

# A Filter Algorithm for Nonlinear Semidefinite Programming

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## Abstract

This paper proposes a filter method for solving nonlinear semidefinite programming problems. Our method extends to this setting the filter SQP (sequential quadratic programming) algorithm, recently introduced for solving nonlinear programming problems, obtaining their respective global convergence results.

## 1 Introduction

We consider in this paper the following nonlinear semidefinite programming problem (NLSDP)

$$\min_{x \in \mathbb{R}^n} \{f(x); h(x) = 0, G(x) \preceq 0\}, \quad (\text{P})$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$  and  $G : \mathbb{R}^n \rightarrow S^m$  are  $C^2$ -functions,  $S^m$  denotes the linear space of  $m \times m$  real symmetric matrices equipped with the inner product  $A \cdot B := \text{trace}(AB) = \sum_{i,j=1}^m A_{ij}B_{ij}$  for all matrices  $A = (A_{ij}), B = (B_{ij}) \in S^m$ , and  $\preceq$  denotes the negative semidefinite order, that is,  $A \preceq B$  iff  $A - B$  is a negative semidefinite matrix. The order relations  $\prec$ ,  $\succeq$  and  $\succ$  are defined similarly.

A particular important case of the above model is the *Linear SemiDefinite Programming* (LSDP) problem, which appears when  $h(x) = 0$  for all  $x$ , and  $f(x)$  and  $G(x)$  are linear mappings (e.g. [45, 48, 51]). The LSDP model is one of the most active research field in the last 15 years with a huge amount of applications in control theory, optimal design, among others, and links to relaxation methods for combinatorial problems. See for instance [5, 10, 42].

The mentioned research effort has produced a lot of different approaches for solving this problem. Let us just mention the fundamental approach of interior point polynomial methods [1, 5, 38], penalty/barrier and augmented Lagrangian methods [24, 36], spectral bundle methods [20], cutting plane methods [2, 19, 26], among others.

The LSDP model is however insufficient for solving important application problems, see e.g. [37, 41]. The more general nonlinear problem NLSDP has been then also used for modelling in, for instance, feedback control, structural optimization and trust design problems. See [4, 21], and more recently [3, 25] and references therein.

The first- and second-order optimality conditions for NLSDP have been widely characterized, see for instance [9] and the references therein. Concerning numerically efficient solution algorithms the situation for NLSDP is completely different to the one of the LSDP case. In fact, the few existing approaches have been recently proposed and are mainly still in a development stage.

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Leibfritz and Mostafa developed in [33, 34] an interior point constrained trust region algorithm for a special class of NLSDP problems.

Another approach was developed by Jarre [22]. It generalizes the predictor-corrector interior-point method to nonconvex semidefinite programming.

An augmented Lagrangian algorithm and a code (PENNOM) supporting NLSDP problems without equality constraints were developed by Kočvara and Stingl [23, 24]. The code was successfully tested for LSDP but recent experiments with nonconvex NLSDP [25] reported still unsatisfactory performance.

In a serie of papers, Apkarian, Noll and others have suggested different approaches for solving NLSDP problems. For instance, the partially augmented Lagrangian approach [3, 12, 41], the spectral bundle methods [39, 40], and the sequential semidefinite programming algorithm [13]. In this paper we are mainly interested in the last idea.

The sequential semidefinite programming (SSDP) method [13] is an extension of the classical *Sequential Quadratic Programming* (SQP) algorithm to the cone  $S_+^p$  of symmetric positive semidefinite matrices. Here a sequence of Karush-Kuhn-Tucker points (see the definition in Section 2) is generated by solving at each step a tangent problem to NLSDP. The subproblems can be stated as LSDP and efficiently solved. Extending the results from [7, 8] to the non-polyhedral case  $S_+^p$ , it was shown in [13] the local fast convergence of the SSDP method.

The SSDP method was later deduced from a different viewpoint by Freund and Jarre in [17]. They also proved local quadratic convergence, using a sensitivity result for NLSDP, when the strict complementarity condition holds. Correa and Ramírez proved in [11] the global convergence of SSDP using a merit function (called Han penalty function) and a line search strategy.

Motivated by the local convergence properties obtained for the SSDP algorithm and the global ones obtained for the filter SQP, the aim of this paper is to exploit this recent filter methodology in order to propose a globally convergent algorithm for solving NLSDP.

Let us say some words about the filter methods. The filter methods were first introduced by Fletcher and Leyffer in [15]. In this technique the trial points are accepted when they improve either the objective function or a constraint violation measure. These criteria are compared to previous iterates collected in a filter. This idea offers an efficient alternative to traditional merit functions with penalty terms which adjustment can be problematic.

The filter method approach was first used as a tool for proving global convergence of algorithms for NLP, but soon it was successfully exploited in many ways for quite different algorithmic ideas in NLP. For instance, global convergence results of trust region filter SQP methods were established in [14, 16]. Local convergence of a particular filter trust region SQP, avoiding the Maratos effect, was given in [47]. Global and local convergence results for filter SQP algorithms using line search were also presented in [49, 50]. Furthermore, filter methods have been successfully combined with many approaches like, for instance, interior point [6, 46], etc.

The paper is organized as follows. In the next section some notations and preliminary results are introduced. In the third section the filter SDP algorithm is given in details and some first auxiliary results are stated. The fourth section is devoted to the proof of the main global convergence results. The conclusions are given in a final section.

## 2 Preliminaries

We say that  $(\bar{\lambda}, \bar{Y})$  is a Lagrange multiplier associated with  $\bar{x}$ , if  $(\bar{x}, \bar{\lambda}, \bar{Y}) \in \mathbb{R}^n \times \mathbb{R}^p \times S^m$  satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$\nabla_x L(\bar{x}, \bar{\lambda}, \bar{Y})^\top = \nabla f(\bar{x})^\top + Dh(\bar{x})^\top \bar{\lambda} + DG(\bar{x})^\top \bar{Y} = 0, \quad (1a)$$

$$G(\bar{x}) \bar{Y} = 0, \quad (1b)$$

$$G(\bar{x}) \preceq 0, h(\bar{x}) = 0, \bar{Y} \succeq 0, \quad (1c)$$

where  $L : \mathbb{R}^n \times \mathbb{R}^p \times S^m \rightarrow \mathbb{R}$  is the Lagrangian function of problem (P)

$$L(x, \lambda, Y) := f(x) + h(x)^\top \lambda + Y \cdot G(x). \quad (2)$$

Here and throughout the paper  $Dh(x) := \left[ \frac{\partial h_i(x)}{\partial x_j} \right]_{ij}$  denotes the  $p \times n$  Jacobian matrix of  $h$  and  $\nabla f(x) \in \mathbb{R}^n$  the row gradient vector of  $f$  at  $x$ . Note that, for a linear operator  $Ay := \sum_{i=1}^m y_i A_i$  with  $A_i \in S^m$ , as  $DG(x)$ , we have for its adjoint operator  $A^\top$  the formula:

$$A^\top Z = (A_1 \cdot Z, \dots, A_n \cdot Z)^\top, \quad \forall Z \in S^m. \quad (3)$$

A pair  $(\bar{x}, \bar{\lambda}, \bar{Y})$  satisfying (1) will be also called critical point or KKT-point of problem (P), and the set of Lagrange multipliers associated with  $\bar{x}$  will be denoted by  $\Lambda(\bar{x})$ . Finally,  $\bar{x}$  is called a critical point or critical solution of (P) if  $\Lambda(\bar{x}) \neq \emptyset$ .

Also, let  $q = m - r$  with  $r := \text{rank}(G(\bar{x}))$ , and denote by  $E \in \mathbb{R}^{m \times q}$  a matrix whose columns are an orthonormal basis of  $\text{Ker } G(\bar{x})$ . Using this notation, KKT conditions (1) can be equivalent written in terms of  $\Phi := E^\top \bar{Y} E \in S^q$  as follows

$$\nabla f(\bar{x})^\top + Dh(\bar{x})^\top \bar{\lambda} + DG(\bar{x})^\top (E \Phi E^\top) = 0, \quad (4a)$$

$$G(\bar{x}) \preceq 0, h(\bar{x}) = 0, \bar{\Phi} \succeq 0. \quad (4b)$$

Hence  $\bar{\Phi}$  can be seen as a *reduced* Lagrangian multiplier associated with the SDP constraint of problem (P) that obviously satisfies that  $\bar{Y} = E \bar{\Phi} E^\top$ .

In this paper, we will use *Robinson's constraint qualification condition* [43] defined at a feasible point  $\bar{x}$  of (P) as

$$0 \in \text{int} \left\{ \begin{pmatrix} G(\bar{x}) \\ h(\bar{x}) \end{pmatrix} + \begin{pmatrix} DG(\bar{x}) \\ Dh(\bar{x}) \end{pmatrix} \mathbb{R}^n - \begin{pmatrix} S_-^m \\ \{0\} \end{pmatrix} \right\}, \quad (5)$$

where  $\text{int } C$  denotes the topological interior of the set  $C$ , and  $S_-^m = \{A \in S^m \mid A \preceq 0\}$ . It is easy to see that condition (5) is equivalent to *Mangasarian-Fromovitz constraint qualification condition (MFCQ)*

$$\{\nabla h_j(\bar{x})\}_{j=1}^p \quad \text{is linearly independent, and} \quad (6a)$$

$$\exists \bar{d} \in \mathbb{R}^n \text{ s. t. } \begin{cases} Dh(\bar{x}) \bar{d} = 0 \\ \text{and } G(\bar{x}) + DG(\bar{x}) \bar{d} \prec 0. \end{cases} \quad (6b)$$

Also, it can be shown that under (5) the set of Lagrange multipliers  $\Lambda(\bar{x})$  is nonempty and also bounded [27]. Actually, when  $\bar{x}$  is assumed to be a local solution of (P), condition (5) is equivalent to saying that  $\Lambda(\bar{x})$  is nonempty and compact.

Given a matrix  $A = (A_{ij}) \in S^m$  we denote by  $\|A\|_{Fr}$  its norm associated with the mentioned (trace) inner product in  $S^m$ . This is also called the Frobenius norm of  $A$  and is given by

$$\|A\|_{Fr} = \sqrt{\text{trace}(A^2)} = \sqrt{\sum_{i,j=1}^m A_{ij}^2} = \sqrt{\sum_{i=1}^m \lambda_i(A)^2},$$

where  $(\lambda_1(A), \dots, \lambda_m(A))$  denotes the vector of eigenvalues of  $A$  in non-increasing order. In particular it holds that

$$|\lambda_j(A)| \leq \|A\|_{Fr}, \quad j = 1, \dots, m. \quad (7)$$

With the linear operator  $DG(x)$  we can also associate a Frobenius norm defined as follows

$$\|DG(x)\|_{Fr} = \sqrt{\sum_{i=1}^n \left\| \frac{\partial G(x)}{\partial x_i} \right\|_{Fr}^2},$$

which satisfies the relation

$$\|DG(x)d\|_{Fr} \leq \|d\|_2 \|DG(x)\|_{Fr}. \quad (8)$$

Here and from now on  $\|v\|_2$  denotes the Euclidean norm of a given vector  $v$ .

The operator  $D^2G(x) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow S^m$ , obtained from the second derivative of  $G(x)$ , is defined by

$$D^2G(x)(d, \hat{d}) = d^\top D^2G(x)\hat{d} = \sum_{i,j=1}^n d_i \hat{d}_j \frac{\partial^2 G(x)}{\partial x_i \partial x_j}.$$

The corresponding Frobenius norm can be analogically intended as

$$\|D^2G(x)\|_{Fr} = \sqrt{\sum_{i,j=1}^n \left\| \frac{\partial^2 G(x)}{\partial x_i \partial x_j} \right\|_{Fr}^2}.$$

Hence, the following relationship holds trivially for all  $j = 1, \dots, m$

$$|\lambda_j(d^\top D^2G(x)d)| \leq \|d^\top D^2G(x)d\|_{Fr} \leq \|d\|_2 \|D^2G(x)\|_{Fr}. \quad (9)$$

Indeed, it follows from the equality  $\|d^\top D^2G(x)d\|_{Fr} = \sqrt{\sum_{k,l=1}^m (d^\top D^2G_{kl}(x)d)^2}$ , where  $G_{kl}(x)$  is the  $(k, l)$  entry of  $G(x)$ .

**Lemma 1 (Debreu's Lemma).** *Let  $A \preceq 0$  be negative semidefinite. There exists  $r > 0$  such that  $B + rA \prec 0$  if and only if  $B|_{Ker A} \prec 0$ .*

In this Lemma the notation  $B|_{Ker A}$  states for a matrix  $V^\top BV$ , where the columns of  $V$  build a linear basis of  $Ker A$ .

**Remark 2.** *A simple consequence of Debreu's Lemma is that: if  $A \preceq 0$  and  $B|_{Ker A} \prec 0$  then there exists a  $\bar{\tau} > 0$  such that  $A + \tau B \prec 0$  for all  $\tau \in (0, \bar{\tau}]$ .*

The following useful lemma relates the fulfillment of the KKT optimality conditions and the MFCQ constraint qualifications.

**Lemma 3.** *Let us suppose that the point  $\bar{x}$  is feasible for (P) satisfying the MFCQ condition (6). If  $\bar{x}$  is not a critical solution of (P), there exist then a unitary vector  $\bar{s} \in \mathbb{R}^p$  and some  $\bar{\eta} > 0$  such that for all  $\eta \in (0, \bar{\eta}]$  we have*

$$\nabla f(\bar{x})\bar{s} < 0, \quad (10a)$$

$$Dh(\bar{x})\bar{s} = 0, \quad (10b)$$

$$G(\bar{x}) + \eta DG(\bar{x})\bar{s} \prec 0. \quad (10c)$$

**Proof.** Note that the vectors  $\{\nabla h_j(\bar{x})\}_{j=1}^p$  are linearly independent due to the fact that MFCQ is fulfilled at  $\bar{x}$ .

Since KKT conditions are not satisfied at  $\bar{x}$ , we can separate the point  $0 \in \mathbb{R}^n$  from the convex and closed set  $\mathcal{L}(\mathbb{R}^p \times S_+^q)$ , where  $\mathcal{L}(\lambda, \Phi) := \nabla f(\bar{x})^\top + Dh(\bar{x})^\top \lambda + DG(\bar{x})^\top (E\Phi E^\top)$  is a

linear function on  $(\lambda, \Phi) \in \mathbb{R}^p \times S_+^q$ ,  $q$  is the dimension of  $\text{Ker } G(\bar{x})$ , and  $E \in \mathbb{R}^{m \times q}$  is a matrix whose columns are an orthonormal basis of  $\text{Ker } G(\bar{x})$  (see (4)).

We obtain therefore the existence of a unitary vector  $\bar{s} \in \mathbb{R}^n$  such that

$$\begin{aligned} 0 &> \nabla f(\bar{x})\bar{s} + \lambda^\top Dh(\bar{x})\bar{s} + DG(\bar{x})\bar{s} \cdot (E\Phi E^\top) \\ &= \nabla f(\bar{x})\bar{s} + \lambda^\top Dh(\bar{x})\bar{s} + (E^\top DG(\bar{x})\bar{s}E) \cdot \Phi, \quad \forall (\lambda, \Phi) \in \mathbb{R}^p \times S_+^q. \end{aligned}$$

Since it is valid for all  $(\lambda, \Phi) \in \mathbb{R}^p \times S_+^q$  and  $\bar{x}$  satisfies MFCQ, it directly follows that conditions (10a) and (10b) hold, as well as  $E^\top DG(\bar{x})\bar{s}E \prec 0$  (by redefining, if necessary,  $\bar{s}$  as  $\bar{s} + \delta\bar{d}$  with  $\delta > 0$  small enough and  $\bar{d}$  given by (6b)). This last inequality together with Debreu's lemma (see also remark 2) finally implies (10c) for all positive  $\eta$  smaller than some  $\bar{\eta} > 0$ .  $\blacksquare$

### 3 The Filter-SDP Algorithm

Let us first adapt the filter SQP algorithm of Fletcher et al. [16] to problem (P).

For a given point  $x \in \mathbb{R}^n$  and a positive radius  $\rho > 0$  we define the following trust region local semidefinite approximation of the problem (P):

$$\begin{aligned} QP(x, \rho) \quad &\min_{d \in \mathbb{R}^n} \quad \nabla f(x)d + \frac{1}{2}d^\top Bd \\ \text{s.t.} \quad &h(x) + Dh(x)d = 0 \\ &G(x) + DG(x)d \preceq 0 \\ &\|d\|_\infty \leq \rho. \end{aligned} \tag{11}$$

Here  $B \in S^n$  is a matrix containing second order information of the problem (P) at  $x$ . Since its definition is not crucial for the analysis of the global convergence, it is not specified in detail.

The filter algorithm deals simultaneously with the optimality and feasibility aspects of (P) using a dominance strategy typical of multiobjective problems. In order to quantify the feasibility let us define

$$\theta(x) := \|h(x)\|_2 + \lambda_1(G(x))_+. \tag{12}$$

where  $\lambda_1(A)$  is the largest eigenvalue of the matrix  $A$ , and  $(\alpha)_+ := \max\{0, \alpha\}$  denotes the positive part of the real number  $\alpha$ . This function is obviously positive and vanishes exactly on the feasible points of (P).

A filter, denoted by  $\mathcal{F}$ , is a finite collection of two dimensional vectors. In the vectors of the filter the first and second component refer to the value of the feasibility function,  $\theta$ , and of the objective function,  $f$ , respectively.

The new candidate  $(\bar{\theta}, \bar{f})$  is called *acceptable* to the filter  $\mathcal{F} = \{(\theta_j, f_j)\}_{j=1}^N$ , if for each  $j = 1 \dots N$  at least one of the following two conditions is fulfilled:

$$\bar{\theta} \leq \beta\theta_j, \tag{13a}$$

$$\bar{f} + \gamma\bar{\theta} \leq f_j, \tag{13b}$$

where  $\beta \in (0, 1)$  and  $\gamma \in (0, \beta)$  are two fixed parameters.

Let us suppose that the point  $(\bar{\theta}, \bar{f})$  is acceptable for  $\mathcal{F}$ . The new filter  $\text{Add}((\bar{\theta}, \bar{f}), \mathcal{F})$  is defined as

$$\text{Add}((\bar{\theta}, \bar{f}), \mathcal{F}) = (\mathcal{F} \cup \{(\bar{\theta}, \bar{f})\}) \setminus \{(\theta_j, f_j) \in \mathcal{F} \mid \bar{\theta} \leq \theta_j, \bar{f} \leq f_j\}$$

Note that a two dimensional vector is acceptable to  $\text{Add}((\bar{\theta}, \bar{f}), \mathcal{F})$  if and only if it is acceptable for  $\mathcal{F} \cup \{(\bar{\theta}, \bar{f})\}$ . Also, the acceptability is in this way considered uses smaller filters because all the *dominated* pairs are removed.

Let us now fix the parameters of our filter version of the SSDP algorithm as follows:  $\beta \in (0, 1)$ ,  $\gamma \in (0, \beta)$ ,  $\bar{\rho} > 0$ ,  $\sigma \in (0, 1)$  and  $u > 0$ .

### Filter-SDP Algorithm

Step 0 Define  $k = 1$ ,  $\mathcal{F}^0 = \{(u, -\infty)\}$ .

Step 1 Find some vector  $x_k$  and a corresponding trust region radius  $\rho > \bar{\rho}$  such that

(A1)  $(\theta(x_k), f(x_k))$  is acceptable to  $\mathcal{F}^{k-1}$

(B1)  $QP(x_k, \rho)$  is feasible.

Go to Step 3.

Step 2 If  $QP(x_k, \rho)$  is not feasible then put  $\mathcal{F}^k = Add((\theta(x_k), f(x_k)), \mathcal{F}^{k-1})$ ,  $k = k + 1$  and go to Step 1.

Step 3 If  $d = 0$  is a critical solution of  $QP(x_k, \rho)$  then STOP. Otherwise, fix an optimal solution  $d \neq 0$  of  $QP(x_k, \rho)$ .

Step 4 If  $(\theta(x_k + d), f(x_k + d))$  is not acceptable to the filter  $\mathcal{F}^{k-1} \cup \{(\theta(x_k), f(x_k))\}$  then put  $\rho = \rho/2$  and go to Step 2.

Step 5 If the following conditions are fulfilled

$$\nabla f(x_k)d + \frac{1}{2}d^\top Bd < 0, \quad (14a)$$

$$f(x_k) + \sigma(\nabla f(x_k)d + \frac{1}{2}d^\top Bd) < f(x_k + d), \quad (14b)$$

then put  $\rho = \rho/2$  and go to Step 2.

Step 6 If  $\nabla f(x_k)d + \frac{1}{2}d^\top Bd \geq 0$  then  $\mathcal{F}^k = Add((\theta(x_k), f(x_k)), \mathcal{F}^{k-1})$ , otherwise set  $\mathcal{F}^k = \mathcal{F}^{k-1}$ .

Step 7 Define  $\rho_k = \rho$ ,  $d_k = d$ . Put  $x_{k+1} = x_k + d_k$ ,  $k = k + 1$ . Finally, initialize  $\rho \geq \bar{\rho}$  and go to Step 2.

In what follows, Step 1 is called *restoration phase* and the loop between the Steps 2 and 5 is defined as the *inner loop*.

**Remark 4.** At Step 3 we have supposed that we can solve exactly the auxiliary problem  $QP(x_k, \rho)$ . This is an acceptable assumption when, for instance,  $B$  is taken positive semidefinite.

In the above algorithm the iterates  $x_k$  are generated at Steps 2 and 7. Once Step 7 is achieved and a new iterate  $x_{k+1}$  is computed, the filter remains the same only if the following condition holds:

$$f(x_k) - f(x_{k+1}) \geq -\sigma(\nabla f(x_k)d + \frac{1}{2}d^\top Bd) > 0. \quad (15)$$

In this case  $x_k$  is called an *f-type iteration*, because the above condition guarantees an adequate improvement of the objective function  $f$ . Otherwise, when  $(\theta(x_k), f(x_k))$  is entered to the filter, either at Step 2 or at Step 6, we say that a *θ-type iteration*  $x_k$  was accomplished.

**Lemma 5.** Let us suppose that the MFCQ condition is satisfied at every feasible point of (P). If  $\mathcal{F}^k$  is a filter generated by the above Algorithm, then

$$\tau^k = \min_{(\theta, f) \in \mathcal{F}^k} \theta > 0.$$

**Proof.** It suffices to show that every point added to the Filter must be infeasible. By contradiction, suppose that  $(\theta(x_k), f(x_k))$  is added to the filter and that  $x_k$  is a feasible point of (P). There are only two steps where a new member can be added to the filter: Steps 2 or 6.

Firstly, the feasibility of  $x_k$  implies that  $d = 0$  is a feasible point of  $QP(x_k, \rho)$  for all  $\rho \geq 0$ . Consequently  $(\theta(x_k), f(x_k))$  could not be added to the filter at step 2.

Secondly, at Step 6 the problem  $QP(x_k, \rho)$  is supposed to be feasible but with a nonnegative optimal value. According to Step 3, vector  $d = 0$  is a feasible but not a critical point of  $QP(x_k, \rho)$ . Now since at  $d = 0$  the norm inequality is not active, the KKT conditions for (P) at  $x_k$  and for  $QP(x_k, \rho)$  at  $d = 0$  coincide. We thus deduce that  $x_k$  is not a critical point of (P). Then Lemma 3 implies the existence of a nonzero vector  $s \in \mathbb{R}^n$  such that for all  $t \in (0, 1)$

$$\begin{aligned}\nabla f(x_k)s &< 0, \\ h(x_k) + Dh(x_k)ts &= 0, \\ G(x_k) + DG(x_k)ts &\preceq 0.\end{aligned}$$

Therefore  $s$  is a local feasible descent direction of  $QP(x_k, \rho)$  at  $d = 0$ . This contradicts the fact that the optimal value of this auxiliary problem is nonnegative. ■

**Remark 6.** The above lemma ensures that Step 1 is always realizable when the original problem (P) is feasible.

The following two lemmas concerning sequences generated by the Filter can be found in [16, pp. 47–48]. Both results are independent to the SDP constraint appearing in (P). Nevertheless, we present for the sake of completeness a slightly different proof of the second result.

**Lemma 7.** Consider a sequence  $\{(\theta_k, f_k)\}_{k=1}^\infty$  such that  $\theta_k \geq 0$ ,  $\{f_k\}_{k=1}^\infty$  is decreasing and bounded below, and  $(\theta_{k+1}, f_{k+1})$  is acceptable to  $(\theta_k, f_k)$  for all  $k$ . Then  $\theta_k$  converges to zero.

**Lemma 8.** Consider a sequence of iterations  $\{(\theta_k, f_k)\}_{k=1}^\infty$  entered into the filter in the above algorithm and such that  $\{f_k\}_{k=1}^\infty$  is bounded below. It follows that  $\theta_k$  converges to zero.

**Proof.** Since the initial filter is given by  $\mathcal{F}^0 = \{(u, -\infty)\}$ , Lemma 5 implies that  $\{\theta_k\}_{k=1}^\infty \subset (0, u]$ . If we now suppose that  $\{\theta_k\}_{k=1}^\infty$  does not converge to zero, then there must exist a minimal index  $\bar{l} \geq 0$  such that

$$\text{card}(\{k \in \mathbb{N} \mid \beta^{\bar{l}+1}u < \theta_k \leq \beta^{\bar{l}}u\}) = \infty, \quad (16a)$$

$$\text{card}(\{k \in \mathbb{N} \mid \beta^{\bar{l}+1}u < \theta_k \leq \beta^l u\}) < \infty, \quad \forall 0 \leq l < \bar{l}, \quad (16b)$$

where  $\text{card}(S)$  denotes the cardinality of a given set  $S$ . Note that  $(\beta^{\bar{l}+1}u, \beta^{\bar{l}}u]$  contains a subsequence of  $\{\theta_k\}_{k=1}^\infty$  converging to  $\limsup_k \theta_k$ . As a consequence we can find an index  $\bar{k} > 0$  such that

$$\theta_k \leq \beta^{\bar{l}}u, \quad \forall k \geq \bar{k}, \quad (17a)$$

$$\text{card}(\{k \in \mathbb{N} \mid k \geq \bar{k}, \beta^{\bar{l}+1}u < \theta_k \leq \beta^{\bar{l}}u\}) = \infty. \quad (17b)$$

We proceed to define recursively the following subsequence  $\{\theta_{k(i)}\}_{i \in \mathbb{N}}$ . First  $\theta_{k(1)} = \theta_{\bar{k}}$ . Then, given  $\theta_{k(i)}$  we define  $\theta_{k(i+1)}$  by taking

$$k(i+1) = \min\{k > k(i) \mid \theta_k > \beta\theta_{k(i)}\}.$$

Note that  $k(i+1) > k(i)$  is well defined due to (17). It follows also from the above definition that

$$\theta_k \leq \beta\theta_{k(i)}, \quad \forall k(i) < k < k(i+1). \quad (18)$$

Since all iterates are entered to the filter, it follows

$$f(x_{k(i)}) - f(x_{k(i+1)}) \geq \gamma \theta_{k(i+1)} > 0, \quad \forall i = 1, 2, \dots$$

This implies that  $\{f(x_{k(i)})\}_{i \in \mathbb{N}}$  is decreasing, and taking into account that  $\{f_k\}_{k=1}^\infty$  is bounded below it is then obtained  $\sum_{i=1}^\infty \theta_{k(i)} < \infty$ . Obtaining that  $\theta_{k(i)} \rightarrow 0$ . The convergence of  $\theta_k$  to zero follows then finally from (18).  $\blacksquare$

## 4 Convergence Results

In the reminder of this article we assume the following hypotheses:

- (i) The points sampled by the algorithm lie in a nonempty compact set  $X \subset \mathbb{R}^n$ .
- (ii) The MFCQ condition is satisfied at each feasible point of (P) lying into the set  $X$ .
- (iii) There exists a positive constant  $M > 0$  such that  $\|B^k\|_2 \leq M$  for all  $k$ .

This set of hypotheses will be called (H) in the sequel. Without loss of generality we suppose that the quantities  $\|\nabla f(x)\|_2$ ,  $\|D^2 f(x)\|_2$ ,  $\|Dh_i(x)\|_2$ ,  $\|D^2 h_i(x)\|_2$ ,  $\|DG(x)\|_{Fr}$  and  $\|D^2 G(x)\|_{Fr}$  are all bounded by  $M$  on a sufficiently large compact convex set containing  $X$ . In what follows we just denote  $B = B^k$  avoiding the index notation.

The first lemma of this section provides useful bounds for the variation of the feasibility and the objective functions, when feasible points of the local approximated problem are considered.

**Lemma 9.** *Let the above assumptions (H) be fulfilled. If  $d$  is a feasible point of  $QP(x_k, \rho)$ , then the following inequalities hold true.*

$$f(x_k + d) - (f(x_k) + \nabla f(x)d + \frac{1}{2}d^\top Bd) \leq n\rho^2 M, \quad (19a)$$

$$|h_i(x_k + d)| \leq \frac{1}{2}n\rho^2 M, \quad i = 1, \dots, p \quad (19b)$$

$$G(x_k + d) \preceq \frac{1}{2}(n\rho^2 M)I \quad (19c)$$

**Proof.** The first inequality is a direct consequence of a second order approximation of  $f$  at  $x_k$

$$f(x_k + d) = f(x_k) + \nabla f(x)d + \frac{1}{2}d^\top D^2 f(x_k + \xi d)d,$$

where  $\xi \in (0, 1)$ . Indeed,

$$\begin{aligned} f(x_k + d) - (f(x_k) + \nabla f(x)d + \frac{1}{2}d^\top Bd) &= \frac{1}{2}d^\top (D^2 f(x_k + \xi d) - B)d \\ &\leq \frac{1}{2}\|d\|_2^2 \|D^2 f(x_k + \xi d) - B\|_2 \\ &\leq \|d\|_2^2 M \leq n\|d\|_\infty^2 M \leq n\rho^2 M. \end{aligned}$$

The second inequality follows in the same way, since

$$|h_i(x_k + d)| = |\frac{1}{2}d^\top D^2 h_i(x_k + \xi d)d| \leq \frac{1}{2}\|d\|_2^2 \|D^2 h_i(x_k + \xi d)\|_2 \leq \frac{1}{2}n\rho^2 M.$$

The same idea used for  $G$  leads to

$$G(x_k + d) = G(x_k) + DG(x_k)d + \frac{1}{2}d^\top D^2 G(x_k + \xi d)d \preceq \frac{1}{2}d^\top D^2 G(x_k + \xi d)d.$$

Using now (9) we obtain

$$\lambda_1(d^\top D^2G(x_k + \xi d)d) \leq \|d\|_2^2 \|D^2G(x_k + \xi d)\|_{Fr} \leq n\rho^2 M.$$

This is equivalent to the condition

$$D^2G(x_k + \xi d) - (n\rho^2 M)I \preceq 0,$$

that trivially implies (19c).  $\blacksquare$

The next result gives a condition on  $\rho$  that ensures the acceptability of new iterates for previous filters as required in Step 4.

**Lemma 10.** *Let the assumptions (H) be satisfied. If  $\rho^2 \leq \frac{2\beta\tau^{k-1}}{(p+1)nM}$  and  $d$  is a feasible point of  $QP(x_k, \rho)$ , then  $(\theta(x_k + d), f(x_k + d))$  is acceptable to the filter  $\mathcal{F}^{k-1}$ .*

**Proof.** It suffices to prove the inequality

$$\theta(x_k + d) \leq \beta\tau^{k-1}. \quad (20)$$

In virtue of the definition of  $\theta$  and inequalities (19b) and (19c), it follows that

$$\theta(x_k + d) = \|h(x_k + d)\|_2 + \lambda_1(G(x_k + d))_+ \leq (p+1)\frac{1}{2}n\rho^2 M.$$

Inequality (20) is now straightforwardly obtained from the assumption  $\rho^2 \leq \frac{2\beta\tau^{k-1}}{(p+1)nM}$ .  $\blacksquare$

Now we proceed to extend the main lemma of [16].

**Lemma 11.** *Let us assume the set of hypotheses (H) concerning the problem (P). Let us further suppose  $\bar{x}$  to be a feasible point that is not a critical one. Then there exist a neighborhood  $\mathcal{N}$  of  $\bar{x}$  and strictly positive constants  $\varepsilon$ ,  $\mu$ , and  $\kappa$  such that if  $x \in \mathcal{N} \cap X$ ,  $\rho > 0$  and*

$$\mu\theta(x) \leq \rho \leq \kappa, \quad (21)$$

*then there exists a feasible point  $d$  of the problem  $QP(x, \rho)$  satisfying the following inequalities*

$$f(x + d) + \gamma\theta(x + d) \leq f(x), \quad (22a)$$

$$f(x + d) - \sigma(\nabla f(x)d + \frac{1}{2}d^\top Bd) \leq f(x), \quad (22b)$$

$$\nabla f(x)d + \frac{1}{2}d^\top Bd < -\frac{1}{3}\rho\epsilon. \quad (22c)$$

*Moreover, the above inequalities hold also true at any optimal solution of  $QP(x, \rho)$ .*

**Proof.** Due to hypotheses (H) the MFCQ condition holds at the feasible point  $\bar{x}$ . Then, since  $\bar{x}$  is not a critical point, it follows from Lemma 3 the existence of a unitary vector  $\bar{s}$  and a real number  $\eta > 0$  such that conditions (10) are satisfied.

Let  $x \in \mathcal{N}$ , with  $\mathcal{N}$  a neighborhood of  $\bar{x}$ . Denote by  $P_x(y)$  the orthogonal projection of a vector  $y \in \mathbb{R}^n$  into  $\text{Ker } Dh(x)$ . Notice that, since  $Dh(\cdot)$  is continuous, vectors  $\nabla h_j(x)$ , with  $j = 1, \dots, p$ , are linear independent when  $x$  is close enough to  $\bar{x}$ . Thus, we can write  $P_x(y) = (I - A_x^\dagger A_x)y$  for all  $y$ , where  $A_x := Dh(x)$  and  $A^\dagger$  denotes the Moore-Penrose inverse of the matrix  $A$ . Recall that when the matrix  $A$  is surjective it is shown that  $A^\dagger = A^\top(AA^\top)^{-1}$ .

Denote by  $q = q(x) := -A_x^\dagger h(x)$  and  $\phi = \phi(x) := \|q\|_2$ . In order to directly deal with conditions related to equality constraints  $h(x) = 0$ , we project the vector  $\bar{s}$  onto  $\text{Ker } Dh(x)$ , and then we normalize it. Obtaining then  $s = s(x) := P_x(\bar{s})/\|P_x(\bar{s})\|_2$ . Note that  $s$  depends

continuously on  $x \in \mathcal{N}$ , and  $s = \bar{s}$  when  $x = \bar{x}$ . Thus, it follows from conditions (10a) and (10c) that there exist a constant  $\varepsilon > 0$  and a (smaller) bounded neighborhood  $\mathcal{N}$  of  $\bar{x}$  such that

$$\nabla f(x)s < -\varepsilon \quad \text{and} \quad G(x) + \eta DG(x)s \prec -\varepsilon I, \quad \forall x \in \mathcal{N}. \quad (23)$$

Let us now fix some  $\kappa > 0$  satisfying the following inequality

$$\kappa < \min \left\{ \eta, \frac{\varepsilon}{M}, \frac{1}{3} \frac{(1-\sigma)\varepsilon}{nM}, \frac{2}{3} \frac{\sigma\varepsilon}{\gamma(p+1)nM} \right\}. \quad (24)$$

The continuity properties of functions  $f$ ,  $h$ , and  $G$  imply the existence of a constant  $C > 0$  such that the following three inequalities hold on the (eventually reduced) neighborhood  $\mathcal{N}$ :

$$\phi \leq C\theta(x), \quad \lambda_1(DG(x)q) \leq C\theta(x), \quad \text{and} \quad (\|\nabla f(x)\|_2 + \varepsilon + \frac{1}{2}M\phi + \kappa M)\phi \leq C\theta(x), \quad \forall x \in \mathcal{N}. \quad (25)$$

Let us finally fix some  $\mu > 0$  so that

$$\mu > \max \left\{ (1+C)(1+\eta/\varepsilon), \frac{6C}{\varepsilon} \right\}. \quad (26)$$

In what follows, we consider a fixed value  $\rho > 0$  satisfying (21). Note that such a  $\rho$  exists, since the neighborhood  $\mathcal{N}$  can be taken small enough in order to ensure the inequality

$$\mu\theta(x) < \kappa, \quad \forall x \in \mathcal{N}.$$

Inequality (26) in particular implies that  $\mu > C$ , which together with (21) yields to

$$\phi < \rho \quad \text{and} \quad 0 < \frac{\rho - \phi}{\eta} < 1. \quad (27)$$

We proceed to prove that  $d_1 = q + (\rho - \phi)s$  is a feasible vector of problem  $QP(x, \rho)$ . Due to the definition of  $q$  and  $s$  the equality constraint  $h(x) + Dh(x)d_1 = 0$  is satisfied. Also, since  $q$  and  $s$  are orthogonal, we get

$$\|d_1\|_2 = \sqrt{\phi^2 + (\rho - \phi)^2} = \sqrt{\rho^2 - 2\rho\phi + 2\phi^2} \leq \rho.$$

Hence,  $d_1$  satisfies the trust region constraint  $\|d_1\|_\infty \leq \rho$  of  $QP(x, \rho)$ . It is just left to verify the inequality constraint  $G(x) + DG(x)d_1 \preceq 0$ . Condition (23) and the first inequality in (27) imply

$$\begin{aligned} G(x) + DG(x)d_1 &= G(x) + DG(x)q + (\rho - \phi)DG(x)s \\ &= \left(1 - \frac{\rho - \phi}{\eta}\right)G(x) + DG(x)q + \frac{\rho - \phi}{\eta}[G(x) + \eta DG(x)s] \\ &\preceq \left(1 - \frac{\rho - \phi}{\eta}\right)G(x) + DG(x)q - \varepsilon \left(\frac{\rho - \phi}{\eta}\right)I. \end{aligned}$$

By applying the largest eigenvalue function  $\lambda_1(\cdot)$  to this last matrix inequality, Weyl's theorem (e.g. [18]), and by using inequalities (27) and (25), respectively, we obtain

$$\begin{aligned}
\lambda_1(G(x) + DG(x)d_1) &\leq \lambda_1\left(\left(1 - \frac{\rho - \phi}{\eta}\right)G(x) + DG(x)q\right) - \varepsilon\left(\frac{\rho - \phi}{\eta}\right) \\
&\leq \left(1 - \frac{\rho - \phi}{\eta}\right)\lambda_1(G(x)) + \lambda_1(DG(x)q) - \varepsilon\left(\frac{\rho - \phi}{\eta}\right) \\
&\leq \left(1 - \frac{\rho - \phi}{\eta}\right)\lambda_1(G(x))_+ + \lambda_1(DG(x)q) - \varepsilon\left(\frac{\rho - \phi}{\eta}\right) \\
&\leq \lambda_1(G(x))_+ + \lambda_1(DG(x)q) - \varepsilon\left(\frac{\rho - \phi}{\eta}\right) \\
&\leq (1 + C)\theta(x) + \frac{\varepsilon}{\eta}\phi - \frac{\varepsilon}{\eta}\rho \leq (1 + C + \frac{\varepsilon}{\eta}C)\theta(x) - \frac{\varepsilon}{\eta}\rho \\
&\leq \frac{\varepsilon}{\eta}\mu\theta(x) - \frac{\varepsilon}{\eta}\rho \leq 0.
\end{aligned}$$

The last two inequalities follow directly from (26) and (21), respectively. We thus conclude that  $d_1$  is feasible for problem  $QP(x, \rho)$ .

It is left to prove that inequalities (22) hold at  $d = d_1$ . Let us start by showing (22c). By definition of  $d_1$  we have

$$\nabla f(x)d_1 + \frac{1}{2}d_1^\top Bd_1 = \nabla f(x)q + (\rho - \phi)\nabla f(x)s + \frac{1}{2}q^\top Bq + (\rho - \phi)q^\top Bs + \frac{1}{2}(\rho - \phi)^2 s^\top Bs.$$

Thanks to the upper bounds given in (H), and inequalities (23) and (27), it follows that

$$\nabla f(x)d_1 + \frac{1}{2}d_1^\top Bd_1 < \|\nabla f(x)\|_2\phi - (\rho - \phi)\varepsilon + \frac{1}{2}M\phi^2 + \rho M\phi + \frac{1}{2}\rho^2 M.$$

Using (21), (24), (25) and (26) leads then to

$$\begin{aligned}
\nabla f(x)d_1 + \frac{1}{2}d_1^\top Bd_1 &< \|\nabla f(x)\|_2\phi - \rho\varepsilon + \phi\varepsilon + \frac{1}{2}M\phi^2 + \kappa M\phi + \frac{1}{2}\rho\varepsilon \\
&\leq -\frac{1}{2}\rho\varepsilon + C\theta(x) \\
&\leq -\frac{1}{2}\rho\varepsilon + \frac{1}{6}\varepsilon\rho = -\frac{1}{3}\rho\varepsilon.
\end{aligned}$$

The relation (22c) is then proved. Let us finally show (22a) and (22b) for any feasible point  $d$  of  $QP(x, \rho)$  satisfying the inequalities

$$\nabla f(x)d + \frac{1}{2}d^\top Bd \leq \nabla f(x)d_1 + \frac{1}{2}d_1^\top Bd_1 < -\frac{1}{3}\rho\varepsilon, \quad (28)$$

which in particular implies (22c) for  $d$ . This analysis obviously includes the point  $d_1$  and any optimal solution of  $QP(x, \rho)$ .

The relation (28) together with inequality (19a) of Lemma 9 provide

$$\begin{aligned}
f(x + d) - f(x) - \sigma(\nabla f(x)d + \frac{1}{2}d^\top Bd) &\leq n\rho^2 M + (1 - \sigma)(\nabla f(x)d_1 + \frac{1}{2}d_1^\top Bd_1) \\
&\leq n\rho^2 M - (1 - \sigma)\frac{1}{3}\rho\varepsilon.
\end{aligned}$$

Applying (21) and (24) to the right hand side of the above inequality leads easily to (22b).

In order to prove (22a) note that the already proved relations (22b) and (28) imply

$$f(x+d) - f(x) \leq -\sigma \frac{1}{3} \rho \varepsilon.$$

On the other hand, due to (19b) and (19c) of Lemma 9 we have

$$\gamma \theta(x+d) \leq \gamma(p+1) \frac{1}{2} n \rho^2 M.$$

Adding up both inequalities, it is obtained

$$f(x+d) - f(x) + \gamma \theta(x+d) \leq \rho \left( -\sigma \frac{1}{3} \varepsilon + \rho \gamma(p+1) \frac{1}{2} n M \right).$$

Then (22a) follows easily from (21) and (24).  $\blacksquare$

**Corollary 12.** *Let us consider an iteration point  $x_k$  of the algorithm lying in the neighborhood  $\mathcal{N}$  defined in Lemma 11 above. If the inequality*

$$\mu \theta(x_k) < \frac{1}{2} \min \left\{ \sqrt{\frac{2\beta\tau^{k-1}}{(p+1)nM}}, \bar{\rho}, \kappa \right\} \quad (29)$$

*is satisfied, then  $x_k$  is an iteration of  $f$ -type satisfying  $\rho_k \geq \frac{1}{2} \min \left\{ \sqrt{\frac{2\beta\tau^{k-1}}{(p+1)nM}}, \bar{\rho}, \kappa \right\}$ .*

**Proof.** In the algorithm a sequence of auxiliary problem  $QP(x_k, \rho_l)$  is solved for the decreasing values  $\rho_l = \frac{\rho_0}{2^l}$ , where  $\rho_0 > \bar{\rho}$ . By inequality (29) there must be an index  $\bar{l}$  such that

$$\mu \theta(x_k) < \frac{1}{2} \min \left\{ \sqrt{\frac{2\beta\tau^{k-1}}{(p+1)nM}}, \bar{\rho}, \kappa \right\} \leq \rho_{\bar{l}} < \min \left\{ \sqrt{\frac{2\beta\tau^{k-1}}{(p+1)nM}}, \bar{\rho}, \kappa \right\}. \quad (30)$$

By definition,  $x_k$  is a  $\theta$ -type iteration if the inner loop ends adding a new point to the filter at Steps 2 or 6. Otherwise, the inner loop finishes at Step 7 with the same filter and  $x_k$  is of  $f$ -type. It suffices then to show twofold. 1) First, that for the values  $\rho_l > \rho_{\bar{l}}$  the loop cannot end defining  $x_k$  as of  $\theta$ -type. 2) Second, if the value  $\rho_{\bar{l}}$  is achieved, then the inner loop ends with  $x_k$  as  $f$ -type iteration.

1) We proceed to prove the first case. From the definition of  $QP(x_k, \rho)$ , it is obvious that, when  $\rho$  decreases to  $\rho_{\bar{l}}$ , the feasible set of  $QP(x_k, \rho)$  shrinks and its optimal value (denoted by  $\text{val}(QP(x_k, \rho))$ ) increases. Since  $QP(x_k, \rho_{\bar{l}})$  is feasible (which is a consequence of Lemma 11 and (30)), it follows that the inner loop will not stop at Step 2 for any  $\rho_l > \rho_{\bar{l}}$ .

Note that the solution selected at Step 3, denoted by  $d^l$ , provides also the optimal value  $\text{val}(QP(x_k, \rho_l)) = \nabla f(x_k) d^l + \frac{1}{2} d^l \top B d^l$ .

As mentioned before, the sequence  $\text{val}(QP(x_k, \rho_l))$  increases when  $\rho_l$  decreases to  $\rho_{\bar{l}}$ . Taking now into account (30) we can conclude from (22c) in Lemma 11, that  $\text{val}(QP(x_k, \rho_l)) < 0$ . Consequently the filter will not be enlarged at Step 6 for any  $\rho_l > \rho_{\bar{l}}$ , since in fact this happens only for nonnegative optimal values.

2) Let us finally see that when  $\rho_{\bar{l}}$  is achieved the inner loop stops without enlarging the filter. Since  $d^{\bar{l}}$  is an optimal solution of  $QP(x_k, \rho_{\bar{l}})$ , Lemma 10 together with (22a) and the third inequality in (30) implies the acceptability of  $(\theta(x_k + d^{\bar{l}}), f(x_k + d^{\bar{l}}))$  to the filter  $\mathcal{F}^{k-1} \cup \{(\theta(x_k), f(x_k))\}$  at Step 4. Note also that the inner loop is not restarted at Step 5, due to the condition (22b). Furthermore, (22c) ensures that  $(\theta(x_k), f(x_k))$  does not enter to the filter  $\mathcal{F}^{k-1}$  at Step 6.

Finally, cases 1) and 2) imply that  $\rho_k \geq \rho_{\bar{l}} \geq \frac{1}{2} \min \left\{ \sqrt{\frac{2\beta\tau^{k-1}}{(p+1)nM}}, \bar{\rho}, \kappa \right\}$ .  $\blacksquare$

Let us study in the last partial result the finite termination of the inner loop.

**Lemma 13.** *Let the assumptions (H) be satisfied. Given a fixed iterate  $x_k$ , the inner loop generated between Steps 2 and 5 finishes after finitely many steps.*

**Proof.** If  $x_k$  is a critical point of (P), then  $d = 0$  is also a critical point for  $QP(x_k, \rho)$  and the inner loop stops at Step 3. We argue by contraction, that is, we suppose that  $x_k$  is not a critical point and that the loop goes infinitely. In this case  $\rho$  converges to zero, because it is divided by two at those Steps (4 and 5) where the inner loop restarts.

In the case when  $x_k$  is not feasible, one of the following two inequalities occurs: either  $\lambda_1(G(x_k)) > 0$  or  $\|h(x_k)\|_2 > 0$ . The second one implies that there exists some  $i \in \{1, \dots, p\}$  with  $|h_i(x_k)| > 0$ . Thus, in virtue of the inequality

$$|h_i(x_k) + \nabla h_i(x_k)d| \geq |h_i(x_k)| - \rho \|\nabla h_i(x_k)\|_2,$$

it follows that  $QP(x_k, \rho)$  becomes infeasible when  $\rho$  is sufficiently close to zero. In this case the inner loop is finished at Step 2.

On the other hand, by using Weyl's theorem, and inequalities (7) and (8), it is obtained

$$\begin{aligned} \lambda_1(G(x_k) + DG(x_k)d) &\geq \lambda_1(G(x_k)) + \lambda_n(DG(x_k)d) \\ &\geq \lambda_1(G(x_k)) - \|DG(x_k)d\|_{Fr} \\ &\geq \lambda_1(G(x_k)) - n\rho \|DG(x_k)\|_{Fr}. \end{aligned}$$

Therefore also in the case when  $\lambda_1(G(x_k)) > 0$ , the problem  $QP(x_k, \rho)$  becomes infeasible for any sufficiently small  $\rho$ .

It remains to consider the case of a feasible  $x_k$ . Due to assumptions (H), the MFCQ condition holds at  $x_k$ . From Lemma 3 we obtain the existence of a unitary vector  $\bar{s}$  and a positive number  $\bar{\eta}$  such that the following conditions are satisfied for all  $\eta \in (0, \bar{\eta}]$

$$\nabla f(x_k)\bar{s} < 0, \quad (31a)$$

$$Dh(x_k)\bar{s} = 0, \quad (31b)$$

$$G(x_k) + \eta DG(x_k)\bar{s} \prec 0. \quad (31c)$$

Let us denote  $\delta = -\nabla f(x_k)\bar{s}$ . We now proceed to show that the inner loop finishes for any  $\rho > 0$  satisfying the following relation

$$\rho \leq \min \left\{ \bar{\eta}, \frac{\delta}{M}, \frac{(1-\sigma)\delta}{2nM}, \frac{\sigma\delta}{\gamma(p+1)nM}, \sqrt{\frac{2\beta\tau^{k-1}}{(p+1)nM}} \right\}. \quad (32)$$

Recall that Lemma 5 provides the positivity of  $\tau^{k-1}$ .

Let us define  $\bar{d} = \rho\bar{s}$ . Since  $x_k$  is feasible, it follows from (31b), (31c) and (32) that  $\bar{d}$  is a feasible point of  $QP(x_k, \rho)$ . Let us now estimate the value of the objective function of the auxiliary problem  $QP(x_k, \rho)$  at  $\bar{d}$ . From the assumption (H) we obtain

$$\nabla f(x_k)\bar{d} + \frac{1}{2}\bar{d}^\top B\bar{d} = -\rho\delta + \frac{1}{2}\rho^2\bar{s}^\top B\bar{s} \leq -\rho\delta + \frac{1}{2}\rho^2M.$$

It follows from (32) that

$$\nabla f(x_k)\bar{d} + \frac{1}{2}\bar{d}^\top B\bar{d} \leq -\frac{1}{2}\rho\delta.$$

Now let us suppose that  $d$  is the solution of  $QP(x_k, \rho)$  selected at Step 3. Due to the feasibility of  $\bar{d}$  we get

$$\nabla f(x_k)d + \frac{1}{2}d^\top Bd \leq -\frac{1}{2}\rho\delta < 0. \quad (33)$$

According to Lemma 9, the following inequality holds

$$f(x_k) - f(x_k + d) \geq -(\nabla f(x_k)d + \frac{1}{2}d^\top Bd) - n\rho^2 M.$$

Then, relations (32) and (33) yield to the following inequalities

$$\begin{aligned} f(x_k) - f(x_k + d) &\geq -\sigma(\nabla f(x_k)d + \frac{1}{2}d^\top Bd) + \frac{1}{2}(1-\sigma)\rho\delta - n\rho^2 M \\ &\geq -\sigma(\nabla f(x_k)d + \frac{1}{2}d^\top Bd), \end{aligned}$$

where the parameter  $\sigma \in (0, 1)$  was used in Step 5 of the algorithm. Therefore, we have already deduced that

$$f(x_k) + \sigma(\nabla f(x_k)d + \frac{1}{2}d^\top Bd) \geq f(x_k + d), \quad (34a)$$

$$f(x_k) - f(x_k + d) \geq \sigma \frac{1}{2}\rho\delta. \quad (34b)$$

Consequently, (34a) implies that the inner loop is not restarted at Step 5.

It remains only to prove that the algorithm goes from Step 4 to Step 5 (i.e. the inner loop does not restart at Step 4). This situation happens when  $(\theta(x_k + d), f(x_k + d))$  is acceptable to the filter  $\mathcal{F}^{k-1} \cup \{(\theta(x_k), f(x_k))\}$ . By Lemma 10 and condition (32), it follows that  $(\theta(x_k + d), f(x_k + d))$  is acceptable for  $\mathcal{F}^{k-1}$ . On the other hand, it can be seen from Lemma 9 that

$$\theta(x_k + d) \leq p(\frac{1}{2}n\rho^2 M) + \frac{1}{2}n\rho^2 M = \frac{1}{2}(p+1)n\rho^2 M.$$

This together with (34b) leads to

$$f(x_k) - f(x_k + d) - \gamma\theta(x_k + d) \geq \sigma \frac{1}{2}\rho\delta - \gamma \frac{1}{2}(p+1)n\rho^2 M.$$

The right hand side of the last expression is positive due to the definition of  $\rho$  in (32), obtaining that  $f(x_k + d) + \gamma\theta(x_k + d) \leq f(x_k)$ . This actually means that  $(\theta(x_k + d), f(x_k + d))$  is also acceptable for  $\{(\theta(x_k), f(x_k))\}$ , which completes the proof. ■

We can now state and proof the global convergence theorem for the algorithm proposed.

**Theorem 14.** Suppose that assumptions (H) holds true. Consider the sequence  $\{x_k\}$  generated by the Filter-SDP algorithm defined above. Then one of the following situations occurs:

1. The restoration phase (Step 1) fails to find a vector  $x_k$  satisfying (A1) and (B1).
2. A critical point of (P) is found, that is,  $d = 0$  solves the auxiliary problem  $QP(x_k, \rho)$  for some iteration  $k$ .
3. There exists an accumulation point of  $\{x_k\}$  that is a critical point of (P).

**Proof.** We suppose that situations 1. and 2. do not happen, and proceed to prove that situation 3. occur.

Due to hypothesis (i) in (H) the sequence  $\{x_k\}_{k=1}^\infty$  sampled by the algorithm lies in a compact set  $X \subset \mathbb{R}^n$  and has therefore at least one accumulation point.

Firstly, consider the case when the algorithm generates infinitely many  $\theta$ -type iterations. Let us denote by  $\{x_{k(j)}^\theta\}_{j=1}^\infty$  the respective subsequence of  $\theta$ -type iterates. From Lemma 8 it follows  $\theta(x_{k(j)}^\theta) \rightarrow 0$ , and consequently  $\tau^k \rightarrow 0$ . Since the decreasing sequence  $\tau^k$  changes only at iterations of  $\theta$ -type, we can suppose passing to a subsequence if necessary that for all  $j$

$$\theta(x_{k(j)}^\theta) = \tau^{k(j)} < \tau^{k(j)-1}, \quad (35)$$

and that  $\{x_{k(j)}^\theta\}$  converges to a feasible point  $\bar{x}$ . Let us argue by contradiction and assume that  $\bar{x}$  is not a critical point of (P).

From hypothesis (ii) in (H) we know that MFCQ is satisfied at  $\bar{x}$ . Hence, Lemma 11 can be applied, obtaining the existence of the corresponding neighborhood  $\mathcal{N}$  of  $\bar{x}$ , and of the strictly positive constants  $\varepsilon$ ,  $\mu$ , and  $\kappa$ . Note that the iterate  $x_{k(j)}^\theta$  will belong to  $\mathcal{N}$  for sufficiently large  $j$ . It suffices then to show that the condition

$$\mu\theta(x_{k(j)}^\theta) < \frac{1}{2} \min \left\{ \sqrt{\frac{2\beta\tau^{k(j)-1}}{(p+1)nM}}, \bar{\rho}, \kappa \right\} \quad (36)$$

is also satisfied for sufficiently large  $j$ . Indeed, condition (36) and Corollary 12 imply that  $x_{k(j)}^\theta$  is an iteration of  $f$ -type, which is a contradiction.

Since  $\theta(x_{k(j)}^\theta)$  and  $\tau^{k(j)-1}$  converge both to zero, for all  $j$  large enough, condition (36) actually reduces to the following one

$$\theta(x_{k(j)}^\theta) < \frac{1}{2\mu} \sqrt{\frac{2\beta\tau^{k(j)-1}}{(p+1)nM}}. \quad (37)$$

We thus conclude by noting that (37) is obtained from (35) when  $j$  is taken large enough such that  $\tau^{k(j)-1} \leq \frac{\beta}{2\mu^2(p+1)nM}$ . This proves the desired result for the first case.

Now, consider the case when only a finite number of  $\theta$ -type iterations happen. Therefore, there exists an index  $K$  so that all iterations  $k \geq K$  are of  $f$ -type.

In particular, it follows from (15) that  $\{f(x_k)\}_{k \geq K}$  is strictly decreasing. This together with the fact that  $(\theta(x_{k+1}), f(x_{k+1}))$  is acceptable to  $(\theta(x_k), f(x_k))$  (cf. Step 4) implies, via Lemma 7, that  $\theta(x_k) \rightarrow 0$ . Let us suppose, without loss of generality and arguing by contradiction, that  $\{x_k\}$  converges to a feasible vector  $\bar{x}$ , which is not a critical point of (P).

Analogically to the first case, hypotheses of Lemma 11 hold, providing the existence of the corresponding neighborhood  $\mathcal{N}$  of  $\bar{x}$ , and suitable constants  $\varepsilon$ ,  $\mu$ , and  $\kappa$ . Since  $x_k \rightarrow \bar{x}$ , it can be assumed that  $x_k \in \mathcal{N}$  for all  $k \geq K$ .

By definition of an  $f$ -type iteration, no filter updates are made for  $k \geq K$ . Consequently,  $\tau^k = \tau^K$  is constant, for all  $k \geq K$ . Notice also that condition (29) is satisfied for sufficiently large  $k$ , since its right hand side is constant and  $\theta(x_k)$  converges to zero. Then, due to Corollary 12, it holds that  $\rho_k \geq \hat{\rho} = \frac{1}{2} \min \left\{ \sqrt{\frac{2\beta\tau^K}{(p+1)nM}}, \bar{\rho}, \kappa \right\}$ , for all  $k \geq K$ .

By using (15) and the decreasing monotonicity on  $\rho$  of  $\text{val}(QP(x, \rho))$ , it yields to

$$f(x_k) - f(x_{k+1}) \geq -\sigma(\nabla f(x_k)d_k + \frac{1}{2}d_k^\top Bd_k) = -\sigma \text{val}(QP(x_k, \rho_k)) \geq -\sigma \text{val}(QP(x_k, \hat{\rho})). \quad (38)$$

Taking now into account that  $\hat{\rho}$  satisfies (21) and using (22c) of Lemma 11, we obtain for all  $k$  large enough,

$$f(x_k) - f(x_{k+1}) \geq -\sigma \text{val}(QP(x_k, \hat{\rho})) = -\sigma(\nabla f(x_k)\hat{d} + \frac{1}{2}\hat{d}^\top B\hat{d}) \geq \frac{1}{3}\sigma\varepsilon\hat{\rho}, \quad (39)$$

where  $\hat{d}$  is some optimal solution of  $QP(x_k, \hat{\rho})$ .

On the other hand, since  $\{f(x_k)\}_{k \geq K}$  is strictly decreasing, and since  $\{x_k\}$  lies in a compact  $X$ , it follows that  $f(x_k)$  converges. This is a contradiction because the right hand side of (39) is a strictly positive constant. The theorem is concluded.  $\blacksquare$

**Remark 15.** As it was pointed out in Ulbrich [47, Remark 4], our global convergence result stated above remains true if, at Step 3, we only determine the local solution  $d$  of  $QP(x_k, d)$  closest to  $x_k$ , instead of a global one. In particular, this gives another possibility to avoid a positive semidefiniteness assumption on  $B$  (see Remark 4).

## 5 Numerical Experiments

A MATLAB® code (version 6.0) was written for the filter algorithm presented in Section 3. We used Jos Sturm' SeDuMi code (cf. [44]) for testing the feasibility of problem QP and for solving it at Steps 1 and 3 of the algorithm. The link between the MATLAB® code and the SeDuMi was provided by the parser YALMIP (see [35]).

The restoration phase described at Step 1 has been implemented in a very simple way. It is just tried to obtain an (almost) feasible point by minimizing the nondifferentiable merit function  $\theta$ . The routine employed for this purpose neither use derivative information nor exploit in any way the structure of the problems solved. Consequently in this implementation the restoration step is not supposed to be very reliable. We expect to improve this issue in future works.

In order to make a preliminary test of the algorithm we selected some examples of the publicly available benchmark collection COMPlieib [30]. For details about the library we refer to [31, 32]. With the data contained in COMPlieib it is possible to construct particular nonlinear semidefinite optimization problems arising in feedback control design (see [30]). We considered in our numerical tests only the basic *Static (or reduced order) Output Feedback*, SOF- $H_2$  problem. The reader can find more details on the motivation of this problem, for instance, in [21, 30, 33, 34].

The following two NLSDP formulations of the SOF- $H_2$  problem were considered (see, for instance, [30, 32]).

$$\min_{F, L} \{Tr(LQ_F); A_F L + LA_F^T + P = 0, A_F L + LA_F^T \prec 0, L \succ 0\} \quad (\text{SOFP1})$$

$$\min_{F, L, V} \{Tr(LB_1 B_1^T); A_F^T L + LA_F + C_F^T C_F = 0, A_F^T V + VA_F \prec 0, V \succ 0\} \quad (\text{SOFP2})$$

Here  $A_F = A + BFC$ ,  $Q_F = C^T F^T RFC + Q$  and  $C_F = C_1 + D_{12}FC$ . The matrices  $A, B_1, B, C_1, C$  and  $D_{12}$  are extracted from COMPlieib. The values for  $P, Q, R$  were taken from the examples in [30, 33, 34]. All the matrices involved are real and of the appropriate dimension. The variables  $L, V$  are symmetric and real, but the matrix  $F$  of the SOF control law is in general not square.

The first data set used was one corresponding to a reactor model introduced in [21]. It is called REA1 in the COMPlieib library. The exactly values for the matrices involved can be found in [30, 33, 34] and [31]. The REA1 data were used in [33, 34] to solve the problem SFOP1. The following starting point  $(F_0, L_0)$  and solution  $(F_*, L_*)$ , together with some values for  $R, P, Q$ , were also suggested in [33, 34].

$$F_0 = \begin{bmatrix} 1.3473 & -1.7793 & 0.1184 \\ 2.9842 & 1.1992 & -0.3040 \end{bmatrix}, \quad L_0 = \begin{bmatrix} 0.9557 & 0.1160 & -0.1798 & 0.0900 \\ 0.1160 & 0.9673 & -0.0742 & 0.0985 \\ -0.1798 & -0.0742 & 1.0771 & 0.2301 \\ 0.0900 & 0.0985 & 0.2301 & 0.6671 \end{bmatrix}, \quad (40)$$

$$F_* = \begin{bmatrix} 0.7434 & -2.0831 & 0.3276 \\ 2.3677 & 0.4788 & -1.0977 \end{bmatrix}, \quad L_* = \begin{bmatrix} 0.4492 & 0.0420 & -0.1244 & 0.0502 \\ 0.0420 & 0.0415 & 0.0091 & 0.0073 \\ -0.1244 & 0.0091 & 0.4762 & 0.3805 \\ 0.0502 & 0.0073 & 0.3805 & 0.4615 \end{bmatrix}.$$

Due to the fact that the algorithm in [33] required an interior starting point regarding the SDP-constraints, the initial matrices (40) were obtained therein by using the (SLPMM) algorithm stated in [28, 29]. These initial matrices satisfy then the SDP-inequalities of (SOFP1).

We solved the (SOFP1) problem for the above data set REA1 as in [33], that is, we started at the point (40) and came to the same result. The next different initial point (41) was also tried and the algorithm successfully found the solution (see Table 1 below).

$$F_0 = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 2 & -5 \end{bmatrix}, \quad L_0 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \quad (41)$$

Finally, we have used the data set REA1 also for solving the problem (SOFP2), starting at the same initial points (taking  $V_0$  as the identity). Note that, except in the case of the starting point (40) for (SOFP1), the initial points considered are not interior ones, regarding the SDP-inequalities.

The second data set used was also described in [21] and is called AC1 in the COMPLEib library. It corresponds to an aircraft model. The exact values of the data (also for the matrices  $P, Q, R$ ) can be found in the same papers [30, 31, 33, 34].

For this second data set AC1 we also run both problems, (SOFP1) and (SOFP2), using the starting point proposed in [33] (and conveniently extended for (SOFP2)), which is given by:

$$F_0 = \begin{bmatrix} 0.8190 & -0.0180 & 0.3200 \\ 0.0247 & -1.7164 & 0.0356 \\ 2.1739 & 0.0236 & 1.8139 \end{bmatrix}, L_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (42)$$

Here again the initial point is not an interior point for (SOFP2), regarding the SDP-inequalities.

For all these numerical tests, we employed a common tolerance value of  $\varepsilon_{tol} = 10^{-4}$ . At each iteration the matrix  $B$  was selected as the identity. This implies that the algorithm is not using second order information in this implementation. Thus, a linear convergence behavior can be expected. Nevertheless the algorithm took no longer than 2 minutes for solving each problem (usually less than a minute).

The results obtained for (SOFP1) are given in the next table.

Data set	$x_0$	iter	$f$ -iter	$\theta$ -iter	$f(x_*)$	$\theta(x_0)$	$\ h(x_0)\ $	$\lambda_1(G_0)_+$	$\lambda_1(-L_0)_+$	rest	Filter
REA1	(40)	51	49	2	3.5036	37.1097	37.1095	0.0002	0.0000	0	2
REA1	(41)	106	101	5	3.5036	91.4359	75.0862	15.9355	0.4142	0	3
AC1	(42)	922	921	1	0.0592	4.5836	4.5836	1.2521	0.0000	0	1

Table 1: Numerical results for (SOFP1)

where

- $x_0$  = initial point  $(F_0, L_0)$
- iter = number of iterations
- $f$ -iter = number of  $f$ -iterations
- $\theta$ -iter = number of  $\theta$ -iterations
- $f(x_*)$  = value of  $f$  at the optimum
- $\theta(x_0)$  = value of  $\theta$  at the initial point  $(F_0, L_0)$
- $\|h(x_0)\|$  = value of  $\|h(\cdot)\|$  at the initial point  $(F_0, L_0)$
- $\lambda_1(G_0)_+$  = positive part of the largest eigenvalue of  $G_0 := -(A_{F_0}L_0 + L_0A_{F_0}^T)$
- $\lambda_1(-L_0)_+$  = positive part of the largest eigenvalue of  $L_0$
- rest = number of times the restoration phase was used
- Filter = final number of vectors in the Filter.

The following last table summarize the results for (SOFP2).

Data set	$x_0$	iter	$f$ -iter	$\theta$ -iter	$f(x_*)$	$\theta(x_0)$	$\ h(x_0)\ $	$\lambda_1(G_0)_+$	$\lambda_1(-V_0)_+$	rest	Filter
REA1	(40)	134	127	7	3.3335	57.3903	50.3244	7.0659	0.0000	3	6
REA1	(41)	161	145	16	3.3335	81.4324	65.1029	16.3295	0.0000	12	12
AC1	(42)	935	930	5	0.0013	4.9984	3.7463	1.2521	0.0000	0	3

Table 2: Numerical results for (SOFP2)

here

- $x_0$  = initial point  $(F_0, L_0, V_0)$
- $\lambda_1(G_0)_+$  = positive part of the largest eigenvalue of  $G_0 := -(A_{F_0}^T V_0 + V_0 A_{F_0})$
- $\lambda_1(-V_0)_+$  = positive part of the largest eigenvalue of  $-V_0$

## 6 Conclusions

The filter SQP algorithm have been extended from [16] to the case of nonlinear semidefinite constraints. The global convergence of the filter method proposed was obtained under quite mild assumptions, like MFCQ, boundedness, etc.

The trust region subproblems at each step of the algorithm are actually linear semidefinite programming problems. In this sense the approach selected is also related to the SSDP algorithm presented in [13].

We have done some numerical experiences applied to optimal SOF problems. However, this aspect is still in a development stage. In a future work we plane to study local convergence properties and to report new numerical results.

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