A Simpler and Tighter Redundant Klee-Minty Construction

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Abstract

By introducing redundant Klee-Minty examples, we have previously shown that the central path can be bent along the edges of the Klee-Minty cubes, thus having $2^n - 2$ sharp turns in dimension n. In those constructions the redundant hyperplanes were placed parallel with the facets active at the optimal solution. In this paper we present a simpler and more powerful construction, where the redundant constraints are parallel with the coordinate-planes. An important consequence of this new construction is that one of the sets of redundant hyperplanes is touching the feasible region, and N, the total number of the redundant hyperplanes is reduced by a factor of n^2, further tightening the gap between iteration-complexity upper and lower bounds.

Key words: Linear programming, Klee-Minty example, interior point methods, worst-case iteration complexity, central path.

1 Introduction

Introduced by Dantzig [1] in 1947, the *simplex method* is a powerful algorithm for linear optimization problems. In 1972 Klee and Minty [7] showed that the simplex method may take an exponential number of iterations. More precisely, they presented an optimization problem over an n-dimensional squashed cube, and proved that a variant of the simplex method visits all of its 2^n vertices. The actual pivot rule in [7] was the most negative reduced cost pivot rule that is frequently referred to as "Dantzig rule". Thus, its iteration-complexity is exponential in the worst case, as $2^n - 1$ iterations are needed for solving this n-dimensional linear optimization problem. Variants of the Klee-Minty n-cube have been used to prove exponential running time for most pivot rules, see [11] and the references therein for details. Stimulated mostly by the Klee-Minty worst-case example, the search for a polynomial algorithm for solving linear optimization problems has been started. In 1979, Khachiyan [6] introduced the polynomial *ellipsoid method* for linear optimization problems. In spite of its polynomial complexity bound, the ellipsoid method turned out to be inefficient in computational practice.

In 1984 in his seminal work, Karmarkar [5] proposed a more efficient polynomial time algorithm that sparked the research on polynomial *interior point methods (IPMs)*. Unlike the simplex method which goes along the edges of the polyhedron corresponding to the feasible region, interior point methods pass through the interior of this polyhedron. Starting at the *analytic center*, most IPMs follow the so-called *central path*[1] and converge to the analytic center of the optimal face; see e.g., [9, 14]. It is well known that the number of iterations needed to have the duality gap smaller than ϵ is upper-bounded by $O(\sqrt{N} \ln \frac{v_0}{\epsilon})$, where N and v_0 respectively denote the number of inequalities and the

[1]See page 4 for the definition of the analytic center and the central path in case of redundant Klee-Minty cubes.

duality gap at the starting point. Then, the standard rounding procedure [9] can be used to compute an exact optimal solution.

In 2004, Deza, Nematollahi, Peyghami and Terlaky [2] showed that, the central path of a redundant representation of the Klee-Minty n-cube may trace the path followed by the simplex method. More precisely, an exponential number of redundant constraints parallel to the facets passing through the optimal vertex are added to the Klee-Minty n-cube to force the central path to visit an arbitrary small neighborhood of all the vertices of that cube, thus having $2^n - 2$ sharp turns. In this construction, uniform distances for the redundant constraints have been chosen and consequently the number of the inequalities for the highly redundant Klee-Minty n-cube becomes $N = O(n^2 2^{6n})$, which is further improved to $N = O(n2^{3n})$ in [4] by a meticulous analysis. In [3], by allowing the distances of the redundant constraints to the corresponding facets to decay geometrically, the number of the inequalities N is significantly reduced to $O(n^3 2^{2n})$. As shown in [3], a standard rounding procedure can be employed after $O(\sqrt{N}n)$ iterations that gives the optimal solution. This result also underlines that the reduction of the number of the redundant inequalities will further tighten the iteration-complexity lower and upper bounds.

In this paper, we simplify the construction in [3] by putting the redundant constraints parallel to the coordinate hyperplanes at geometrically decaying distances, and show that fewer redundant inequalities, only in the order of $N = O(n2^{2n})$, is needed to bend the central path along the edges of the Klee-Minty n-cube. This yields a tighter iteration-complexity upper bound $O(n^{\frac{3}{2}} 2^n)$ while retaining the same lower bound $\Omega(2^n)$. In other words, the number of iterations is bounded below by $\Omega(\sqrt{\frac{N}{\ln N}})$ and above by $O(\sqrt{N} \ln N)$.

The rest of the paper is organized as follows. In Section 2 we first introduce some notations and then provide main results in the form of some propositions. The proofs of the propositions are given in Section 3.

2 Notations and the Main Results

We consider the following variant of the Klee-Minty problem[7], with the convention $x_0 = 0$, where τ is a small positive factor by which the unit cube $[0,1]^n$ is squashed.

$$\begin{array}{lll} \min & x_n & \\ \text{subject to} & \tau\, x_{k-1} \le x_k \le 1 - \tau\, x_{k-1} & \text{for } k = 1, \ldots, n. \end{array}$$

This optimization problem has $2n$ constraints, n variables and the feasible region is a squashed n-dimensional cube denoted by $\mathcal{C}$. Variants of the simplex method may take $2^n - 1$ iterations to solve this problem since starting from $(0, \ldots, 0, 1)^T$ they may visit all the vertices ordered by the decreasing value of the last coordinate x_n till the optimal point, which is the origin.

We consider redundant constraints induced by the hyperplanes $H_k : d_k + x_k = 0$ repeated h_k times, for $k = 1, \ldots, n$. The analytic center of the Klee-Minty cube and the central path will be affected by the addition of the redundant constraints, while the feasible region remains unchanged. Having denoted the repetition-vector by $h = (h_1, \ldots, h_n)^T$ and the distance-vector by $d = (d_1, \ldots, d_n)^T$, we define the redundant linear optimization problem $\mathcal{C}_\tau^h$ as

$$\begin{array}{lll} \min & x_n & \\ \text{subject to} & \tau\, x_{k-1} \le x_k \le 1 - \tau\, x_{k-1} & \text{for } k = 1, \ldots, n, \\ & 0 \le d_k + x_k & \text{repeated } h_k \text{ times, for } k = 1, \ldots, n. \end{array}$$

By analogy with the unit cube $[0,1]^n$, we denote the vertices of the Klee-Minty cube $\mathcal{C}$ by using subsets S of $\{1,\dots,n\}$. For $S \subseteq \{1,\dots,n\}$, a vertex v^S of $\mathcal{C}$ is defined, see Figure 1, by

$$v_1^S = \begin{cases} 1, & \text{if } 1 \in S \\ 0, & \text{otherwise,} \end{cases}$$

$$v_k^S = \begin{cases} 1-\varepsilon v_{k-1}^S, & \text{if } k \in S \\ \varepsilon v_{k-1}^S, & \text{otherwise,} \end{cases} \qquad k = 2,\dots,n.$$

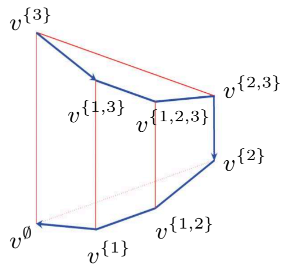

Figure 1: The vertices of the Klee-Minty 3-cube and the simplex path P_0.

The δ-neighborhood of a vertex v^S, see Figure (2), is defined, with the convention $x_0 = 0$, by

$$\mathcal{N}_\delta(v^S) = \left\{ x \in \mathcal{C} : \begin{cases} 1 - x_k - \varepsilon x_{k-1} \le \delta, & \text{if } k \in S \\ x_k - \varepsilon x_{k-1} \le \delta, & \text{otherwise} \end{cases} \qquad k = 1,\dots,n \right\}.$$

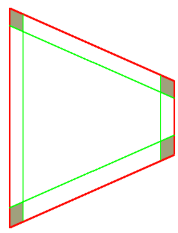

Figure 2: The δ-neighborhoods of the 4 vertices of the Klee-Minty 2-cube.

In this paper we focus on $\mathcal{C}_\tau^h$ with the following parameters:

$\tau = \frac{n}{2(n+1)}$, $\delta \le \frac{1}{4(n+1)}$,
$d = \left(\frac{1}{\sqrt{\tau^{n-1}}}, \frac{1}{\sqrt{\tau^{n-2}}}, \dots, \frac{1}{\sqrt{\tau}}, 0\right)$,
$h = \left(\left\lfloor \frac{4(1+\sqrt{\tau^{n-1}})}{\sqrt{\tau^{n-1}}\delta} \right\rfloor, \left\lfloor \frac{4(1+\sqrt{\tau^{n-2}})(2+\sqrt{\tau^{n-1}})}{\tau\sqrt{\tau^{n-2}}\delta} \right\rfloor, \dots, \left\lfloor \frac{4(1+\sqrt{\tau})\prod_{i=2}^{n-1}(2+\sqrt{\tau^i})}{\tau^{n-2}\sqrt{\tau}\delta} \right\rfloor, \left\lfloor \frac{4\prod_{i=1}^{n-1}(2+\sqrt{\tau^i})}{\tau^{n-1}\delta} \right\rfloor\right)$.

Although the geometrically decaying sequence of the components of d would give $d_n = 1$, the new construction allows us to choose $d_n = 0$ that lets many of the redundant constraints to be active at optimality. Consequently, h_n will not obey the sequence rule for the first $n-1$ components of h. The derivation of h is discussed on page 6.

To ensure that the δ-neighborhoods of the 2^n vertices are non-overlapping, τ and δ are chosen to satisfy $\tau + \delta < \frac{1}{2}$. Note that h depends linearly on δ^{-1}. The following results could be stated in term of δ but, for simplicity and to exhibit the sharpest bounds, we set $\delta = \frac{1}{4(n+1)}$. The corresponding n-dimensional linear optimization problem depends only on n, as $\tau = \frac{n}{2(n+1)}$, and is denoted by $\mathcal{C}^n$.

The analytic center χ^n of $\mathcal{C}^n$ is the unique solution to the problem consisting of maximizing the product of the slack variables, or equivalently the sum of the logarithms of the slack variables, i.e., it is the solution of the following maximization problem:

$$\max_x \sum_{k=1}^{n} (\ln s_k + \ln \bar{s}_k + h_k \ln \tilde{s}_k),$$

where $s_k = x_k - \tau x_{k-1}$, $\bar{s}_k = 1 - x_k - \tau x_{k-1}$, and $\tilde{s}_k = d_k + x_k$, for $k = 1, \dots, n$.

The optimality conditions (the gradient is equal to zero at optimality) for this strictly concave maximization problem give:

$$\begin{cases} \frac{1}{s_k} - \frac{\tau}{s_{k+1}} - \frac{1}{\bar{s}_k} - \frac{\tau}{\bar{s}_{k+1}} + \frac{h_k}{\tilde{s}_k} = 0 & \text{for } k = 1, \dots, n-1, \\ \frac{1}{s_n} - \frac{1}{\bar{s}_n} + \frac{h_n}{\tilde{s}_n} = 0 & \\ s_k > 0, \bar{s}_k > 0, \tilde{s}_k > 0 & \text{for } k = 1, \dots, n. \end{cases} \tag{1}$$

It is important to note that any point on the central path satisfies all the above equalities, except the last one, since the central path $\mathcal{P} = \{x(\mu) : \ \mu > 0\}$ is the set of strictly feasible solution $x(\mu)$ that are the unique solutions of

$$\max_x \ -x_n + \mu \sum_{k=1}^{n} (\ln s_k + \ln \bar{s}_k + h_k \ln \tilde{s}_k).$$

To ease the analysis, we give a mathematical definition for the edge-path followed by the simplex method (or simply the simplex path) and its δ-neighborhood in $\mathcal{C}$. For this purpose, we first define the following sets, for $k = 2, \dots, n$,

$$\begin{aligned} T_\delta^k &= \{x \in \mathcal{C} : \ \bar{s}_k < \delta\} \\ C_\delta^k &= \{x \in \mathcal{C} : \ \bar{s}_k \geq \delta, s_k \geq \delta\} \\ B_\delta^k &= \{x \in \mathcal{C} : \ s_k < \delta\} \end{aligned}$$

and $\hat{C}_\delta^k = \{x \in \mathcal{C} : \ \bar{s}_k < \delta, s_{k-1} < \delta, \dots, s_1 < \delta\}$, for $k = 1, \dots, n$. Visually, the sets T_δ^k, C_δ^k, and B_δ^k can be considered as the top, central, and bottom parts of $\mathcal{C}$, and obviously $\mathcal{C} = T_\delta^k \cup C_\delta^k \cup B_\delta^k$ for $k = 1, \dots, n$. By defining the set $A_\delta^k = T_\delta^k \cup \hat{C}_\delta^{k-1} \cup B_\delta^k$ for $k = 2, \dots, n$, a δ-neighborhood of the simplex path might be given as $P_\delta = \bigcap_{k=2}^n A_\delta^k$, see Figure 3. The simplex path itself is precisely determined by $P_0 = \bigcap_{k=2}^n A_0^k$, see Figure 1. Proposition 2.1 is to ensure that the analytic center of

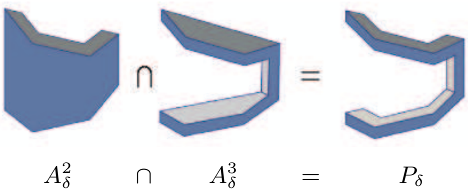

Figure 3: The set P_δ for the Klee-Minty 3-cube.

$\mathcal{C}^n$ is in the δ-neighborhood of the vertex $v^{\{n\}}$, which is precisely $\hat{C}_\delta^n$. The proof of the proposition presented in Section 3.2.

Proposition 2.1. *The analytic center χ^n is in the δ-neighborhood of $v^{\{n\}}$, i.e., $\chi^n \in \hat{C}_\delta^n$.*

Proposition 2.2 states that, for $\mathcal{C}^n$, the central path takes at least $2^n - 2$ turns before converging to the origin as it stays in the δ-neighborhood of the simplex path; in particular it passes through the δ-neighborhood of all the 2^n vertices of the Klee-Minty n-cube. See Section 3.3 for the proof.

Proposition 2.2. *The central path $\mathcal{P}$ stays in the δ-neighborhood of the simplex path of $\mathcal{C}^n$, i.e., $\mathcal{P} \subset P_\delta$.*

As discussed in [3, 4], the number of iterations required by path-following interior point methods is at least the number of sharp turns the central path takes. Proposition 2.2 yields a theoretical lower bound for the iteration-complexity of central path-following IPMs when solving this n-dimensional linear optimization problem.

Corollary 2.3. *For $\mathcal{C}^n$, the iteration-complexity lower bound of path-following interior point methods is $\Omega(2^n)$.*

Since the theoretical iteration-complexity upper bound for path-following interior point methods is $O(\sqrt{N}L)$, where N and L respectively denote the number of inequalities and the bit-length of the input-data, we have:

Proposition 2.4. *For $\mathcal{C}^n$, the iteration-complexity upper bound of path-following interior point methods is $O(2^n\sqrt{n}L)$; that is, $O(2^{3n}n^{\frac{5}{2}})$.*

Proof. We have $N = 2n + \sum_{k=1}^{n} h_k$. Using the fact that $e^{\frac{x}{2}} \geq 1 + \frac{x}{2}$, we obtain

$$h_k = \frac{\prod_{i=n-k}^{n-1}(2+\sqrt{\tau^i})}{\tau^{k-1}\sqrt{\tau^{n-k}}\delta} \leq \frac{2^k e^{\frac{1}{2}\sum_{i=n-k}^{n-1}\sqrt{\tau^i}}}{\tau^{k-1}\sqrt{\tau^{n-k}}\delta}.$$

Since $e^{\frac{1}{2}\sum_{i=n-k}^{n-1}\sqrt{\tau^i}} \leq \frac{\sqrt{\tau^{n-k}}}{1-\sqrt{\tau}}$ and $\tau < \frac{1}{2}$, we get $h_k \leq \frac{2^k}{\tau^{k-1}(1-\sqrt{\tau})\delta} \leq \frac{2^{k+1}}{\tau^k \delta}$. Therefore, with $\tau = \frac{n+1}{2n}$ and $\delta = \frac{1}{4(n+1)}$, we have

$$N \leq 2n + \sum_{k=1}^{n} 2^{2k+3}\left(\frac{n}{n+1}\right)^k (n+1) \leq 2n + (n+1)\sum_{k=1}^{n} 2^{2k+3} = 2n + \frac{32}{3}(n+1)(2^{2n}-1).$$

As a result, we obtain $N = O(n2^{2n})$ and $L \leq N\lceil \ln d_1 \rceil = O(n^2 2^{2n})$. □

Since the last two vertices $v^{\{1\}}$ and $v^{\emptyset}$ are decreasingly ordered by their last components that are $v_n^{\{1\}} = \tau^{n-1}$ and $v_n^{\emptyset} = 0$, the ϵ-precision stopping criterion $N\mu^* < \epsilon$ can be replaced by $N\mu^* < \frac{1}{2}\tau^{n-1}$. Then, a standard rounding procedure, see e.g. [9], can be used to compute the exact optimal point. Additionally, one can check (see Section 3.4) that the starting point can be chosen the point on the central path corresponding to the central path parameter $\mu^0 = \tau^{n-1}\delta$ that belongs to the interior of the δ-neighborhood of the vertex $v^{\{n\}}$. These remarks yield an input-length-independent iteration-complexity bound $O(\sqrt{N}\ln\frac{N\mu^0}{N\mu^*}) = O(\sqrt{N}n)$, and Proposition 2.4 can therefore be strengthened to:

Proposition 2.5. *For $\mathcal{C}^n$, the iteration-complexity upper bound of path-following interior point methods is $O(n^{\frac{3}{2}}2^n)$.*

Remark 2.6. For $\mathcal{C}^n$, by Corollary 2.3 and Proposition 2.5, the order of the iteration-complexity of path-following interior point methods is between $O(2^n)$ and $O(n^{\frac{3}{2}}2^n)$ or, equivalently, between $O(\sqrt{\frac{N}{\ln N}})$ and $O(\sqrt{N}\ln N)$. For further discussion about the implications of this result, the reader is referred to [3, 4].

Remark 2.7. Unlike the redundant examples of [2, 3], a set of the redundant constraints in the new construction touch the feasible region. Therefore, most preprocessors would not detect easily these redundant constraints.

3 Proofs of Propositions 2.1, 2.2, and 2.5

3.1 Preliminary Lemmas and the Main Theorem

The following n inequalities are key settings for problem $\mathcal{C}_\tau^h$ in order to bend the central path along the simplex path:

$$\begin{aligned} \frac{h_1}{d_1+1} &\geq \frac{2}{\delta}, \\ \frac{h_k\tau^{k-1}}{d_k+1} - \frac{h_{k-1}\tau^{k-2}}{d_{k-1}} - \dots - \frac{h_1}{d_1} &\geq \frac{2}{\delta}, \quad \text{for } \; k = 2, \dots, n. \end{aligned} \tag{2}$$

In fact, for $k = 1, \dots, n$, inequality k ensures that the central path $\mathcal{P}$ is pushed enough toward the set $\hat{C}_\delta^k$. The inequalities of (2) can also be written as $Ah \geq b$, where $b = (\frac{2}{\delta}, \dots, \frac{2}{\delta})^T$ and

$$A = \begin{pmatrix} \frac{1}{d_1+1} & & & & & \\ \frac{-1}{d_1} & \frac{\tau}{d_2+1} & & & & \\ \vdots & \vdots & \ddots & & & \\ \frac{-1}{d_1} & \frac{-\tau}{d_2} & \cdots & \frac{\tau^{k-1}}{d_k+1} & & \\ \vdots & \vdots & & \vdots & \ddots & \\ \frac{-1}{d_1} & \frac{-\tau}{d_2} & \cdots & \frac{-\tau^{k-1}}{d_k} & \cdots & \frac{\tau^{n-1}}{d_n+1} \end{pmatrix}.$$

In this section, we assume that τ, d, and δ are defined as specified on page 3. The following lemma proves that the system $Ah \geq b$ has a solution.

Lemma 3.1. *The system $Ah = b$ has the following unique solution:*

$$\tilde{h} = \left(\frac{2(1+\sqrt{\tau^{n-1}})}{\sqrt{\tau^{n-1}}\delta}, \frac{2(1+\sqrt{\tau^{n-2}})(2+\sqrt{\tau^{n-1}})}{\tau\sqrt{\tau^{n-2}}\delta}, \dots, \frac{2\prod_{i=1}^{n-1}(2+\sqrt{\tau^i})}{\tau^{n-1}\delta} \right).$$

Proof. It can be easily checked that the triangular system $Ah = b$ has the following unique solution

$$\tilde{h} = \left(\frac{2(1+d_1)}{\delta}, \dots, \frac{2\prod_{i=1}^{k-1}\left(2+\frac{1}{d_i}\right)(1+d_k)}{\tau^{k-1}\delta}, \dots, \frac{2\prod_{i=1}^{n-1}\left(2+\frac{1}{d_i}\right)(1+d_n)}{\tau^{n-1}\delta} \right).$$

As $d = \left(\frac{1}{\sqrt{\tau^{n-1}}}, \frac{1}{\sqrt{\tau^{n-2}}}, \dots, \frac{1}{\sqrt{\tau}}, 0\right)$, the result immediately follows. □

Remark 3.2. We look for an integer-valued solution of $Ah \geq b$ since it represents the number of the redundant inequalities. Analogous to the discussion in [3], the integer-valued vector $h = \lfloor 2\tilde{h} \rfloor$ satisfies $Ah \geq b$, where $\tilde{h}$ is given by Lemma 3.1.

The following lemma presents a large set of inequalities that is implied by $Ah \geq b$. Those inequalities play a crucial role in Lemmas 3.4 and 3.5 that characterize how close the central path $\mathcal{P}$ moves along the boundary of $\mathcal{C}$.

Lemma 3.3. *If* $\frac{h_k\tau^{k-1}}{d_k+1} - \frac{h_{k-1}\tau^{k-2}}{d_{k-1}} - \ldots - \frac{h_1}{d_1} \geq \frac{2}{\delta}$*, then for all* $k = 2, \ldots, n$ *and* $m = 1, \ldots, k$*, the inequality* $\frac{h_k\tau^{k-m}}{d_k+1} - \frac{h_{k-1}\tau^{k-m-1}}{d_{k-1}} - \ldots - \frac{h_m}{d_m} \geq \frac{2}{\delta}$ *holds.*

Proof. Since $\frac{h_k\tau^{k-m}}{d_k+1} - \frac{h_{k-1}\tau^{k-m-1}}{d_{k-1}} - \ldots - \frac{h_m}{d_m} \geq \frac{h_k\tau^{k-1}}{d_k+1} - \frac{h_{k-1}\tau^{k-2}}{d_{k-1}} - \ldots - \frac{h_m\tau^{m-1}}{d_m}$ and $\frac{h_k\tau^{k-1}}{d_k+1} - \frac{h_{k-1}\tau^{k-2}}{d_{k-1}} - \ldots - \frac{h_m\tau^{m-1}}{d_m} \geq \frac{h_k\tau^{k-1}}{d_k+1} - \frac{h_{k-1}\tau^{k-2}}{d_{k-1}} - \ldots - \frac{h_1}{d_1}$, the result follows. ∎

In the next lemma, we study how close the central path is pushed to the boundary of $\mathcal{C}$ for problem $\mathcal{C}^n$, assuming that inequalities (2) hold. In particular, we show the effect of the k-th inequality of (2) on the slack variables $s_1, \ldots, s_{k-1}$, and $\bar{s}_k$.

Lemma 3.4. *Assume that for all* $k = 1, \ldots, n-1$ *and* $m = 1, \ldots, k$*, the inequality* $\frac{h_k\tau^{k-m}}{d_k+1} - \frac{h_{k-1}\tau^{k-m-1}}{d_{k-1}} - \ldots - \frac{h_m}{d_m} \geq \frac{2}{\delta}$ *holds. Then, for the central path we have*

1. *If* $s_{k+1} \geq \delta$ *and* $\bar{s}_{k+1} \geq \delta$*, then for* $m < k$ *the inequality* $s_m < \delta$ *holds.*

2. *If* $s_{m+1} \geq \delta$ *and* $\bar{s}_{m+1} \geq \delta$*, then the inequality* $\bar{s}_m < \delta$ *holds.*

Proof. Recall that every point on the central path of problem $\mathcal{C}^n$ satisfies all equalities of (1) except the last one.
Case 1: Adding the k-th equation of (1) multiplied by τ^{k-m} to the j-th equation of (1) multiplied by $-\tau^{j-m-1}$ for $j = m \ldots, k-1$, we have, for $k = 1, \ldots, n-1$,

$$\begin{aligned}\frac{h_k\tau^{k-m}}{\tilde{s}_k} - \frac{h_{k-1}\tau^{k-m-1}}{\tilde{s}_{k-1}} - \ldots - \frac{h_m}{\tilde{s}_m} &= \frac{1}{s_m} - \frac{1}{\bar{s}_m} - \sum_{i=m+1}^{k} \frac{2\tau^{i-m}}{\bar{s}_i} + \frac{\tau^{k-m+1}}{s_{k+1}} + \frac{\tau^{k-m+1}}{\bar{s}_{k+1}} \\ &\leq \frac{1}{s_m} + \frac{2\tau^{k-m+1}}{\delta},\end{aligned}$$

which implies, since $\tilde{s}_k \leq d_k + 1$, $\tilde{s}_j \geq d_j$, that

$$\frac{h_k\tau^{k-m}}{d_k+1} - \frac{h_{k-1}\tau^{k-m-1}}{d_{k-1}} - \ldots - \frac{h_m}{d_m} \leq \frac{1}{s_m} + \frac{2\tau^{k-m+1}}{\delta} < \frac{1}{s_m} + \frac{1}{\delta}.$$

Since by assumption the left-hand-side expression is larger than or equal to $\frac{2}{\delta}$, we have $s_m < \delta$.
Case 2: From the m-th equation of (1), we have

$$\frac{h_m}{\tilde{s}_m} = -\frac{1}{s_m} + \frac{1}{\bar{s}_m} + \frac{\tau}{s_{m+1}} + \frac{\tau}{\bar{s}_{m+1}} \leq \frac{1}{\bar{s}_m} + \frac{2\tau}{\delta},$$

which implies, since $\tilde{s}_m \leq d_m + 1$, that $\frac{h_m}{d_m+1} \leq \frac{1}{\bar{s}_m} + \frac{2\tau}{s_{m+1}} < \frac{1}{\bar{s}_m} + \frac{1}{\delta}$. Since by assumption $\frac{h_m}{d_m+1} \geq \frac{2}{\delta}$, we have $\bar{s}_m < \delta$. ∎

The position of the analytic center of $\mathcal{C}^n$ is determined in the following lemma. Specifically, we show that the last inequality of (2) ensures that the analytic center of problem $\mathcal{C}^n$ satisfies $s_1 < \delta, \ldots, s_{n-1} < \delta$, and $\bar{s}_n < \delta$.

Lemma 3.5. *Assume that* $\frac{h_n\tau^{n-m}}{d_n+1} - \frac{h_{n-1}\tau^{n-m-1}}{d_{n-1}} - \ldots - \frac{h_m}{d_m} \geq \frac{2}{\delta}$*, for* $m = 1, \ldots, n$*. Then, for the analytic center* χ^n *of* $\mathcal{C}^n$ *we have* $s_m < \delta$ *for all* $m < n$ *and* $\bar{s}_n < \delta$*.*

Proof. Recall that the analytic center of problem $\mathcal{C}^n$ satisfies all the equalities of (1).
Case 1: Adding the n-th equation of (1) multiplied by τ^{n-m} to the j-th equation of (1) multiplied by $-\tau^{j-m-1}$ for $j = m \dots, n-1$, we have

$$\frac{h_n \tau^{n-m}}{\tilde{s}_n} - \frac{h_{n-1}\tau^{n-m-1}}{\tilde{s}_{n-1}} - \dots - \frac{h_m}{\tilde{s}_m} = \frac{1}{s_m} - \frac{1}{\bar{s}_m} - \sum_{i=m+1} \frac{2\tau^{i-m}}{\bar{s}_i} \leq \frac{1}{s_m},$$

which implies, since $\tilde{s}_n < d_k + 1$, $\tilde{s}_j > d_j$, that $\frac{h_n \tau^{k-m}}{d_n+1} - \dots - \frac{h_m}{d_m} < \frac{1}{s_m}$, implying $s_m < \delta$.

Case 2: From the n-th equation of (1), we have $\frac{h_n}{\tilde{s}_n} = -\frac{1}{s_n} + \frac{1}{\bar{s}_n} \leq \frac{1}{\bar{s}_n}$, which implies, since $\tilde{s}_n < d_n + 1$, that $\frac{h_n}{d_n+1} < \frac{1}{\bar{s}_n}$, implying $\bar{s}_n < \delta$. □

In the following theorem we show that for $\mathcal{C}^n$ in the central part C^k_δ, the central path $\mathcal{P}$ is forced to be in $\hat{C}^{k-1}_\delta$, which is a subset of C^k_δ.

Theorem 3.6. *For $\mathcal{C}^n$, we have: $C^k_\delta \cap \mathcal{P} \subset \hat{C}^{k-1}_\delta$ for $k = 2, \dots, n$.*

Proof. The result immediately follows from Lemmas 3.1, 3.3, and 3.4, because h satisfies (2) and Lemma 3.3 proves that the assumptions of Lemma 3.4 are satisfied. □

3.2 Proof of Proposition 2.1

Since h satisfies (2), by Lemma 3.3 the assumptions of Lemma 3.5 are satisfied that implies the claim of Proposition 2.1. □

3.3 Proof of Proposition 2.2

For $0 < \delta \leq \frac{1}{4(n+1)}$, we show that the set $P_\delta = \bigcap_{k=2}^n A^k_\delta$ contains the central path $\mathcal{P}$. In other words, for $\mathcal{C}^n$, the central path $\mathcal{P}$ is bent along the simplex path P_0. By Proposition 2.1, the starting point χ^n of $\mathcal{P}$, which is the analytic center of $\mathcal{C}^n$, belongs to $\hat{C}^n_\delta = \mathcal{N}_\delta(v^{\{n\}})$. Since $\mathcal{C} = \bigcap_{k=2}^n (T^k_\delta \cup C^k_\delta \cup B^k_\delta)$, we have:

$$\mathcal{P} = \bigcap_{k=2}^{n} (T^k_\delta \cup C^k_\delta \cup B^k_\delta) \cap \mathcal{P} = \bigcap_{k=2}^{n} (T^k_\delta \cup (C^k_\delta \cap \mathcal{P}) \cup B^k_\delta) \cap \mathcal{P},$$

i.e., by Theorem 3.6, $\mathcal{P} \subset \bigcap_{k=2}^n (T^k_\delta \cup \hat{C}^{k-1}_\delta \cup B^k_\delta) = \bigcap_{k=2}^n A^k_\delta = P_\delta$, which completes the proof. □

3.4 Proof of Proposition 2.5

We consider the point $\bar{x}$ of the central path which lies on the boundary of the δ-neighborhood of the highest vertex $v^{\{n\}}$. This point is defined by: $s_1 = \delta, s_k \leq \delta$ for $k = 2, \dots, n-1$, and $\bar{s}_n \leq \delta$. Let $\bar{\mu}$ denote the central path parameter corresponding to $\bar{x}$. To prove Proposition 2.5, it suffices to show that $\bar{\mu} \leq \tau^{n-1}\delta$ (see Theorem 3.7) which implies that any point of the central path with corresponding parameter $\mu \geq \bar{\mu}$ belong to the interior of the δ-neighborhood of the highest vertex $v^{\{n\}}$.

Theorem 3.7. *Let $\bar{x}$ be a point on the central path $\mathcal{P}$ of problem $\mathcal{C}^n$ that satisfies $s_1 = \delta, s_k \leq \delta$ for $k = 2, \ldots, n-1$, and $\bar{s}_n \leq \delta$. Then, the central path parameter $\bar{\mu}$ corresponding to $\bar{x}$ satisfies $\bar{\mu} < \tau^{n-1}\delta$.*

Proof. We consider the dual formulation of the problem $\mathcal{C}^n$. For $k = 1, \ldots, n$ let us denote by y_k and $\bar{y}_k$ the dual variables corresponding to the inequalities with the slack variables s_k and $\bar{s}_k$ respectively. Since the inequality with the slack variable $\tilde{s}_k$ is repeated h_k times, we denote the dual variable corresponding to each inequality by $\tilde{y}_{kj}$, $j = 1, \ldots, h_k$. Note that points on the central path satisfy $y_k s_k = \mu$, $\bar{y}_k \bar{s}_k = \mu$, and $\tilde{y}_{kj}\tilde{s}_k = \mu$, for $j = 1, \ldots, h_k$ and $k = 1, \ldots, n$, where μ is the central path parameter. The formulation of the dual problem of $\mathcal{C}^n$ is:

$$\begin{aligned}
\max \quad & -\sum_{k=1}^{n}\left(\bar{y}_k + d_k \sum_{j=1}^{h_k} \tilde{y}_{kj}\right) \\
\text{subject to} \quad & \bar{y}_k - y_k + \tau\bar{y}_{k+1} + \tau y_{k+1} - \sum_{j=1}^{h_k} \tilde{y}_{kj} = 0, \qquad \text{for } k = 1, \ldots, n-1, \\
& \bar{y}_n - y_n - \sum_{j=1}^{h_n} \tilde{y}_{nj} = -1, \\
& y \geq 0.
\end{aligned}$$

For $k = 1, \ldots, n-1$, let us multiply the k-th equation by τ^{k-1} and the n-th equation by $-\tau^{n-1}$. Then, by summing them up we have:

$$\bar{y}_1 - y_1 + 2\sum_{k=2}^{n} \tau^{k-1} y_k + \sum_{j=1}^{h_n} \tau^{n-1}\tilde{y}_{nj} - \sum_{k=1}^{n-1}\sum_{j=1}^{h_k} \tau^{k-1}\tilde{y}_{kj} = \tau^{n-1},$$

which implies that $2\tau^{n-1}y_n \leq y_1 - \sum_{j=1}^{h_n} \tau^{n-1}\tilde{y}_{nj} + \sum_{k=1}^{n-1}\sum_{j=1}^{h_k} \tau^{k-1}\tilde{y}_{kj} + \tau^{n-1}$.
Since $y_1 = \frac{\bar{\mu}}{\delta}$, $\tilde{y}_{nj} \geq \frac{\bar{\mu}}{d_n+1}$, and $\tilde{y}_{kj} \leq \frac{\bar{\mu}}{d_k}$, for $j = 1, \ldots, h_k$ and $k = 1, \ldots, n-1$, we obtain

$$2\tau^{n-1}y_{2n} \leq \left(\frac{1}{\delta} - \frac{h_n\tau^{n-1}}{d_n+1} + \sum_{k=1}^{n-1}\frac{h_k\tau^{k-1}}{d_k}\right)\bar{\mu} + \tau^{n-1},$$

implying by the last inequality of $Ah \geq b$ that $2\tau^{n-1}y_{2n} \leq -\frac{\bar{\mu}}{\delta} + \tau^{n-1}$. Therefore, the right-hand-side has to be positive implying that $\bar{\mu} \leq \tau^{n-1}\delta$. □

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