Copositive and Semidefinite Relaxations of the Quadratic Assignment Problem †
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Janez Povh ‡ Franz Rendl §

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Abstract

Semidefinite relaxations of the quadratic assignment problem (QAP) have recently turned out to provide good approximations to the optimal value of QAP. We take a systematic look at various conic relaxations of QAP. We first show that QAP can equivalently be formulated as a linear program over the cone of completely positive matrices. Since it is hard to optimize over this cone, we also look at tractable approximations and compare with several relaxations from the literature. We show that several of the well-studied models are in fact equivalent. It is still a challenging task to solve the strongest of these models to reasonable accuracy on instances of moderate size.

Key words: quadratic assignment problem, copositive programming, semidefinite relaxations, lift-and-project relaxations.

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1 Introduction

The quadratic assignment problem (QAP) is a standard problem in location theory and is very famous because of its hardness. Koopmans and Beckmann [12] introduced it in 1957 in the following form:

$$(QAP) \qquad OPT_{QAP} \;=\; \min \{\sum_{i,j} a_{ij} b_{\pi(i)\pi(j)} + \sum_i c_{i,\pi(i)} \colon \pi \text{ a permutation}\},$$

where A, B, C are $n \times n$ matrices. We make the standard assumption that A and B are symmetric. Recent surveys about QAP are given for instance in [6, 21], and most recently in [14].

We may represent each permutation π by a permutation matrix $X \in \{0,1\}^{n \times n}$, defined by $x_{ij} = 1 \iff \pi(i) = j$. If we denote the set of all permutation matrices by Π, then we may formulate QAP as follows

$$(QAP) \qquad OPT_{QAP} \;=\; \min \{\langle X,\, AXB + C\rangle \colon X \in \Pi\},$$

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‡Institute of mathematics, physics and mechanics Ljubljana, Slovenia (e-mail: janez.povh@fis.unm.si).

§Alpe-Adria University Klagenfurt, Institute for Mathematics, Universitätsstraße 65-67, 9020 Klagenfurt, Austria (e-mail: franz.rendl@uni-klu.ac.at).

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product, i.e. $\langle X, Y \rangle = \mathrm{trace}(X^T Y)$ for $X, Y \in \mathbb{R}^{m\times n}$.

QAP is known to be very hard from a theoretical and practical point of view. Problems of size $n \geq 25$ are currently still considered as difficult. Sahni and Gonzales [23] showed that even finding an ε-approximate solution for QAP is NP-hard. Solving QAP in practice is usually based on the Branch and Bound (B&B) algorithm. The performance of B&B algorithms depends on the computational quality and efficiency of lower bounds (see [1] for a summary of recent advances in the solution of QAP by B&B). The study of lower bounds for QAP is therefore very important for the development of B&B algorithms.

The most recent and promising trends of research for the bounding methods for QAP are based on semidefinite programming. Zhao et al., Sotirov and Rendl [24, 22, 26] lifted the problem from the vector space $\mathbb{R}^{n\times n}$ to the cone of positive semidefinite matrices of order n^2+1 and formulated several semidefinite relaxations which give increasingly tight lower bounds for QAP. They used interior point methods [26] and the bundle method [22] to solve these programs. The computational results show that these lower bounds are among the strongest known but also the most expensive to compute (state-of-the-art computers could compute the strongest of these bounds only for $n \leq 35$).

Recently Burer and Vandenbussche [4] applied the lift-and-project technique, introduced by Lovász and Schrijver [15] to QAP. They used the Augmented Lagrangian method to solve the resulting semidefinite programs and this way obtained lower bounds for QAP, which are somewhat tighter than the bounds from [22], but the practical upper bound for solving the tighter semidefinite lower bound remains $n = 35$.

Our contribution in this paper to the literature on semidefinite programming lower bounds consists of the following results:

- In Section 2 we show that solving QAP amounts to solving a linear program over the cone of completely positive matrices of order n^2. This linear program is actually the Lagrangian dual of the Lagrangian dual of the QAP, if we rewrite QAP as a quadratically constrained quadratic problem with some additional redundant quadratic constraints. This does not make the problem tractable since optimization over the cone of completely positive matrices is intractable, but this result shows new possibilities on how to solve QAP, approximately.

- In Section 3 we consider the semidefinite relaxations of QAP, obtained from the copositive representation of QAP from Section 2. We suggest two new semidefinite programs, denoted by QAP_{ZKRW1} and QAP_{AW+}, which both follow from the copositive representation of QAP. The relaxation QAP_{AW+} is a simple improvement of the Anstreicher-Wolkowicz relaxation [2] and can be computed efficiently. It has the same computational cost as the bound QAP_{R0} from [22], but is often much tighter.

- After describing various previously published relaxations in Sections 4 and 5, we compare these relaxations in Section 6. We show that the strongest model $QAP_{\mathcal{K}_n^{0*}}$ introduced in the present paper is equivalent to the strongest relaxations from [4, 22, 26]. We also show that QAP_{ZKRW1} is in fact equivalent to the model QAP_{R_2} from [22, 26].

1.1 Notation

We denote the ith standard unit vector by e_i and when we index components by $0, 1, \ldots, n$, then e_0 is the first unit vector. The vector of all ones is $u_n \in \mathbb{R}^n$ (or u if the dimension n is obvious). The square matrix of all ones is J_n (or J), the identity matrix is I and $E_{ij} = e_i e_j^T$.

In this paper we consider the following sets of matrices:

- The vector space of real symmetric $n \times n$ matrices: $\mathcal{S}_n = \{X \in \mathbb{R}^{n\times n} : X = X^T\}$,
- the cone of $n \times n$ symmetric nonnegative matrices: $\mathcal{N}_n = \{X \in \mathcal{S}_n : x_{ij} \geq 0, \forall i, j\}$,
- the cone of $n \times n$ positive semidefinite matrices: $\mathcal{S}_n^+ = \{X \in \mathcal{S}_n : y^T X y \geq 0, \forall y \in \mathbb{R}^n\}$,
- the cone of $n \times n$ copositive matrices: $\mathcal{C}_n = \{X \in \mathcal{S}_n : y^T X y \geq 0, \forall y \in \mathbb{R}^n_+\}$,
- the cone of $n \times n$ completely positive matrices: $\mathcal{C}_n^* = \mathrm{conv}\{yy^T : y \in \mathbb{R}^n_+\}$, where $\mathrm{conv}(A)$ stands for the convex hull of A.

The dual cone $\mathcal{K}^*$ of a given cone $\mathcal{K} \subset \mathbb{R}^{m\times n}$ is define as follows: $\mathcal{K}^* = \{Y \in \mathbb{R}^{m\times n} : \langle X, Y\rangle \geq 0,\ \forall X \in \mathcal{K}\}$. Note that the cone of completely positive matrices is dual to the cone of copositive matrices. This justifies the notation $\mathcal{C}_n^*$. The cones of symmetric non-negative matrices and positive semidefinite matrices are self-dual, i.e. $\mathcal{N}^* = \mathcal{N}$ and $(\mathcal{S}_n^+)^* = \mathcal{S}_n^+$.

We also use $X \succeq 0$ for $X \in \mathcal{S}_n^+$ and $X \geq 0$ for $X \in \mathcal{N}$. A linear program over $\mathcal{S}_n^+$ is called a semidefinite program while a linear program over $\mathcal{C}_n$ or $\mathcal{C}_n^*$ is called a copositive program.

The sign $\otimes$ stands for Kronecker product. When we consider matrix $A \in \mathbb{R}^{m\times n}$ as a vector from $\mathbb{R}^{mn}$ obtained from A columnwise, we write this vector as $\mathrm{vec}(A)$ or a. For matrix columns and rows we use the matlab notation. Hence $X(i,:)$ and $X(:,i)$ stand for ith row and column, respectively, and $X(i:j, p:q)$ is a submatrix of X, which consists of elements x_{st}, for $i \leq s \leq j$ and $p \leq t \leq q$. If $a \in \mathbb{R}^n$, then $\mathrm{Diag}(a)$ is a $n \times n$ diagonal matrix with a on the main diagonal and $\mathrm{diag}(X)$ is a vector containing the main diagonal of a square matrix X.

For a matrix $Z \in \mathcal{S}_{k^2+1}$, with $k \geq 1$, we often use the following block notation:

$$Z = \left[\begin{array}{c|cccc} Z^{00} & Z^{01} & \cdots & Z^{0k} \\ \hline Z^{10} & Z^{11} & \cdots & Z^{1k} \\ \vdots & \vdots & \ddots & \vdots \\ Z^{k0} & Z^{k1} & \cdots & Z^{kk} \end{array}\right], \tag{1}$$

where $Z^{i0} \in \mathbb{R}^k$, $1 \leq i \leq k$ and $Z^{ij} \in \mathbb{R}^{k\times k}$, $1 \leq i, j \leq k$. Since $Z^{00} \in \mathbb{R}$, we denote it also by Z_{00}. Similarly we address component blocks of a matrix $Z \in \mathcal{S}_{k^2}$ via

$$Z = \begin{bmatrix} Z^{11} & \cdots & Z^{1k} \\ \vdots & \ddots & \vdots \\ Z^{k1} & \cdots & Z^{kk} \end{bmatrix}, \tag{2}$$

where $Z^{ij} \in \mathbb{R}^{k\times k}$.

When P or $P_{subscript}$ is the name of the optimization problem, then OPT_P or $OPT_{subscript}$, respectively, denote their optimal values.

1.2 Technical preliminaries

In the proofs of Theorems 3, 7 and 8, which contain the main results of the paper, we need the following technical lemmas.

Lemma 1 *Let $Y \in \mathcal{S}_k^+$ with $\mathrm{diag}(Y) = a$ and $\sum_{i,j} Y_{ij} = (\sum_i \sqrt{a_i})^2$. Then $Y_{ij} = \sqrt{a_i a_j}$, for $1 \leq i, j \leq k$, or equivalently, $Y = yy^T$ for $y_i = \sqrt{a_i}$, $1 \leq i \leq k$.*

Proof: Since $Y \succeq 0$ we know that $|Y_{ij}| \leq \sqrt{Y_{ii}Y_{jj}} = \sqrt{a_i a_j}$ and $\sum_{i,j} Y_{ij} \leq \sum_{i,j} |Y_{ij}| \leq \sum_{i,j} \sqrt{a_i a_j} = (\sum_i \sqrt{a_i})^2$. The equality holds throughout if and only if $Y_{ij} = \sqrt{a_i a_j}$. $\square$

Lemma 2 *Let*

$$\tilde{Y} = \begin{bmatrix} Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \end{bmatrix} \in \mathcal{S}_{2n}^+$$

with $Y^{11} = \mathrm{Diag}(a) \in \mathcal{S}_n^+$, $Y^{12} \in \mathbb{R}^{n \times n}$ *and* $Y^{22} = \mathrm{Diag}(b) \in \mathcal{S}_n^+$. *If* $u^T a = \alpha^2$, $u^T b = \beta^2$ *and* $u^T Y^{12} u = \alpha\beta$, *then* $Y^{12}u = \beta/\alpha \cdot a$ *and* $u^T Y^{12} = \alpha/\beta \cdot b$.

Proof: Without loss of generality we can assume $a_i > 0$ and $b_i > 0$, for all i. From $Y \succeq 0$ it follows by using Schur complement [10, Theorem 7.7.6] that $Y^{11} - Y^{12}(Y^{22})^{-1}Y^{21} \succeq 0$, hence $u^T(Y^{11} - Y^{12}(Y^{22})^{-1}Y^{21})u \geq 0$. But

$$\begin{aligned} u^T(Y^{11} - Y^{12}(Y^{22})^{-1}Y^{21})u \;&=\; \alpha^2 - \sum_{i=1}^n \frac{(Y^{21}(i,:)u)^2}{b_i} \\ = \alpha^2 - \sum_{i=1}^n \left(\frac{Y^{21}(i,:)u}{b_i}\right)^2 b_i \;&\leq\; \alpha^2 - \frac{\left(\sum_{i=1}^n Y^{21}(i,:)u\right)^2}{\sum_i b_i} \\ = \alpha^2 - \frac{\alpha^2\beta^2}{\beta^2} \;&=\; 0 \end{aligned}$$

with equality holding if and only if $Y^{21}(i,:)u/b_i = Y^{21}(j,:)u/b_j,\ \forall i,j$. Since

$$\begin{aligned} \alpha\beta \;&=\; \sum_i Y^{21}(i,:)u \;=\; \sum_i \frac{Y^{21}(i,:)u}{b_i} b_i \\ &=\; \frac{Y^{21}(1,:)u}{b_1} \sum_i b_i \;=\; \frac{Y^{21}(1,:)u}{b_1} \beta^2 \end{aligned}$$

it follows $Y^{21}(1,:)u/b_1 = \alpha/\beta$ and consequently $Y^{21}u = \alpha/\beta \cdot b$. The second part of the lemma follows by using $Y^{22} - Y^{21}(Y^{11})^{-1}Y^{12} \succeq 0$. $\square$

2 QAP as a copositive program

In this section, we first formulate QAP as a quadratically constrained quadratic program. A restricted Lagrangian dual is a copositive program. Our main result shows that there is a zero duality gap between this copositive program and its dual.

Every permutation matrix has in each row and column exactly one non-zero element, which is equal to 1. Therefore the rows and columns are orthonormal. In fact, this is already a complete characterization of the set of permutation matrices: $\Pi = \{X \in \mathbb{R}^{n \times n} \colon X^T X = I,\ X \geq 0\}$.

Anstreicher and Wolkowicz [2] added to this description of Π the redundant constraint $XX^T = I$ and showed that the Lagrangian dual of the resulting quadratic program (with the sign constraints omitted) yields a semidefinite program with the optimal value equal to the well-known Hoffman-Wielandt eigenvalue lower bound (see also [20] for further reading about this topic).

We add one more redundant constraint. Since the sum of all elements in X is n, $u^T X u = n$, we include the constraint $\langle X,\, JXJ\rangle = n^2$, which follows from $\langle X,\, JXJ\rangle \;=\; (u^T X u)^2$. Every permutation matrix obviously satisfies this constraint. We can therefore represent QAP as a quadratically constrained quadratic program

$$OPT_{QAP} \;=\; \min\,\{\langle X,\, AXB + C\rangle\colon X^T X = XX^T = I,\; \langle X,\, JXJ\rangle = n^2,\; X \geq 0\}. \tag{3}$$

In the sequel we use the facts that $\langle C,\, X\rangle = \sum_{i,j} c_{ij}x_{ij} = \sum_{i,j} c_{ij}x_{ij}^2 = \langle \mathrm{Diag}(c),\, xx^T\rangle$ for $X \in \Pi$ and $\langle X,\, PXQ\rangle = \langle Q^T \otimes P, xx^T\rangle$, for any X, where $x = \mathrm{vec}(X)$ and $c = \mathrm{vec}(C)$.

We dualize (3) as follows:

$$\begin{aligned}
&OPT_{QAP} \;= \\
&= \min_{X\geq 0}\; \Big\{\langle B\otimes A + \mathrm{Diag}(c), xx^T\rangle + \max_{S,T\in\mathcal{S}_n, v\in\mathbb{R}}\; \{\langle S,\, I - XX^T\rangle + \langle T,\, I - X^T X\rangle + v(n^2 - \langle X,\, JXJ\rangle)\}\Big\} \\
&\geq \max_{S,T\in\mathcal{S}_n, v\in\mathbb{R}} \Big\{\mathrm{trace}(S) + \mathrm{trace}(T) + n^2\, v + \min_{x\in\mathbb{R}^{n^2}_+}\; \{x^T(B\otimes A + \mathrm{Diag}(c) - I\otimes S - T\otimes I - vJ_{n^2})x\}\Big\} \\
&= \max\; \Big\{\mathrm{trace}(S) + \mathrm{trace}(T) + n^2\, v\colon S,\; T\in\mathcal{S}_n,\; B\otimes A + \mathrm{Diag}(c) - I\otimes S - T\otimes I - vJ_{n^2} \in \mathcal{C}_{n^2}\Big\} \\
&= \min\; \Big\{\langle B\otimes A + \mathrm{Diag}(c),\, Y\rangle\colon \textstyle\sum_i Y^{ii} = I,\; \langle I,\, Y^{ij}\rangle = \delta_{ij},\; \forall i,j,\; \langle J_{n^2}, Y\rangle = n^2,\; Y\in\mathcal{C}^*_{n^2}\Big\}.
\end{aligned}$$

We denote the last problem by QAP_{CP}. In the equations in QAP_{CP} we use the block description of Y, introduced in (2). Note that the equality constraints based on the blocks of Y first appeared in [26].

The first inequality above is due to exchanging min and max. The second equality follows from the fact that the inner minimization problem is bounded from below on the nonnegative orthant if and only if the matrix $B\otimes A + \mathrm{Diag}(c) - I\otimes S - T\otimes I - vJ_{n^2}$ is copositive (this is exactly the definition of copositive matrices). The last two problems are conic duals to each other. The last equality above follows from strict feasibility of the last but one problem, i.e., for $T = S = -\alpha I$ and $u = 0$ the matrix $B\otimes A + \mathrm{Diag}(c) - I\otimes S - T\otimes I$ is positive definite for α sufficiently large, hence in the interior of $\mathcal{C}_{n^2}$ and therefore strictly feasible. So, strong duality holds.

By construction it follows that $OPT_{QAP} \geq OPT_{CP}$, but we will see below that we have in fact equality. First we study the feasible set $\mathcal{F}$ for QAP_{CP}:

$$\mathcal{F} := \{Y\in\mathcal{C}^*_{n^2} : Y \text{ feasible for } QAP_{CP}\}.$$

We have the following description of $\mathcal{F}$.

Theorem 3

$$\mathcal{F} = \mathrm{conv}\{xx^T\colon\;\; x = \mathrm{vec}(X),\;\; X\in\Pi\}.$$

Proof: The " $\supseteq$ " part is obvious. Let us consider now the opposite direction. Let Y be arbitrary from $\mathcal{F}$. From the definition of the cone $\mathcal{C}^*_{n^2}$ it follows that there exists $r \geq 1$ and non-zero vectors $y^1,\ldots,y^r \in \mathbb{R}^{n^2}_+$ such that $Y = \sum_{k=1}^r y^k(y^k)^T$. We will find numbers $\lambda_k \in [0,1]$ and vectors $x^k \in \mathbb{R}^{n^2}_+$ such that $y^k(y^k)^T = \lambda_k x^k(x^k)^T$, $1 \leq k \leq r$, $\sum_{k=1}^r \lambda_k = 1$ and each x^k is a vector representation of some permutation matrix X^k. This will prove the theorem.

We consider each vector y^k as $\mathrm{vec}(Y^k)$ for some $Y^k \in \mathbb{R}^{n\times n}$, therefore we index components of each y^k by two indices such that $y^k(i,j)$ is (i,j)th component of Y^k. We will also call, by abuse of notation, the components $y^k(1,i),\ldots,y^k(n,i)$ by "ith column" of y^k and components $y^k(j,1),\ldots,y^k(j,n)$ by "jth row" of y^k.

From $Y \in \mathcal{C}^*_{n^2}$ it follows $Y \geq 0$ and $Y \succeq 0$. Constraints $\sum_i Y^{ii} = I$ and $\langle I, Y^{ij} \rangle = \delta_{ij}$ therefore imply that Y^{ii} is diagonal for all i and $\mathrm{diag}(Y^{ij}) = 0$ for $i \neq j$. We know that $y^k \geq 0$, hence if we had in the ith column of y^k two non-zero components, then their product would be positive and would lie out of the main diagonal of the (i,i)th block. This is not possible, since the sum of (i,i) blocks of $y^k (y^k)^T$ is a diagonal matrix. Therefore we have in each column of y^k one non-zero element at most. Similar arguments imply that each row of y^k has one non-zero element at most.

We may write $Y = \sum_{i,j} E_{ij} \otimes Y^{ij}$ and $Y \succeq 0$ implies that the matrix $\tilde{Y}$, defined by

$$\tilde{Y} = (I \otimes u_n^T) Y (I \otimes u_n^T)^T = \sum_{i,j} \langle J, Y^{ij} \rangle E_{ij},$$

is positive semidefinite and satisfies the assumptions of Lemma 1. Therefore we have $\tilde{Y}_{ij} = \langle J, Y^{ij} \rangle = 1$.

Let us fix i and j, $i \neq j$, and denote by a_k and b_k the maximum of the components in the ith and jth column of y^k, respectively. The (i,i)th block of $y^k (y^k)^T$ is therefore diagonal and has one non-zero component at most, which is exactly a_k^2 and lies on the main diagonal of the block. Similarly the (i,j)th block of $y^k (y^k)^T$ has one non-zero component $a_k b_k$ at most. If it is not 0, then it is off-diagonal in the block. For chosen $i \neq j$ the matrix Y therefore satisfies:

$$\begin{array}{rcccl} 1 & = & \langle I, Y^{ii} \rangle & = & \sum_{k=1}^r a_k^2 \\ 1 & = & \langle I, Y^{jj} \rangle & = & \sum_{k=1}^r b_k^2 \\ 1 & = & \langle J, Y^{ij} \rangle & = & \sum_{k=1}^r a_k b_k. \end{array}$$

The Cauchy-Schwarz inequality [10, p. 15], applied to vectors $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$, implies $a_k = b_k$ for $1 \leq k \leq r$. Since i and j were arbitrary and none of y^k is a zero vector, we have that each y^k has in each "row" and "column" exactly one non-zero component and all non-zeros are equal (we keep the notation and denote them by a_k). Vectors $x^k = y^k / a_k$ therefore correspond to permutation matrices. Let $\lambda_k = a_k^2$. Then $Y = \sum_k y^k (y^k)^T = \sum_k \lambda_k x^k (x^k)^T$ and $\sum_k \lambda_k = \sum_k a_k^2 = 1$. $\square$

Clearly QAP may be written equivalently as

$$OPT_{QAP} \;=\; \min \; \Big\{ \langle B \otimes A + \mathrm{Diag}(c), Y \rangle \colon Y \in \mathrm{conv}\{ x x^T \colon \; x = \mathrm{vec}(X) \; \mathit{for} \; X \in \Pi \} \Big\}.$$

The following corollary therefore follows immediately.

Corollary 4 *The optimal value of QAP is equal to the optimal value of QAP_{CP}.*

Remark 1 *This copositive representation again confirms the importance of copositive programming in combinatorial optimization which was first suggested by De Klerk and Pasechnik [11] on the stability number problem and further illustrated by Povh and Rendl for the graph partitioning problem [19].*

De Klerk and Pasechnik proved that computing the stability number of a graph is equivalent to solving a copositive program and then presented a hierarchy of linear and positive semidefinite relaxations, which follow from this approach and are strongly connected with the ϑ-function. Povh and Rendl reformulated the 3-partitioning problem as a copositive program, then showed that the simplest (semidefinite) relaxation of the copositive program is exactly the eigenvalue lower bound from [8] and suggested stronger relaxations which are still efficiently computable.

Remark 2 *While this paper was under peer review, Burer [3] presented an interesting paper, in which he generalized results from [11, 19]. He proved that any quadratic programming problem with linear* *constraints, possibly containing binary variables, can be rewritten as a copositive program. Since QAP can be restated in the following way*

$$OPT_{QAP} \;=\; \min\,\{\langle X,\, AXB+C\rangle \colon X\in\{0,1\}^{n\times n},\; Xu=u,\; X^Tu=u\}, \tag{4}$$

we can apply Burer's result to obtain a different copositive representation of QAP:

$$(QAP_{CP1})\qquad \begin{array}{rrcl} OPT_{QAP} = & \min & \langle L, Z\rangle & \\ \text{s. t.} & Z^{0i}u & = & 1, \forall i, \quad \sum_i Z^{0i} \;=\; u^T, \\ & \langle J,\, Z^{ii}\rangle & = & 1, \forall i, \quad \operatorname{diag}(\sum_{i,j} Z^{ij}) \;=\; u, \\ & \operatorname{diag}(Z^{ii}) & = & Z^{i0}, \forall i, \quad Z \;\in\; \mathcal{C}^*_{n^2+1}, \end{array}$$

where we use the block notation from (1) and the matrix L is from (9) below.

Burer used a similar proof technique as we did in the present paper and in [19]. His proof, reduced to QAP, implies that the feasible solutions for QAP_{CP1} are exactly those $Z\in\mathcal{S}_{n^2+1}$ which can be written as

$$Z=\sum_k \lambda_k \begin{bmatrix}1\\ z^k\end{bmatrix}\cdot\begin{bmatrix}1\\ z^k\end{bmatrix}^T, \tag{5}$$

for some $z^k=\operatorname{vec}(Z^k)$, $Z^k\in\Pi$, where $\sum_k\lambda_k=1$, $\lambda_k\geq 0$, $\forall k$. Using Theorem 3 we see that $Z\in\mathcal{S}_{n^2+1}$ is feasible for QAP_{CP1} if and only if $Y=\sum_k\lambda_k z^k(z^k)^T$ is feasible for QAP_{CP}, where λ_k and z^k are from (5), hence QAP_{CP} and QAP_{CP1} are equivalent in the sense that the feasible sets are in bijective correspondence and each pair of corresponding solutions gives the same objective values.

3 A hierarchy of semidefinite relaxations for QAP

In this section we take the formulation QAP_{CP} as a starting point for tractable relaxations. A simple relaxation is obtained by changing $Y\in\mathcal{C}^*_{n^2}$ to the weaker condition $Y\succeq 0$.

$$(QAP_{AW+})\qquad \begin{array}{rrcl} \min & \langle B\otimes A+\operatorname{Diag}(c), Y\rangle & & \\ \text{s. t.} & \sum_i Y^{ii} & = & I, \\ & \langle I,\, Y^{ij}\rangle & = & \delta_{ij},\ \forall i,j, \\ & \langle J_{n^2},\, Y\rangle & = & n^2, \\ & Y & \in & \mathcal{S}^+_{n^2}. \end{array}$$

This relaxation corresponds to the Anstreicher-Wolkowicz relaxation for QAP [2], modified by the single equation

$$\langle J_{n^2},\, Y\rangle = n^2.$$

We therefore denote it by QAP_{AW+}. It is remarkable that this single additional equation often yields a substantial improvement of the bound. We note in particular that this semidefinite program has only $O(n^2)$ equality constraints.

A systematic way to replace the intractable constraint $Y\in\mathcal{C}^*_{n^2}$ with weaker tractable constraints was recently suggested by Parrilo [16, 11]. He pointed out the fact that a given matrix $X\in\mathcal{S}_n$ is copositive is equivalent to the condition that the polynomial $P(z)=\sum_{i,j=1}^n x_{ij}z_i^2z_j^2$ in variables $(z_1,\ldots,z_n)$ is non-negative. While checking whether this is true is also intractable, we can efficiently check by semidefinite programming whether this polynomial is a sum of squares (SOS), i.e. if there exist (real) polynomials $q_1,q_2,\ldots$ such that $P(z)=\sum_i q_i(z)^2$.

If a symmetric matrix X yields a SOS polynomial, then X is copositive while the converse is not necessarily true. We can further weaken this constraint by demanding only that the polynomial $P_r(z) = P(z)(\sum_i z_i^2)^r$, $r \in \mathbb{N}$, is SOS.

This gives the following hierarchy of cones $\mathcal{K}_n^0 \subset \mathcal{K}_n^1 \subset \cdots \subset \mathcal{C}_n$, where

$$\mathcal{K}_n^r = \{X \in \mathcal{S}_n \colon (\sum_{i,j=1}^{n} x_{ij} z_i^2 z_j^2)(\sum_{i=1}^{n} z_i^2)^r \text{ is a SOS}\},$$

which approximates the copositive cone $\mathcal{C}_n$ from the inside arbitrarily closely, point-wise.

The hierarchy of dual cones $\mathcal{K}_n^{r*}$ approximates the cone $\mathcal{C}_n^*$ from the outside. We will focus on these cones. The first member in this hierarchy is the cone of symmetric doubly nonnegative matrices $\mathcal{K}_n^{0*} = \mathcal{N}_n \cap \mathcal{S}_n^+$ (see [16]). Our next model will therefore be:

$$(QAP_{\mathcal{K}_n^{0*}}) \qquad \begin{array}{lrcl} \min & \multicolumn{3}{l}{\langle B \otimes A + \mathrm{Diag}(c), Y \rangle} \\ \text{s. t.} & \sum_i Y^{ii} & = & I, \\ & \langle I, Y^{ij} \rangle & = & \delta_{ij}, \ \forall i, j, \\ & \langle J_{n^2}, Y \rangle & = & n^2, \\ & Y & \in & \mathcal{N}_{n^2} \cap \mathcal{S}_{n^2}^+, \end{array}$$

We have to emphasize that this is already a computationally expensive model since the constraint $Y \in \mathcal{N}_{n^2}$ implies $O(n^4)$ linear inequalities.

Trading quality of the relaxation for more computational efficiency, we follow the approach from Zhao et al. [26], and observe the following zero pattern for matrices feasible for $QAP_{\mathcal{K}_n^{0*}}$:

$$Y_{jk}^{ii} = 0, \quad Y_{ii}^{jk} = 0 \quad \forall j \neq k, \forall i.$$

Collecting all these $O(n^3)$ equations symbolically in the map $\mathcal{G}(Y) = 0$, we get the relaxation QAP_{ZKRW1}:

$$(QAP_{ZKRW1}) \qquad \begin{array}{lrclrcl} \min & \multicolumn{6}{l}{\langle B \otimes A + \mathrm{Diag}(c), Y \rangle} \\ \text{s. t.} & \sum_i Y_{jj}^{ii} & = & 1, & \langle I, Y^{jj} \rangle & = & 1, \ \forall j, \\ & \langle J_{n^2}, Y \rangle & = & n^2, & \mathcal{G}(Y) & = & 0 \\ & Y & \in & \mathcal{S}_{n^2}^+ & & & \end{array}$$

We use the acronym ZKRW1 to emphasize that this model is inspired by Zhao et al. [26]. We will show in the following sections that this model is in fact equivalent to the 'gangster-model' from [26].

The relaxation QAP_{ZKRW1} has 'only' $O(n^3)$ constraints, but solving it is still a computational challenge. We address this issue in some more detail in Section 7 below.

4 Other semidefinite relaxations for QAP

In this section we review the semidefinite relaxations introduced in [26] and further investigated in [22]. The key features of this approach consist in lifting the problem into the space of matrices of order $n^2 + 1$ and using a parametrization which reflects the assignment constraints. To be specific, the polytope

$$\mathcal{P} := \mathrm{conv}\Big\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \end{bmatrix}^T : x = \mathrm{vec}(X), \ X \in \Pi \Big\} \tag{6}$$

is replaced by a larger convex set

$$\hat{\mathcal{P}} := \left\{ Y \in \mathcal{S}_{n^2+1} \colon \exists Z \in \mathcal{S}^+_{(n-1)^2+1} \text{ s. t. } Z_{00} = 1 \text{ and } Y = \hat{V} Z \hat{V}^T \right\} \subset \mathcal{S}^+_{n^2+1},$$

where

$$\hat{V} = \begin{bmatrix} e_0^T \\ W \end{bmatrix}, \; W = \left[\frac{1}{n} u_{n^2} \mid V \otimes V \right] \text{ and } V = \left[\frac{I_{n-1}}{-u_{n-1}^T} \right]. \tag{7}$$

It is crucial to understand the semidefinite programs which are based on $\hat{\mathcal{P}}$, therefore we add an explicit description of $\hat{\mathcal{P}}$ here. We note however, that the 'only if' part of the result below was already proved in [26].

Lemma 5 *A matrix $Y \in \mathcal{S}^+_{n^2+1}$ is in $\hat{\mathcal{P}}$ if and only if Y satisfies*

(i) $Y_{00} = 1$, $Y^{0i} u = 1$, $1 \le i \le n$, $\sum_{i=1}^n Y_{0i} = u^T$.

(ii) $Y^{0j} = u^T Y^{ij}$, $1 \le i, j \le n$.

(iii) $\sum_{i=1}^n Y^{ij} = u Y^{0j}$, $1 \le j \le n$.

Proof: The "only if " part is done in [26]. We add it here for the sake of completeness. Let $Y = \hat{V} Z \hat{V}^T$. From $Z_{00} = 1$ it follows that $Y_{00} = 1$. Let us define the operator $\mathcal{T} \colon \mathbb{R}^{(n^2+1)\times(n^2+1)} \to \mathbb{R}^{2n\times(n^2+1)}$ as $\mathcal{T}(X) = \hat{T} X$, where

$$\hat{T} = \begin{bmatrix} -u_n & I \otimes u_n^T \\ -u_n & u_n^T \otimes I \end{bmatrix} \in \mathbb{R}^{2n\times(n^2+1)}.$$

A short calculation shows that $\mathcal{T}(\hat{V}) = 0$, hence $\hat{T} Y = 0$. The second and third property from (i) are just the equations $\hat{T} Y(:,0) = 0$ in explicit form. The equations from (ii) are exactly the equations $[-u_n \,|\, I \otimes u_n^T] \cdot Y(:, 1 : n^2) = 0$ and similarly the equations from (iii) are obtained by expanding the constraint $[-u_n \,|\, u_n^T \otimes I] \cdot Y(:, 1 : n^2) = 0$.

Let us consider the opposite direction. Let $Y \in \mathcal{S}^+_{n^2+1}$ satisfy (i)–(iii). Then we have $\hat{T} Y = 0$, hence the columns of Y are in $\mathrm{Ker}(\hat{T})$. Since $\mathrm{Ker}(\hat{T})$ is spanned by the columns of $\hat{V}$, which are also linearly independent (see [26, Theorem 3.1]), there exists $\Lambda \in \mathbb{R}^{((n-1)^2+1)\times(n^2+1)}$ such that $Y = \hat{V}\Lambda = \Lambda^T \hat{V}^T$ (Y is also symmetric). In addition, we can find the matrix $\hat{V}^{-1} \in \mathbb{R}^{((n-1)^2+1)\times(n^2+1)}$ such that $\hat{V}^{-1}\hat{V} = I_{(n-1)^2+1}$. Therefore we have $\Lambda = \hat{V}^{-1}\Lambda^T \hat{V}^T$ and $Y = \hat{V}\Lambda = \hat{V}(\hat{V}^{-1}\Lambda^T)\hat{V}^T$, which means that Y is equal to $\hat{V} Z \hat{V}^T$ for $Z = \hat{V}^{-1}\Lambda^T$. From $Y \succeq 0$ and $Y_{00} = 1$ it follows that $Z \succeq 0$ and $Z_{00} = 1$. □

Remark 3 *In Lemma 5 we used $Y \succeq 0$ only to prove $Z \succeq 0$. Hence if $Y \in \mathcal{S}_{n^2+1}$ is not positive semidefinite and satisfies (i)–(iii) from Lemma 5, then we can still write it as $Y = \hat{V} Z \hat{V}^T$ for $Z \in \mathcal{S}_{(n-1)^2+1}$ with $Z_{00} = 1$.*

In the sequel we list some semidefinite relaxations, summarized from [22, 26]. They are obtained by considering QAP, lifted into the space $\mathcal{S}_{n^2+1}$ as follows:

$$OPT_{QAP} \;=\; \min \,\{\langle L, Y\rangle \colon Y \in \mathcal{P}\}, \tag{8}$$

where

$$L = \begin{bmatrix} 0 & \frac{1}{2}c^T \\ \frac{1}{2}c & B \otimes A \end{bmatrix}. \tag{9}$$

To get relaxations, the constraint $Y \in \mathcal{P}$ is replaced by $Y \in \hat{\mathcal{P}}$ and some cutting planes are added.

We follow the notation from [22] and denote them by QAP_{R_0}, QAP_{R_2} and QAP_{R_3}:

$$\begin{array}{ll} (QAP_{R_0}) & \min \{\langle L, Y\rangle \colon Y \in \hat{\mathcal{P}},\ \mathrm{Arrow}(Y) = 0\}, \\ (QAP_{R_2}) & \min \{\langle L, Y\rangle \colon Y \in \hat{\mathcal{P}},\ \mathrm{Arrow}(Y) = 0,\ \mathcal{G}(Y) = 0\}, \\ (QAP_{R_3}) & \min \{\langle L, Y\rangle \colon Y \in \hat{\mathcal{P}},\ \mathrm{Arrow}(Y) = 0,\ \mathcal{G}(Y) = 0,\ Y \geq 0\}. \end{array}$$

All optimization problems above are semidefinite programs. The constraint $\mathrm{Arrow}(Y) = 0$ demands that in the matrix $Y \in \mathcal{S}_{n^2+1}$ with the block structure from (1) the first row must be equal to the diagonal, i. e. $Y^{0i} = \mathrm{diag}(Y^{ii})$, $1 \leq i \leq n$. The constraint $\mathcal{G}(Y) = 0$ is exactly the same as in QAP_{ZKRW1} and assures the right zero pattern in the right-lower block of Y. The new constraint in QAP_{R_3} is due to the observation that any matrix from $\mathcal{P}$ has only non-negative components.

5 Lovász-Schrijver relaxation for QAP

In this section we briefly review the Lovász-Schrijver hierarchy of relaxations applied to QAP and recall the semidefinite approximations for QAP from Burer and Vandenbussche [4]. Burer and Vandenbussche [4] report good computational results in solving relaxations for QAP, based on the Lovász-Schrijver hierarchy of relaxations for general 0-1 polyhedra [13, 15].

Let $K \subset \mathbb{R}^{n^2}$ be the convex set of doubly stochastic matrices in a vector representation:

$$K = \{x \colon x = \mathrm{vec}(X),\ Xu = u,\ X^T u = u,\ X \geq 0\}.$$

We may express K also as $K = \{x \colon Ax = u_{2n},\ x \geq 0\}$ for

$$A = \begin{bmatrix} I \otimes u_n^T \\ u_n^T \otimes I \end{bmatrix} \in \mathbb{R}^{2n \times n^2}. \tag{10}$$

The intersection $K \cap \{0, 1\}^{n^2}$ is exactly the set of all permutation matrices in a vector form.

Following Lovász and Schrijver we may get a hierarchy of linear and semidefinite relaxations for the following 0-1 polytope

$$P := \mathrm{conv}\{K \cap \{0, 1\}^{n^2}\} = \mathrm{conv}\{x \colon x = \mathrm{vec}(X),\ X \in \Pi\}.$$

The first members of these hierarchies are

$$\begin{aligned} N(K) &:= \left\{x \in \mathbb{R}^n \colon \begin{bmatrix} 1 \\ x \end{bmatrix} = \mathrm{diag}(Y) \text{ for some } Y \in M(K)\right\} \\ N_+(K) &:= \left\{x \in \mathbb{R}^n \colon \begin{bmatrix} 1 \\ x \end{bmatrix} = \mathrm{diag}(Y) \text{ for some } Y \in M_+(K)\right\}, \end{aligned}$$

where

$$\begin{aligned} M(K) &:= \{Y \in \mathcal{S}_{n^2+1} \colon Ye_0 = \mathrm{diag}(Y),\ Ye_i \in \hat{K},\ Y(e_0 - e_i) \in \hat{K},\ i = 1, \ldots, n^2\} \\ M_+(K) &:= M(K) \cap \mathcal{S}^+_{n^2+1} \quad \text{and} \\ \hat{K} &:= \{\lambda \begin{bmatrix} 1 \\ x \end{bmatrix} \colon \lambda \geq 0,\ x \in K\}. \end{aligned} \tag{11}$$

We can get higher order linear and semidefinite relaxations for K recursively:

$$\begin{aligned} N^k(K) &:= N(N^{k-1}(K)), \text{ with } N^1(K) = N(K), \\ N_+^k(K) &:= N_+(N_+^{k-1}(K)), \text{ with } N_+^1(K) = N_+(K). \end{aligned}$$

In the case of QAP we have a quadratic objective function, hence we are not interested in linear and semidefinite relaxations for P but we need relaxations for $\mathcal{P}$ from (6). The hierarchy from above yields the following linear relaxation for $\mathcal{P}$:

$$\{Y \in \mathcal{S}_{n^2+1} \colon Y \in M(K),\, Y_{00} = 1\}$$

and the semidefinite relaxation for $\mathcal{P}$:

$$\{Y \in \mathcal{S}_{n^2+1} \colon Y \in M_+(K),\, Y_{00} = 1\}.$$

The semidefinite program from [4] (we denote it by QAP_{LS}) is obtained by taking the semidefinite relaxation from above

$$(QAP_{LS}) \qquad \min\,\{\langle L,\, Y\rangle \colon Ye_i \in \hat{K},\ Y(e_0 - e_i) \in \hat{K},\ i = 1, \ldots, n^2,\ Y_{00} = 1,\ Y \in \mathcal{S}^+_{n^2+1}\},$$

where L is from (9).

Remark 4 *For our particular K we may replace the constraints $Ye_i \in \hat{K},\ Y(e_0 - e_i) \in \hat{K}$, $i = 1, \ldots, n^2$, by the following linearly independent set of constraints $Ye_i \in \hat{K},\ i = 0, 1, \ldots, n^2$.*

6 Comparing the relaxations

In this section we show that QAP_{ZKRW1} and QAP_{R_2} are equivalent and $QAP_{\mathcal{K}_n^{0*}}$, QAP_{R_3} and QAP_{LS} are also equivalent. The difference in favor of QAP_{LS}, noticed in [4], is therefore due to computational reasons (Sotirov [22] used the bundle method which is known to be very slowly converging close to the optimum, making accurate computation of optimal values difficult).

We need the following lemma.

Lemma 6 *A matrix $Y \in \mathcal{S}^+_{n^2}$ is feasible for QAP_{ZKRW1} if and only if Y satisfies*

(i) $\mathcal{G}(Y) = 0$, $\operatorname{trace}(Y^{ii}) = 1$ *for* $1 \le i \le n$, $\sum_i \operatorname{diag}(Y^{ii}) = u$,

(ii) $u^T Y^{ij} = \operatorname{diag}(Y^{jj})^T$ *for* $1 \le i, j \le n$,

(iii) $\sum_i Y^{ij} = u \operatorname{diag}(Y^{jj})^T$ *for* $1 \le j \le n$.

Proof: If Y satisfies (i)–(iii), then obviously Y is feasible for QAP_{ZKRW1} (feasibility for all but the last constraint follows from (i), while the last is a simple corollary of (i) and (ii)).

The opposite direction is less obvious. Let $Y \in \mathcal{S}^+_{n^2}$ be feasible for QAP_{ZKRW1}. Property (i) contains only constraints from QAP_{ZKRW1}, hence is satisfied.

The property (ii) follows from the fact that the matrix

$$\tilde{Y} = \sum_{i,j} \langle J,\, Y^{ij}\rangle \cdot E_{ij} = (I \otimes u^T)\, Y\, (I \otimes u^T)^T$$

is positive semidefinite and satisfies all assumptions from Lemma 1, therefore we have $\tilde{Y}_{ij} = 1$ (we used this fact also in the proof of Theorem 3). This implies that for any $i \neq j$ the matrix

$$\begin{bmatrix} Y^{ii} & Y^{ij} \\ Y^{ji} & Y^{jj} \end{bmatrix}$$

satisfies the assumptions of Lemma 2, hence the property (ii) follows.

We prove (iii) by considering $\hat{Y} = \sum_{i,j} Y^{ij} = (u^T \otimes I)\, Y\, (u^T \otimes I)^T$ and repeating the arguments from the previous paragraph. $\square$

We now have the tools to compare the semidefinite and Lovász–Schrijver relaxations for QAP.

Theorem 7 *The semidefinite program QAP_{R_2} is equivalent to QAP_{ZKRW1} in the sense that feasible sets are in bijective correspondence and $OPT_{R_2} = OPT_{ZKRW1}$.*

Proof: First we show that for any feasible $Y \in \mathcal{S}^+_{n^2+1}$ for QAP_{R_2} we can find exactly one matrix $Z = Z(Y) \in \mathcal{S}^+_{n^2}$, feasible for QAP_{ZKRW1} and vice versa. We address the components of Y and Z via the block structure, described in (1) and (2). The correspondence is as follows:

$$Y \mapsto Z = Z(Y) = [Y^{ij}]_{1 \le i,j \le n} \ \text{ and } \ Z \mapsto Y = Y(Z) = \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix}, \quad z = \operatorname{diag}(Z). \tag{12}$$

If Y is feasible for QAP_{R_2}, then $Z = Z(Y) \succeq 0$ and Lemma 5 implies that Z satisfies (i)–(iii) from Lemma 6. Hence Z is feasible for QAP_{ZKRW1}.

Let Z be feasible for QAP_{ZKRW1}. Then we have $\operatorname{Arrow}(Y) = 0$ and from Lemma 6 it follows that Y satisfies (i)–(iii) from Lemma 5, hence by using Remark 3 we have $Y = \hat{V} R \hat{V}^T$ for some $R \in \mathcal{S}_{(n-1)^2+1}$ with $R_{00} = 1$. It remains to show that $Y \succeq 0$. From the block structure of Y and $\hat{V}$ it follows that

$$Y = \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix} = \begin{bmatrix} e_0^T R e_0 & e_0^T R W^T \\ W R e_0 & W R W^T \end{bmatrix},$$

where W is from (7). Since $Z = WRW^T \succeq 0$, we have $R \succeq 0$ and consequently $Y \succeq 0$.

For any pair (Y, Z) of feasible solutions for QAP_{R_2} and QAP_{ZKRW1}, which satisfy (12), we have

$$\langle B \otimes A + \operatorname{Diag}(c), Z \rangle = \langle L, Y \rangle.$$

The equality $OPT_{R_2} = OPT_{ZKRW1}$ is therefore an immediate consequence of the first part of the proof. $\square$

In the following theorem we compare semidefinite programs QAP_{R_3}, $QAP_{\mathcal{K}_n^{0*}}$ and QAP_{LS} .

Theorem 8 *The semidefinite programs $QAP_{\mathcal{K}_n^{0*}}$, QAP_{R_3} and QAP_{LS} are equivalent, i.e. the feasible sets are in bijective correspondence and $OPT_{R_3} = OPT_{\mathcal{K}_n^{0*}} = OPT_{LS}$.*

Proof: The programs QAP_{R_3} and $QAP_{\mathcal{K}_n^{0*}}$ are obtained from the models QAP_{R_2} and QAP_{ZKRW1} by adding the sign constraint $Y \geq 0$. The equivalence therefore follows from the equivalence of models QAP_{R_2} and QAP_{ZKRW1}, proven in Theorem 7.

It remains to show equivalence of models $QAP_{\mathcal{K}_n^{0*}}$ and QAP_{LS}. Let $Y \in \mathcal{S}^+_{n^2+1}$ be feasible for QAP_{LS} and Z the matrix, obtained from Y by deleting the first row and column. According to Remark 4 we know that $Y \geq 0$, $Ye_0 = \operatorname{diag}(Y)$ and $Ye_i \in \hat{K}$ for $0 \le i \le n^2$ and $\hat{K}$ from (11). Therefore

$Z \geq 0$ and if we prove that Z is feasible for QAP_{ZKRW1}, then Z is feasible also for $QAP_{\mathcal{K}_n^{0*}}$, since QAP_{ZKRW1} was obtained from $QAP_{\mathcal{K}_n^{0*}}$ by omitting the sign constraint for non-diagonal entries in the non-diagonal blocks.

It is sufficient to show that Z satisfy properties (i)–(iii) from Lemma 6. Constraints $Ye_0 \in \hat{K}$ together with $Y_{00} = 1$ implies that $AY(1 : n^2, 0) = u_{2n}$, where A is from (10). This equations can be written equivalently as $u^T Y^{i,0} = 1$ for $1 \leq i \leq n$ and $\sum_i Y^{i0} = u$. By using $Ye_0 = \operatorname{diag}(Y)$ we get that the main diagonal of Z satisfies the property (i) from Lemma 6. Similarly we see that for any $1 \leq i \leq n^2$ the matrix Y satisfy $Ye_i \in \hat{K}$, or equivalently $AY(1 : n^2, i) = Y(0, i)u_{2n}$. Expanding this terms yields exactly the properties (ii) and (iii) from Lemma 6 for the matrix Z. For feasibility for QAP_{ZKRW1} it remains to show $\mathcal{G}(Z) = 0$. Let us consider the diagonal block Z^{ii}. Since we know that components of Z are non-negative, the property $u^T Z^{ii} = \operatorname{diag}(Z^{ii})$ implies that $Z^{ii}_{jk} = 0$ for $j \neq k$. Similarly we get from $\sum_i Z^{ij} = u \operatorname{diag}(Z^{jj})^T$ that $\sum_i \operatorname{diag}(Z^{ij}) = \operatorname{diag}(Z^{jj})$ which means that all diagonal elements in any non-diagonal block must be zero.

Therefore Z is feasible for $QAP_{\mathcal{K}_n^{0*}}$ and gives the same value of objective function. The reverse direction, i. e. proving that the feasible solution Z for $QAP_{\mathcal{K}_n^{0*}}$ yields a feasible solution Y for QAP_{LS} via

$$Y = \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix}, \quad z = \operatorname{diag}(Z),$$

is easy and again relies on Lemma 6. □

7 Summary and practical implications

We now summarize the equivalences shown in the previous sections and draw some practical conclusions. In Table 1, we collect in the same line problems which we showed to be equivalent. The last column refers to the theorem which shows the equivalence. All relaxations are formulated in the space of symmetric matrices of order n^2 (or $n^2 + 1$), hence each relaxation has $O(n^4)$ variables. The weakest relaxation has $O(n^2)$ constraints, while the strongest ones all have $O(n^4)$ constraints.

The two weakest, but computationally cheapest models can be solved easily by interior point methods. The other models (with at least $O(n^3)$ constraints) cannot be solved by interior point methods. In [22] the bundle method is suggested to solve both QAP_{R_2} and QAP_{R_3} with low accuracy by considering the Lagrangian dual, which is obtained by dualizing all constraints except those from QAP_{R_0}. Thus a function evaluation of the Lagrangian amounts to solving QAP_{R_0}. In [22] it is reported that after about 150 bundle iterations, i.e. function evaluations of QAP_{R_0}, one has a rough estimate of the respective relaxations.

The strongest models are still considered a computational challenge. Currently, the augmented Lagrangian method proposed by Burer and Vandenbussche leads to moderately accurate solutions of QAP_{LS}. The bundle method [22] seems to be slightly faster, but gives less accurate results. During the revision process of this paper, Zhao, Sun and Toh [27] proposed a new method, again based on the Augmented Lagrangian method, and provide accurate solutions of the strongest relaxation QAP_{R3}.

In Table 2 we summarize the currently strongest bounds on some standard test instances from the Nugent collection, and indicate the size of each instance in column 2. The bounds in column 3 and 4 are reasonably tight, but their efficient computation still has to be considered a serious computational challenge. The values of QAP_{R2} are taken from [22], those of QAP_{R3} from [27]. We emphasize that we considered only SDP based bounds in this paper. The QAPLIB website [5] maintains the development of all bounds available for QAP.

problems			hardness	equivalence
QAP	QAP_{CP}		NP-hard	Theorem 3
QAP_{R_3}	$QAP_{\mathcal{K}_n^{0*}}$	QAP_{LS}	$O(n^4)$	Theorem 7
QAP_{R_2}	QAP_{ZKRW1}		$O(n^3)$	Theorem 8
	QAP_{AW+}		n^2+n	
QAP_{R_0}			n^2+1	

Table 1: Problems in the same line are equivalent

name	n	OPT_{R_2}	OPT_{R_3}	OPT_{QAP}
nug15	15	1069	1139	1150
nug20	20	2386	2503	2570
nug25	25	3454	3662	3744
nug30	30	5695	5944	6124

Table 2: The strongest SDP bounds for some instances of the Nugent collection.

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