Metric regularity and systems of generalized equations

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Abstract

The paper is devoted to a revision of the metric regularity property for mappings between metric or Banach spaces. Some new concepts are introduced: uniform metric regularity and metric multi-regularity for mappings into product spaces, when each component is perturbed independently. Regularity criteria are established based on a nonlocal version of Lyusternik-Graves theorem due to Milyutin. The criteria are applied to systems of generalized equations producing some "error bound" type estimates.

Key words: Variational analysis, Lyusternik-Graves theorem, regularity, set-valued mapping, constraint qualification

1 Introduction

The property of metric regularity has proved to be one of the central concepts of the contemporary variational analysis, playing an extremely important role both in theory and its numerous applications to generalized equations, variational inequalities, optimization, etc. Being well-established and recognized, this concept still continues its expansion into new areas of mathematical analysis (see the recent monographs [19, 22] and survey papers [2, 13]).

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A set-valued mapping $F : X \rightrightarrows Y$ between metric spaces is said to be *metrically regular* near $(x^\circ, y^\circ) \in \operatorname{gph} F$ if there exist $\kappa \geq 0$ and $\delta > 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)), \qquad \forall x \in B_\delta(x^\circ),\ \forall y \in B_\delta(y^\circ). \tag{1}$$

Here F^{-1} denotes the *inverse* mapping: $F^{-1}(y) = \{u \in X :\ F(u) \ni y\}$. An important interpretation of this property is in terms of *error bounds*: given a solution x° of the inclusion (generalized equation) $F(u) \ni y^\circ$, any x and y close to x° and y°, respectively, and $\kappa' > \kappa$, it guarantees the existence of a solution x' of the perturbed inclusion $F(u) \ni y$ satisfying the error estimate $d(x, x') \leq \kappa' d(y, F(x))$.

In the case of a single-valued mapping f between Banach spaces, strictly differentiable at x°, the main regularity criterion is provided by the famous Lyusternik-Graves theorem, which basically reduces the problem of regularity of f to that of its linear approximation $\nabla f(x^\circ)$, the criterion being the surjectivity of $\nabla f(x^\circ)$ (see [7, 13]). This result is recognized as one of the main theorems of (smooth) nonlinear analysis (see [13]).

In the general case, some extensions of Lyusternik-Graves theorem have been developed based on the same idea: to reduce the problem of regularity of F to that of another mapping close to F in a certain sense. Such extensions provide, in particular, estimates of the *radius of metric regularity* [11] (with respect to Lipschitz continuous perturbations): the extent to which a regular near some point mapping can be perturbed by adding a locally Lipschitz continuous function before metric regularity is lost. This idea can be traced back to the original papers by Lyusternik (see the extensions of Lyusternik theorem in [14] and [5]) and Graves (see its extension in [7]).

Despite its great generality, Lyusternik-Graves theorem in its standard form as well as its traditional extensions does not cover some important cases. For instance, when considering a system of inclusions (generalized equations)

$$f_i(u) + Q_i \ni 0, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $f_i : X \to Y_i$ is a function between Banach spaces and Q_i is a subset (usually a cone) of Y_i, it can be important to have error bounds estimates when the variables in each equation are perturbed independently. More precisely, if x° is a solution of (2), conditions are needed guaranteeing the existence of $\kappa > 0$ and $\delta > 0$ such that for any $x_i \in B_\delta(x^\circ)$ and $y_i \in B_\delta$, $i = 1, 2, \ldots, n$, there exists $x \in X$ satisfying

$$f_i(x_i + x) + Q_i \ni y_i, \quad i = 1, 2, \ldots, n, \tag{3}$$

$$\|x\| \leq \kappa \max_{1 \leq i \leq n} d(y_i - f_i(x_i), Q_i). \tag{4}$$

Conditions of this type do not follow directly from the standard Lyusternik-Graves theorem. Fortunately, once again classical results happen to be richer than they are usually cited. It was noticed by Milyutin in early seventies (see e.g. the survey paper [5]) that the original proof of Lyusternik's result was applicable to the "nonlocal" metric regularity, when the estimate of type (1) held on a set. Although well known to the variational analysis community, the importance of this feature of Milyutin's extension of Lyusternik-Graves theorem seems to be still underestimated. At the same time this is exactly the property needed to prove the "parametric" version of Lyusternik-Graves theorem (see Theorem 2) and deduce the above estimates (see Theorem 3). It can also be useful for other applications (see [4]).

The paper is organized as follows. Section 2 is devoted to the metric regularity property and Lyusternik-Graves theorem. It contains a nonlocal (due to Milytin) version of the theorem as well as some discussions. All regularity statements formulated in the rest of the paper are corollaries of this fundamental result. In Section 3 *uniform metric regularity* is defined and a *parametric* version of Lyusternik-Graves theorem (for a family of mappings depending on a parameter) is formulated and proved. In Section 4 we define *metric multi-regularity* for mappings into product spaces when each component is perturbed independently, and consider systems of generalized equations. The application of the regularity criteria produces some "error bound" type estimates. In the final Section 5 regularity properties of collections of sets are considered. Based on the results from Section 4, some regularity conditions are established.

Mainly standard notations are used throughout the paper. A closed ball of radius ρ centered at x in a metric space is denoted by $B_\rho(x)$. We write B_ρ if $x = 0$. Distance is denoted by $d(\cdot, \cdot)$. The same notation is used for point-to-set distances: $d(x, \Omega) = \inf_{\omega \in \Omega} d(x, \omega)$. When considering products of metric spaces, we always assume that they are equipped with the maximum-type distance: $d((x_1, y_1), (x_2, y_2)) = \max \{d(x_1, x_2), d(y_1, y_2)\}$. The notation $F : X \rightrightarrows Y$ is used for set-valued mappings (the term "set-valued" is usually omitted) between metric spaces as opposed to single-valued functions $f : X \to Y$.

2 Metric regularity and Lyusternik-Graves theorem

For a mapping $F : X \rightrightarrows Y$ between metric spaces and a point $(x^\circ, y^\circ) \in \operatorname{gph} F$ (the graph of F) define a (possibly infinite) constant

$$r[F](x^\circ, y^\circ) = \liminf_{x \to x^\circ,\, y \to y^\circ} \left[\frac{d(y, F(x))}{d(x, F^{-1}(y))} \right]_\infty . \tag{5}$$

Here $[\cdot/\cdot]_\infty$ is the "extended" division operation, which differs from the usual one in the additional rule $[0/0]_\infty = +\infty$. This allows one not to worry about the points (x, y) with $y \in F(x)$ when evaluating the lower limit in (5).

The constant (5) characterizes the metric regularity property: $r[F](x^\circ, y^\circ) > 0$ if and only if F is metrically regular near (x°, y°). Moreover, if positive, this constant provides a quantitative characteristic of metric regularity. It is also known as the *rate* or *modulus of surjection* or *covering* (see [13,19]) and is often denoted $\operatorname{sur} F(x^\circ | \, y^\circ)$ or $\operatorname{cov} F(x^\circ | \, y^\circ)$.

It is a well known fact (see [13,19,22]) that

$$r[F](x^\circ, y^\circ) = \frac{1}{\operatorname{reg} F(x^\circ | \, y^\circ)},$$

where $\operatorname{reg} F(x^\circ | \, y^\circ)$ is the *modulus of metric regularity* defined as the infimum of all κ for which (1) holds.

We prefer $r[F](x^\circ, y^\circ)$ to $\operatorname{reg} F(x^\circ | \, y^\circ)$ when characterizing metric regularity because of its explicit "derivative-like" definition (5) and because it leads to simpler estimates in the statement of the extended Lyusternik-Graves theorem (see Theorem 1 below).

Similarly to (5), one can define another constant:

$$l[F](x^\circ, y^\circ) = \limsup_{x \to x^\circ, \, y \to y^\circ} \left[\frac{d(y, F(x))}{d(x, F^{-1}(y))} \right]_0, \tag{6}$$

where $[\cdot / \cdot]_0$ stands for another "extended" division operation with the additional rule $[0/0]_0 = 0$. Formula (6) can be rewritten equivalently as

$$l[F](x^\circ, y^\circ) = \limsup_{\substack{x, x' \to x^\circ, \, y \to y^\circ \\ y \in F(x')}} \left[\frac{d(y, F(x))}{d(x, x')} \right]_0. \tag{7}$$

This is exactly the *Lipschitz modulus* $\operatorname{lip} F(x^\circ | \, y^\circ)$ [22]. This constant is convenient for characterizing the Aubin's *Lipschitz-like* (*pseudo-Lipschitz*) property [1,19,22]. It follows immediately from the definition, that

$$l[F](x^\circ, y^\circ) = \frac{1}{r[F^{-1}](y^\circ, x^\circ)}.$$

If $f : X \to Y$ is a usual (single-valued) function we shall speak about metric regularity near x° and write $r[f](x^\circ)$ and $l[f](x^\circ)$ instead of $r[f](x^\circ, y^\circ)$ and $l[f](x^\circ, y^\circ)$, respectively. In this case, (7) takes a simpler form:

$$l[f](x^\circ) = \limsup_{x, x' \to x^\circ} \left[\frac{d(f(x), f(x'))}{d(x, x')} \right]_0. \tag{8}$$

Along with (5), (8), consider their "approximate" counterparts corresponding to some positive δ:

$$r_\delta[F](x^\circ,y^\circ) = \inf_{x\in B_\delta(x^\circ),\, y\in B_\delta(y^\circ)} \left[\frac{d(y,F(x))}{d(x,F^{-1}(y))}\right]_\infty, \tag{9}$$

$$l_\delta[f](x^\circ) = \sup_{x,x'\in B_\delta(x^\circ)} \left[\frac{d(f(x),f(x'))}{d(x,x')}\right]_0. \tag{10}$$

Obviously,

$$r[F](x^\circ,y^\circ) = \lim_{\delta\to 0} r_\delta[F](x^\circ,y^\circ), \quad l[f](x^\circ) = \lim_{\delta\to 0} l_\delta[f](x^\circ).$$

Moreover, $r_\delta[F](x^\circ,y^\circ) \geq \alpha > 0$ if and only if (1) holds with $\kappa = 1/\alpha$.

Using (9) and (10), one can formulate the following extension of Lyusternik-Graves theorem.

Theorem 1 *Consider mappings $F : X \rightrightarrows Y$ and $g : X \to Y$ from a complete metric space into a normed linear space. Suppose that $\operatorname{gph} F$ is closed in $X \times Y$ and $(x^\circ, y^\circ) \in \operatorname{gph} F$. Then for any $\delta > 0$,*

$$r_\delta[F+g](x^\circ, y^\circ + g(x^\circ)) \geq r_\delta[F](x^\circ,y^\circ) - l_\delta[g](x^\circ). \tag{11}$$

The single-valued version of Theorem 1 can essentially be found in [5]. The proof given in [5] can be easily extended to cover the set-valued case (see also [13] for set-valued statements and historical comments). Taking $\delta \to 0$ in (11) one immediately obtains the *extended Lyusternik-Graves theorem* as it is formulated in [10, 12] (in the Banach space setting).

Theorem 1 guarantees metric regularity of the perturbed mapping $F+g$ when F is metrically regular and $l_\delta[g](x^\circ) < r_\delta[F](x^\circ,y^\circ)$ for some $\delta > 0$. Note the "nonlocal" character of this theorem: metric regularity of $F+g$ is guarantied in the same δ-neighborhood of (x°, y°), where F is metrically regular and g is Lipschitz continuous. (The corresponding statement in [5] is formulated for an arbitrary set.) This is important for some applications (see Theorem 2 below).

Remark 1 *The assumption that the graph of F is closed can be weakened: it is sufficient to assume that it is locally closed near (x°, y°): if $\operatorname{gph} F \cap B_{\delta_0}(x^\circ,y^\circ)$ is closed for some $\delta_0 > 0$ then (11) holds for any positive $\delta < \delta_0$.*

Applying Theorem 1 to the sum of the set-valued mapping $F+g$ and the function $-g$, one immediately obtains an upper estimate for $r_\delta[F+g](x^\circ, y^\circ + g(x^\circ))$.

Corollary 1.1 *Under the conditions of Theorem 1, for any $\delta > 0$,*

$$r_\delta[F](x^\circ,y^\circ) - l_\delta[g](x^\circ) \leq r_\delta[F+g](x^\circ, y^\circ + g(x^\circ)) \leq r_\delta[F](x^\circ,y^\circ) + l_\delta[g](x^\circ).$$

The main application of this result is when $\lim_{\delta\to 0} l_\delta[g](x^\circ) = 0$, that is g is *strictly stationary* at x° [7, 8], or, in other words, F is a *strict first-order approximation* [12, 19] to $F + g$ (and vice versa) at x°. It guarantees that a perturbation of a mapping by a strictly stationary function does not affect its metric regularity. This result allows one to reduce examining metric regularity of a complicated mapping to that of a simpler (usually linear) one. For instance, in the case of a strictly differentiable at x° function $f : X \to Y$ between Banach spaces, it follows that f is metrically regular near x° if and only if $\nabla f(x^\circ)$ is regular (Lyusternik-Graves theorem, see [11]), which by Banach open mapping theorem is equivalent to its surjectivity.

Based on Corollary 1.1, it is possible to consider a more general case of "partial strict linearization" for the sum $f + F$ of a strictly differentiable function f and an arbitrary set-valued mapping F with closed graph (see [12]). It is also possible to consider mappings, which admit nonlinear (in particular, positively homogenous) approximations.

3 Uniform metric regularity

Theorem 1 (or its Corollary 1.1) makes it possible to extend the metric regularity estimates to the important for applications case of mappings depending on a parameter.

Consider a mapping $F : P \times X \rightrightarrows Y$, where X and Y are metric spaces and P is a topological space. Denote $F_p = F(p, \cdot) : X \rightrightarrows Y$. Let $(p^\circ, x^\circ, y^\circ) \in \operatorname{gph} F$. Similarly to (5), (6) define (possibly infinite) constants

$$r[F](p^\circ; x^\circ, y^\circ) = \liminf_{\substack{x\to x^\circ,\, y\to y^\circ \\ p\to p^\circ}} \left[\frac{d(y, F(p, x))}{d(x, F_p^{-1}(y))}\right]_\infty, \tag{12}$$

$$l[F](p^\circ; x^\circ, y^\circ) = \limsup_{\substack{x\to x^\circ,\, y\to y^\circ \\ p\to p^\circ}} \left[\frac{d(y, F(p, x))}{d(x, F_p^{-1}(y))}\right]_0.$$

(The approximate δ-versions of the above constants can also be of interest.)

We shall say that F is *uniformly metrically regular* near $(p^\circ, x^\circ, y^\circ)$ with respect to (x, y) if $r[F](p^\circ; x^\circ, y^\circ) > 0$, that is if there exist $\kappa \geq 0$ and $\delta > 0$ such that

$$d(x, F_p^{-1}(y)) \leq \kappa d(y, F(p, x)), \quad \forall x \in B_\delta(x^\circ),\ y \in B_\delta(y^\circ),\ p \in B_\delta(p^\circ).$$

If $f : P \times X \to Y$ is single-valued we write $r[f](p^\circ; x^\circ)$ and $l[f](p^\circ; x^\circ)$. In this case

$$l[f](p^\circ; x^\circ) = \limsup_{\substack{x,x' \to x^\circ \\ p \to p^\circ}} \left[\frac{d(f(p,x), f(p,x'))}{d(x,x')} \right]_0 . \tag{13}$$

The condition $l[f](p^\circ; x^\circ) < \infty$ means that f is locally *uniformly Lipschitz* [6, 9] (*equi-Lipschitz* [3]) in x at (p°, x°).

When X and Y are normed spaces, a function $f : P \times X \to Y$ is said to be *partially strictly differentiable* in x [20] (see also [3]) at (p°, x°) if there exists a linear and continuous mapping $\nabla_x f(p^\circ, x^\circ) : X \to Y$ such that

$$\lim_{\substack{x,x' \to x^\circ,\, p \to p^\circ \\ x' \neq x}} \frac{f(p,x') - f(p,x) - \nabla_x f(p^\circ, x^\circ)(x' - x)}{\|x' - x\|} = 0,$$

or, in other words, $l[f - \nabla_x f(p^\circ, x^\circ)](p^\circ; x^\circ) = 0$, that is the linear mapping $x \to \nabla_x f(p^\circ, x^\circ)x$ *strictly approximates* f in x at (p°, x°) (see [21] where the term "strongly" is used). Clearly this is a direct generalization of the notion of *strict differentiability* [20] (see [19, 22]) which corresponds to the case when f does not depend on p.

Theorem 2 *Consider a mapping $F : X \rightrightarrows Y$ from a complete metric space into a normed linear space and a function $g : P \times X \to Y$, where P is a topological space. Suppose that* $\operatorname{gph} F$ *is closed in $X \times Y$, $(x^\circ, y^\circ) \in \operatorname{gph} F$ and $l[g](p^\circ; x^\circ) = 0$. Then*

$$r[F + g](p^\circ; x^\circ, y^\circ + g(p^\circ, x^\circ)) = r[F](x^\circ, y^\circ). \tag{14}$$

The short proof provided below illustrates standard arguments used when deducing this type of "parametric" statements from the extended Lyusternik-Graves theorem. Note that one needs to use the full "approximate" Theorem 1 (or Corollary 1.1); the limiting statement is not sufficient.

Proof. Take an arbitrary $\varepsilon > 0$. It follows from (13), that there exists $\delta > 0$ such that for any $p \in B_\delta(p^\circ)$ one has $l_\delta[g_p](x^\circ) \le \varepsilon$, where $g_p = g(p, \cdot)$. Corollary 1.1 implies the estimates

$$r_\delta[F](x^\circ, y^\circ) - \varepsilon \le r_\delta[(F + g)_p](x^\circ, y^\circ + g(p, x^\circ)) \le r_\delta[F](x^\circ, y^\circ) + \varepsilon. \tag{15}$$

Evidently

$$\begin{aligned} r[F + g](p^\circ; x^\circ, y^\circ + g(p^\circ, x^\circ)) &\le \liminf_{p \to p^\circ} r[(F + g)_p](x^\circ, y^\circ + g(p, x^\circ)) \\ &= \liminf_{p \to p^\circ} \lim_{\delta \to 0} r_\delta[(F + g)_p](x^\circ, y^\circ + g(p, x^\circ)), \end{aligned}$$

and the second inequality in (15) yields

$$r[F + g](p^\circ; x^\circ, y^\circ + g(p^\circ, x^\circ)) \le r[F](x^\circ, y^\circ) + \varepsilon. \tag{16}$$

On the other hand, it follows from the first inequality in (15) that

$$d(y,(F+g)(p,x)) \geq (r_\delta[F](x^\circ,y^\circ)-\varepsilon)d(x,(F+g)_p^{-1}(y))$$

for all $x \in B_\delta(x^\circ)$, $y \in B_\delta(y^\circ)$, $p \in B_\delta(p^\circ)$, and consequently

$$r[F+g](p^\circ;x^\circ,y^\circ+g(x^\circ)) \geq \lim_{\delta\to 0} r_\delta[F](x^\circ,y^\circ)-\varepsilon = r[F](x^\circ,y^\circ)-\varepsilon. \tag{17}$$

Since ε is arbitrary, (16) and (17) imply (14). $\square$

4 Systems of generalized equations

Consider a mapping $F : X \rightrightarrows Y$, where X is a normed linear space and $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ is a Cartesian product of $n \geq 1$ normed linear spaces. Suppose that F can be represented as $F = (F_1, F_2, \ldots, F_n)$, where each F_i is a mapping from X into Y_i. This means that for any $x \in X$ its image $F(x)$ under F is the product of the images: $F(x) = F_1(x) \times F_2(x) \times \ldots \times F_n(x)$. If F is single-valued this assumption is fulfilled automatically.

Let $(x^\circ, y^\circ) \in \operatorname{gph} F$ and $y^\circ = (y_1^\circ, y_2^\circ, \ldots, y_n^\circ)$. Thus, $(x^\circ, y_i^\circ) \in \operatorname{gph} F_i$, $i = 1, 2, \ldots, n$. Along with constant (5) define another regularity constant:

$$\hat{r}[F](x^\circ,y^\circ) = \liminf_{\substack{x_i\to x^\circ,\, y_i\to y_i^\circ\\ i=1,2,\ldots,n}} \left[\frac{\max\limits_{1\leq i\leq n} d(y_i,F_i(x_i))}{d(0,\bigcap\limits_{i=1}^{n}(F_i^{-1}(y_i)-x_i)}\right]_\infty. \tag{18}$$

Note that (5) corresponds to taking $x_1 = x_2 = \ldots = x_n$ in the above limit. Thus, in general,

$$\hat{r}[F](x^\circ,y^\circ) \leq r[F](x^\circ,y^\circ),$$

and (18) gives rise to a stronger regularity concept: one can say that F is *metrically multi-regular* near $(x^\circ, y^\circ) \in \operatorname{gph} F$ if $\hat{r}[F](x^\circ, y^\circ) > 0$, that is if there exist $\kappa \geq 0$ and $\delta > 0$ such that

$$d(0,\bigcap_{i=1}^{n}(F_i^{-1}(y_i)-x_i)) \leq \kappa \max_{1\leq i\leq n} d(y_i,F_i(x_i)) \tag{19}$$

for all $x_i \in B_\delta(x^\circ)$, $y_i \in B_\delta(y_i^\circ)$, $i = 1, 2, \ldots, n$. The infimum of all such κ equals $1/\hat{r}[F](x^\circ, y^\circ)$.

Now we turn to system (2) from Section 1. Let x° be a solution of this system. We say that (2) is *metrically multi-regular* at x° if there exist $\kappa > 0$ and $\delta > 0$ such that for any $x_i \in B_\delta(x^\circ)$ and $y_i \in B_\delta$, $i = 1, 2, \ldots, n$, there exists $x \in X$ such that (3), (4) hold true.

The metric multi-regularity property guarantees the existence of a solution of the perturbed system (3) and provides the estimate (4) of the norm of the solution which can be interpreted as an *error bound*. Note that both the right-hand sides and variables are perturbed in (3), and the variables in different generalized equations are perturbed independently.

One can easily check that metric multi-regularity of (2) at x° as it is defined above is nothing else but metric multi-regularity near (x°, y°) (with $y^\circ = 0$) of the mapping $F : X \rightrightarrows Y$ defined by $F(x) = f(x) + Q$, where $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$ and $Q = Q_1 \times Q_2 \times \dots \times Q_n$.

Remark 2 *Conditions* (3), (4) *can be viewed as a realization of* (19). *However, the constant* κ *in* (4) *is in general greater than (but can be taken arbitrarily close to) the correspondent constant in* (19), *because the distance in the left-hand side of* (19) *does not have to be achieved.*

Theorem 3 *Let* $x^\circ \in X$ *be a solution of system* (2), *where the functions* $f_i : X \to Y_i$ *between Banach spaces are strictly differentiable at* x° *and the sets* $Q_i \subset Y_i$ *are closed and convex. Then system* (2) *is metrically multi-regular at* x° *if and only if*

$$0 \in \operatorname{int} \{f(x^\circ) + \nabla f(x^\circ)X + Q\}. \tag{20}$$

Proof. Consider the Banach spaces $P = X^n$ and $Y = Y_1 \times Y_2 \times \dots \times Y_n$, and define the multifunction $\Phi : P \times X \rightrightarrows Y$ by the relation $\Phi(p, x) = \phi(p, x) + Q$, where

$$\phi(p, x) = (f_1(x_1 + x), f_2(x_2 + x), \dots, f_n(x_n + x))$$

and $p = (x_1, x_2, \dots, x_n)$. Denote $p^\circ = (x_1^\circ, x_2^\circ, \dots, x_n^\circ)$. Then metric multi-regularity of (2) at x° means the existence of $\kappa > 0$ and $\delta > 0$ such that for any $p \in B_\delta(p^\circ)$, $y \in B_\delta$ there exists $x \in X$ such that $\Phi(p, x) \ni y$ and $\|x\| \le \kappa d(y, \Phi(p, 0))$. Notice that Φ possesses the following property: for any $x, x' \in X$, $p \in P$ one has $\Phi(p, x + x') = \Phi(p + p', x)$, where $p' = (x', x', \dots, x')$. This property allows us to conclude that metric multi-regularity of (2) at x° is equivalent to the existence of $\kappa > 0$ and $\delta > 0$ such that for any $p \in B_\delta(p^\circ)$, $x \in B_\delta$, $y \in B_\delta$ there exists an $x' \in \Phi_p^{-1}(y)$ such that $\|x - x'\| \le \kappa d(y, \Phi(p, x))$. In its turn, the last condition is equivalent to the uniform metric regularity of Φ near $(p^\circ, 0, 0)$.

The function ϕ is partially strictly differentiable in x at $(p^\circ, 0)$ and its derivative coincides with that of $f = (f_1, f_2, \dots, f_n)$ at x°: $\nabla f(x^\circ) = (\nabla f_1(x^\circ), \nabla f_2(x^\circ), \dots, \nabla f_n(x^\circ)) : X \to Y$. Define the mapping $F(x) = \nabla f(x^\circ)x + Q$ and the function $g(p, x) = \phi(p, x) - \nabla f(x^\circ)x$. Then $\operatorname{gph} F$ is closed and convex, $l[g](p^\circ; 0) = 0$ and $\Phi(p, x) = F(x) + g(p, x)$. By Theorem 2 uniform metric regularity of Φ near $(p^\circ, 0, 0)$ is equivalent to metric regularity of F at $(0, -f(x^\circ))$. Due to convexity of $\operatorname{gph} F$ the criterion of metric regularity of F is provided by Robinson-Ursescu theorem (see [1, 11, 19]): $-f(x^\circ) \in \operatorname{int} F(X)$ which is exactly condition (20). $\square$

Note that (20) implies the following well known qualification condition:

$$[y^* \in Y^*, \quad (\nabla f(x^\circ))^* y^* = 0, \quad \langle y^*, y \rangle \geq 0, \ \forall y \in f(x^\circ) + Q] \quad \Rightarrow \quad y^* = 0.$$

If Q is convex and $\operatorname{int}\{\nabla f(x^\circ)X + Q\} \neq \emptyset$ (the last requirement can be weakened) both conditions are equivalent.

If the functions f_i, $i = 1, 2, \dots, n$, are continuous near x° then the maximum in (4) can obviously be taken over the set of *active constraints*

$$I(x^\circ) = \{i \in \{1, 2, \dots, n\} : \ -f_i(x^\circ) \in \operatorname{bd} Q_i\}.$$

Here $\operatorname{bd} Q_i = Q_i \backslash \operatorname{int} Q_i$ is the boundary of Q_i. (The case $\operatorname{int} Q_i = \emptyset$ is not excluded.)

The next proposition gives an estimate for the solution of the system of perturbed generalized equations, in which only the variables are perturbed, not the right-hand sides of the equations. Here, the variable in each equation is perturbed not relative to the solution of the whole system, but relative to any sufficiently close to it solution of this particular equation.

Proposition 1 *Let $x^\circ \in X$ be a solution of system* (2). *Suppose that, in the neighborhood of x°, the functions f_i are Lipschitz continuous for $i \in I(x^\circ)$ and continuous for $i \notin I(x^\circ)$. If system* (2) *is metrically multi-regular at x°, then there exist constants $L > 0$ and $\delta > 0$ such that for any $x_i \in B_\delta(x^\circ)$ satisfying the individual inclusions*

$$f_i(x_i) + Q_i \ni 0, \qquad i = 1, 2, \dots, n,$$

and any variations $u_i \in B_\delta$, $i = 1, 2, \dots, n$, one can find $x \in X$ such that

$$f_i(x_i + u_i + x) + Q_i \ni 0, \quad i = 1, 2, \dots, n, \tag{21}$$

$$\|x\| \leq L \max_{i \in I(x^\circ)} \|u_i\|. \tag{22}$$

Proof. By the definition of metric multi-regularity, there exist constants $\kappa > 0$ and $\delta > 0$ such that for any $x_i \in B_{2\delta}(x^\circ)$, $i = 1, 2, \dots, n$, there exists $x \in X$ satisfying

$$f_i(x_i + x) + Q_i \ni 0, \quad i = 1, 2, \dots, n,$$

$$\|x\| \leq \kappa \max_{1 \leq i \leq n} d(f_i(x_i), -Q_i). \tag{23}$$

Choosing a sufficiently small δ, one can assume that, for $i \in I(x^\circ)$, f_i is Lipschitz continuous on $B_{2\delta}(x^\circ)$ with a modulus l, and $f_i(x_i) \in -Q_i$ if $x_i \in B_{2\delta}(x^\circ)$ and $i \notin I(x^\circ)$. So, the maximum in (23) is actually over $I(x^\circ)$. Take an arbitrary $x_i \in B_\delta(x^\circ)$ satisfying $f_i(x_i) \in -Q_i$, and $u_i \in B_\delta$. Then $x_i + u_i \in B_{2\delta}(x^\circ)$ and

$$d(f_i(x_i + u_i), -Q_i) \leq \|f_i(x_i + u_i) - f_i(x_i)\| \leq l \, \|u_i\|.$$

Combining this with (23), one gets the existence of $x \in X$ satisfying (21), (22) with $L = \kappa l$. $\square$

It follows from Theorem 3 that in the case when all the spaces are Banach and the functions are strictly differentiable, the regularity condition (20) is sufficient for the conclusions of Proposition 1.

5 Regularity of collections of sets

In this section we consider a collection of sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ in a normed space X with a common point $x^\circ \in \cap_{i=1}^n \Omega_i$. For any $\rho > 0$ one can define the constant

$$\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = \sup \left\{ r \geq 0 : \left(\bigcap_{i=1}^n (\Omega_i - a_i) \right) \bigcap B_\rho(x^\circ) \neq \emptyset, \ \forall a_i \in B_r \right\} \tag{24}$$

characterizing the mutual arrangement of these sets. It shows how far they can be "pushed apart" independently while still intersecting in a given neighborhood of x°.

Obviously, $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) \geq 0$. If $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$ for some $\rho > 0$, then the collection of sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ is *locally extremal* [17] at x°.

A "limiting" constant can be defined based on (24):

$$\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) = \liminf_{\substack{\omega_i \to x^\circ, \ \omega_i \in \Omega_i \\ \rho \to +0}} \frac{\theta_\rho[(\Omega_1 - \omega_1), \ldots, (\Omega_n - \omega_n)](0)}{\rho}. \tag{25}$$

The constant (25) is in a sense a derivative-like object. It possesses some properties of the strict derivative as it accumulates information about local properties of the sets not only at the given point but also at all nearby points.

If $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) > 0$, then the collection of sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ is called *strongly regular* [15, 16] at x°.

The last condition means that there exist constants $\alpha > 0$ and $\delta > 0$ such that

$$\theta_\rho[\Omega_1, \ldots, \Omega_n](\omega_1, \ldots, \omega_n) > \alpha\rho$$

for any positive $\rho \leq \delta$ and any $\omega_i \in B_\delta(x^\circ)$, $i = 1, 2, \ldots, n$, or, in other words, the sets $(\Omega_i - \omega_i - a_i)$ have a common point in B_ρ for any $\rho \leq \delta$ and any $\omega_i \in \Omega_i \cap B_\delta(x^\circ)$, $a_i \in B_{\alpha\rho}$, $i = 1, 2, \ldots, n$. The supremum of all such α equals $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ)$.

The concept of strong regularity of a collection of sets seems to be useful in variational analysis, e.g. when formulating qualification conditions. It can also have algorithmic applications (see [18]).

Different primal and dual characterizations of strong regularity of collections of sets can be found in [15, 16]. There exist strong relations between regularity properties of collections of sets and the corresponding properties of appropriate multifunctions (see e.g. [13]). The next assertion is basically Proposition 8 from [16].

Proposition 2 *The collection of sets $\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_n$ is strongly regular at x° if and only if the multifunction $x \to (\Omega_1 - x)\times(\Omega_2 - x)\times\ldots\times(\Omega_n - x)$ from X into subsets of X^n is metrically regular at $(0, x^\circ, \ldots, x^\circ)$.*

Consider now the case when the sets Ω_i are of the form

$$\Omega_i = \{x \in X:\ F_i(x) \ni 0\}, \quad i = 1, 2, \ldots, n, \tag{26}$$

where F_i is a (set-valued) mapping from X into a normed space Y_i. Then the definition of strong regularity takes the following form.

Proposition 3 *The collection of sets (26) is strongly regular at x° if and only if there exist constants $\alpha > 0$ and $\delta > 0$ such that for any positive $\rho \le \delta$ and any $x_i \in B_\delta(x^\circ)$ with $F_i(x_i) \ni 0$, $u_i \in B_{\alpha\rho}$, $i = 1, 2, \ldots, n$, one can find an $x \in B_\rho$, such that $F_i(x_i + u_i + x) \ni 0$, $i = 1, 2, \ldots, n$.*

The case of particular interest is when each F_i is of the form $F_i(x) = f_i(x) + Q_i$, where $f_i : X \to Y_i$ is a (single-valued) function and $Q_i \subset Y_i$. The application of Theorem 3 and Propositions 1, 3 yields the following sufficient regularity condition.

Theorem 4 *Let the conditions of Theorem 3 be satisfied. Suppose also that* (20) *holds true, where $f = (f_1, f_2, \ldots, f_n)$ and $Q = Q_1 \times Q_2 \times \ldots \times Q_n$. Then the collection of sets*

$$\Omega_i = \{x \in X:\ f_i(x) + Q_i \ni 0\},\ i = 1, 2, \ldots, n,$$

is strongly regular at x°.

For the collection of sets (26), along with the linear shifts in (24), one can also consider their nonlinear transformations related to variations of the right-hand sides of the inclusions in (26). Namely, for any $y_i \in Y_i$ consider the "perturbed" sets

$$\Omega_i(y_i) = \{x \in X:\ F_i(x) \ni y_i\} = F_i^{-1}(y_i), \quad i = 1, 2, \ldots, n. \tag{27}$$

The initial sets (26) correspond to $y_i = 0$, $i = 1, 2, \ldots, n$.

Similarly to (24)–(25), the following "analytical" constants can be defined:

$$\sigma_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = \sup\left\{ r \ge 0 : \left(\bigcap_{i=1}^{n} \Omega_i(y_i)\right) \bigcap B_\rho(x^\circ) \ne \emptyset,\ \forall y_i \in B_r \right\}, \tag{28}$$

$$\hat\sigma[\Omega_1, \ldots, \Omega_n](x^\circ) = \liminf_{\substack{\omega_i \to x^\circ,\ \omega_i \in \Omega_i \\ \rho \to +0}} \frac{\sigma_\rho[(\Omega_1 - \omega_1), \ldots, (\Omega_n - \omega_n)](0)}{\rho}. \tag{29}$$

Note that $\Omega_i - \omega_i = \{u:\ F_i(u + \omega_i) \ni 0\}$.

Defining the mapping $F : X \rightrightarrows Y = Y_1 \times Y_2 \times \ldots \times Y_n$ by $F(x) = F_1(x) \times F_2(x) \times \ldots \times F_n(x)$ one can easily see that

$$\sigma_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = \sup\{r \geq 0 : \ F(B_\rho(x^\circ)) \supset B_r\} \tag{30}$$

and (29) is actually the covering constant for F at $(x^\circ, 0)$. At the same time (29) characterizes a kind of joint regularity for the collection of mappings $F_1, F_2, \ldots, F_n$ near $(x^\circ, 0)$, related to metric multi-regularity defined in terms of constant (18).

Proposition 4 $\hat{r}[F](x^\circ, 0) \leq \hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ)$.

Proof. Let $0 < r < \hat{r}[F](x^\circ, 0)$. It is sufficient to show that $r \leq \hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ)$. By (18) there exists $\delta > 0$ such that for any $x_i \in B_\delta(x^\circ)$, $y_i \in B_{\delta r}$, $i = 1, 2 \ldots, n$, one can find $x \in X$ such that $y_i \in F_i(x_i + x)$, $i = 1, 2 \ldots, n$, and

$$\|x\| < \frac{1}{r} \max_{1 \leq i \leq n} d(y_i, F_i(x_i)).$$

Take arbitrary positive $\rho < \delta$ and $\omega_i \in \Omega_i \cap B_\delta(x^\circ)$, $y_i \in B_{\rho r}$, $i = 1, 2 \ldots, n$. Then $y_i \in B_{\delta r}$, $F_i(\omega_i) \ni 0$ and $d(y_i, F_i(\omega_i)) \leq \|y_i\| \leq \rho r$. Consequently, there exists $x \in X$ such that $F_i(\omega_i + x) \ni y_i$, $i = 1, 2 \ldots, n$, and $\|x\| < \rho$. In other words, $\left(\bigcap_{i=1}^n (\Omega_i(y_i) - \omega_i)\right) \bigcap B_\rho(x^\circ) \neq \emptyset$ for any $y_i \in B_{\rho r}$. This yields $\sigma_\rho[\Omega_1, \ldots, \Omega_n](\omega_1, \ldots, \omega_n) \geq \rho r$ and $\hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ) \geq r$. $\square$

Thus, metric multi-regularity of F implies $\hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ) > 0$. Note that the inequality in Proposition 4 is not reversible in general, since in the definition (29) of $\hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ)$ the sequences $\{\omega_i\}$ are limited to the corresponding sets Ω_i. Some partial inversion is possible if the mappings are single-valued.

Theorem 5 *Let $f_i : X \to Y_i$, $i = 1, 2, \ldots, n$, be functions between Banach spaces,*

$$\Omega_i = \{x \in X : \ f_i(x) = 0\}, \ i = 1, 2, \ldots, n, \tag{31}$$

and $x^\circ \in \cap_{i=1}^n \Omega_i$. Suppose that $f = (f_1, f_2, \ldots, f_n)$ is strictly differentiable at x°. Then $\hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ) > 0$ if and only if $\nabla f(x^\circ)$ is surjective. Under these conditions the collection of sets (31) *is strongly regular at x°.*

Proof. Apply Theorem 3 with $Q = \{0\}$. Condition (20) takes the form $\nabla f(x^\circ)X = Y$. Thus, surjectivity of ∇f is equivalent to $\hat{r}[f](x^\circ) > 0$. Due to Proposition 4 the last inequality implies $\hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ) > 0$.

Suppose now that $\nabla f(x^\circ)X \neq Y$. We need to show that $\hat{\sigma}[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$. Take an arbitrary $\varepsilon > 0$. It is known (see Lemma 1 below) that in this case there exists an element $y^* \in Y^*$ with $\|y^*\| = 1$ such that $|\langle y^*, \nabla f(x^\circ)x\rangle| \leq \varepsilon \|x\|$ for all $x \in X$. Since $f(x^\circ) = 0$, there exists a $\delta > 0$ such that for any $x \in B_\delta$ one has $\|f(x^\circ + x) - \nabla f(x^\circ)x\| \leq \varepsilon \|x\|$, and consequently $|\langle y^*, f(x^\circ + x)\rangle| \leq 2\varepsilon \|x\|$. Take any positive numbers $\rho \leq \delta$ and $r < \sigma_\rho[\Omega_1, \ldots, \Omega_n](x^\circ)$ (if $\sigma_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$, then $\hat{\sigma}[\Omega_1,\dots,\Omega_n](x^\circ) = 0$ trivially holds). According to (30), $F(B_\rho(x^\circ)) \supset B_r$, hence

$$r = \sup_{y\in B_r} \langle y^*, y\rangle \le \sup_{x\in B_\rho} \langle y^*, f(x^\circ + x)\rangle \le 2\varepsilon\rho,$$

and consequently $\sigma_\rho[\Omega_1,\dots,\Omega_n](x^\circ) \le 2\varepsilon\rho$. According to definition (29) one has

$$\hat{\sigma}[\Omega_1,\dots,\Omega_n](x^\circ) \le \liminf_{\rho\to+0} \frac{\sigma_\rho[\Omega_1,\dots,\Omega_n](x^\circ)}{\rho} \le 2\varepsilon,$$

and consequently $\hat{\sigma}[\Omega_1,\dots,\Omega_n](x^\circ) = 0$.

The last assertion follows from Theorem 4. □

Lemma 1 *Let $A : X \to Y$ be a linear bounded mapping between Banach spaces. If A is not surjective, then for any $\varepsilon > 0$ there exists a $y^* \in Y^*$ with $\|y^*\| = 1$ such that $|\langle y^*, Ax\rangle| \le \varepsilon$ for all $x \in B_1$.*

Proof. If $\overline{AX} \ne Y$, the assertion is trivial. Suppose that $\overline{AX} = Y$. Take any $\varepsilon > 0$. If $\overline{A(B_1)} \supset B_\varepsilon$, then $A(B_1)$ is dense for B_ε, and so, by Theorem 1.5 from [5] (or by the second part of the proof of the Banach open mapping theorem) $A(B_1) \supset B_{\varepsilon/2}$, whence $AX = Y$, so A is surjective, a contradiction. Thus, the closed convex set $C = \overline{A(B_1)}$ does not contain some point $y_0 \in B_\varepsilon$. By the separation theorem there exists $y^* \in Y^*$ with $\|y^*\| = 1$ such that $\langle y^*, y\rangle \le \langle y^*, y_0\rangle \le \varepsilon$ for all $y \in C$. Hence, for any $x \in B_1$ one has $\langle y^*, Ax\rangle \le \varepsilon$ and $-\langle y^*, Ax\rangle = \langle y^*, A(-x)\rangle \le \varepsilon$. □

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References

[1] J.-P. Aubin, I. Ekeland, Applied Nonlinear Analysis, John Wiley, New York, 1984.

[2] D. Azé, A unified theory for metric regularity of multifunctions, J. Convex Anal. 13 (2) (2006) 225–252.

[3] J. M. Borwein, Stability and regular points of inequality systems, J. Optim. Theory Appl. 48 (1) (1986) 9–52.

[4] A. V. Dmitruk, On a nonlocal metric regularity of nonlinear operators, Control Cybernet. 34 (3) (2005) 723–746.

[5] A. V. Dmitruk, A. A. Milyutin, N. P. Osmolovsky, Lyusternik's theorem and the theory of extrema, Russian Math. Surveys 35 (1980) 11–51.

[6] A. L. Dontchev, Implicit function theorems for generalized equations, Math. Program., Ser. A 70 (1) (1995) 91–106.

[7] A. L. Dontchev, The Graves theorem revisited, J. Convex Anal. 3 (1) (1996) 45–53.

[8] A. L. Dontchev, W. W. Hager, An inverse mapping theorem for set-valued maps, Proc. Amer. Math. Soc. 121 (2) (1994) 481–489.

[9] A. L. Dontchev, W. W. Hager, Implicit functions, Lipschitz maps, and stability in optimization, Math. Oper. Res. 19 (3) (1994) 753–768.

[10] A. L. Dontchev, A. S. Lewis, Perturbations and metric regularity, Set-Valued Anal. 13 (4) (2005) 417–438.

[11] A. L. Dontchev, A. S. Lewis, R. T. Rockafellar, The radius of metric regularity, Trans. Amer. Math. Soc. 355 (2) (2003) 493–517 (electronic).

[12] A. L. Dontchev, R. T. Rockafellar, Regularity and conditioning of solution mappings in variational analysis, Set-Valued Anal. 12 (1-2) (2004) 79–109.

[13] A. D. Ioffe, Metric regularity and subdifferential calculus, Russian Math. Surveys 55 (2000) 501–558.

[14] A. D. Ioffe, V. M. Tikhomirov, Theory of Extremal Problems, North Holland, Amsterdam, 1979.

[15] A. Y. Kruger, Stationarity and regularity of set systems, Pac. J. Optim. 1 (1) (2005) 101–126.

[16] A. Y. Kruger, About regularity of collections of sets, Set-Valued Anal. 14 (2) (2006) 187–206.

[17] A. Y. Kruger, B. S. Mordukhovich, Extremal points and the Euler equation in nonsmooth optimization, Dokl. Akad. Nauk BSSR 24 (8) (1980) 684–687, in Russian.

[18] A. S. Lewis, D. R. Luke, J. Malick, Local convergence for alternating and averaged nonconvex projections, preprint (2007); available from Optimization Online, at http://www.optimization-online.org/DB_HTML/2007/09/1766.html; submitted to Foundations of Computational Mathematics.

[19] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation. I: Basic Theory, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 330, Springer-Verlag, Berlin, 2006.

[20] A. Nijenhuis, Strong derivatives and inverse mappings, Amer. Math. Monthly 81 (1974) 969–980.

[21] S. M. Robinson, An implicit-function theorem for a class of nonsmooth functions, Math. Oper. Res. 16 (2) (1991) 292–309.

[22] R. T. Rockafellar, R. J.-B. Wets, Variational Analysis, Springer-Verlag, New York, 1998.