Asymptotics of minimax stochastic programs

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Abstract. We discuss in this paper asymptotics of the sample average approximation (SAA) of the optimal value of a minimax stochastic programming problem. The main tool of our analysis is a specific version of the infinite dimensional Delta Method. As an example, we discuss asymptotics of SAA of risk averse stochastic programs involving the absolute semideviation risk measure.

Key words: sample average approximation, infinite dimensional delta method, functional Central Limit Theorem, minimax stochastic programming, absolute semideviation risk measure.

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1 Introduction

Consider the following minimax stochastic problem

$$\min_{x\in X}\max_{y\in Y}\big\{f(x,y):=\mathbb{E}\big[F(x,y,\xi)\big]\big\} \tag{1.1}$$

where $X\subset\mathbb{R}^m$, $Y\subset\mathbb{R}^n$, ξ is a random vector having probability distribution P supported on a set $\Xi\subset\mathbb{R}^d$ and $F: X\times Y\times\Xi\to\mathbb{R}$. (Unless stated otherwise we assume throughout the paper that the expectations are taken with respect to the probability distribution P.) Suppose that we have an iid sample $\xi^1,...,\xi^N$ of N realizations of the random vector ξ. Then we can approximate problem (1.1) by the so-called sample average approximation (SAA) minimax problem:

$$\min_{x\in X}\max_{y\in Y}\left\{\hat f_N(x,y):=\frac{1}{N}\sum_{j=1}^{N}F(x,y,\xi^j)\right\}. \tag{1.2}$$

We discuss in this paper asymptotic properties of the optimal value $\hat v_N$ of the SAA problem (1.2) viewed as an estimator of the optimal value v^* of the "true" problem (1.1).

Of course, if the set Y is a singleton, say $Y=\{y^*\}$, then problem (1.1) becomes a stochastic programming problem of minimization of the expectation function $f(x,y^*)$ over $x\in X$. Statistical properties of sample average approximations of such (minimization) problems were studied extensively (see, e.g., [10] and references therein). This study is motivated by the following example. Consider the following risk averse stochastic problem

$$\min_{x\in X}\rho_\lambda[G(x,\xi)], \tag{1.3}$$

where $G:\mathbb{R}^m\times\Xi\to\mathbb{R}$ and $\rho_\lambda[Z]:=\mathbb{E}[Z]+\lambda\mathbb{E}\big\{[Z-\mathbb{E}(Z)]_+\big\}$ is the so-called absolute semideviation risk measure with $\lambda\in[0,1]$ being the weight constant and $[a]_+:=\max\{0,a\}$. Such risk averse stochastic problems were investigated in a number of recent publications (e.g., [1],[7] and references therein). It is possible to show (cf., [7]) that

$$\rho_\lambda[Z]=\inf_{t\in\mathbb{R}}\sup_{\alpha\in[0,1]}\mathbb{E}\big\{Z+\lambda\alpha[Z-t]_++\lambda(1-\alpha)[t-Z]_+\big\}, \tag{1.4}$$

and hence to represent problem (1.3) as a minimax stochastic problem of the form (1.1).

We assume that the sets X and Y are nonempty, closed and convex, and for every $\xi\in\Xi$ the function $F(\cdot,\cdot,\xi)$ is *convex-concave*, i.e., for all y the function $F(\cdot,y,\xi)$ is convex, and for all x the function $F(x,\cdot,\xi)$ is concave. The main result of this paper is that, under mild additional conditions,

$$\hat v_N=\min_{x\in X_f^*}\max_{y\in Y_f^*}\hat f_N(x,y)+o_p(N^{-1/2}), \tag{1.5}$$

where $X_f^*\times Y_f^*$ is the set of saddle points of the minimax problem (1.1). The asymptotic result (1.5) can be viewed as an extension of a theory presented in [9] where a similar result was established for minimization type stochastic problems. The main tool of our analysis will be the infinite dimensional Delta Method, which was initiated in works of Gill [3], Grübel [5], King [6] and Shapiro [9] (for a recent survey see Römisch [8]).

The remainder of this paper is organized as follows. In the next section we derive a form of a Delta Theorem giving asymptotics of the optimal value of minimax stochastic programming problems. In section 3 we employ this theorem to describe asymptotics of the optimal value of minimax SAA problems and, in particular, asymptotics of SAA risk averse stochastic programs involving the absolute semideviation risk measure.

2 Minimax Delta Theorem

We assume in this section that the sets X and Y are nonempty convex and *compact*. Consider the space $C(X,Y)$, of continuous functions $\phi : X \times Y \to \mathbb{R}$ equipped with the corresponding sup-norm, and set $\mathcal{K} \subset C(X,Y)$ formed by convex-concave functions, i.e., $\phi \in \mathcal{K}$ iff $\phi(\cdot, y)$ is convex for every $y \in Y$ and $\phi(x, \cdot)$ is concave for every $x \in X$. It is not difficult to see that the set $\mathcal{K}$ is a closed (in the norm topology of $C(X,Y)$) and convex cone. Consider the optimal value function $V : C(X,Y) \to \mathbb{R}$ defined as

$$V(\phi) := \inf_{x\in X} \sup_{y\in Y} \phi(x,y). \tag{2.1}$$

Note that the function $x \mapsto \sup_{y\in Y} \phi(x,y)$ is continuous, and hence the minimax problem in the right hand side of (3.1) has a finite optimal value and attains its minimal optimal value at a nonempty compact subset X^*_ϕ of X. Similarly, the function $y \mapsto \inf_{x\in X} \phi(x,y)$ attains its maximal optimal value at a nonempty compact subset Y^*_ϕ of Y. If $\phi \in \mathcal{K}$, then $X^*_\phi \times Y^*_\phi$ forms the set of saddle points of $\phi(x,y)$ and

$$\inf_{x\in X} \sup_{y\in Y} \phi(x,y) = \sup_{y\in Y} \inf_{x\in X} \phi(x,y).$$

Note also that for any $\phi_1, \phi_2 \in C(X,Y)$,

$$\big|V(\phi_1) - V(\phi_2)\big| \le \sup_{x\in X} \Big| \sup_{y\in Y} \phi_1(x,y) - \sup_{y\in Y} \phi_2(x,y)\Big| \le \sup_{(x,y)\in X\times Y} \big|\phi_1(x,y) - \phi_2(x,y)\big|,$$

i.e., the function $V(\cdot)$ is Lipschitz continuous (with Lipschitz constant one).

We denote by $T_{\mathcal{K}}(\phi)$ the tangent cone to the set $\mathcal{K}$ at a point $\phi \in \mathcal{K}$. Recall that $\gamma \in T_{\mathcal{K}}(\phi)$ iff the distance $\mathrm{dist}(\phi + t\gamma, \mathcal{K}) = o(t)$ for $t \ge 0$. It is said that $V(\cdot)$ is Hadamard directionally differentiable at $\phi \in \mathcal{K}$ tangentially to the set $\mathcal{K}$ if the following limit exists for any $\gamma \in T_{\mathcal{K}}(\phi)$:

$$V'_\phi(\gamma) := \lim_{\substack{t\downarrow 0 \\ \mathcal{K}\ni\gamma'\to\gamma}} \frac{V(\phi + t\gamma') - V(\phi)}{t}. \tag{2.2}$$

We will need the following which is an adaptation of a result going back to Gol'shtein [4].

Proposition 2.1 *The optimal value function $V(\cdot)$ is Hadamard directionally differentiable at any $\phi \in \mathcal{K}$ tangentially to the set $\mathcal{K}$ and*

$$V'_\phi(\gamma) = \inf_{x \in X^*_\phi} \sup_{y \in Y^*_\phi} \gamma(x, y), \tag{2.3}$$

for any $\gamma \in T_\mathcal{K}(\phi)$.

Proof. Consider a sequence $\gamma_k \in C(X, Y)$ converging (in the norm topology) to $\gamma \in T_\mathcal{K}(\phi)$ and such that $\zeta_k := \phi + t_k\gamma_k \in \mathcal{K}$ for some sequence $t_k \downarrow 0$. For a point $x^* \in X^*_\phi$ we have

$$V(\phi) = \sup_{y \in Y} \phi(x^*, y) \;\text{ and }\; V(\zeta_k) \le \sup_{y \in Y} \zeta_k(x^*, y).$$

Since Y is compact and $\zeta_k(x^*, \cdot)$ is continuous, we have that the set $\arg\max_{y \in Y} \zeta_k(x^*, y)$ is nonempty. Let $y_k \in \arg\max_{y \in Y} \zeta_k(x^*, y)$. We have that $\arg\max_{y \in Y} \phi(x^*, y) = Y^*_\phi$ and, since ζ_k tends to ϕ, we have that y_k tends in distance to Y^*_ϕ (i.e., the distance from y_k to Y^*_ϕ tends to zero as $k \to \infty$). By passing to a subsequence if necessary we can assume that y_k converges to a point $y^* \in Y$ as $k \to \infty$. It follows that $y^* \in Y^*_\phi$, and of course we have that $\sup_{y \in Y} \phi(x^*, y) \ge \phi(x^*, y_k)$. We obtain

$$V(\zeta_k) - V(\phi) \le \zeta_k(x^*, y_k) - \phi(x^*, y_k) = t_k\gamma_k(x^*, y_k) = t_k\gamma(x^*, y^*) + o(t_k).$$

We obtain that for any $x^* \in X^*_\phi$ there exists $y^* \in Y^*_\phi$ such that

$$\limsup_{k \to \infty} \frac{V(\zeta_k) - V(\phi)}{t_k} \le \gamma(x^*, y^*).$$

It follows that

$$\limsup_{k \to \infty} \frac{V(\zeta_k) - V(\phi)}{t_k} \le \inf_{x \in X^*_\phi} \sup_{y \in Y^*_\phi} \gamma(x, y). \tag{2.4}$$

In order to prove the converse inequality we proceed as follows. Consider a sequence $x_k \in \arg\min_{x \in X} \theta_k(x)$, where $\theta_k(x) := \sup_{y \in Y} \zeta_k(x, y)$. We have that $\theta_k : X \to \mathbb{R}$ are continuous functions converging uniformly in $x \in X$ to the function $\vartheta(x) := \sup_{y \in Y} \phi(x, y)$. Consequently x_k converges in distance to the set $\arg\min_{x \in X} \vartheta(x)$, which is equal to X^*_ϕ. By passing to a subsequence if necessary we can assume that x_k converges to a point $x^* \in X^*_\phi$. For any $y \in Y^*_\phi$ we have $V(\phi) \le \phi(x_k, y)$. Since $\zeta_k \in \mathcal{K}$, i.e., $\zeta_k(x, y)$ is convex-concave, it has a nonempty set of saddle points $X^*_{\zeta_k} \times Y^*_{\zeta_k}$. We have that $x_k \in X^*_{\zeta_k}$, and hence $V(\zeta_k) \ge \zeta_k(x_k, y)$ for any $y \in Y$. It follows that for any $y \in Y^*_\phi$ the following holds

$$V(\zeta_k) - V(\phi) \ge \zeta_k(x_k, y) - \phi(x_k, y) = t_k\gamma_k(x^*, y) + o(t_k),$$

and hence

$$\liminf_{k \to \infty} \frac{V(\zeta_k) - V(\phi)}{t_k} \ge \gamma(x^*, y).$$

Since y was an arbitrary element of Y^*_ϕ, we obtain that

$$\liminf_{k\to\infty} \frac{V(\zeta_k) - V(\phi)}{t_k} \geq \sup_{y\in Y^*_\phi} \gamma(x^*, y),$$

and hence

$$\liminf_{k\to\infty} \frac{V(\zeta_k) - V(\phi)}{t_k} \geq \inf_{x\in X^*_\phi} \sup_{y\in Y^*_\phi} \gamma(x, y). \tag{2.5}$$

The assertion of the theorem follows from (2.4) and (2.5). □

We use the following version of (infinite dimensional) Delta Theorem, [9, Theorem 2.1].

Proposition 2.2 *Let $\mathcal{X}$ be a separable Banach space, equipped with its Borel sigma algebra, $\mathcal{C}$ be a closed convex subset of $\mathcal{X}$, $g : \mathcal{X} \to \mathbb{R}$ be a measurable and Hadamard directionally differentiable at a point $\mu \in \mathcal{C}$ tangentially to the set $\mathcal{C}$ function, $\tau_N \to \infty$ be a sequence of positive numbers, and $\{\psi_N\}$ be a sequence of random elements of $\mathcal{X}$ such that $\psi_N \in \mathcal{C}$ w.p.1 and $\tau_N(\psi_N - \mu)$ converges in distribution to a random element Z, written $\tau_N(\psi_N - \mu) \xrightarrow{\mathcal{D}} Z$. Then*

$$g(\psi_N) = g(\mu) + g'_\mu(\psi_N - \mu) + o_p\left(\tau_N^{-1}\right) \tag{2.6}$$

and

$$\tau_N\big(g(\psi_N) - g(\mu)\big) \xrightarrow{\mathcal{D}} g'_\mu(Z). \tag{2.7}$$

Propositions 2.1 and 2.2 imply the following result.

Theorem 2.1 *Consider the space $\mathcal{X} := C(X, Y)$ and the set $\mathcal{K} \subset \mathcal{X}$ of convex-concave functions. Let τ_N be a sequence of positive numbers tending to infinity, $\psi \in \mathcal{K}$ and $\{\hat{\psi}_N\}$ be a sequence of random elements of $\mathcal{X}$ such that $\hat{\psi}_N \in \mathcal{K}$ w.p.1 and $\tau_N(\hat{\psi}_N - \psi)$ converges in distribution to a random element Ψ of $\mathcal{X}$. Denote $\vartheta^* := \inf_{x\in X} \sup_{y\in Y} \psi(x, y)$ and $\hat{\vartheta}_N := \inf_{x\in X} \sup_{y\in Y} \hat{\psi}_N(x, y)$. Then*

$$\hat{\vartheta}_N = \inf_{x\in X^*_\psi} \sup_{y\in Y^*_\psi} \hat{\psi}_N(x, y) + o_p\left(\tau_N^{-1}\right) \tag{2.8}$$

and

$$\tau_N(\hat{\vartheta}_N - \vartheta^*) \xrightarrow{\mathcal{D}} \inf_{x\in X^*_\psi} \sup_{y\in Y^*_\psi} \Psi(x, y). \tag{2.9}$$

Proof. Consider the optimal value function $V : \mathcal{X} \to \mathbb{R}$. This function is continuous and hence measurable. Clearly we have that $\vartheta^* = V(\psi)$ and $\hat{\vartheta}_N = V(\hat{\psi}_N)$. By Proposition 2.1, the optimal value function $V(\cdot)$ is Hadamard directionally differentiable at ψ tangentially to the set $\mathcal{K}$ and formula (2.3) holds. Together with formula (2.7) of the above Delta Theorem this implies (2.9). Formula (2.8) follows by (2.6) and by noting that $\psi(x^*, y^*) = \vartheta^*$ for any $(x^*, y^*) \in X^*_\psi \times Y^*_\psi$. □

3 Applications to minimax stochastic problems

Consider the minimax stochastic problem (1.1) and its sample average approximation (1.2) based on an iid sample. Let us make the following assumptions.

(A1) The sets X and Y are nonempty, convex and closed, and for every $\xi \in \Xi$ the function $F(\cdot, \cdot, \xi)$ is convex-concave on $X \times Y$.

(A2) Problem (1.1) and its dual

$$\max_{y \in Y} \min_{x \in X} f(x, y) \tag{3.1}$$

have nonempty and bounded sets of optimal solutions $X_f^* \subset X$ and $Y_f^* \subset Y$, respectively.

(A3) For every $(x, y) \in X \times Y$, the function $F(x, y, \cdot)$ is measurable.

(A4) For some point $(x, y) \in X \times Y$, the expectation $\mathbb{E}[F(x, y, \xi)^2]$ is finite.

(A5) There exists a measurable function $c : \Xi \to \mathbb{R}_+$ such that $\mathbb{E}[c(\xi)^2]$ is finite and the inequality

$$|F(x', y', \xi) - F(x, y, \xi)| \le c(\xi)\big(\|x' - x\| + \|y' - y\|\big) \tag{3.2}$$

holds for all $(x, y), (x', y') \in X \times Y$ and a.e. $\xi \in \Xi$.

It follows that the expected value function $f(x, y)$ is well defined, finite valued and Lipschitz continuous with Lipschitz constant $\mathbb{E}[c(\xi)]$. By assumption (A1) we have that the function $f(x, y)$ is convex-concave, and by (A2) that the optimal values of problems (1.1) and (3.1) are equal to each other and $X_f^* \times Y_f^*$ forms the set of saddle points of these problems. Moreover, optimal solutions of the respective SAA problems converge (in distance) w.p.1 to X_f^* and Y_f^*, respectively (e.g., [10, Theorem 4, p.360]). Therefore, without loss of generality we can assume that the sets X and Y are compact.

The above assumptions (A3) - (A5) are sufficient for the Central Limit Theorem to hold in the Banach space $C(X, Y)$. That is, the sequence $N^{1/2}(\hat{f}_N - f)$, of random elements of $C(X, Y)$, converges in distribution to a (Gaussian) random element Ψ (see, e.g., [2, Chapter 7, Corrollary 7.17]). For any fixed point $(x, y) \in X \times Y$, $\Psi(x, y)$ is a real valued random variable having normal distribution with zero mean and variance $\sigma^2(x, y)$ equal to the variance of $F(x, y, \xi)$, i.e.,

$$\sigma^2(x, y) = \mathbb{E}_P\left[F(x, y, \xi)^2\right] - f(x, y)^2. \tag{3.3}$$

As a consequence of Theorem 2.1 we have the following result.

Theorem 3.1 *Consider the minimax stochastic problem* (1.1) *and the SAA problem* (1.2) *based on an iid sample. Suppose that assumptions* (A1) - (A5) *hold and let v^* and $\hat{v}_N$ be the optimal values of* (1.1) *and* (1.2), *respectively. Then*

$$\hat{v}_N = \min_{x \in X_f^*} \max_{y \in Y_f^*} \hat{f}_N(x, y) + o_p(N^{-1/2}). \tag{3.4}$$

Moreover, if the sets $X_f^* = \{x^*\}$ and $Y_f^* = \{y^*\}$ are singletons, then $N^{1/2}(\hat{v}_N - v^*)$ converges in distribution to normal with zero mean and variance $\sigma^2 = \mathbb{V}\mathrm{ar}[F(x^*, y^*, \xi)]$.

Let us finally discuss application of the above results to problem (1.3). We assume that the set X is nonempty convex and closed, the expectation $\mathbb{E}[G(x,\xi)]$ is well defined and finite valued for every $x \in X$, for a.e. $\xi \in \Xi$ the function $G(\cdot,\xi)$ is convex and problem (1.3) has a nonempty and bounded set X^* of optimal solutions.

For a random variable Z having finite first order moment we have that

$$\sup_{\alpha\in[0,1]} \mathbb{E}\big\{Z+\lambda\alpha[Z-t]_+ +\lambda(1-\alpha)[t-Z]_+\big\} = \mathbb{E}[Z]+\lambda \max\big\{\mathbb{E}([Z-t]_+),\mathbb{E}([t-Z]_+)\big\}, \quad (3.5)$$

and $\mathbb{E}([Z-t]_+) = \mathbb{E}([t-Z]_+)$ if $t = \mathbb{E}[Z]$. Consequently the minimum, over $t \in \mathbb{R}$, of the right hand side of (3.5) is attained at $t^* = \mathbb{E}[Z]$, and hence formula (1.4) follows. Because of (1.4) we can write problem (1.3) in the following equivalent form

$$\min_{(x,t)\in X\times\mathbb{R}} \max_{\alpha\in[0,1]} \mathbb{E}\left[H_\lambda(G(x,\xi)-t,\alpha)+t\right], \quad (3.6)$$

where

$$H_\lambda(z,\alpha) := z + \lambda\alpha[z]_+ + \lambda(1-\alpha)[-z]_+.$$

The function $H_\lambda(z,\alpha)$ is convex in z and linear (and hence concave) in α. Moreover, for any $\alpha \in [0,1]$ and $\lambda \in [0,1]$, this function is monotonically nondecreasing in z. It follows that the function $H_\lambda(G(x,\xi)-t,\alpha)+t$ is convex in (x,t) and concave (linear) in α.

The sample average approximation of problem (1.3) is obtained by replacing the "true" distribution P with its sample approximation $P_N := N^{-1}\sum_{j=1}^N \Delta(\xi^j)$, where $\Delta(\xi)$ denotes measure of mass one at point $\xi \in \Xi$. That is, the SAA problem associated with problem (1.3) is

$$\min_{x\in X} \frac{1}{N}\sum_{j=1}^N \left\{G(x,\xi^j)+\lambda[G(x,\xi^j)-\hat{g}_N(x)]_+\right\}, \quad (3.7)$$

where $\hat{g}_N(x) := N^{-1}\sum_{j=1}^N G(x,\xi^j)$. Alternatively this SAA problem can be written in the minimax form:

$$\min_{(x,t)\in X\times\mathbb{R}} \max_{\alpha\in[0,1]} \left\{\frac{1}{N}\sum_{j=1}^N H_\lambda(G(x,\xi^j)-t,\alpha)+t\right\}. \quad (3.8)$$

The sets of optimal solutions the minimax problem (3.6) and its dual can be described as follows. We have that if (x^*,t^*) is an optimal solution of (3.6), then $t^* = \mathbb{E}[G(x^*,\xi)]$ and the set of optimal solutions of the dual of the problem (3.6) is interval $A^* = [\alpha', \alpha^*]$, where

$$\alpha' := \mathrm{Prob}\{G(x^*,\xi) < \mathbb{E}[G(x^*,\xi)]\} \text{ and } \alpha^* := \mathrm{Prob}\{G(x^*,\xi) \le \mathbb{E}[G(x^*,\xi)]\}. \quad (3.9)$$

It follows that this interval A^* is the same for any optimal solution $x^* \in X^*$. Theorem 3.1 can be applied now in a straightforward way. That is, under additional assumptions

(A3)–(A5) adapted to the considered problem, we have that

$$\hat{v}_N = \min_{\substack{x\in X^* \\ t=\mathbb{E}[G(x,\xi)]}} \max_{\alpha\in A^*} \frac{1}{N}\sum_{j=1}^{N} \Big\{ G(x,\xi^j) \quad +\lambda\alpha\,[G(x,\xi^j)-t]_+ \\ \qquad\qquad +\lambda(1-\alpha)\,[t-G(x,\xi^j)]_+ \Big\} + o_p(N^{-1/2}). \tag{3.10}$$

In particular, if $X^* = \{x^*\}$ is a singleton and $\text{Prob}\{G(x^*,\xi) = \mathbb{E}[G(x^*,\xi)]\} = 0$, then $A^* = \{\alpha^*\}$ is a singleton. In that case $N^{1/2}(\hat{v}_N - v^*)$ converges in distribution to normal with zero mean and variance σ^2 given by

$$\sigma^2 = \mathbb{V}\text{ar}\Big\{ G(x^*,\xi) \quad +\lambda\alpha^*\big[G(x^*,\xi) - \mathbb{E}[G(x^*,\xi)]\big]_+ \\ \qquad +\lambda(1-\alpha^*)\big[\mathbb{E}[G(x^*,\xi)] - G(x^*,\xi)\big]_+ \Big\}. \tag{3.11}$$

References

[1] S. Ahmed, Convexity and decomposition of mean-risk stochastic programs, *Mathematical Progamming*, 106 (2006), 433-446.

[2] A. Araujo and E. Giné, *The Central Limit Theorem for Real and Banach Valued Random Variables*. Wiley, New York, 1980.

[3] R.D. Gill, Non-and-semiparametric maximum likelihood estimators and the von Mises method (Part I), *Scandinavian Journal of Statistics*, 16 (1989), 97-124.

[4] E.G. Gol'shtein, *Theory of Convex Programming*, Translations of Mathematical Monographs, volume 36, American Mathematical Society, Providence, 1972.

[5] R. Grübel, The length of the short, *Ann. Statist.*, 16 (1988), 619-628.

[6] A.J. King, Generalized delta theorems for multivalued mappings and measurable selections, *Mathematics of Operations Research*, 14 (1989), 720-736.

[7] W.Ogryczak and A.Ruszczyński, Dual stochastic dominance and related mean-risk models, *SIAM Journal of Optimization*, 13 (2002), 60-78.

[8] W. Römisch, Delta method, infinite dimensional, *Encyclopedia of Statistical Sciences*, S. Kotz, C.B. Read, N. Balakrishnan, B. Vidakovic eds., Second Edition, Wiley 2005.

[9] A. Shapiro. Asymptotic analysis of stochastic programs. *Annals of Operations Research*, 30 (1991), 169–186.

[10] A. Shapiro, Monte Carlo sampling methods, in: Ruszczyński, A. and Shapiro, A., (Eds.), *Stochastic Programming*, Handbook in OR & MS, Vol. 10, North-Holland Publishing Company, Amsterdam, 2003.