An integer programming approach to the OSPF weight setting problem*

Amandeep Parmar, Shabbir Ahmed† and Joel Sokol
H. Milton Stewart School of Industrial and Systems Engineering,
Georgia Institute of Technology, 765 Ferst Drive, Atlanta GA 30332

October 18, 2006

Abstract

Under the Open Shortest Path First (OSPF) protocol, traffic flow in an Internet Protocol (IP) network is routed on the shortest paths between each source and destination. The shortest path is calculated based on pre-assigned weights on the network links. The OSPF weight setting problem is to determine a set of weights such that, if the flow is routed based on the OSPF protocol, some measure of network congestion is minimized. A variety of optimization approaches for this strongly NP-hard problem have been proposed. However, the existing studies develop heuristic solution methods without any quality guarantees. In this paper we propose an integer programming based solution strategy for the OSPF weight setting problem so as to obtain solutions with quality guarantees. We develop a family of valid inequalities for a mixed-integer linear programming formulation of the problem. These inequalities are incorporated within a branch-and-cut algorithm. Computational experiments using some randomly generated test problems and some problems taken from the literature indicate that the proposed approach is able to provide feasible solutions with significantly smaller optimality gaps than those provided by the state-of-the-art integer programming solver CPLEX.

1 Introduction

Open Shortest Path First (OSPF) is the most commonly used intra-domain routing protocol in IP networks. Under this protocol every link in the network is assigned a weight, and based on these weights shortest paths between each source and destination are computed. The OSPF protocol then routes flow on shortest paths. If there are multiple shortest paths from some source to destination then the flow is split equally among all the outgoing arcs that are on the shortest paths. This equal splitting of flow is usually referred to as the Equal-Cost Multipath (ECMP) principle. The weights assigned to the links completely determine the flow in the network and hence the load on each link. The OSPF weight setting problem is to determine a set of weights such that, if the flow is routed based on the OSPF protocol, network congestion is minimized.

A variety of optimization based approaches for the OSPF weight setting problem have been proposed. Fortz and Thorup [5] considered a version of the problem involving a piecewise linear convex cost function to measure congestion. They showed that it is NP-hard to determine an optimal set of weights, and even to approximate an optimal solution within a constant factor. They proposed a local search technique to get good solutions for the problem. Pioro et.al. [8] considered the maximum load on any link in the network as the measure of congestion. They showed that the OSPF weight setting problem with this objective is NP-hard even for a single-source destination pair. They presented a mixed-integer linear programming (MILP) formulation for the problem, and proposed some heuristic methods for its solution based on local search, simulated annealing and Lagrangian relaxation. Srivastava et.al. [10] used a hybrid of the two objectives as a measure of congestion, and suggested heuristic solution methods. Srivastava et.al. [10] and Lin and

*This research has been supported in part by the Office of Naval Research (Award # N00014-05-1-0312) and the National Science Foundation (Award # DMI-0457066).
†Corresponding author: Shabbir Ahmed (sahmed@isye.gatech.edu)

Wang [6] proposed heuristic algorithms based on Lagrangian relaxation to determine feasible solutions for the weight setting problem. Ericsson et.al. [4] presented a genetic algorithm-based heuristic. Buriol et.al. [3] extended the genetic algorithm proposed in [4] to a memetic algorithm by adding a local search procedure. Bley and Koch [1] considered a more restricted version of the problem where the aim is to determine a set of weights such that these weights define unique shortest paths for each source-destination pair. Hence when the flow is routed on these paths there will be no splitting of flows. The authors proposed an integer programming approach for this restricted problem. Bley [2] showed that there does not exist a constant factor approximation for this restricted problem unless $P{=}NP$.

The existing studies on the general OSPF weight setting problem develop heuristic solution methods without any quality guarantees. In this paper we propose an integer programming based solution strategy for this problem so as to obtain solutions along with qualitative guarantees. We consider the MILP formulation for the OSPF weight setting problem with a maximum load objective presented by Pioro et.al. [8]. Unfortunately this MILP is extremely difficult to solve as-is by state-of-the-art solvers. We develop a family of valid inequalities and incorporate these within a branch-and-cut algorithm. We provide numerical results on the performance of our solution methodology on some randomly generated test problems and some problems taken from the literature. Unlike the heuristic methods the proposed integer programming based approach is able to provide a bound on the worst case optimality gap for the obtained solution. Moreover, the solutions generated by our approach have significantly smaller optimality gaps than those produced by the state-of-the-art MILP solver CPLEX.

The remainder of the paper is organized as follows. In the next section, we discuss an MILP formulation of the OSPF weight setting problem. In Section 3 we develop a family of valid inequalities to improve the MILP formulation. In Section 4 we describe a branch-and-cut implementation for the problem and present numerical results on some randomly generated problems and some problems taken from literature. Finally, conclusions and further research issues are discussed in Section 5.

2 A mixed integer linear programming model

In this section we present the MILP formulation of the OSPF weight setting problem from Pioro et.al. [8]. The underlying IP network is represented as a directed graph $G = (V, E)$, where the nodes represent routers and edges represent the links between them. Each edge $e \in E$ has a capacity c_e. We also assume that we have a demand matrix D, where D_{st} represents the flow to be sent from a source node $s \in V$ to a destination node $t \in V$. We denote $V_d \subseteq V$ as the set of destination nodes. Congestion in the network is measured as the maximum load over all the edges in the network. More precisely, if l_e denotes the total load (flow) on edge e, then the congestion is measured as the maximum of (l_e/c_e) over all edges $e \in E$. This objective allows the flow on an edge to exceed its capacity and hence any set of weight vectors is feasible for the problem. The problem is to determine weights w_e for every edge such that if the flow is routed, as per the OSPF protocol, between each source destination pair satisfying the demand, it minimizes the congestion in the network. The decision variables and constraints of the MILP formulation are presented next.

Decision variables:

x_e^t	binary variable denoting whether e is on some shortest path to destination t.
f_e^t	flow on edge e for destination t.
w_e	weight on edge e.
d_v^t	shortest path distance from v to destination t.
$f_{d_v}^t$	dummy flow variables for splitting the flow.
L	the maximum load in the network.

Flow conservation constraints:

$$\sum_{e:e=(-,t)} f_e^t - \sum_{e:e=(t,-)} f_e^t = \sum_{v \in V} D_{vt} \quad \forall\, t \in V_d \tag{1}$$

$$\sum_{e:e=(-,v)} f_e^t - \sum_{e:e=(v,-)} f_e^t = -D_{vt} \qquad \forall\, v \in V \setminus \{t\}\ \forall\, t \in V_d \tag{2}$$

$$f_e^t \leq M x_e^t \quad \forall e \ \ \forall\, t \in V_d \tag{3}$$

Constraints (1) and (2) ensure that total inflow is equal to total outflow. Constraints (3) force the flow to be sent on only arcs that are chosen to be on the shortest path. Here M is a large number that can be set to $\sum_{v \in V} D_{vt}$.

Feasible distance label constraints:

$$d_u^t \leq d_v^t + w_e \ \ \forall e = (u, v) \ \ \forall\, t \in V_d \tag{4}$$

$$d_v^t - d_u^t + w_e \leq M(1 - x_e^t) \ \ \forall\, e = (u, v) \ \ \forall\, t \in V_d \tag{5}$$

$$(1 - x_e^t) \leq M(d_v^t - d_u^t + w_e) \ \ \forall\, e = (u, v) \ \ \forall\, t \in V_d \tag{6}$$

Constraints (4), (5) and (6) ensure that x_e^t is 1 if and only if e is on a shortest path to destination t.

Flow splitting constraints:

$$f_e^t \leq f_{d_v}^t \ \ \forall\, e : e = (v, -) \ \ \forall\, v,\ t \in V_d \tag{7}$$

$$f_{d_v}^t - f_e^t \leq M(1 - x_e^t) \ \ \forall\, e : e = (v, -) \ \ \forall\, v,\ t \in V_d \tag{8}$$

Constraints (7) and(8) ensure that flow is equally split amongst all outgoing arcs on the shortest path to each destination.

Congestion constraints:

$$\sum_{t \in V_d} f_e^t \leq L c_e \ \ \forall\, e \tag{9}$$

Constraint (9) relate the total flow on an edge to the congestion measure L.

Using the variables and constraints defined above, a MILP model for the OSPF weight setting problem is:

$$\begin{array}{lll} \min & L & \\ \text{s.t.} & (1),\ (2),\ (3),\ (4),\ (5) & \\ & (6),\ (7),\ (8),\ (9) & (P) \\ & x_e^t \in \{0,1\},\ f_e^t \geq 0,\ w_e \geq 0 & \\ & d_v^t \geq 0,\ f_{d_v}^t \geq 0 & \end{array}$$

For a network with m edges, n nodes and k destinations, formulation (P) has $6mk + nk + m$ constraints and $2k(n + m) + m + 1$ variables, out of which mk are binary. As has been discussed by Pioro et.al. [8], even for a small sized network, the above MILP is very difficult to solve using a state-of-art MILP solver such as CPLEX. In the following section, we describe a family of valid inequalities to strengthen the MILP formulation.

3 A family of valid inequalities

In this section we develop valid inequalities for simple subsystems of the OSPF weight setting problem (P). Since the subsystem is a relaxation of (P) the proposed inequalities are also valid for (P). We consider four different subsystems.

3.1 A simple single node system

Consider a single node system which has n outgoing edges and flow of $b > 0$ units coming in. OSPF requires that flow has to be equally split amongst the edges which carry positive flow. Denote f_i to be flow on edge i, x_i to be 1 if and only if edge i is used to send flow out, and f_d to be a dummy variable used to ensure that we split the flows equally. The system is given by:

$$
\begin{aligned}
X^1 \quad = \Big\{ \quad & (f_d, f, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \{0,1\}^n : \\
& \sum_{i=1}^n f_i = b, \\
& f_i \le b x_i \quad i = 1, \dots, n, \\
& f_i \le f_d \quad i = 1, \dots, n, \\
& f_d \le f_i + b(1 - x_i) \quad i = 1, \dots, n \quad \Big\}.
\end{aligned}
$$

The first constraint balances flow in to flow out; the second constraint relates flows to arcs chosen; and the last two sets of constraints equate the flow on chosen arcs. Let $\text{co}(\cdot)$ denotes the convex hull of the set $\cdot$, we can now state the following result for convex hull of X^1, $\text{co}(X^1)$.

Proposition 1. *The dimension of* $\text{co}(X^1)$ *is given by:*

$$
dim(\text{co}(X^1)) = \begin{cases} 0 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 2n & \text{if } n \ge 3 \end{cases}
$$

Proof. See Appendix. □

Let us project out the variables f_i, and then aggregate the x_i variables to a general integer variable z, i.e., $z = \sum_{i=1}^n x_i$. Note that $z \in \{1, \dots, n\}$. Under this projection, the set X^1 is

$$
Y = \{(f_d, z) \in \mathbb{R}_+ \times \mathbb{Z}_+ : \ f_d z = b, \ 1 \le z \le n\}.
$$

We can now define the convex hull of Y.

Theorem 1. *The following inequalities define* $\text{co}(Y)$*:*

$$
k(k+1) f_d + bz \ge (2k+1)b \quad k = 1, \dots, n-1, \tag{10}
$$

$$
n f_d + bz \le b(n+1). \tag{11}
$$

The proof of the theorem (omitted here) is constructed by obtaining the convex hull of n points in a two dimensional plane.

The above defined valid inequalities (10) with z replaced by $\sum_{i=1}^n x_i$ are also valid for X^1 and indeed are facet defining. The following proposition states the result.

Proposition 2. *The following inequalities are valid for* X^1

$$
k(k+1) f_d + b \sum_{i=1}^n x_i \ge (2k+1)b \quad k = 1, \dots, n-1, \tag{12}
$$

and are facet defining for $1 \le k \le n-2$.

Proof. Let $\sum x_i = l \ge 1$, then $f_d = \frac{b}{l}$. We have $k(k+1)\frac{b}{l} + bl - (2k+1)b = \frac{b}{l}\left((l - \frac{2k+1}{2})^2 - \frac{1}{4}\right) \ge 0 \ \forall l$ integer. □

Proposition 3. *The family of valid inequalities (12) for* X^1 *are facet defining for* $1 \le k \le n-2$.

Proof. See Appendix. □

For the weight setting problem, for every node that has positive flow coming in we can generate the above family of cuts. Let l_v^t be a lower bound on the flow incoming to node v going to destination t ($l_v^t > 0$ for all source nodes v which have flow going to destination t) and let us denote O_v as the set of outgoing edges out of node v, then repeating the above procedure we get the following set of cuts :

$$k(k+1)f_{d_v}^t + l_v^t \sum_{e \in O_v} x_e^t \geq (2k+1)l_v^t \quad k = 1, \ldots, |O_v| - 1, \ \forall v, \ \forall t \in V_d \tag{13}$$

We now present some more valid inequalities for X^1.

Proposition 4. *Let $S \subseteq \{1, \ldots, n\}$ and $\{f^{(1)}, \ldots, f^{(|S|)}\}$ denote a permutation of $\{f_i\colon i \in S\}$. The following inequalities are valid for X^1:*

$$k(k+1)f_d + b\sum_{i \notin S} x_i - b\sum_{i \in S} x_i + 2\sum_{i=1}^{|S|}(k+i)f^{(i)} \geq (2k+1)b \tag{14}$$
$$\forall S : |S| \leq n-2, 1 \leq k \leq n - |S| - 1$$

Proof. Let $\sum x_i = l$ and $|S| = m$. We know that $f_d = \frac{b}{l}$ and flow will be equal to $\frac{b}{l}$ for all the arcs with $x = 1$. We need to show that $k(k+1)\frac{b}{l} + b(l-m) - bm + \frac{2kmb}{l} + \frac{m(m+1)b}{l} - (2k+1)b \geq 0$.
Note that the above condition is equivalent to

$$\frac{b}{l}\left[l^2 - \frac{(2m+2k+1)l}{2} + 2km + m^2 + m + k^2 + k\right] \geq 0 \Leftrightarrow$$

$$\frac{b}{l}\left[\left(l - \frac{(2m+2k+1)}{2}\right)^2 + \left(2km + m^2 + m + k^2 + k - \left(\frac{(2m+2k+1)}{2}\right)^2\right)\right] \geq 0 \Leftrightarrow$$

$$\frac{b}{l}\left[\left(l - \frac{(2m+2k+1)}{2}\right)^2 - \frac{1}{4}\right] \geq 0$$

which is true since l and m are integers. This completes the validity proof. □

Proposition 5. *The family of valid inequalities (14) for X^1 are facet defining for all S with $|S| = 1$.*

Proof. See Appendix. □

The following proposition provides another set of facet defining valid inequalities.

Proposition 6. *Let $S \subseteq \{1, \ldots, n\}$, then the following inequalities are valid for X^1:*

$$f_d + b\sum_{j \in S} x_j + \sum_{j \notin S} f_j \leq (|S|+1)b \quad \forall |S| \geq 1 \tag{15}$$

Proof. Let $\sum x_j = l$ and $|S| = m$. We need to show that $\frac{b}{l} + bm + (l-m)\frac{b}{l} - (m+1)b \leq 0$. Note that this condition is equivalent to $\frac{b}{l}(1-m) \leq 0$ which is true for all $m \geq 1$. This completes the proof. □

Proposition 7. *The family of valid inequalities (15) for X^1 are facet defining for $|S| = 1$ and $n \geq 3$.*

Proof. See Appendix. □

For the OSPF weight setting problem we can generate all the above families of valid inequalities, for every node that has a positive exogenous flow coming in, similar to the family of inequalities (13).

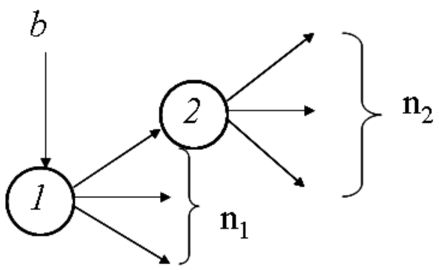

Figure 1: A two node system

3.2 A two node system

We now consider a two node system of a network flow problem with equal splits among chosen arcs as shown in Figure 1. Node 1 has n_1 outgoing arcs with arc 1 connecting to node 2 which has n_2 outgoing arcs. There is a positive flow b entering node 1 and there is no exogenous flow entering node 2.
We shall use the following notation:
Node 1 variables

f_i	Flow on arc i
x_i	Binary variable indicating if arc i is used or not
f_d	Dummy variable to equate flow on chosen arcs
b	Flow entering arc i (we assume $b > 0$)

Node 2 variables

g_i	Flow on arc i
y_i	Binary variable indicating if arc i is used or not
g_d	Dummy variable to equate flow on chosen arcs

The two node system is defined as X^2:

$$
\begin{aligned}
X^2 \quad = \Big\{ \quad & (f_d, g_d, f, g, x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^{n_1+n_2}_+ \times \{0,1\}^{n_1+n_2} : \\
& \sum_{i=1}^{n_1} f_i = b, \\
& \sum_{i=1}^{n_2} g_i = f_1, \\
& f_i \leq b x_i \quad i = 1, \ldots, n_1, \\
& g_i \leq f_1 y_i \quad i = 1, \ldots, n_2, \\
& f_i \leq f_d \quad i = 1, \ldots, n_1, \\
& g_i \leq g_d \quad i = 1, \ldots, n_2, \\
& f_d \leq f_i + b(1 - x_i) \quad i = 1, \ldots, n \\
& g_d \leq g_i + f_1(1 - y_i) \quad i = 1, \ldots, n \Big\}
\end{aligned}
$$

As before the flow has to be equally split amongst the chosen arcs. In this system even though the arc from node 1 to 2 is not chosen, some of the outgoing arcs from node 2 can still have y equal to 1 but the flow corresponding to those arcs will always be zero. Let $X^1(n)$ be defined as a single node system with n outgoing arcs, so if $n_1 = 0$ or $n_2 = 0$ our system X^2 can be written as $X^1(n_2)$ or $X^1(n_1)$ respectively. We have the following result for the dimension of $\text{co}(X^2)$, the convex hull of X^2.

Proposition 8. *The dimension of* $\text{co}(X^2)$ *is given by:*

$$dim(\text{co}(X^2)) = \begin{cases} dim(\text{co}(X^1(n_2))) & \text{if } n_1 = 1, n_2 \geq 1 \\ dim(\text{co}(X^1(n_1))) + 1 & \text{if } n_2 = 1, n_1 \geq 2 \\ 2 + 2n_2 & \text{if } n_1 = 2, n_2 \geq 2 \\ 2n_1 + 2n_2 & o.w. \end{cases}$$

Proof. See Appendix. □

We now present some valid inequalities for X^2.

Proposition 9. *The following inequalities are valid for* X^2*:*

$$-bx_1 + b\sum_{j\neq 1} x_j + k(k+1)f_d + 2by_i - 2(k+1)(n_2-1)g_i + 2(k+1)(n_2-1)g_d \geq (2k+1)b \ \forall i, k \quad (16)$$

$$-bx_1 + b\sum_{j\neq 1} x_j + k(k+1)f_d - 2by_i + 4(k+1)g_i + 2(k+1)\sum_{j\neq i} g_j - 2(k+1)g_d \geq (2k-1) \ \forall i, k \quad (17)$$

$$bx_1 + by_i - 2n_1 g_i - n_1\sum_{j\neq i} g_j + n_1 g_d \leq b \ \forall i \quad (18)$$

$$bx_1 - by_i + (n_2-1)n_1 g_i - (n_2-1)n_1 g_d \leq 0 \ \forall i \quad (19)$$

Proof. To prove the validity of the inequalities (16) we will consider the following four cases. Take any point in X^2 and let $\sum x_j = l$ and $\sum y_j = m$. Denote the difference between the left hand side and right hand side by (LHS - RHS).

(i). $x_1 = 0$ and $y_i = 0$. We have (LHS - RHS) equal to

$$\frac{b}{l}\left(l^2 - (2k+1)l + k(k+1)\right) = \frac{b}{l}\left(\left(l - \frac{(2k+1)}{2}\right)^2 - \frac{1}{4}\right) \geq 0$$

for all l integer.

(ii). $x_1 = 0$ and $y_i = 1$. Since $x_1 = 0$ even though $y_i = 1$ we still have $g_i = 0$. We have (LHS - RHS) equal to $\frac{b}{l}\left(\left(l - \frac{(2k+1)}{2}\right)^2 - \frac{1}{4}\right) + 2b \geq 0$

(iii). $x_1 = 1$ and $y_i = 0$. We have (LHS - RHS) equal to

$$\left(-b + b(l-1) + k(k+1)\frac{b}{l} + 2(k+1)(n_2-1)\frac{b}{lm} - (2k+1)b\right)$$

$$\geq \frac{b}{l}\left(l^2 - \frac{(2k+3)l}{2} + k^2 + 3k + 2\right) = \frac{b}{l}\left(\left(l - \frac{(2k+3)}{2}\right)^2 - \frac{1}{4}\right) \geq 0$$

for all l integer.

(iv). $x_1 = 1$ and $y_i = 1$. We have (LHS - RHS) equal to $\left(-b + b(l-1) + k(k+1)\frac{b}{l} + 2b - (2k+1)b\right)$ $=\frac{b}{l}\left(\left(l - \frac{(2k+1)}{2}\right)^2 - \frac{1}{4}\right) \geq 0$ for all l integer.

Similar to above to prove the validity for (17) we also consider four cases. Take any point in X^2 and let $\sum x_j = l$ and $\sum y_j = m$.

(i). $x_1 = 0$ and $y_i = 0$. We have (LHS - RHS) equal to

$$\frac{b}{l}\left(l^2 - (2k-1)l + k(k+1)\right) = \frac{b}{l}\left(\left(l - \frac{(2k-1)}{2}\right)^2 - \frac{1}{4} + 2k\right) \geq 0$$

for all l integer.

(ii). $x_1 = 0$ and $y_i = 1$. Since $x_1 = 0$ even though $y_i = 1$ we still have $g_i = 0$. We have (LHS - RHS) equal to $\left(bl + k(k+1)\frac{b}{l} - 2b - (2k-1)b\right) = \frac{b}{l}\left(\left(l - \frac{(2k+1)}{2}\right)^2 - \frac{1}{4}\right) \geq 0$.

(iii). $x_1 = 1$ and $y_i = 0$. We have (LHS - RHS) equal to

$$\left(-b + b(l-1) + k(k+1)\frac{b}{l} + 2(k+1)\frac{b}{l} - 2(k+1)\frac{b}{lm} - (2k-1)b\right)$$

$$\geq \frac{b}{l}\left(l^2 - \frac{(2k+1)l}{2} + k(k+1)\right) = \frac{b}{l}\left(\left(l - \frac{(2k+1)}{2}\right)^2 - \frac{1}{4}\right) \geq 0$$

for all l integer.

(iv). $x_1 = 1$ and $y_i = 1$. We have (LHS - RHS) equal to

$$\left(-b + b(l-1) + k(k+1)\frac{b}{l} - 2b + 4(k+1)\frac{b}{lm} + +2(k+1)\frac{b(m-1)}{lm} - 2(k+1)\frac{b}{lm} - (2k-1)b\right)$$

$$= \frac{b}{l}\left(l^2 - \frac{(2k+3)l}{2} + k(k+1) + 2(k+1)\right) = \frac{b}{l}\left(\left(l - \frac{(2k+3)}{2}\right)^2 - \frac{1}{4}\right) \geq 0$$

for all l integer.

We will prove the validity of the set of inequalities (18) by considering the following four cases. Take any point in X^2 and let $\sum x_j = l$ and $\sum y_j = m$.

(i). $x_1 = 0$ and $y_i = 0$. We have the (LHS - RHS) equal to $0 - b \leq 0$.

(ii). $x_1 = 0$ and $y_i = 1$. We have (LHS - RHS) equal to $b - b \leq 0$.

(iii). $x_1 = 1$ and $y_i = 0$. We have (LHS - RHS) equal to $b - n_1\frac{b}{l} + n_1\frac{b}{lm} \leq 0$, since $m \geq 1$.

(iv). $x_1 = 1$ and $y_i = 1$. We have (LHS - RHS) equal to $b + b - 2n_1\frac{b}{lm} - n_1\frac{b(m-1)}{lm} + n_1\frac{b}{lm} = 2b - 2n_1\frac{b}{l} \leq 0$ (since $l \leq n_1$).

Similarly to prove the validity of (19) we consider the following four cases. Take any point in X^2 and let $\sum x_j = l$ and $\sum y_j = m$.

(i). $x_1 = 0$ and $y_i = 0$. We have left hand side equal to $0 \leq 0$.

(ii). $x_1 = 0$ and $y_i = 1$. We have left hand side equal to $-b \leq 0$.

(iii). $x_1 = 1$ and $y_i = 0$. We have left hand side equal to $b - (n_2 - 1)n_1\frac{b}{lm} \leq 0$ ($m \leq (n_2 - 1)$ since $y_i = 0$).

(iv). $x_1 = 1$ and $y_i = 1$. We have left hand side equal to $b - b - (n_2 - 1)n_1\frac{b}{lm} - (n_2 - 1)n_1\frac{b}{lm} = 0 \leq 0$.

□

3.3 An alternate two node system

We consider a slight variation of the above mentioned two node system where we assume that there is positive incoming flow into both nodes as shown in Figure 2. Let the incoming flow into node 1 and 2 be b_1 and b_2 respectively. The rest of the variables are defined similarly as in the previous section.

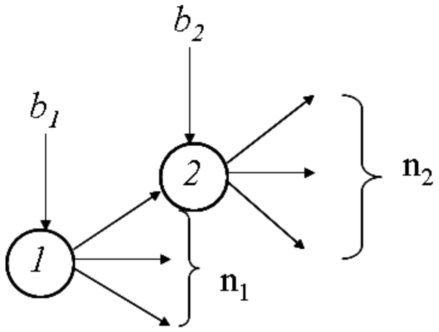

Figure 2: An alternate two node system

We can represent the set of feasible points as

$$
\begin{aligned}
X^{2+} \quad = \Big\{ \quad & (f_d, g_d, f, g, x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^{n_1+n_2}_+ \times \{0,1\}^{n_1+n_2} : \\
& \sum_{i=1}^{n_1} f_i = b_1, \\
& \sum_{i=1}^{n_2} g_i = f_1 + b_2, \\
& f_i \leq b_1 x_i \quad i = 1, \dots, n_1, \\
& g_i \leq (f_1 + b_2) y_i \quad i = 1, \dots, n_2, \\
& f_i \leq f_d \quad i = 1, \dots, n_1, \\
& g_i \leq g_d \quad i = 1, \dots, n_2, \\
& f_d \leq f_i + b_1(1 - x_i) \quad i = 1, \dots, n \\
& g_d \leq g_i + (b_2 + f_1)(1 - y_i) \quad i = 1, \dots, n \Big\}
\end{aligned}
$$

We have the following result for the dimension of $\text{co}(X^{2+})$, the convex hull of X^{2+}, .

Proposition 10. *The dimension of* $\text{co}(X^{2+})$ *is given by:*

$$
dim(\text{co}(X^{2+})) = \begin{cases} dim(\text{co}(X^1(n_2)) & \text{if } n_1 = 1, n_2 \geq 1 \\ dim(\text{co}(X^1(n_1)) & \text{if } n_2 = 1, n_1 \geq 2 \\ 2 + 2n_2 & \text{if } n_1 = 2, n_2 \geq 2 \\ 2n_1 + 2n_2 & o.w. \end{cases}
$$

The proof of the above proposition is similar to that of Proposition 8 and is not repeated here.

In the following proposition we present some valid inequalities for X^{2+}

Proposition 11. *The following inequalities are valid for* X^{2+}*:*

$$
\begin{aligned}
&-b_1x_1 + b\sum_{j\neq 1} x_j + k(k+1)f_d + (2b_1 + 2(k+1)b_2)y_i - 2(k+1)(n_2-1)g_i \\
&+ 2(k+1)(n_2-1)g_d \geq (2k+1)b_1 + 2(k+1)b_2 && \forall i\ \forall k && (20)\\
&-b_1x_1 + (b_1 + n_1b_2)y_i - n_1(n_2-1)g_i + n_1(n_2-1)g_d \geq n_1b_2 && \forall i && (21)\\
&-b_1x_1 + b\sum_{j\neq 1} x_j + k(k+1)f_d + 2(k+1)\sum g_j \geq (2k+1)b_1 + 2(k+1)b_2 && \forall k && (22)\\
&b_1x_1 + (n_1b_2 + b_1)y_i - 2n_1g_i - n_1\sum_{j\neq i} g_j + n_1g_d \leq b_1 && \forall i && (23)\\
&b_2\sum y_i - n_2\sum g_i + n_2g_d \leq b_2 && && (24)
\end{aligned}
$$

Proof. We will prove the validity of the set of inequalities (20) by considering the following four cases. Take any point in X^{2+} and let $\sum x_j = l$ and $\sum y_j = m$.

(i). $x_1 = 0$ and $y_i = 0$. We have (LHS - RHS) equal to

$$
\begin{aligned}
&b_1l + k(k+1)\frac{b_1}{l} + 2(n_2-1)(k+1)\frac{b_2}{m} - (2k+1)b_1 - 2(k+1)b_2\\
&\geq \frac{b_1}{l}\left(\left(l - \frac{2k+1}{2}\right)^2 - \frac{1}{4}\right) \geq 0 \quad \text{(since } m \leq n_2 - 1 \text{ , as } y_i = 0)
\end{aligned}
$$

for all l integer.

(ii). $x_1 = 0$ and $y_i = 1$. We have (LHS - RHS) equal to

$$
\begin{aligned}
&b_1l + k(k+1)\frac{b_1}{l} + 2b_1 + 2(k+1)b_2 - (2k+1)b_1 - 2(k+1)b_2\\
&\geq \frac{b_1}{l}\left(\left(l - \frac{2k+1}{2}\right)^2 - \frac{1}{4}\right) + 2b_1 \geq 0
\end{aligned}
$$

for all l integer.

(iii). $x_1 = 1$ and $y_i = 0$. We have (LHS - RHS) equal to

$$
\begin{aligned}
&-b_1 + b_1(l-1) + k(k+1)\frac{b_1}{l} + 2(k+1)(n_2-1)(\frac{b_1}{lm} + \frac{b_2}{m}) - (2k+1)b_1 - 2(k+1)b_2\\
&\geq b_1(l-2) + k(k+1)\frac{b_1}{l} + 2(k+1)\frac{b_1}{l} - (2k+1)b_1 \quad \text{(since } m \leq n_2 - 1\text{, as } y_i = 0)\\
&= \frac{b_1}{l}\left(\left(l - \frac{2k+3}{2}\right)^2 - \frac{1}{4}\right) \geq 0
\end{aligned}
$$

for all l integer.

(iv). $x_1 = 1$ and $y_i = 1$. We have (LHS - RHS) equal to

$$
\begin{aligned}
&-b_1 + b_1(l-1) + k(k+1)\frac{b_1}{l} + 2b_1 + 2(k+1)b_2 - (2k+1)b_1 - 2(k+1)b_2\\
&= \frac{b_1}{l}\left(\left(l - \frac{2k+1}{2}\right)^2 - \frac{1}{4}\right) \geq 0
\end{aligned}
$$

for all l integer.

Similarly to prove the validity of (21) we consider the following four cases. Take any point in X^{2+} and let $\sum x_j = l$ and $\sum y_j = m$.

(i). $x_1 = 0$ and $y_i = 0$. We have (LHS - RHS) equal to $n_1(n_2-1)\frac{b_2}{m} - n_1 b_2 \geq 0$ (since $m \leq n_2 - 1$).

(ii). $x_1 = 0$ and $y_i = 1$. We have (LHS - RHS) equal to $n_1 b_2 - n_1(n_2-1)\frac{b_2}{m} + n_1(n_2-1)\frac{b_2}{m} - n_1 b_2 = 0$.

(iii). $x_1 = 1$ and $y_i = 0$. We have (LHS - RHS) equal to $-b_1 + n_1(n_2-1)\left(\frac{b_1}{lm} + \frac{b_2}{m}\right) - n_1 b_2 \geq -b_1 + \frac{n_1 b_1}{l}$ (since $m \leq n_2$) and is ≥ 0 (since $l \leq n_1$).

(iv). $x_1 = 1$ and $y_i = 1$. We have (LHS - RHS) equal to $-b_1 + b_1 + n_1 b_2 - n_1 b_2 = 0 \geq 0$.

To prove the validity of (22) we consider the following two cases. Take any point in X^{2+} and let $\sum x_j = l$ and $\sum y_j = m$.

(i). $x_1 = 0$. We have the (LHS - RHS) equal to $b_1 l + k(k+1)\frac{b_1}{l} + 2(k+1)b_2 - 2(k+1)b_2 - (2k+1)b_1 = \frac{b_1}{l}\left(\left(l - \frac{2k+1}{2}\right)^2 - \frac{1}{4}\right) \geq 0$ for all l integer.

(ii). $x_1 = 1$. We have (LHS - RHS) equal to

$$-b + b(l-1) + k(k+1)\frac{b}{l} + 2(k+1)(\frac{b_1}{l} + b_2) - (2k+1)b_1 - 2(k+1)b_2$$

$$\frac{b_1}{l}\left(\left(l - \frac{2k+3}{2}\right)^2 - \frac{1}{4}\right) \geq 0 \text{ for all } l \text{ integer.}$$

To prove the validity of (23) we consider the following four cases. Take any point in X^{2+} and let $\sum x_j = l$ and $\sum y_j = m$.

(i). $x_1 = 0$ and $y_i = 0$. We have (LHS - RHS) equal to $-n_1\frac{b_2}{m} + n_1\frac{b_2}{m} - b_1 \leq 0$.

(ii). $x_1 = 0$ and $y_i = 1$. We have (LHS - RHS) equal to $n_1 b_2 + b_1 - 2n_1\frac{b_2}{m} - n_1\frac{(m-1)b_2}{m} + n_1\frac{b_2}{m} - b_1 = 0$.

(iii). $x_1 = 1$ and $y_i = 0$. We have (LHS - RHS) equal to $b_1 - n_1\left(\frac{b_1}{l} + b_2\right) + n_1\left(\frac{b_1}{lm} + \frac{b_2}{m}\right) - b_1 \leq 0$ (since $m \geq 1$).

(iv). $x_1 = 1$ and $y_i = 1$. We have (LHS - RHS) equal to

$$b_1 + n_1 b_2 + b_1 - 2n_1\left(\frac{b_1}{lm} + \frac{b_2}{m}\right) - n_1(m-1)\left(\frac{b_1}{lm} + \frac{b_2}{m}\right) + n_1\left(\frac{b_1}{lm} + \frac{b_2}{m}\right) - b_1$$

$$= b_1 + n_1 b_2 - n_1\left(\frac{b_1}{l} + b_2\right) = b_1 - n_1\frac{b_1}{l} \leq 0 \text{ (since } l \leq n_1.)$$

For the proof of validity of (24) note that it is sufficient to show that the inequality is valid for the case when $x_1 = 0$. Let $\sum y_i = m$, then (LHS - RHS) is equal to $b_2 m - n_2 b_2 + n_2\frac{b_2}{m} - b_2 = (m-1)(b_2 - n_2\frac{b_2}{m}) \leq 0$ (since $m \leq n_2$).

□

3.4 A multiple node system

Let us now consider a set of nodes $\mathcal{P}_i = \{1, 2, \ldots, P\}$ on a path to a node i as depicted in Figure 3 (assume that $b_i > 0$). For each $p = 1, \ldots, P$, let n_p denote the number of outgoing arcs from node p and $b_p (\geq 0)$ denote the exogenous flow into node p. The binary variable x_p for $p = 1, \ldots, P-1$ corresponds to arcs going from p to $p+1$, and for $p = P$, corresponds to the arc from node P to i. If $x_p = 1$ for all $p = 1, \ldots, P$, then a lower bound on the flow into node i is $b_i + l(\mathcal{P}_i)$ where

$$l(\mathcal{P}_i) = \frac{b_1 + n_1 b_2 + n_1 n_2 b_3 + \cdots + n_1 n_2 .. n_{P-1} b_{P-1}}{n_1 n_2 \ldots n_P}.$$

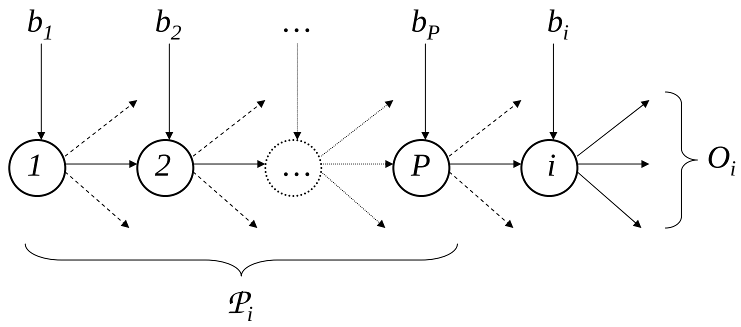

Figure 3: Path to i

Thus, the following inequality is valid

$$f_{d_i} \geq \frac{b_i + l(\mathcal{P}_i)}{z_i} + l(\mathcal{P}_i)\Big(\sum_{p\in\mathcal{P}_i} x_p - |\mathcal{P}_i|\Big),$$

where f_{d_i} is the flow on any used arc from i and z_i is the number of out-going used arcs. We can now apply the idea in (10) to linearize the above inequality and obtain the following result.

Proposition 12. *The following inequalities are valid for the multiple node system defined above:*

$$k(k+1)f_{d_i} + (b_i + l(\mathcal{P}_i))\sum_{j\in O_i} x_j - k(k+1)l(\mathcal{P}_i)\sum_{p\in\mathcal{P}_i} x_p \geq (2k+1)(b_i + l(\mathcal{P}_i)) - k(k+1)l(\mathcal{P}_i)|\mathcal{P}_i|$$
$$k = 1, \ldots, n_i - 1 \qquad (25)$$

where O_i is the set of outgoing arcs from i.

The proof of the validity (omitted here) of (25) is constructed by obtaining the convex hull of n points in a two dimensional plane, exactly similar to that of (10).

3.5 Separation

In order to use the valid inequalities in a branch and cut fashion we need to separate them, i.e., find out which inequalities are violated by a given fractional solution. Lets look at the family of inequalities given by (12). Given a fractional solution $(\widehat{\mathbf{x}}, \widehat{f_d})$, let us define $h(k)$ as

$$h(k) = k(k+1)\widehat{f_d} + b\sum_{i=1}^{n}\widehat{x_i} - (2k+1)b \qquad (26)$$

In order to find a violated inequality of the form (12) we need to find a $0 \leq k \leq n-1$ such that $h(k) < 0$. From the definition of $h(k)$, it is clear that it is a convex quadratic function in k (since $\widehat{f_d} > 0$). Let k^* be the point where $h(k)$ reaches its minimum. We know from simple calculus that $k^* = \frac{b}{\widehat{f_d}} - \frac{1}{2}$. Now if $h(k^*) \geq 0$ then we know that $h(k) \geq 0\ \forall k$, otherwise we can find the two roots k^1 and k^2 of $h(k)$ and $h(k) < 0$ for all the integers between (k^1, k^2). Above is true only if both the roots of $h(k)$ are real, if both the roots are not real then $h(k) < 0\ \forall k$. Similarly if one of them is real, say k^1, then we know if $k^1 < k^*$, then $h(k) < 0$ for all $k > k^1$ and if $k^1 > k^*$, then $h(k) < 0$ for all $k < k^1$. Based on this information can check for what values of $k \in [0, n-1]$, the valid inequality is violated. This gives a constant time separation routine for family of valid inequalities (12).

We can use the same procedure as described for family of valid inequalities (14) and (14) for the case when $|S| = 1$. This is the case for which we know that the inequalities are indeed facet defining. It is not clear how to separate efficiently for sets S with size strictly bigger than 1.

The family of valid inequalities described for the two node subsystems as described in Propositions 9 and 11 can be easily separated using the procedure described above. However it is not clear how to separate the inequalities (25) described for the multiple node system.

4 A branch-and-cut implementation

We incorporated the cuts derived in the previous section within a branch-and-cut algorithm to solve the OSPF weight setting problem. We used CPLEX 9.0 callable libraries to implement the branch and cut algorithm. Cuts were added using the cut callback function as they were violated. Computations were done on a Intel Pentium 4 Zeon machine with 2.4 GHz speed and 2.0 GB RAM running Linux kernel 2.4.27.

4.1 Test Problems

We experimented using some problems taken from literature and some randomly generated problems. To generate random networks we first fixed the number of nodes and number of demand pairs, and then an edge between two nodes was randomly added with a probability of 0.5. If the random graph obtained was not connected, more edges were added to ensure connectedness. Capacities and demands were also randomly generated. Table 1 provides the number of nodes, edges, demand pairs and number of binary variables in the MILP models for the problems considered. The problem names starting with 'n' are randomly generated. The problems 'Pioro7,' 'Pioro12w1' and 'Pioro12w2' are taken from [8]. The problem 'snh' is obtained from [9]. The remaining problems, i.e., 'graph1' and 'caldata' are are proprietary telecommunication networks.

Prob	Nodes	Edges	Demand pairs	Binary Vars
n5d10	5	12	10	60
n10d5	10	40	5	200
n10d6	10	48	6	288
n10d11	10	42	11	336
n10d12	10	38	12	266
n10d15	10	42	15	378
n15d5	15	98	5	392
n15d6	15	82	6	410
n15d7	15	96	7	672
n15d8	15	84	8	504
n15d9	15	88	9	792
n15d10	15	98	10	784
graph1	10	28	10	168
caldata	14	36	33	504
snh	11	46	9	322
Pioro7	7	24	21	144
Pioro12w1	12	36	66	396
Pioro12w2	12	36	66	396

Table 1: Size of the problems

4.2 Cuts

We identified single node subsystems (X^1) and added the corresponding valid inequalities. It is easily seen that the cuts (12) are a special case of (25) when the path length is zero (i.e., $\mathcal{P} = \emptyset$). There are exponentially many cuts of type (25) and it is not clear how to separate these efficiently. Moreover, the cuts become weak, in general, as we increase the length of the the path $\mathcal{P}$. So in our branch-and-cut procedure we only use the cuts for which the path length $|\mathcal{P}| \leq 2$. For the family of inequalities (14) we observed after doing computational experiments that the inequalities become weaker as the size of set S increases, so we just use valid inequalities for $|S| = 1, 2$. The family of valid inequalities (15) are facet defining for only $|S| = 1$ (Proposition 7), so these are the only ones that are used in our branch-and-cut procedure. The valid inequalities that can not be separated efficiently as discussed in Section 3.5 are added to the CPLEX cut pool.

We also identified two node subsystems, as shown in Figure 1, in our OSPF model and used the cuts (16),(17),(18) and (19). Similarly we identified subsystems as shown in Figure 2 and added the valid inequalities (20),(21),(22),(23) and (24). All these family of inequalities are polynomial in number and can be easily separated as discussed in Section 3.5.

Inequalities that could not be separated efficiently were added directly to the CPLEX cut pool. The rest of them were added, using the cut callback function of CPLEX, as they were violated. At any given fractional node all violated valid inequalities are added (through cut callback).

4.3 Heuristics

We implemented some basic heuristic methods to start CPLEX with a good initial solution.

Local search

We know that any set of weights (on edges) is feasible for the OSPF weight setting problem. So we implemented a weight adjustment heuristic in which we start with a set of weights (all edge weights are set to 1) and after routing the flow based on these weights we adjust the weights, increasing for the edges with larger load and decreasing for edges with smaller load, in hope of decreasing the maximum load in the network. This procedure is repeated a fixed number of times and the best solution obtained over all the iterations is stored.

IP heuristic

We realized that the problem with objective as the sum of the loads rather than maximum deviation is much easier to minimize. And since the feasible region remains the same this IP provides a feasible solution to our problem with objective as minimizing the maximum deviation. So in the heuristic stage we solved an IP with objective as sum of loads on each arc and in order to get better starting solution, we also added the maximum load variable (with certain penalty) to our objective. The more we increased the penalty on the maximum load variable, harder the problem became to solve. If the penalty on the maximum load variable is much larger then the problem basically reduces to our original problem. We did some computational experiments to determine the best penalty. Since we did not want so spend too much time on the heuristic IP, so we put a time limit of 100 seconds on this procedure and collect the best solution obtained. We also kept track of the best solution obtained in terms of the maximum load amongst all the feasible solutions obtained.

We got the best solution from both the heuristics and started our weight setting problem with the one with better objective function value. It was observed that the IP heuristic outperformed the local search heuristic most of the times.

An advantage of getting a good starting solution is that the MILP model does not have any capacity constraints on the flow that can be routed on any edge, and good heuristic solutions provide an upper bound on the capacity. This facilitates generation of general flow cover type of cuts which help in the branch-and-cut procedure.

4.4 Symmetry

The integer programming formulation for the weight setting problem has a lot of symmetry issues. Given a feasible solution with certain load, another feasible solution with the same objective can be obtained by changing a x variable from 0 to 1 and keeping the flow zero. In order to get rid of this issue we added lower bounding inequalities, which imply that if the arc is chosen then there has to be a certain positive flow on that arc. These valid are easy to obtain for the arcs emanating from the nodes with positive exogenous incoming flow. For the simple single node system we can add the following inequalities $f_i \geq \frac{b_i}{n} x_i$. For the other arcs we find a path from the closest source node and add an inequality similar to the one obtained for the multiple node system. This does not completely remove the symmetry problem but adding these inequalities help in the computations.

4.5 Results

In Table 2 we provide the optimality gap of the best feasible solution returned after 800 seconds. The reason we chose to stop our computations after 800 seconds is because after around 800 seconds the gaps did not change must at all when the problem is given to CPLEX, so either the problem would be solved to optimality before that time or it would be solved at all. The first column is the name of the problem. The second column provides optimality gaps for the solutions returned by the default CPLEX solver. The third column provides optimality gaps for the solutions for our branch-and-cut procedure when all the above defined cuts are added. The fourth column presents results after adding the heuristic methods and adding all the cuts obtained in the previous sections for penalty of 5 on the maximum load objective(see discussion in Section 4.3). The fifth and the sixth columns present results when penalty is set to 10 and 20 respectively.

				Heuristic + Cuts	
Prob	*CPLEX*	*Cuts*	*Penalty(5)*	*Penalty(10)*	*Penalty(20)*
n5d10	**0.00%**	**0.00%**	**0.00%**	**0.00%**	**0.00%**
n10d5	**0.00%**	**0.00%**	**0.00%**	**0.00%**	**0.00%**
n10d6	22.06%	6.00%	10.02%	7.35%	3.15%
n10d11	13.01%	3.49%	1.18%	1.18%	0.40%
n10d12	19.63%	12.89%	12.89%	10.04%	9.87%
n10d15	39.71%	29.15%	32.14%	19.92%	17.71%
n15d5	8.15%	23.37%	16.42%	4.71%	0.02%
n15d6	44.74%	33.33%	16.82%	7.89%	3.16%
n15d7	79.83%	35.36%	30.87%	24.45%	7.83%
n15d8	54.07%	36.72%	33.47%	8.14%	**0.00%**
n15d9	53.28%	8.33%	8.33%	**0.00%**	**0.00%**
n15d10	76.54%	61.23%	60.88%	38.54%	30.21%
graph1	12.76%	**0.00%**	**0.00%**	**0.00%**	**0.00%**
caldata	-	4.84%	4.84%	4.83%	2.82%
snh	-	35.18%	20.49%	19.92%	19.92%
Pioro7	5.17%	4.36%	**0.00%**	**0.00%**	**0.00%**
Pioro12w1	-	6.32%	6.32%	6.32%	**0.00%**
Pioro12w2	98.36%	15.58%	12.25%	10.88%	4.21%
Average Gap*	35.15%	17.56%	14.83%	9.12%	5.52%

*Computed over the 14 problems for which CPLEX found a feasible solution.

Table 2: Percentage optimality gap after 800 secs.

As is evident from Table 2 the OSPF weight setting MILP is quite difficult to solve as-is. Within the allotted 800 second time limit, the default CPLEX solver produced solutions with an average optimality gap of 35.7%, and for three of the problems, it was not even able to find a feasible solution. When cuts are used in the branch and bound procedure we see that there is a significant reduction in the optimality gaps for the problems considered as seen in column 3. When the proposed branch-and-cut scheme is used with a heuristic to hot start it produced feasible solutions with optimality gaps significantly reduced from those of default CPLEX. The penalty setting of 20 for the heuristic procedure seems to be the best in terms of computational results as in Table 2. We can see from these results that the cuts and heuristic methods are both equally important for getting good solutions with smaller optimality gaps.

5 Conclusions

We proposed an integer programming approach to obtain provably good solutions to the OSPF weight setting problem. The key contribution is to strengthen a mixed-integer linear programming formulation of the problem using cutting planes and integrate these cuts with local search heuristics within a branch-and-cut algorithm. Computational results indicate that the proposed method performs significantly better than straight-forward use of the commercial solver CPLEX. Even though we are still not able to solve most of the problems to optimality within a reasonable time limit, there is evidence that the proposed methodology helped to get significantly tighter bounds. Integrating the proposed scheme with more sophisticated heuristic schemes in the literature may provide significant additional benefits.

References

[1] A. Bley and T. Koch. Integer programming approaches to access and backbone IP-network planning. *preprint ZIB ZR-02-41*, 2002.

[2] A. Bley. On the approximability of the minimum congestion unsplittable shortest path routing problem. *Proceedings of 11th Conference on Integer Programming and Combinatorial Optimization (IPCO 2005)*, 2005.

[3] L.S. Buriol, M.G.C. Resende, C.C. Rebeiro and M. Thorup. A memetic algorithm for OSPF routing. *Proceedings of the 6th INFORMS Telecom*, pp. 187-188, 2002.

[4] M. Ericsson, M. Resende and P. Pardolas. A genetic algorithm for weight setting problem in OSPF routing. *Journal of Combinatorial Optimization*, pp. 299-333, vol. 6, 2002.

[5] B. Fortz and M. Thorup. Increasing internet capacity using local search. *Computational Optimization and Applications*, pp. 13-48, vol. 29, 2004.

[6] F. Lin and J. Wang. Minimax open shortest path first routing algorithms in networks supporting the smds services. *Proceedings of IEEE International Conference on Communications*, pp. 666-670, vol. 2, 1993.

[7] G.L. Nemhauser and L.A. Wolsey. *Integer and Combinatorial Optimization.* Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, 1989.

[8] M. Pioro, A. Szentsi, J. Harmatos, A. Juttner, P. Gajownicczek and S. Kozdrowski. On open shortest path first related network optimization problems. *Performance Evaluation*, pp. 201-223, vol. 48(4), 2002.

[9] H. Sakauchi, Y. Nichimura and S. Hasegawa. A self-healing network with an economical spare channel assignment. *Proceedings of IEEE Global Telecommunications Conference*, pp. 438-443, 1990.

[10] S. Srivastava, G. Agarwal, D. Medhi and M. Pioro. Determining feasible link weight systems under various objectives for OSPF networks. *IEEE eTransactions on Network and Service Management*, vol. 2(1), 2005.

Appendix

Proposition 1. *The dimension of* $\mathrm{co}(X^1)$ *is given by:*

$$dim(\mathrm{co}(X^1)) = \begin{cases} 0 & \textit{if } n = 1 \\ 2 & \textit{if } n = 2 \\ 2n & \textit{if } n \geq 3 \end{cases}$$

Proof. For $n = 1$ there is only one feasible point and for $n = 2$ there are only three feasible points (it can be easily verified that these points are affinely independent), so the result easily follows for $n = 1, 2$. For $n \geq 3$, look at the following set of $2n + 1$ points in X^1;

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n & f_1 & f_2 & \dots & f_n & f_d \\ 1 & 0 & \dots & 0 & b & 0 & \dots & 0 & b \\ 0 & 1 & \dots & 0 & 0 & b & \dots & 0 & b \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & b & b \\ 0 & 1 & \dots & 1 & 0 & \frac{b}{n-1} & \dots & \frac{b}{n-1} & \frac{b}{n-1} \\ 1 & 0 & \dots & 1 & \frac{b}{n-1} & 0 & \dots & \frac{b}{n-1} & \frac{b}{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & \frac{b}{n-1} & \frac{b}{n-1} & \dots & 0 & \frac{b}{n-1} \\ 1 & 1 & \dots & 1 & \frac{b}{n} & \frac{b}{n} & \dots & \frac{b}{n} & \frac{b}{n} \end{pmatrix}$$

To see that these points are affinely independent, let $\lambda_1, \dots, \lambda_{2n+1}$ be the multipliers. Then we have the following set of equalities,

$$\lambda_i + \sum_{j \neq i} \lambda_{n+j} + \lambda_{2n+1} = 0 \quad \forall i = 1, \dots, n \tag{27}$$

$$b\lambda_i + \left(\frac{b}{n-1}\right) \sum_{j \neq i} \lambda_{n+j} + \left(\frac{b}{n}\right) \lambda_{2n+1} = 0 \quad \forall i = 1, \dots, n \tag{28}$$

$$b \sum_i \lambda_i + \left(\frac{b}{n-1}\right) \sum_j \lambda_{n+j} + \left(\frac{b}{n}\right) \lambda_{2n+1} = 0 \tag{29}$$

$$\sum_{i=1}^{2n+1} \lambda_i = 0. \tag{30}$$

Subtracting (27) from (30) and (28) from (29) yields $\sum_{j \neq i} \lambda_j + \lambda_{n+i} = 0$ and $b \sum_{j \neq i} \lambda_j + (\frac{b}{n-1})\lambda_{n+i} = 0$ respectively . Subtracting the above two we get $\lambda_{n+i} = 0\ \forall i$. Plugging back values of λ_{n+i}, we get $\lambda_i = 0\ \forall i$, which yields $\lambda_{2n+1} = 0$. Hence the points are affinely independent and the result follows. □

Proposition 3. *The family of valid inequalities (12) for* X^1 *are facet defining for* $1 \leq k \leq n-2$.

Proof. Look at all the points in X^1 that satisfy (12) at equality. It is easy to see that these points are such that $x_i = 1\ \forall i \in I$, $x_i = 0$ otherwise, where $I \subseteq \{1, \dots, n\}$ and $|I| = k$ or $k + 1$.

Let $\lambda x = \lambda_0$ for all the points x which satisfy (12) at equality, then we know that (λ, λ_0) should satisfy the following set of equalities;

$$\sum_{i \in I} \lambda_i + \frac{b}{k} \sum_{i \in I} \lambda_{n+i} + \frac{b}{k} \lambda_{2n+1} = \lambda_0 \quad \forall I \subseteq \{1, \dots, n\},\ |I| = k \tag{31}$$

$$\sum_{i \in I} \lambda_i + \frac{b}{k+1} \sum_{i \in I} \lambda_{n+i} + \frac{b}{k+1} \lambda_{2n+1} = \lambda_0 \quad \forall I \subseteq \{1, \dots, n\},\ |I| = k+1 \tag{32}$$

From (31), taking appropriate sets I and subtracting we get $\lambda_i + \frac{b}{k}\lambda_{n+i} = \lambda_j + \frac{b}{k}\lambda_{n+j}$. Similarly from (32) we get $\lambda_i + \frac{b}{k+1}\lambda_{n+i} = \lambda_j + \frac{b}{k+1}\lambda_{n+j}$. From these two equalities we get $\lambda_i = \lambda_j$ and $\lambda_{n+i} = \lambda_{n+j}$. Plugging it back into (31) and (32) we get two equalities with four unknowns (we denote $\lambda_i = \lambda_1\ \forall i$ and $\lambda_{n+i} = \lambda_{n+1}\ \forall i$)

$$k\lambda_1 + b\lambda_{n+1} + \frac{b}{k}\lambda_{2n+1} = \lambda_0 \tag{33}$$

$$(k+1)\lambda_1 + b\lambda_{n+1} + \frac{b}{k+1}\lambda_{2n+1} = \lambda_0 \tag{34}$$

A general solution to above equalities (33) and (34)is given by:

$$\begin{pmatrix} \lambda_1 \\ \lambda_{n+1} \\ \lambda_{2n+1} \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ k(k+1) \\ (2k+1)b \end{pmatrix} \mu_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ b \end{pmatrix} \mu_2$$

Hence the general solution to (31) and (32) can be written as:

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \lambda_{n+1} \\ \vdots \\ \lambda_{2n} \\ \lambda_{2n+1} \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} b \\ \vdots \\ b \\ 0 \\ \vdots \\ 0 \\ k(k+1) \\ (2k+1)b \end{pmatrix} \mu_1 + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ b \end{pmatrix} \mu_2$$

The first vector is the vector defining the valid inequality and the second vector is the one defining the equality in the description of X^1. Hence the result follows from Theorem 3.6 in [7]. We also require $2 \leq k \leq n-2$ to get adequate number of affinely independent points. ☐

Proposition 5. *The family of valid inequalities (14) for X^1 are facet defining for all S with $|S| = 1$.*

Proof. We prove the facet defining property when $|S| = 1$. Let $S = \{j\}$, then the inequality (14) becomes,

$$k(k+1)f_d + b\sum_{i\neq j} x_i - bx_j + 2(k+1)f_j \geq (2k+1)b \quad \forall j, 1 \leq k \leq n-2 \tag{35}$$

To prove the facet defining property, look at all the points in X^1 that satisfy (35) at equality. The points are;

$$x_j = 0, x_i = 1 \forall i \in I, 0 \text{ o.w.}, I \subseteq \{1, \ldots, j-1, j+1, \ldots, n\}, |I| = k \text{ or } k+1$$

$$x_j = 1, x_i = 1 \forall i \in I, 0 \text{ o.w.}, I \subseteq \{1, \ldots, j-1, j+1, \ldots, n\}, |I| = k \text{ or } k+1$$

Let $\lambda x = \lambda_0$ for all the points x which satisfy (35) at equality, then we know that (λ, λ_0) should satisfy the following set of equalities;

$$\sum_{i\in I} \lambda_i + \frac{b}{k}\sum_{i\in I} \lambda_{n+i} + \frac{b}{k}\lambda_{2n+1} = \lambda_0 \quad \forall I \subseteq \{1, \ldots, n\},\ |I| = k \tag{36}$$

$$\sum_{i\in I} \lambda_i + \frac{b}{k+1}\sum_{i\in I} \lambda_{n+i} + \frac{b}{k+1}\lambda_{2n+1} = \lambda_0 \quad \forall I \subseteq \{1, \ldots, n\},\ |I| = k+1 \tag{37}$$

$$\sum_{i\in I} \lambda_i + \lambda_j + \frac{b}{k+1}\sum_{i\in I} \lambda_{n+i} + \frac{b}{k+1}\lambda_{n+j} + \frac{b}{k+1}\lambda_{2n+1} = \lambda_0 \quad \forall I \subseteq \{1, \ldots, n\},\ |I| = k \tag{38}$$

$$\sum_{i\in I} \lambda_i + \lambda_j + \frac{b}{k+2}\sum_{i\in I} \lambda_{n+i} + \frac{b}{k+2}\lambda_{n+j} + \frac{b}{k+2}\lambda_{2n+1} = \lambda_0 \quad \forall I \subseteq \{1, \ldots, n\},\ |I| = k+1 \tag{39}$$

It is easy to see that $\lambda_i = \lambda_l$ & $\lambda_{n+i} = \lambda_{n+l}$ $\forall i \neq l \neq j$. Plugging it back into (36)-(39) we get the following four inequalities with six unknowns(denote $\lambda_i = \lambda_1, \lambda_{n+i} = \lambda_{n+1}$ $i \neq j$);

$$k\lambda_1 + b\lambda_{n+1} + \frac{b}{k}\lambda_{2n+1} = \lambda_0 \tag{40}$$

$$(k+1)\lambda_1 + b\lambda_{n+1} + \frac{b}{k+1}\lambda_{2n+1} = \lambda_0 \tag{41}$$

$$k\lambda_1 + \lambda_j + \frac{bk}{k+1}\ \lambda_{n+1} + \frac{b}{k+1}\lambda_{n+j} + \frac{b}{k+1}\lambda_{2n+1} = \lambda_0 \tag{42}$$

$$(k+1)\lambda_1 + \lambda_j + \frac{b(k+1)}{k+2}\ \lambda_{n+1} + \frac{b}{k+2}\lambda_{n+j} + \frac{b}{k+2}\lambda_{2n+1} = \lambda_0 \tag{43}$$

A general solution to above equalities (40)-(43) is given by:

$$\begin{pmatrix} \lambda_1 \\ \lambda_j \\ \lambda_{n+1} \\ \lambda_{n+j} \\ \lambda_{2n+1} \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} b \\ -b \\ 0 \\ 2 \\ k(k+1) \\ (2k+1)b \end{pmatrix} \mu_1 \quad + \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ b \end{pmatrix} \mu_2$$

The first vector is the vector defining the valid inequality (35) and the second vector is the vector defining the equality in the description of X^1. Hence the result follows from Theorem 3.6 in [7]. We also require $2 \leq k \leq n-2$ to get adequate number of affinely independent points. □

Proposition 7. *The family of valid inequalities (15) for X^1 are facet defining for $|S| = 1, n \geq 3$.*

Proof. We prove the facet defining property for the case when $|S| = 1$. Let $S = \{j\}$, then the inequality becomes,

$$f_d + bx_j + \sum_{i \neq j} f_i \leq 2b \tag{44}$$

Now look at all the points that satisfy (44) at equality. These points are precisely,

- For each k, $x_k = 1, x_i = 0\ \forall i \neq k$
- For each $k \neq j$, $x_j = 1, x_k = 1, x_i = 0\ \forall i \neq j, k$
- For each $k, l \neq j$, $x_j = 1, x_k = 1, x_l = 1, x_i = 0\ \forall i \neq j, k, l$

Let $\lambda x = \lambda_0$ for all the points x which satisfy (44) at equality, then we know that (λ, λ_0) should satisfy the following set of equalities;

$$\lambda_k + b\lambda_{n+k} + b\lambda_{2n+1} = \lambda_0\ \forall k \tag{45}$$

$$\lambda_k + \lambda_j + \frac{b}{2}\lambda_{n+k} + \frac{b}{2}\lambda_{n+j} + \frac{b}{2}\lambda_{2n+1} = \lambda_0\ \forall k \neq j \tag{46}$$

$$\lambda_k + \lambda_j + \lambda_l + \frac{b}{3}\lambda_{n+k} + \frac{b}{3}\lambda_{n+j} + \frac{b}{3}\lambda_{n+l} + \frac{b}{3}\lambda_{2n+1} = \lambda_0\ \forall k, l,\ k \neq l \neq j \tag{47}$$

We can easily derive from (45)-(47) that $\lambda_k = \lambda_l$ & $\lambda_{n+k} = \lambda_{n+l}$ $\forall k, l\ k \neq l \neq j$. Plugging the values back into (45)-(47) we get four equations with six unknowns (denote $\lambda_k = \lambda_1$ & $\lambda_{n+k} = \lambda_{n+1}$ $\forall k \neq j$):

$$\lambda_1 + b\lambda_{n+1} + b\lambda_{2n+1} = \lambda_0 \tag{48}$$

$$\lambda_j + b\lambda_{n+j} + b\lambda_{2n+1} = \lambda_0 \tag{49}$$

$$\lambda_1 + \lambda_j + \frac{b}{2}\lambda_{n+1} + \frac{b}{2}\lambda_{n+j} + \frac{b}{2}\lambda_{2n+1} = \lambda_0 \tag{50}$$

$$2\lambda_1 + \lambda_j + \frac{2b}{3}\lambda_{n+1} + \frac{b}{3}\lambda_{n+j} + \frac{b}{3}\lambda_{2n+1} = \lambda_0 \tag{51}$$

A general solution to (48)-(51) is given by :

$$\begin{pmatrix} \lambda_1 \\ \lambda_j \\ \lambda_{n+1} \\ \lambda_{n+j} \\ \lambda_{2n+1} \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 1 \\ 0 \\ 1 \\ 2b \end{pmatrix} \mu_1 \quad + \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ b \end{pmatrix} \mu_2$$

The first vector is the vector defining the valid inequality in question and the second vector is the vector defining the equality in the description of X^1. Hence the result follows from Theorem 3.6 in [7]. $\square$

Proposition 8. *The dimension of* $\text{co}(X^2)$ *is given by:*

$$dim(\text{co}(X^2)) = \begin{cases} dim(\text{co}(X^1(n_2))) & \text{if } n_1 = 1, n_2 \geq 1 \\ dim(\text{co}(X^1(n_1))) + 1 & \text{if } n_2 = 1, n_1 \geq 2 \\ 2 + 2n_2 & \text{if } n_1 = 2, n_2 \geq 2 \\ 2n_1 + 2n_2 & o.w. \end{cases}$$

Proof. Case $n_1 = 1$. The system reduces to $X^1(n_2)$ since the arc 1 outgoing from node 1 has to be chosen and it serves as an exogenous flow entering node 2. Hence it just reduces to a single node system with positive incoming flow b and n_2 leaving arcs.

Case $n_2 = 1$. If $n_1 = 2$ we can easily check that there are only four points in X^2 all of which are affinely independent, which proves the result for this case. For $n_1 \geq 3$, first note that since $n_2 = 1$ all the feasible solutions satisfy $g_1 = g_d$, hence the rank of the equality set of convex hull of X^2 is at least 3, hence the dimension of convex hull is at most $2n_1 + 1 + 3 - 3 = 2n_1 + 1$. But we have $2n_1 + 2$ affinely independent points in X^2, $2n_1 + 1$ are the same as defined in Proposition 1 (along with y, g and g_d). The last point comes from the fact that when $x_1 = 0$ we can have both $y_1 = 0$ and $y_1 = 1$ and these points are affinely independent, which completes $2n_1 + 2$ points we needed to prove the dimension.

Case $n_1 = 2, n_2 \geq 2$. We know from Proposition 1 that for the case of $n_1 = 2$, the equality set for convex hull of $X^1(n_1)$ is of rank 3(5-2). The same equality set must be present in convex hull of X^2, since X^2 contains all the constraints of $X^1(n_1)$. So the rank of the equality set of convex hull of X^2 must be at least $3 + 1$ (1 comes from the equality for second node). Hence the dimension of convex hull of X^2 is at most $5 + 2n_2 + 1 - 4 = 2 + 2n_2$. We now show $3 + 2n_2$ affinely independent points in X^2.

x_1	x_2	f_1	f_2	f_d	y_1	y_2	$\dots$	y_{n_2}	g_1	g_2	$\dots$	g_{n_2}	g_d
1	0	b	0	b	1	0	$\dots$	0	b	0	$\dots$	0	b
1	0	b	0	b	0	1	$\dots$	0	0	b	$\dots$	0	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
1	0	b	0	b	0	0	$\dots$	1	0	0	$\dots$	b	b
1	0	b	0	b	1	1	$\dots$	0	$\frac{b}{2}$	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$
1	1	$\frac{b}{2}$	$\frac{b}{2}$	$\frac{b}{2}$	1	0	$\dots$	0	$\frac{b}{2}$	0	$\dots$	0	$\frac{b}{2}$
1	1	$\frac{b}{2}$	$\frac{b}{2}$	$\frac{b}{2}$	0	1	$\dots$	0	0	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
1	1	$\frac{b}{2}$	$\frac{b}{2}$	$\frac{b}{2}$	0	0	$\dots$	1	0	0	$\dots$	$\frac{b}{2}$	$\frac{b}{2}$
1	1	$\frac{b}{2}$	$\frac{b}{2}$	$\frac{b}{2}$	1	1	$\dots$	0	$\frac{b}{4}$	$\frac{b}{4}$	$\dots$	0	$\frac{b}{4}$
0	1	0	b	b	0	0	$\dots$	0	0	0	$\dots$	0	0

The matrix above gives us $2n_2 + 3$ points, it can be easily verified that the points are affinely independent.

Case $n_1 \geq 3, n_2 \geq 2$. In this case we have the rank of equality set of convex hull of X^2 is at least 2, hence dimension of convex hull of X^2 is at most $2n_1 + 1 + 2n_2 + 1 - 2 = 2n_1 + 2n_2$. We now show $2n_1 + 2n_2 + 1$ affinely independent points in x^2.

x_1	x_2	$\dots$	x_{n_1}	f_1	f_2	$\dots$	f_{n_1}	f_d	y_1	y_2	$\dots$	y_{n_2}	g_1	g_2	$\dots$	g_{n_2}	g_d
1	1	$\dots$	0	$\frac{b}{2}$	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$	1	0	$\dots$	0	$\frac{b}{2}$	0	$\dots$	0	$\frac{b}{2}$
1	1	$\dots$	0	$\frac{b}{2}$	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$	0	1	$\dots$	0	0	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
1	1	$\dots$	0	$\frac{b}{2}$	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$	0	0	$\dots$	1	0	0	$\dots$	$\frac{b}{2}$	$\frac{b}{2}$
1	0	$\dots$	1	$\frac{b}{2}$	0	$\dots$	$\frac{b}{2}$	$\frac{b}{2}$	1	0	$\dots$	0	$\frac{b}{2}$	0	$\dots$	0	$\frac{b}{2}$
1	0	$\dots$	1	$\frac{b}{2}$	0	$\dots$	$\frac{b}{2}$	$\frac{b}{2}$	0	1	$\dots$	0	0	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
1	0	$\dots$	1	$\frac{b}{2}$	0	$\dots$	$\frac{b}{2}$	$\frac{b}{2}$	0	0	$\dots$	1	0	0	$\dots$	$\frac{b}{2}$	$\frac{b}{2}$
1	0	$\dots$	0	b	0	$\dots$	0	b	1	1	$\dots$	0	$\frac{b}{2}$	$\frac{b}{2}$	$\dots$	0	$\frac{b}{2}$
0	1	$\dots$	0	0	b	$\dots$	0	b	1	1	$\dots$	0	0	0	$\dots$	0	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\dots$	1	0	0	$\dots$	b	b	1	1	$\dots$	0	0	0	$\dots$	0	0
1	0	$\dots$	0	b	0	$\dots$	0	b	1	0	$\dots$	1	$\frac{b}{2}$	0	$\dots$	$\frac{b}{2}$	$\frac{b}{2}$
0	1	$\dots$	0	0	b	$\dots$	0	b	1	0	$\dots$	1	0	0	$\dots$	0	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\dots$	1	0	0	$\dots$	b	b	1	0	$\dots$	0	0	0	$\dots$	0	0
1	1	$\dots$	1	$\frac{b}{n_1}$	$\frac{b}{n_1}$	$\dots$	$\frac{b}{n_1}$	$\frac{b}{n_1}$	1	1	$\dots$	1	$\frac{b}{n_1 n_2}$	$\frac{b}{n_1 n_2}$	$\dots$	$\frac{b}{n_1 n_2}$	$\frac{b}{n_1 n_2}$

We have $2n_1 + 2n_2 + 1$ points above, which can be easily seen to be affinely independent. This completes the proof. $\square$