u \in U$, we deduce $\delta^1_{mm} = 0$ for all $m \in S(t_{m_0})$ and $K_{t_{m_0}} = 0$.

If $S(t_{m_0}) = \emptyset$ the left-hand side of (7) is then positive for any point δ of $\mathcal{F}$ so that $\delta_{m_0m_0} = 0$. Since $K_{t_{m_0}} = 0$, we then have $\mathcal{F} \subsetneq \{\delta \in P^M_{\text{PMP}} : \delta_{m_0m_0} = 1\} \subsetneq P^M_{\text{PMP}}$. Therefore, we can suppose $S(t_{m_0}) \neq \emptyset$. Let $\delta^2 \in \mathcal{F}$ so that $\delta^2_{m_0m_0} = 0$. Such a point exists since $\mathcal{F}$ is a facet of the full-dimensional polytope P^M_{PMP}. For any $m_1 \in S(t_{m_0})$, the valid inequalities $\delta^2_{m_0m_1} + \delta^2_{m_1m_0} + \delta^2_{m_1m_1} \leq 1$ and $\delta^2_{m_0m_1} + \delta^2_{m_1m_0} + \delta^2_{m_0m_0} + \delta^2_{m_1m_1} \geq 1$ combined with $\delta^2_{m_0m_0} = 0$ give $\delta^2_{m_0m_1} + \delta^2_{m_1m_0} + \delta^2_{m_1m_1} = 1$, that is,

$$\delta^2_{m_0m_1} + \delta^2_{m_1m_0} + \delta^2_{m_0m_0} + \delta^2_{m_1m_1} = 1. \quad (8)$$

We previously have shown that (8) also holds for δ^1 and then for any point of $\mathcal{F} \cap \{\delta \in P^M_{\text{PMP}} : \delta_{m_0m_0} = 1\}$. Therefore, we obtain

$$\mathcal{F} \subsetneq \{\delta \in P^M_{\text{PMP}} : \delta_{m_0m_1} + \delta_{m_0m_0} + \delta_{m_0m_0} + \delta_{m_1m_1} = 1\} \subsetneq P^M_{\text{PMP}},$$

a contradiction. □

3 Facets of the process move program polytope

3.1 Facets from the partial linear ordering polytope

We now give necessary and sufficient conditions for facets of P^n_{PLO} introduced in [5] to be also facets of P^M_{PMP}. From the definition of the polytope P^M_{PMP}, any valid inequality of P^n_{PLO} is obviously valid for P^M_{PMP}. The next proposition states that the k-clique inequalities define facets of P^M_{PMP}. We remind that inequalities (1) correspond to a special case of k-clique inequalities (i.e., $k = 2$) and then, are also facet-defining for P^M_{PMP}.

Proposition 8 *Let $I \subseteq M$ with $|I| = k$. The k-clique inequality*

$$\sum_{i \in I} \sum_{j \in I} \delta_{ij} \geq |I| - 1 \tag{9}$$

is facet-defining for P^M_{PMP}.

Proof. Given $m, m' \in M$ with $m \neq m'$, we have

$$A_{mm'} = \begin{cases} \{[m, m']\} & \text{if } m \in I,\ m' \in I, \\ \{[m, m']\} & \text{if } m \in I,\ m' \notin I, \\ \{[m, m']\} & \text{if } m \notin I,\ m' \in I, \\ \{[m, m'; i] : i \in I\} & \text{if } m \notin I,\ m' \notin I. \end{cases}$$

We clearly have $M_s = I$ and $F_1 = I$. From the full-dimension assumption on P^M_{PMP} and Lemma 1, $A_{mm'}$ contains an admissible program for all $m, m' \in M$ with $m \neq m'$. Moreover for any $m_0 \in M \setminus I$ and $\{i_1, i_2\} \subseteq I$, the incomplete program $[m_0; \{i_1, i_2\}]$ is admissible and corresponds to a point satisfying (9) with equality. It is then obvious that if inequality (9) is facet-defining for P^I_{PMP}, so it is for P^M_{PMP} by Proposition 4.
Let $\mathcal{F}_s$ be the face of P^I_{PMP} induced by inequality (9). Consider the $|I|^2$ points introduced in the proof of Proposition 6 in [5]. None of the programs associated with these points orders more than two process moves. From the assumption that P^I_{PMP} is full-dimensional, these points belong to $\mathcal{F}_s$. Since they are affinely independent, inequality (9) is facet-defining for P^I_{PMP} and for P^M_{PMP} by Proposition 4. □

In the next proposition, we state that the k-unicycle inequalities (and also inequalities (2) that indeed correspond to the 1-unicycle inequalities) define

facets of P^M_{PMP}.

Proposition 9 *Let $I \subseteq M$ with $|I| = k$ and let $i_0 \in M \setminus I$. The k-unicycle inequality*

$$\delta_{i_0 i_0} + \sum_{i \in I} (\delta_{i i_0} + \delta_{i_0 i}) - \sum_{i \in I} \sum_{j \in I \setminus \{i\}} \delta_{ij} \leq 1 \tag{10}$$

is facet-defining for P^M_{PMP} .

Proof. Given $m, m' \in M$ with $m \neq m'$, we have

$$A_{mm'} = \begin{cases} \{[i_0, m']\} & \text{if } m = i_0,\ m' \in I, \\ \{[m, i_0]\} & \text{if } m \in I,\ m' = i_0, \\ \{[m, m'; i_0]\} & \text{if } m \in I,\ m' \in I, \\ \{[m, m']\} & \text{if } m \notin I \cup \{i_0\},\ m' \in I, \\ \{[m, m']\} & \text{if } m \in I,\ m' \notin I \cup \{i_0\}, \\ \{[m, m']\} & \text{if } m \notin I \cup \{i_0\},\ m' \notin I \cup \{i_0\}, \\ \{[i_0, m'; i] : i \in I\} & \text{if } m = i_0,\ m' \notin I, \\ \{[m, i_0; i] : i \in I\} & \text{if } m \notin I,\ m' = i_0. \end{cases}$$

We clearly have $M_s = I \cup \{i_0\}$, $F_1 = I$ and $F_2 = \{i_0\}$. The program consisting of interrupting all the process moves of M corresponds to a point of the face $\mathcal{F}$ of P^M_{PMP} induced by (10). Moreover from Proposition 1, the incomplete programs $[m_0, i_0; i]$ and $[i_0, m_0; i]$ are admissible for any $m_0 \in M \setminus M_s$ and $i \in I$. It is then straightforward that these two incomplete programs induce points of $\mathcal{F}$. The two conditions of Proposition 3 are fulfilled and thus, we only need to focus on showing that inequality (10) is facet-defining for $P^{M_s}_{\text{PMP}}$. Let us consider the $(|I|+1)^2$ points introduced in the proof of Proposition 8 in [5]. The points of the first three sets correspond to programs ordering no more than two process moves. From the full-dimension assumption, they all belong to the face $\mathcal{F}_s$ of $P^{M_s}_{\text{PMP}}$ induced by inequality (10). The fourth set only contains points that correspond to programs $[m, m'; i_0]$ for all $m, m' \in M$ with $m \neq m'$ (i.e., $A_{mm'}$). From Lemma 1, all the points of this last set belong to $\mathcal{F}_s$. We then have found $(|I|+1)^2$ affinely-independent points of $\mathcal{F}_s$. Therefore, inequality (10) defines a facet of $P^{M_s}_{\text{PMP}}$, and by Proposition 3, of P^M_{PMP} as well. □

We nowfocus on the k-l-bicycle inequalities for which we give in the next proposition necessary and sufficient conditions to be facet-defining for P^M_{PMP}.

Proposition 10 *Let $i_0 \in M$, $j_0 \in M \setminus \{i_0\}$, $\emptyset \subset I \subset M \setminus \{i_0, j_0\}$ and $\emptyset \subset J \subset M \setminus \{i_0, j_0\}$ with $|I| = k$, $|J| = l$ and $I \cap J = \emptyset$. The k-l-bicycle inequality*

$$\delta_{i_0 i_0} + \delta_{j_0 j_0} + \delta_{i_0 j_0} + \sum_{i \in I} (\delta_{i i_0} + \delta_{i_0 i} - \delta_{i j_0}) + \sum_{j \in J} (\delta_{j j_0} + \delta_{j_0 j} - \delta_{i_0 j})$$
$$- \sum_{i \in I} \sum_{i' \in I \setminus \{i\}} \delta_{i i'} - \sum_{j \in J} \sum_{j' \in J \setminus \{j\}} \delta_{j j'} - \sum_{i \in I} \sum_{j \in J} \delta_{j i} \leq 2 \tag{11}$$

is facet-defining for P^M_{PMP} if and only if

i) $[j, i_0, j_0, i]$ is admissible for all $i \in I$ and $j \in J$,

ii) for all $i \in I$, there exists $j \in J$ so that $[i, j, i_0, j_0]$ is admissible,

iii) for all $j \in J$, there exists $i \in I$ so that $[i_0, j_0, i, j]$ is admissible,

iv) there exist $i \in I$ and $j \in I$ so that $[j_0, i, j, j_0]$.

Proof. We first prove the necessary conditions. For $i \in I$ and $m \in M \setminus \{i\}$, we have

$$A_{im} = \begin{cases} \{[i, i_0]\} & \text{if } m = i_0, \\ \{[i, j, i_0, j_0] : j \in J\} & \text{if } m = j_0, \\ \{[i, m; i_0]\} & \text{if } m \in I \setminus \{i\}, \\ \{[i, m]\} & \text{if } m \in J, \\ \{[i, m]\} & \text{if } m \notin I \cup J \cup \{i_0, j_0\}. \end{cases}$$

From the full-dimension assumption on P^M_{PMP} and Lemma 1, A_{im} contains an admissible program for all $m \in M \setminus \{j_0\}$. The set A_{ij_0} contains an admissible program if there exists $j \in J$ so that $[i, j, i_0, j_0]$ is admissible. The necessary condition ii) is then proved.
For $j \in J$ and $m \in M \setminus \{j\}$, we have

$$A_{jm} = \begin{cases} \{[j, j_0]\} & \text{if } m = j_0, \\ \{[j, i_0, j_0], \{[i, j, i_0] : i \in I\}\} & \text{if } m = i_0, \\ \{[j, m; j_0]\} & \text{if } m \in J \setminus \{j\}, \\ \{[j, i_0, j_0, m]\} & \text{if } m \in I, \\ \{[j, m]\} & \text{if } m \notin I \cup J \cup \{i_0, j_0\}. \end{cases}$$

As above, we then deduce the necessary condition i), that is, for all $j \in J$ and $i \in I$, $[j, i_0, j_0, i]$ is admissible. Remark that A_{ji_0} implies either that $[j, i_0, j_0]$ is admissible or that there exists $i \in I$ so that $[i, j, i_0]$ is admissible. The admissibility of $[j, i_0, j_0]$ is indeed implied by the one of $[j, i_0, j_0, i]$ for $i \in I$.
Consider now the process moves i_0 and $m \in M \setminus \{i_0\}$. We have

$$A_{i_0 m} = \begin{cases} \{\{[i_0, j_0, i] : i \in I\}, \{[j, i_0, j_0] : j \in J\}\} & \text{if } m = j_0, \\ \{[i_0, m]\} & \text{if } m \in I, \\ \{[i_0, j_0, i, m] : i \in I\} & \text{if } m \in J, \\ \{\{[i_0, m; i] : i \in I\}, \{[j, i_0, m, j_0] : j \in J\}\} & \text{if } m \notin I \cup J \cup \{i_0, j_0\}. \end{cases}$$

As above, we deduce that for all $j \in J$ there must exist $i \in I$ so that $[i_0, j_0, i, j]$ is admissible. Remark that the admissibility of the programs of $A_{i_0 j_0}$ is obviously implied by Condition i). We then obtain necessary condition iii).
Using similar arguments for j_0 and $m \in M \setminus \{j_0\}$ where

$$A_{j_0 m} = \begin{cases} \{[j_0, i, j, i_0] : i \in I,\ j \in J\} & \text{if } m = i_0, \\ \{\{[j_0, m, j] : j \in J\}, \{[i_0, j_0, m\}\} & \text{if } m \in I, \\ \{[j_0, m]\} & \text{if } m \in J, \\ \{[j_0, m; j] : j \in J\} & \text{if } m \notin I \cup J \cup \{i_0, j_0\}, \end{cases}$$

we obtain necessary condition iv).
Finally, let $m \notin I \cup J \cup \{i_0, j_0\}$ and $m' \in M \setminus \{m\}$. We have

$$A_{mm'} = \begin{cases} \{[m, i_0; i] : i \in I\} & \text{if } m' = i_0, \\ \{[m, j_0; j] : j \in J\} & \text{if } m' = j_0, \\ \{[m, m']\} & \text{if } m' \in I, \\ \{[m, m']\} & \text{if } m' \in J, \\ \{[m, m']\} & \text{if } m' \notin I \cup J \cup \{i_0, j_0, m\}. \end{cases}$$

From the full-dimension assumption on P^M_{PMP} and Lemma 1, $A_{mm'}$ contains an admissible program for all $m \notin I \cup J \cup \{i_0, j_0\}$ and $m' \in M$.

We now prove the sufficiency of the conditions. We clearly have $M_s = I \cup J \cup \{i_0, j_0\}$, $F_1 = I \cup J$ and $F_2 = \{i_0, j_0\}$. As we did in the proof of the previous propositions of this section, we will only focus on showing that inequality (11) is facet-defining for $P^{M_s}_{\text{PMP}}$. In fact, condition i) of Proposition 3 is trivially satisfied, and because of Lemma 1, the incomplete programs $[m_0, i_0; i]$, $[m_0, j_0; j]$, $[i_0, m_0; i]$ and $[j_0, m_0; j]$ are admissible and correspond

to points of the face of P^M_{PMP} induced by (11), for any $m_0 \in M \setminus M_s$, $i \in I$ and $j \in J$. Consider then the $|M_s|^2$ affinely-independent points introduced in the proof of Proposition 10 in [5]. These points are sorted into 15 different sets. The first five sets contain points associated with programs ordering at most two process moves. The assumed full dimension of $P^{M_s}_{\text{PMP}}$ implies that all these points belong to $\mathcal{F}_s = \{\delta \in P^{M_s}_{\text{PMP}} : (11) \text{ is tight for } \delta\}$. The admissibility of the programs associated with the points in the sets 9 and 10 (these points correspond to incomplete programs of the form $[m, m'; m'']$) comes directly from Lemma 1, whereas conditions i)-iv) imply the admissibility of the programs associated with the points in the remaining sets. These $|M_s|^2$ points then belong to $\mathcal{F}_s$ and thus, inequality (11) defines a facet of $P^{M_s}_{\text{PMP}}$. From Proposition 3, it also defines a facet of P^M_{PMP}. □

As we proved in the companion paper, the separation problems for inequalities (9), (10) and (11) are NP-hard. The proofs are based on showing that an algorithm solving any of these separation problems would also solve the max-cut problem.

3.2 Cover inequalities

We now turn our attention to capacity-related facet-defining inequalities for P^M_{PMP}. The underlying idea of these inequalities comes from the well-known cover inequalities [1, 2, 6] which represent an important class of facets of the binary knapsack problem. For our problem, we can actually see a *cover* as a set of process moves that cannot be on a same processor at the same time.

Proposition 11 *Let $m_0 \in M$, $\emptyset \subset A \subseteq T(s_{m_0})$ and $B \subseteq S(s_{m_0}) \setminus \{m_0\}$ be so that*

$$\sum_{m \in A} w_m > K_{s_{m_0}} + \sum_{m \in \overline{B}} w_m \tag{12}$$

with $\overline{B} = S(s_{m_0}) \setminus (B \cup \{m_0\})$. The source cover inequality

$$\sum_{m \in A} \delta_{mm_0} + \sum_{m \in B} \delta_{m_0 m} \leq (|A| + |B| - 1)(1 - \delta_{m_0 m_0}) \tag{13}$$

is valid for P^M_{PMP}.

Proof. Inequality (13) expresses the fact that all of the process moves in A cannot be performed (by migrations) if none of the process moves in $B \cup \{m_0\}$

have been performed (by migrations or interuptions), or in other words, that performing all the process moves in $\overline{B}$ does not free enough resources to perform all the ones in A.

Let $\overline{\delta}$ be a point of P^M_{PMP}. Suppose first that m_0 is interrupted, that is, $\overline{\delta}_{m_0m_0} = 1$ and then $\overline{\delta}_{mm_0} = \overline{\delta}_{m_0m} = 0$ for all $m \in M \setminus \{m_0\}$. Inequality (13) can then be rewritten $0 \leq 0$.
Assume now that m_0 is not interrupted, that is, $\overline{\delta}_{m_0m_0} = 0$. If the left-hand side of (13) is equal to $|A|+|B|$, the program associated with $\overline{\delta}$ then migrates all the process moves in A before m_0 and all the process moves in B are performed after m_0. (Remark that $|A| + |B|$ is the maximum possible value for the left-hand side of (13).) Therefore, the process moves in $A \cup B \cup \{m_0\}$ are all together on processor s_{m_0}, and that implies

$$\sum_{m \in A} w_m \leq K_{s_{m_0}} + \sum_{m \in \overline{B}} w_m,$$

contradicting (12). The claim of validity follows. □

In the same manner, we can introduce the following inequalities by considering this time the target processor of a process move $m_0 \in M$ instead of its source processor as in Proposition 11.

Proposition 12 *Let $m_0 \in M$, $A \subseteq T(t_{m_0}) \setminus \{m_0\}$ and $\emptyset \subset B \subseteq S(t_{m_0})$ be so that*

$$w_{m_0} + \sum_{m \in A} w_m > K_{t_{m_0}} + \sum_{m \in \overline{B}} w_m \tag{14}$$

with $\overline{B} = S(t_{m_0}) \setminus B$. The target cover inequality

$$\sum_{m \in A} \delta_{mm_0} + \sum_{m \in B} \delta_{m_0m} \leq (|A| + |B| - 1)(1 - \delta_{m_0m_0}) \tag{15}$$

is valid for P^M_{PMP}.

Proof. Condition (14) expresses the fact that all of the process moves in $A \cup \{m_0\}$ cannot be performed (by migrations) if none of the process moves in B have been performed (by interruptions or migrations), or in other words, that performing all the process moves in $\overline{B}$ does not free enough resources to perform all the ones in $A \cup \{m_0\}$.

The claim of validity follows from an argument similar to the end of the proof of Proposition 11. $\square$

In order to clearly give necessary and sufficient conditions for the cover inequalities (13) and (15) to be facet-defining for P^M_{PMP}, we introduce some additional notation. Given $X \subseteq M$, let $m^X_{\min} = \text{argmin}\{w_m : m \in X\}$ (respectively $m^X_{\max} = \text{argmax}\{w_m : m \in X\}$) be a process move of X consuming the minimum (respectively maximum) amount of resource, that is, $w_m \geq w_{m^X_{\min}}$ (respectively $w_m \leq w_{m^X_{\max}}$) for all $m \in X$. Given $u \in U$, $A \subseteq T(u)$ and $B \subseteq S(u)$, let $\lambda_{A,B} = w(A) - K_u - w(\overline{B})$ with $\overline{B} = S(u) \setminus B$.

Proposition 13 *A source cover inequality* (13) *defines a facet of* P^M_{PMP} *if and only if the following conditions hold*

i) $w_{m^A_{\min}} \geq \lambda_{A,B\cup\{m_0\}}$,

ii) $w_{m^B_{\min}} \geq \lambda_{A,B\cup\{m_0\}}$,

iii) if there exists $a \in A$ *so that* $B \cup \{m_0\} \subseteq T(s_a)$, *then either*

a) $w_{m_0} \geq \lambda_{A,B\cup\{m_0\}}$, *or*

b) $w_{m_0} + w_{m^B_{\min}} \leq K_{s_a} + w(S(s_a)) - w_a$,

iv) $w_m \leq \max\{w_{m^A_{\max}}, w_{m^B_{\max}}\} - \lambda_{A,B\cup\{m_0\}}$ *for all* $m \in (T(s_{m_0}) \cup S(s_{m_0})) \setminus (A \cup B \cup \{m_0\})$.

Proof. We have $M_s = A \cup B \cup \{m_0\}$. We will first prove that inequality (13) defines a facet of $P^{M_s}_{\text{PMP}}$ if and only if conditions i)-iii) hold. We will then use Proposition 3 to prove the necessary and sufficient conditions for (13) to be facet-defining for P^M_{PMP}. We point out that throughout this proof, K_m, $S(s_m)$ and $T(s_m)$, for any $m \in M_s$, are considered with respect to the process move programming problem associated with M_s, that is, all the process moves of $M \setminus M_s$ are supposed interrupted.
Let $\mathcal{F}_s$ be the face of $P^{M_s}_{\text{PMP}}$ induced by a source cover inequality (13). We first prove the necessity of conditions i)-iii). Suppose $\mathcal{F}_s$ defines a facet of $P^{M_s}_{\text{PMP}}$. Given $\overline{a} \in A$, there exists $\delta^1 \in \mathcal{F}_s$ so that $\delta^1_{\overline{a}m_0} = 0$ and $\delta^1_{m_0m_0} = 0$. In fact, if such a point did not exist, we would have $\mathcal{F}_s \subsetneq \{\delta \in P^{M_s}_{\text{PMP}} : \delta_{m_0m_0} + \delta_{\overline{a}m_0} = 1\} \neq P^{M_s}_{\text{PMP}}$ and $\mathcal{F}_s$ would not define a facet. Since $\delta^1_{m_0m_0} = 0$ and $\delta^1 \in \mathcal{F}_s$, we have

$$\sum_{a\in A} \delta^1_{am_0} + \sum_{b\in B} \delta^1_{m_0b} = |A| + |B| - 1.$$

Because of $\delta^1_{\overline{a}m_0} = 0$, we then get $\delta^1_{am_0} = 1$ for all $a \in A \setminus \{\overline{a}\}$ and $\delta^1_{m_0b} = 1$ for all $b \in B$. Therefore, we have $w(A \setminus \{\overline{a}\}) \leq K_{s_{m_0}} + w(\overline{B})$, that is, $w_{\overline{a}} \geq w(A) - (K_{s_{m_0}} + w(\overline{B})) = \lambda_{A,B\cup\{m_0\}}$. Condition i) then directly follows. Using similar arguments, we can prove the necessity of Condition ii).
Consider now a process move $\overline{a} \in A$ so that $B \cup \{m_0\} \subseteq T(s_{\overline{a}})$. Since $\mathcal{F}_s$ defines a facet of $P^{M_s}_{\text{PMP}}$, there exists $\delta^3 \in \mathcal{F}_s$ so that $\delta^3_{m_0\overline{a}} = 1$. We then clearly have $\delta^3_{m_0m_0} = 0$, $\delta^3_{am_0} = 1$ for all $a \in A \setminus \{\overline{a}\}$ and $\delta^3_{m_0b} = 1$ for all $b \in B$. (Remark that right before migrating m_0, all the process moves in $M_s \setminus \{\overline{a}\}$ are together on processor s_{m_0}.) Since $\lambda_{A,B\cup\{m_0\}} > 0$, there exists $B' \subseteq B$ so that $\delta^3_{b\overline{a}} = 1$ for all $b \in B'$. We then have $w(A) \leq K_{s_{m_0}} + w(\overline{B}) + w_{m_0} + w(B')$, that is, $\lambda_{A,B\cup\{m_0\}} - w(B') \leq w_{m_0}$. If $w_{m_0} \geq \lambda_{A,B\cup\{m_0\}}$, the point δ^3 can be chosen so that $B' = \emptyset$. (We remind that the full-dimension assumption on $P^{M_s}_{\text{PMP}}$ implies that the program $[m_0, \overline{a}]$ is admissible.) Suppose now that $w_{m_0} < \lambda_{A,B\cup\{m_0\}}$. Since $B \cup \{m_0\} \subseteq T(s_{\overline{a}})$, we then must have $w_{m_0} + w(B') \leq K_{s_{\overline{a}}} + w(S(s_{\overline{a}})) - w_{\overline{a}}$. We thus directly obtain $w_{m_0} + w_{m^B_{\min}} \leq K_{s_{\overline{a}}} + w(S(s_{\overline{a}})) - w_{\overline{a}}$, and Condition iii) is proved.

We now prove the sufficiency of conditions i)-iii). Suppose they are satisfied. Let $X_1 = \{\delta^1, \ldots, \delta^{n_1}\}$ with $n_1 = n_s^2 - 2(n_s - 1)$ be the set of points so that

- $\delta^1_{mm} = 1$ for all $m \in M_s$, that is, all the process moves of M_s are interrupted,
- $\sum_{m\in M_s} \delta^i_{mm} = n_s - 1$ and $\delta^i_{m_0m_0} = 1$ for $i = 2, \ldots, n_s$, that is, all the process moves of $M_s \setminus \{m_0\}$ but one are interrupted,
- $\sum_{m\in M} \delta^i_{mm} = n_s - 2$ and $\delta^i_{m_0m_0} = 1$ for $i = n_s + 1, \ldots, n_1$, that is, all the process moves of $M_s \setminus \{m_0\}$ but two are interrupted.

The points δ^i for $i = 1, \ldots, n_s$ clearly belong to $\mathcal{F}_s$. Moreover since $P^{M_s}_{\text{PMP}}$ is assumed full-dimensional, we also have $\delta^i \in \mathcal{F}_s$ for $i = n_s + 1, \ldots, n_1$. Furthermore, it is straightforward to see that the points of X_1 are affinely independent.
Consider now points for which $\delta_{m_0m_0} = 0$, that is, points satisfying

$$\sum_{a\in A} \delta_{am_0} + \sum_{b\in B} \delta_{m_0b} = |A| + |B| - 1.$$

Let $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$. Consider the points δ^{n_1+i} for $i = 1, \ldots, |A|$ so that

$$\sum_{a \in A} \delta^{n_1+i}_{am_0} = |A| - 1 \quad \text{and} \quad \delta^{n_1+i}_{a_i a_i} = 1.$$

Let $X_2 = X_1 \cup \{\delta^{n_1+1}, \ldots, \delta^{n_2}\}$ with $n_2 = n_1 + |A|$. By Condition i), the points δ^{n_1+i} for $i = 1, \ldots, |A|$ belong to $\mathcal{F}_s$. Moreover, they are affinely independent since point δ^{n_1+i}, $i = 1, \ldots, |A|$, is the only one for which $\delta_{m_0 m_0} + \delta_{a_i m_0} = 0$. Consider the points δ^{n_2+i} for $i = 1, \ldots, |B|$ so that

$$\sum_{b \in B} \delta^{n_2+i}_{m_0 b} = |B| - 1 \quad \text{and} \quad \delta^{n_2+i}_{b_i b_i} = 1.$$

As above and using Condition ii), we set $X_3 = X_2 \cup \{\delta^{n_2+1}, \ldots, \delta^{n_3}\} \subseteq \mathcal{F}$ where $n_3 = n_2 + |B|$, and the points of X_3 are affinely independent.
Let $X_4 = X_3 \cup \{\delta^{n_3+1}, \ldots, \delta^{n_4}\}$ with $n_4 = n_3 + |A|$ and the points δ^{n_3+i} for $i = 1, \ldots, |A|$ are so that

$$\sum_{a \in A} \delta^{n_3+i}_{am_0} = |A| - 1 \quad \text{and} \quad \delta^{n_3+i}_{m_0 a_i} = 1.$$

(Remark that all the process moves of B are migrated after m_0.) If $w_{m_0} \geq \lambda_{A, B \cup \{m_0\}}$, the points δ^{n_3+i} for $i = 1, \ldots, |A|$ can be chosen so that

$$\sum_{b \in B} \delta^{n_3+i}_{a_i b} = |B|,$$

and they all belong to $\mathcal{F}_s$. If $w_{m_0} < \lambda_{A, B \cup \{m_0\}}$, the points δ^{n_3+i} for $i = 1, \ldots, |A|$ can be chosen so that

$$\sum_{b \in B \setminus \{m^B_{\min}\}} \delta^{n_3+i}_{a_i b} = |B| - 1 \quad \text{and} \quad \delta^{n_3+i}_{m^B_{\min} a_i} = 1.$$

Given $i \in \{1, \ldots, |A|\}$, if $\{m_0, m^B_{\min}\} \subseteq T(s_{a_i})$ conditions ii) and iii.b) then imply that $\delta^{n_3+i} \in \mathcal{F}_s$, and if $\{m_0, m^B_{\min}\} \not\subseteq T(s_{a_i})$ then the full-dimension assumption on $P^{M_s}_{\text{PMP}}$ and Condition ii) imply that $\delta^{n_3+i} \in \mathcal{F}_s$. Therefore $X_4 \subseteq \mathcal{F}_s$ and since a point δ^{n_3+i}, $i = 1, \ldots, |A|$, is the only one of X_4 so that $\delta_{m_0 a_i} = 1$, the points of X_4 clearly are affinely independent.

Let $X_5 = X_4 \cup \{\delta^{n_4+1}, \ldots, \delta^{n_5}\}$ with $n_5 = n_4 + |B|$ and the points δ^{n_4+i} for $i = 1, \ldots, |B|$ are so that

$$\sum_{b \in B} \delta^{n_4+i}_{m_0 b} = |B| - 1, \quad \delta^{n_4+i}_{b_i m_0} = 1 \text{ and } \sum_{a \in A} \delta^{n_3+i}_{a b_i} = |A| - 1.$$

(Remark that all the process moves of A are migrated before m_0.) From conditions i)-ii) and the full-dimension assumption on $P^{M_s}_{\text{PMP}}$, we clearly have $X_5 \subseteq \mathcal{F}_s$. Since the point δ^{n_4+i}, $i = 1, \ldots, |B|$, is the only one having $\delta_{b_i m_0} = 1$ the points of X_5 are affinely independent.
We then have exhibited $n_5 = n_s^2 - 2(n_s - 1) + 2(|A| + |B|) = n_s^2$ affinely independent points of $\mathcal{F}_s$. (Recall that $n_s = |A| + |B| + 1$.) Since $\mathcal{F}_s \subsetneq P^{M_s}_{\text{PMP}}$, inequality (13) defines a facet of $P^{M_s}_{\text{PMP}}$.

We now just need to show that by adding Condition iv), we obtain necessary and sufficient conditions for inequality (13) to be facet-defining for P^{M}_{PMP}. Assume that inequality (13) defines a facet $\mathcal{F}$ of P^{M}_{PMP}. Consider first a process move $x \in M_0 \cap T(s_{m_0})$. There must exist $\delta^t \in \mathcal{F}$ such that $\delta^t_{x m_0} = 1$. We then clearly have $\delta^t_{m_0 m_0} = 0$ and

$$\sum_{a \in A} \delta^t_{a m_0} + \sum_{b \in B} \delta^t_{m_0 b} = |A| + |B| - 1.$$

For some $m_s \in A \cup B$, we then must have $w_m + w(A) \leq K_{s_{m_0}} + w(\overline{B}) + w_{m_s}$, that is,

$$\begin{aligned} w_m &\leq w_{m_s} - (w(A) - K_{s_{m_0}} - w(\overline{B})) \\ &= w_{m_s} - \lambda_{A, B \cup \{m_0\}} \\ &\leq \max\{w_{m^A_{\max}}, w_{m^B_{\max}}\} - \lambda_{A, B \cup \{m_0\}}. \end{aligned}$$

We thus obtain Condition iv) for a process move of $M_0 \cap T(s_{m_0})$. By considering $x \in M_0 \cap S(s_{m_0})$ and a point of $\mathcal{F}$ such that $\delta_{m_0 x} = 1$, we can obtain Condition iii) for a process move of $M_0 \cap S(s_{m_0})$ as well. Therefore, Condition iv) is necessary.
We will use Proposition 3 to prove the sufficiency. We clearly have $F_1 = A \cup B$ and $F_{n_s - 1} = \{m_0\}$. (Remark that $F_h = \emptyset$ for all $h = 2, \ldots, n_s - 2$, and $F_{n_s} = \emptyset$.) We only need to check if for any $x \in M_0$, there exist $X_a \subseteq F_1$ and $X_b \subseteq F_1$ so that the incomplete programs $[x, m_0; X_a]$ and $[m_0, x; X_b]$ are admissible and induce points that belong to $\mathcal{F}' = \{\delta \in P^{M}_{\text{PMP}} : (13) \text{ is tight for } \delta\}$. In fact, the first condition of Proposition 3 is straightforwardly satisfied. Let

$x \in M_0$ and suppose $x \in T(s_{m_0})$. (Using similar ideas, we can prove the claim if $x \in S(s_{m_0})$ or $x \notin T(s_{m_0}) \cup S(s_{m_0})$. Remark that some of the associated programs do not need Condition iv) to be admissible.) Denote by m_s a process move of $A \cup B$ so that $w_{m_s} = \max\{w_{m^A_{\max}}, w_{m^B_{\max}}\}$. From Condition iv), we have

$$\begin{aligned} w(A) + w_x &\leq w(A) + w_{m_s} - \lambda_{A,B\cup\{m_0\}} \\ &= w_{m_s} + K_{s_{m_0}} + w(\overline{B})) \end{aligned}$$

Set $X_a = X_b = (A \cup B) \setminus \{m_s\}$. Using the previous inequality, it can be shown that the incomplete program $[x, m_0; X_a]$ that consists of migrating all the process moves of $A \setminus \{m_s\}$, then migrating x, then migrating m_0, and finally migrating all the process moves of $B \setminus \{m_s\}$ is admissible and its associated point belongs to $\mathcal{F}'$. Similarly, by only switching the order of migrations of x and m_0 (i.e., m_0 is migrated before x this time) we have an admissible program $[m_0, x; X_b]$ inducing a point of $\mathcal{F}'$. Therefore by Proposition 3, conditions i)-iv) are sufficient for inequality (13) to be facet-defining for P^M_{PMP}. Our proof is then complete. □

In the same way, we can give necessary and sufficient conditions for the target cover inequality (15) to be facet-defining for P^M_{PMP}.

Proposition 14 *A target cover inequality* (15) *defines a facet of* P^M_{PMP} *if and only if the following conditions hold:*

i) $w_{m^A_{\min}} \geq \lambda_{A\cup\{m_0\},B}$,

ii) $w_{m^B_{\min}} \geq \lambda_{A\cup\{m_0\},B}$,

iii) if $(A \cup \{m_0\}) \subseteq S(s_{m_0})$ *and there exists* $b \in B$ *so that* $t_b = s_{m_0}$ *either*

- *a)* $w_{m_0} \geq \lambda_{A\cup\{m_0\},B}$, *or*
- *b)* $w_{m_0} + w_{m^A_{\min}} \leq K_{s_{m_0}} + w(S(s_{m_0})) - w_b$,

iv) $w_m \leq \max\{w_{m^A_{\max}}, w_{m^B_{\max}}\} - \lambda_{A\cup\{m_0\},B}$ *for all* $m \in (T(t_{m_0}) \cup S(t_{m_0})) \setminus (A \cup B \cup \{m_0\})$. □

We now turn our attention to the separation problems of inequalities (13) and (15). We actually show that separating either the source or target cover inequalities can be reduced to solving n knapsack problems.

Proposition 15 *The source cover inequalities* (13) *can be separated in pseudo-polynomial time.*

Proof. Let $m_0 \in M$ and $\delta^\star \in \mathbb{R}^{n^2}$. The separation problem for (13) asks for two sets $A \subseteq T(s_{m_0})$ and $B \subseteq S(s_{m_0}) \setminus \{m_0\}$ that satisfy Condition (12) and

$$\sum_{m \in A} \delta^\star_{mm_0} + \sum_{m \in B} \delta^\star_{m_0 m} > (|A| + |B| - 1)(1 - \delta^\star_{m_0 m_0}), \tag{16}$$

if such two sets exist. For $m \in T(s_{m_0}) \cup S(s_{m_0}) \setminus \{m_0\}$, let $x_m = 1$ if and only if either $m \in A$ or $m \in B$. Since $w(\overline{B}) = w(S(s_{m_0})) - w(B) - w_{m_0}$, Condition (12) can be rewritten

$$\sum_{m \in T(s_{m_0})} w_m x_m + \sum_{m \in S(s_{m_0}) \setminus \{m_0\}} w_m x_m \geq K_{s_{m_0}} + \sum_{m \in S(s_{m_0})} w_m - w_{m_0} + 1$$

Since $|A| = \sum_{m \in T(s_{m_0})} x_m$ and $|B| = \sum_{m \in S(s_{m_0}) \setminus \{m_0\}} x_m$, (16) can be rewritten (after some rearrangements)

$$\sum_{m \in T(s_{m_0})} \alpha_m x_m + \sum_{m \in S(s_{m_0}) \setminus \{m_0\}} \beta_m x_m < 1 - \delta^\star_{m_0 m_0}$$

where $\alpha_m = 1 - \delta^\star_{mm_0} - \delta^\star_{m_0 m_0}$ for all $m \in T(s_{m_0})$ and $\beta_m = 1 - \delta^\star_{m_0 m} - \delta^\star_{m_0 m_0}$ for all $m \in S(s_{m_0}) \setminus \{m_0\}$. Letting $y_m = 1 - x_m$ for all $m \in T(s_{m_0}) \cup S(s_{m_0}) \setminus \{m_0\}$ leads to the following knapsack problem

$$\begin{cases} z = \text{Maximize} \displaystyle\sum_{m \in T(s_{m_0})} \alpha_m y_m + \sum_{m \in S(s_{m_0}) \setminus \{m_0\}} \beta_m y_m \\ \text{s. t.} \\ \displaystyle\sum_{m \in T(s_{m_0})} w_m y_m + \sum_{m \in S(s_{m_0}) \setminus \{m_0\}} w_m y_m \leq \sum_{m \in T(s_{m_0})} w_m - K_{s_{m_0}} - 1, \\ y_m \in \{0, 1\} \qquad \text{for all } m \in T(s_{m_0}) \cup S(s_{m_0}) \setminus \{m_0\}. \end{cases}$$

It is obvious that this knapsack problem has a feasible solution only if $w(T(s_{m_0}) - K_{s_{m_0}} - 1 \geq 0$, that is, there is not enough remaining capacity on processor s_{m_0} to migrate all the process moves of $T(s_{m_0})$ without performing a process move of $S(s_{m_0})$. If the optimal value $z^\star$ of this knapsack problem exists and

$$z^\star > \delta^\star_{m_0 m_0} - 1 + \sum_{m \in T(s_{m_0})} \alpha_m + \sum_{m \in S(s_{m_0}) \setminus \{m_0\}} \beta_m,$$

a violated source cover inequality (13) generated by m_0 has then been found. Otherwise, it can be concluded that none exists.

Separating over the entire set of source cover inequalities can (13) therefore be performed by solving n knapsack problems such as the above. Each of this knapsack problem can be solved in pseudo-polynomial time using for instance the well-known Bellman recursion [3] The claim follows. $\square$

Proposition 16 *The target cover inequalities* (15) *can be separated in pseudo-polynomial time.*

Proof. We proceed as in the proof of Proposition 15. The only difference lies in the considered knapsack problem. En fact, given $m_0 \in M$ and $\delta^\star \in \mathbb{R}^{n^2}$, the separation problem for (15) turns out to reduce to the following knapsack problem.

$$\begin{cases} z = \text{Maximize} \displaystyle\sum_{m \in T(t_{m_0}) \setminus \{m_0\}} \alpha_m y_m + \sum_{m \in S(t_{m_0})} \beta_m y_m \\ \text{s. t.} \\ \displaystyle\sum_{m \in T(t_{m_0}) \setminus \{m_0\}} w_m y_m + \sum_{m \in S(t_{m_0})} w_m y_m \leq \sum_{m \in T(t_{m_0})} w_m - K_{t_{m_0}} - 1, \\ y_m \in \{0,1\} \qquad \text{for all } m \in (T(t_{m_0}) \setminus \{m_0\}) \cup S(t_{m_0}), \end{cases}$$

where $\alpha_m = 1 - \delta^\star_{mm_0} - \delta^\star_{m_0 m_0}$ for all $m \in T(s_{m_0}) \setminus \{m_0\}$ and $\beta_m = 1 - \delta^\star_{m_0 m} - \delta^\star_{m_0 m_0}$ for all $m \in S(s_{m_0})$. $\square$

3.3 A cover-inequalities-based formulation

The cover inequalities previously introduced enable us to give an integer linear programming formulation of the PMP problem based only on facet-defining inequalities for P^M_{PMP}.

Lemma 2 *A vector δ of $\mathbb{R}^{n^2}$ is the characteristic vector associated with a solution of the process move programming problem if and only if it belongs to the set defined by the following constraints*

- 2-*clique inequalities (i.e., $\delta_{mm'} + \delta_{m'm} + \delta_{mm} + \delta_{m'm'} \geq 1$ for all $\{m, m'\} \subseteq M$),*
- 1-*unicycle inequalities (i.e., $\delta_{mm'} + \delta_{m'm} + \delta_{mm} \leq 1$ for $m \neq m' \in M$),*

- *extended transitivity inequalities (i.e.,* $\delta_{mm'} + \delta_{m'm''} - \delta_{mm''} + \delta_{m'm'} \leq 1$ *for* $m \neq m' \neq m'' \in M$*),*
- *source cover inequalities* (13)*,*
- *target cover inequalities* (15)*,*
- $\delta_{mm'} \in \{0, 1\}$ *for* $m,\ m' \in M$*.*

Proof. From the results obtained in this paper and its companion one [5], it is obvious that any solution to the PMP problem has its characteristic vector that satisfies all these constraints.

Let $\overline{\delta}$ be a solution of this system of constraints. If this solution does not induce a solution to the PMP problem, there then exists a process moves $\overline{m}$ so that the available capacity on its target processor $t_{\overline{m}}$ is not sufficient when it is migrated. Let A (respectively B) be the set of process moves having $t_{\overline{m}}$ as a target (respectively source) processor and that have already been migrated to $t_{\overline{m}}$ (respectively migrated from $t_{\overline{m}}$ or interrupted). We then have

$$w(A) + w_{\overline{m}} > K_{t_{\overline{m}}} + w(\overline{B})$$

where $\overline{B} = S(s_{t_{\overline{m}}}) \setminus B$. Therefore the target cover inequalities induced by $\overline{m}$, A and B is violated by $\overline{\delta}$, a contradiction. □

4 Conclusion

In this paper, we have studied the polytope P^M_{PMP} associated with the solution of the process move programming problem. We first have obtained necessary and sufficient conditions for the facet-defining inequalities introduced in the companion paper [5] for the partial linear ordering polytope to be facet-defining for P^M_{PMP}. We then have introduced capacity-related facet-defining inequalities for P^M_{PMP} that can be separated in pseudo-polynomial time, and strengthen the integer linear programming formulation of the PMP problem we started with.

In [4], Sirdey and Kerivin have devised a branch-and-cut algorithm based on this strengthened formulation given in Section 3.3. Their extensive computational results show the practical relevance of the formulation in terms of both exact and approximate resolution when the instance size increases. It would be interesting to extend their branch-and-cut algorithm by also considering inequalities (9)-(11).

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