A new adaptive algorithm for linear multiobjective programming problems

S. RADJEF - M.0. BIBI - M.S. RADJEF

Laboratory LAMOS, University of Béjaia, 06000 (Algéria)

sonia.radjef@caramail.com
mobibi@hotmail.com
radjefms@yahoo.fr

Abstract. *In this paper, we present a new adaptive algorithm for defining the solution set of a multiobjective linear programming problem, where the decision variables are upper and lower bounded, using the direct support method [6, 8, 2, 7]. The particularity of this method is the fact that it avoids the preliminary transformation of the decision variables. It handles the bounds such as they are initially presented, and possesses a suboptimal criterion which stops the algorithm with the desired accuracy.*
Key words: *linear program, multiobjective linear programming, bounded variables, mixed variables, direct support method, efficient, weakly efficient, $\epsilon-$weakly efficient, suboptimality.*

1 Introduction

Multicriteria optimization problems are a class of difficult optimization problems in which several different objective functions have to be considered simultaneously. Usually, there is no solution optimizing simultaneously all the several objective functions. Therefore, we search the so-called *efficient* points. The method presented in this paper allows to solve one class of those problems: the multiobjective linear programming problem, where the decision variables are upper and lower bounded. The method is a generalization of the direct support method [7, 6, 2], known for single-objective linear programming. It is based on the adapted methods developed for solution of the linear or quadratic programs in the 70 to 80 years by Professors R Gabassov and F.M. Kirillova at the university of Minsk, Bielorussia.
These methods, unfortunately far from known in occident, are intermediate between the methods of activation constraints and those of the interior points.
This method has the advantage of handling the bounds of the decision variables as they are initially presented. This is very significant for the post-optimal analysis in linear programming and to accurately characterize the big-bang property of an optimal command during the optimization of the linear dynamic systems with linear or quadratic criteria. The algorithm is also equipped with a stop criterion which can give an approximate solution known as ϵ-optimal solution with a selected precision $\epsilon \geq 0$.
It presents a significant gain in computational time and memory space, and it's easily implemented. An algorithm and a numerical example are proposed.

A multiobjective linear program with bounded variables can be presented in the following canonical form:

$$\begin{cases} Cx \longrightarrow \max, \\ Ax = b, \\ d^- \leq x \leq d^+, \end{cases} \tag{1}$$

where C is a $k \times n$-matrix, A a $m \times n$-matrix with $rank(A) = m \leq n$, $b \in \mathbb{R}^m$, $d^- \in \mathbb{R}^n$ and $d^+ \in \mathbb{R}^n$.
We denote by S the set of feasible decisions:

$$S = \{x \in \mathbb{R}^n, Ax = b, d^- \leq x \leq d^+\}.$$

The problem of multiobjective linear programming with bounded variables can then be regarded as the problem of searching for all the feasible solutions which are efficient or weakly efficient.

Definition 1. A feasible decision $x^0 \in \mathbb{R}^n$ is said to be efficient for the problem (1), if there is no other feasible solution $x \in S$ such that $Cx \geq Cx^0$ and $Cx \neq Cx^0$.

Definition 2. A feasible decision $x^0 \in \mathbb{R}^n$ is said to be weakly efficient for the problem (1), if there is no other feasible solution $x \in S$ such that $Cx > Cx^0$.

Definition 3. Let $\epsilon \in \mathbb{R}^k, \epsilon \geq 0$. A feasible decision $x^\epsilon \in S$ is said to be $\epsilon-$weakly efficient for the problem (1), if there is no other feasible solution $x \in S$ such that $Cx - Cx^\epsilon > \epsilon$.

The multiobjective linear programming consists on determining the whole set of the efficient decisions and all weakly efficient decisions of problem (1) for given C, A, b, d^- and d^+.

The solution goes by the following phases of multicriteria simplex method [5, 4]:

- Find an extreme point.
- Find an initial efficient extreme point.
- Generate all efficient extreme points.
- Generate the whole of the efficient solutions.

Phase 1: it is the first ordinary phase of single-objective linear programming. It consists on finding a basic feasible initial solution by solving the constraints system of the program. Phase 4: it is identical to that of the method of multicriteria simplex [5, 4], so we omit its presentation in this work. The phases 2 and 3 are studied in detail in what follows.

The following theorems focus on the conditions of existence of efficient solutions and weakly efficient solutions.

Consider the following sets

$$\Lambda = \left\{ \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k, \ \ \lambda_i > 0, \ i = 1, \ldots, k, \ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

and

$$\overline{\Lambda} = \left\{ \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k, \ \ \lambda_i \geq 0, \ i = 1, \ldots, k, \ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Theorem 1. *A feasible solution $x^0 \in S$ is efficient for the problem (1) if and only if there is $\lambda \in \Lambda$ such that*

$$\max_{x \in S} \lambda^T Cx = \lambda^T Cx^0.$$

Theorem 2. *A feasible solution $x^0 \in S$ is weakly efficient for the problem (1) if and only if there exists $\lambda \in \overline{\Lambda}$ such that*

$$\max_{x \in S} \lambda^T Cx = \lambda^T Cx^0.$$

2 Procedure for finding an initial efficient extreme point

We propose a procedure for finding an initial efficient extreme point, inspired by the one proposed by H.P. Benson [1], taking into account the specificity of the constraints of the program (1). This procedure consists on resolving a particular linear program by the direct support method [7].

Let $\lambda \in \Lambda$ and consider the following linear program:

$$\begin{cases} \lambda^T Cx \longrightarrow \max, \\ Ax = b, \\ d^- \leq x \leq d^+, \end{cases} \tag{2}$$

If we set $y = x - d^-$, then we obtain the L.P

$$\begin{cases} \lambda^T Cy + \lambda^T Cd^- \longrightarrow \max, \\ Ay = b - Ad^-, \\ y \leq d^+ - d^-, \\ y \geq 0. \end{cases} \tag{3}$$

To establish the procedure of resolution, the following L.P is defined where $x^0 \in S$:

$$\begin{cases} u^T(-Cx^0 + Cd^-) + w^T(b - Ad^-) + \gamma^T(d^+ - d^-) \longrightarrow \min, \\ u^T C - w^T A - \gamma^T + \alpha^T = -e^T C, \\ u, \alpha, \gamma \geq 0. \end{cases} \tag{4}$$

The suggested procedure is then given by the following three steps:

Step (1): Find a feasible point $x^0 \in S$.

Step (2): If $S \neq \emptyset$, then find an optimal solution $(u^0, w^0, \gamma^0, \alpha^0)$ for program (4) by using the direct support method [7] and go to step (3).
If not stop.

Step (3): Obtain an extreme optimal point solution of the L.P (2) with $\lambda = (u^0 + e)$ by using the direct support method for the resolution of a L.P with bounded variables [7].

Let x^0 be the feasible solution selected at step (1) of the procedure.

Theorem 3. *If the program (4) admits an optimal solution* $(u^0, w^0, \gamma^0, \alpha^0)$*, then the L.P (2) with* $\lambda = (u^0 + e)$ *admits an optimal extreme point solution.*

Proof. Let $(u^0, w^0, \gamma^0, \alpha^0)$ an optimal solution of (4). With $\lambda = (u^0 + e)$, the dual L.P of program (3) is given by

$$\begin{cases} w^T(b - Ad^-) + \gamma^T(d^+ - d^-) \longrightarrow \min, \\ w^T A + \gamma^T \geq (u^0 + e)^T C, \\ \gamma \geq 0. \end{cases} \tag{5}$$

As $(u^0, w^0, \gamma^0, \alpha^0)$ is an optimal solution of (4), then (w^0, γ^0) is a feasible solution of (5). S is nonempty, then the linear program (2), with $\lambda = (u^0 + e)$, is a feasible problem, and, by the duality theory, it admits an optimal extreme point solution. □

The following theorem brings back the search for an efficient solution of multiobjective problem (1) with the resolution of one linear program with bounded variables.

Theorem 4. *[3] The following linear program:*

$$\begin{cases} e^T Cx \longrightarrow \max, \\ Cx \geq Cx^0, \\ x \in S, \end{cases} \tag{6}$$

admits an optimal solution if, and only if, the multiobjective problem (1) has an efficient solution.

Theorem 5. *The linear program (4) admits an optimal solution if, and only if, the multiobjective problem (1) has an efficient solution.*

Proof. The dual program of (4) is given by

$$\begin{cases} e^T Cy \longrightarrow \max, \\ -Cy \leq -Cx^0 + Cd^-, \\ Ay = b - Ad^-, \\ y \leq d^+ - d^-, \\ y \geq 0. \end{cases} \tag{7}$$

However, as $e^T Cd^-$ is a constant value, then the linear program (7) is equivalent to the following one:

$$\begin{cases} e^T Cy + e^T Cd^- \longrightarrow \max, \\ -Cy \leq -Cx^0 + Cd^-, \\ Ay = b - Ad^-, \\ y \leq d^+ - d^-, \\ y \geq 0. \end{cases} \tag{8}$$

If we set $y + d^- = x$, then we obtain the program (6). By applying the theorem 4 and according to the duality theory, we establish the theorem.

□

3 Computing all Efficient Extreme Points

In this phase, we locate all the efficient extreme points by introducing all non optimal nonbasic variables into basis; and this, by using the direct support method adapted for taking into account the multiobjective aspect of the problem. The principle of the method consists on: starting from an initial efficient extreme point, we determine a neighboring solution, and test whether it is efficient. If it is not, we return to an another efficient point and the process is reiterated. A test of efficiency of a nonbasic variable is then necessary.

3.1 Test of Efficiency of a Nonbasic Variable

In order to test efficiency of a point x^0 in the multiobjective linear program (1), we introduce a k-dimensional column vector s and define the following L.P:

$$\begin{cases} h = es \longrightarrow \max, \\ Ax = b, \\ Cx - s = Cx^0, \\ d^- \leq x \leq d^+, \\ s \geq 0. \end{cases} \tag{9}$$

Theorem 6. *If* $\max h = 0$*, then* x^0 *is efficient.*
If not, x^0 *is non efficient.*

The problem (9) is a linear program with mixed variables (i.e. some variables are non negative and some others are bounded). We solve it by the direct support method adapted for the resolution of a L.P. with mixed variables [7]. For this, we introduce the following notations:

$$\overline{A} = \begin{pmatrix} A \\ C \end{pmatrix}, \overline{H} = \begin{pmatrix} 0 \\ -I \end{pmatrix}, \bar{b} = \begin{pmatrix} b \\ Cx^0 \end{pmatrix},$$

where I is the identity matrix of order k.
The program (9) becomes then

$$\begin{cases} h = es \longrightarrow \max, \\ \bar{A}x + \bar{H}s = \bar{b}, \\ d^- \leq x \leq d^+, \\ s \geq 0. \end{cases} \tag{10}$$

where e is a $k-$vector with elements equal to 1.
However, two particular cases can arise:

Particular case 1: If all the elements of the matrix C are non negative, the obtained test program is:

$$\begin{cases} h = es \longrightarrow \max, \\ Ax = b, \\ Cx - s = Cx^0, \\ d^- \le x \le d^+, \\ Cd^- - Cx^0 \le s \le Cd^+ - Cx^0. \end{cases} \tag{11}$$

With the following notations:

$$y = \begin{pmatrix} x \\ s \end{pmatrix}, \overline{A} = \begin{pmatrix} A & 0 \\ C & -I \end{pmatrix}, \overline{b} = \begin{pmatrix} b \\ Cx^0 \end{pmatrix}, \overline{C} = \begin{pmatrix} 0 & e \end{pmatrix},$$

$$\overline{d}^- = \begin{pmatrix} d^- \\ Cd^- - Cx^0 \end{pmatrix}, \overline{d}^+ = \begin{pmatrix} d^+ \\ Cd^+ - Cx^0 \end{pmatrix},$$

the linear program (11) takes the form:

$$\begin{cases} h = \overline{C}y \longrightarrow \max, \\ \overline{A}y = \overline{b}, \\ \overline{d}^- \le y \le \overline{d}^+. \end{cases} \tag{12}$$

We solve the linear program (12) by the direct support method adapted to the resolution of a L.P. with bounded variables [7].

Particular case 2: If all the elements of the matrix C are non positive, the test program has the form:

$$\begin{cases} h = es \longrightarrow \max, \\ Ax = b, \\ Cx - s = Cx^0, \\ d^- \le x \le d^+, \\ Cd^+ - Cx^0 \le s \le Cd^- - Cx^0. \end{cases} \tag{13}$$

Using the following notations:

$$y = \begin{pmatrix} x \\ s \end{pmatrix}, \overline{A} = \begin{pmatrix} A & 0 \\ C & -I \end{pmatrix}, \overline{b} = \begin{pmatrix} b \\ Cx^0 \end{pmatrix}, \overline{C} = \begin{pmatrix} 0 & e \end{pmatrix},$$

$$\overline{d}^- = \begin{pmatrix} d^- \\ Cd^+ - Cx^0 \end{pmatrix}, \overline{d}^+ = \begin{pmatrix} d^+ \\ Cd^- - Cx^0 \end{pmatrix},$$

we obtain the following test program:

$$\begin{cases} h = \overline{C}y \longrightarrow \max, \\ \overline{A}y = \overline{b}, \\ \overline{d}^- \le y \le \overline{d}^+. \end{cases} \tag{14}$$

which can be solved by the direct support method adapted to the resolution of a L.P. with bounded variables [7].

3.2 Method of computing All the Efficient Extreme Points

Following the direct support method for the resolution of a L.P., we propose a method which consists on generating from an efficient extreme point all the others by using the direct support method modified for the circumstance.
We denote:
$I = \{1, 2, \cdots, m\}$: the set of indices of the lines of A,
$J = \{1, 2, \cdots, n\}$: the set of indices of the columns of A, $J = J_B \cup J_N$ with $J_B \bigcap J_N = \phi$, $|J_B| = m$,
$K = \{1, 2, \cdots, k\}$: the set of indices of the lines of the criteria matrix C.
We can then split the vectors and the matrix in the following way:

$$x = \begin{pmatrix} x_B \\ -- \\ x_N \end{pmatrix}, \quad x_B = x(J_B) = (x_j,\ j \in J_B), \quad x_N = x(J_N) = (x_j,\ j \in J_N),$$

$$C = \begin{pmatrix} C_B \\ -- \\ C_N \end{pmatrix}, \quad C_B = C(K, J_B), \quad C_N = C(K, J_N),$$

$$A = (A_B | A_N), \quad A_B = A(I, J_B), \quad A_N = A(I, J_N).$$

Definition 4.

- A vector x satisfying the constraints of the problem (1) is called *feasible solution* of problem.

- A feasible solution x^0 is said to be *optimal* for the objective i if

$$Z_i(x^0) = C_i x^0 = \max_{x \in S} C_i x.$$

  The solution x^0 is then *weakly efficient* for the problem (1).

- Let x^0 be the optimal solution for the objective function i and the fixed vector $0 \leq \epsilon = (\epsilon_1, \cdots, \epsilon_k) \in \mathbb{R}^k$. A feasible solution x^ϵ is said to be ϵ-*optimal* or *suboptimal* for the objective i if

$$Z_i(x^0) - Z_i(x^\epsilon) = C_i x^0 - C_i x^\epsilon \leq \epsilon_i.$$

  The solution x^ϵ is then ϵ-*weakly efficient* for the problem (1).

- The subset $J_B \subset J$, $|J_B| = m$ is said to be *support* if

$$\det A_B = \det A(I, J_B) \neq 0.$$

- The pair $\{x, J_B\}$ formed by the feasible solution x and the support J_B is said to be *support feasible solution* for the problem.

- The support feasible solution $\{x, J_B\}$ is an extreme point, if $x_j = d_j^-$ or $x_j = d_j^+$, $\forall j \in J_N$.

- The support feasible solution is said to be *non-degenerate* if

$$d_j^- < x_j < d_j^+,\ j \in J_B.$$

3.2.1 Increment Formula of the Objective Function

Let $\{x, J_B\}$ be a support feasible solution for the problem (1) and let's consider another unspecified feasible solution $\bar{x} = x + \Delta x$. The increment of the objective function is:

$$\Delta Z = Z(\bar{x}) - Z(x) = C\bar{x} - Cx = C\Delta x.$$

In addition, we have $\left\{ \begin{array}{l} Ax = b, \\ A\bar{x} = b. \end{array} \right. \Leftrightarrow\ A\bar{x} = A(\Delta x + x) = A\Delta x + Ax = b \Rightarrow\ A\Delta x = 0.$

While setting $\Delta x = \begin{pmatrix} \Delta x_B \\ -- \\ \Delta x_N \end{pmatrix}$, $\Delta x_B = \Delta x(J_B)$, $\Delta x_N = \Delta x(J_N)$, the equality $A\Delta x = 0$ can be written $A_B \Delta x_B + A_N \Delta x_N = 0\ \Rightarrow \Delta x_B = -A_B^{-1} A_N \Delta x_N$.

The increment formula of the objective function becomes then:

$$\Delta Z = -(C_B A_B^{-1} A_N - C_N)\Delta x_N.$$

We define the potential matrix U and the estimation matrix E:

$$U = C_B A_B^{-1},\quad E = E(K, J) = UA - C.$$

We set $E = \begin{pmatrix} E_B \\ --- \\ E_N \end{pmatrix}$, where $E_B = E(K, J_B) = 0,\quad E(K, J_N) = C_B A_B^{-1} A_N - C_N$.

The increment formula takes the following final form:

$$\Delta Z = -E_N \Delta x_N = \left(-\sum_{j \in J_N} E_{ij} \Delta x_j, i = 1, \dots, k \right).$$

3.2.2 Optimality criterion

Theorem 7. *Let $\{(x, y), (J_{x_B}, J_{y_B})\}$ be a support feasible solution of the problem (1). Then the following relations :*

$$\left\{ \begin{array}{lll} E_{ij} \geq 0, & \text{if } x_j = d_j^-, & j \in J_N, \\ E_{ij} \leq 0, & \text{if } x_j = d_j^+, & j \in J_N, \\ E_{ij} = 0, & \text{if } d_j^- < x_j < d_j^+, & j \in J_N, \end{array} \right. \tag{15}$$

are sufficient to the optimality of the feasible solution (x, y) for the i^{th} objective, the point x is then said weakly efficient point for the problem (1).
Those relations are also necessary if the support feasible solution is nondegenerate.

3.2.3 The Suboptimality Condition

The value

$$\beta_i(x, J_B) = \sum_{j \in J_N, E_{ij} > 0} E_{ij}(x_j - d_j^-) + \sum_{j \in J_N, E_{ij} < 0} E_{ij}(x_j - d_j^+)$$

is called *suboptimality formula* of the objective i, $i = 1, \dots, k$.

Theorem 8. *(The suboptimality condition).*
Let $\{x, J_B\}$ be a support feasible solution of the problem (1) and ϵ an arbitrary nonnegative vector of dimension k.
If $\beta(x, J_B) = (\beta_i(x, J_B), i = 1, \dots, k) \le \epsilon$, then $\{x, J_B\}$ is ϵ-optimal.

If there exists $i \in \{1, \dots, k\}$ such as $\beta_i(x, J_B) \le \epsilon_i$, then $\{x, J_B\}$ is ϵ_i-optimal for the objective i; the feasible solution x is then called ϵ_i-weakly efficient for the problem (1).

The method of searching for all the efficient extreme points consists on introducing into the basis, one by one, the nonbasic variables corresponding to the first efficient extreme point found in the first phase.
The construction of the new feasible solution $\overline{x} = x + \theta^0 l$ consists on choosing a vector $l \in \mathbb{R}^n$, called direction of improvement, and a non negative real number θ^0 which is the maximum step along this direction.
Let j_0 be the index candidate to enter in basis and the criterion i_0 is defined by the relation:

$$|E_{i_0 j_0}| = \max_{i=1,\dots,k} |E_{i j_0}|.$$

We set

$$\begin{cases} l_{j_0} = -\text{ sign } E_{i_0 j_0}, \\ l_j = 0, \ j \neq j_0, \ j \in J_N, \\ l_B = A_B^{-1} a_{j_0} \text{sign } E_{i_0 j_0}. \end{cases}$$

In addition, the step θ^0 has to satisfy the following relations:

1. $d_j^- - x_j \le \theta^0 l_j \le d_j^+ - x_j$, $j \in J_B$,
2. $d_{j_0}^- - x_{j_0} \le \theta^0 l_{j_0} \le d_{j_0}^+ - x_{j_0}$.

Consequently, the maximum step θ^0 along the direction l is equal to

$$\theta^0 = \min(\theta_{j_1}, \theta_{j_0}),$$

where

$$\theta_{j_0} = d_{j_0}^+ - d_{j_0}^-,$$

and

$$\theta_{j_1} = \min_{j \in J_B} \theta_j,$$

with

$$\theta_j = \begin{cases} \frac{d_j^+ - x_j}{l_j}, & \text{if } l_j > 0, \\ \frac{d_j^- - x_j}{l_j}, & \text{if } l_j < 0, \\ \infty, & \text{if } l_j = 0. \end{cases}$$

The new feasible solution $\bar{x}$ is thus written $\bar{x} = x + \theta^0 l$.

Calculation of $\beta(\bar{x}, J_B)$.

We have

$$\beta_i(\bar{x}, J_B) = \sum_{E_{ij}>0, j \in J_N} E_{ij}(\bar{x}_j - d_j^-) + \sum_{E_{ij}<0, j \in J_N} E_{ij}(\bar{x}_j - d_j^+),$$

for $i = 1, \dots, k$.

The components $\bar{x}_j$ for $j \in J_N$ are defined as follows:

$$\begin{cases} \bar{x}_j = x_j, & \text{for } j \neq j_0, \\ \bar{x}_{j_0} = x_{j_0} - \theta^0, & \text{if } E_{i_0 j_0} > 0, \\ \bar{x}_{j_0} = x_{j_0} + \theta^0, & \text{if } E_{i_0 j_0} < 0. \end{cases}$$

So

$$\beta_{i_0}(\bar{x}, J_B) = \beta_{i_0}(x, J_B) - \theta^0 |E_{i_0 j_0}|.$$

If $\beta_{i_0}(\bar{x}, J_B) \leq \epsilon_{i_0}$, then the feasible solution $\bar{x}$ is ϵ-optimal for the objective i_0, then we consider all nonbasic variables.

If not, we change J_B in the following way:

If $\theta^0 = \theta_{j_0}$, then $\bar{J}_B = J_B$ and $\bar{x} = x + \theta^0 l$.

If $\theta^0 = \theta_{j_1}$, then $\bar{J}_B = (J_B \backslash j_1) \cup j_0$ and $\bar{x} = x + \theta^0 l$.

The new support feasible solution $\{\bar{x}, \bar{J}_B\}$ will be written

$$\bar{x} = x + \theta^0 l, \ \bar{J}_B = (J_B \backslash j_1) \cup j_0.$$

If $\beta_{i_0}(\bar{x}, \bar{J}_B) > \epsilon_{i_0}$, we start a new iteration with the new support feasible solution $\{\bar{x}, \bar{J}_B\}$. If not, we stop the procedure by having an extreme point. The test program is then used (given in the previous section) to test the efficiency of this extreme point. We start again the process by considering another nonbasic variable. However, the use of this test is not always necessary, since some solutions are clearly efficient or non efficient, according to the following observations :

Observation 1: Let x be a basic feasible solution.

If there is $j \in J_N$ for which for all $i = 1, \dots, k$, we have

$$\begin{cases} \text{Case } E_{ij} \geq 0, & \text{for } x_j = d_j^-, \\ \text{Case } E_{ij} \leq 0, & \text{for } x_j = d_j^+, \end{cases} \tag{16}$$

then, we have $\Delta Z \leq 0$, i.e. $\overline{Z} - Z \leq 0 \Rightarrow \overline{Z} \leq Z$ and $\overline{Z} \neq Z$, therefore the introduction of j in basis leads to a solution $\overline{x}$ dominated by the current solution x. Thus, the introduction of j in basis is useless.

Proof. Set $j \in J_N$ and $i \in \{1, \dots, k\}$.
Let us consider them following cases:

- Case where $E_{ij} \geq 0$ and $x_j = d_j^-$.
  Having $\Delta Z_{ij} = -E_{ij}(\overline{x}_j - x_j) = -E_{ij}(\overline{x}_j - d_j^-)$. As $\overline{x}$ must be feasible, i.e. $\overline{x}_j \geq d_j^-$ then $\overline{x}_j - d_j^- \geq 0$, so $-E_{ij}(\overline{x}_j - d_j^-) \leq 0 \Rightarrow \overline{Z} = Z + \Delta Z \leq Z$.

- Case where $E_{ij} \leq 0$ and $x_j = d_j^+$.
  Having $\Delta Z_{ij} = -E_{ij}(\overline{x}_j - x_j) = -E_{ij}(\overline{x}_j - d_j^+)$. As $\overline{x}$ must be feasible, i.e. $\overline{x}_j \leq d_j^+$ then $\overline{x}_j - d_j^+ \leq 0$, so $-E_{ij}(\overline{x}_j - d_j^+) \leq 0 \Rightarrow \overline{Z} = Z + \Delta Z \leq Z$.

In all the cases, the introduction of j in basis leads to a solution $\overline{x}$ dominated by the current solution x. □

Observation 2: Let x be a basic feasible solution.
If there is $j_0 \in J_N$ such that for all $i \in \{1, \dots, k\}$, the relations (16) are not satisfied, then the introduction of j_0 in basis leads to a solution $\overline{x}$ dominating the current solution x.

Proof. Let us suppose that there is $j_0 \in J_N$ such that for all $i \in \{1, \dots, k\}$, we have

$$E_{ij_0} < 0,\ x_{j_0} = d_{j_0}^- \quad \text{or} \quad E_{ij_0} > 0,\ x_{j_0} = d_{j_0}^+.$$

- Case where $E_{ij_0} < 0$ for $x_{j_0} = d_{j_0}^-$.
  $\Delta Z_{ij_0} = -E_{ij_0}(\overline{x}_{j_0} - x_{j_0}) = -E_{ij_0}\theta^0 l_{j_0}$, since $\overline{x}_{j_0} = x_{j_0} + \theta^0 l_{j_0}$. As $l_{j_0} = -signE_{ij_0}$, then $\Delta Z_{ij_0} = -\theta^0 E_{ij_0} > 0$ so $\overline{Z} = Z + \Delta Z \geq Z$.

- Case where $E_{ij_0} > 0$ for $x_{j_0} = d_{j_0}^+$.
  $\Delta Z_{ij_0} = -E_{ij_0}(\overline{x}_{j_0} - x_{j_0}) = -E_{ij_0}\theta^0 l_{j_0}$, since $\overline{x}_{j_0} = x_{j_0} + \theta^0 l_{j_0}$. As $l_{j_0} = -signE_{ij_0}$, then $\Delta Z_{ij_0} = \theta^0 E_{ij_0} > 0$ so $\overline{Z} = Z + \Delta Z \geq Z$.

In all the cases, the introduction of j_0 in basis leads to a solution $\overline{x}$ dominating the current solution x.

□

Observation 3: Let x be basic feasible solution.
If there is $i \in \{1, \dots, k\}$ such that $\beta_i(x, J_B) \leq \epsilon_i$, then the maximum of the i-th criterion is attained with the precision ϵ_i.
If, $\beta_i(x, J_B) = 0$, then the maximum of the i-th criterion is attained and the solution x is weakly efficient for the problem.

In conclusion, only the remaining columns are candidates for entering into basis. In this case, we can not say nothing on efficiency of the corresponding solution $\overline{x}$. For this, we apply the test of efficiency stated before.

4 Algorithm for Computing All Efficient Extreme Points

The steps of the method of searching for all efficient extreme points are summarized in the following algorithm:

1. Find a feasible solution of the problem, let be x^0.
   - If it exists, then go into 2 .
   - If not, stop, the problem is impossible.
2. Find the first efficient extreme point using the following procedure:
   - Find an optimal solution $(u^0, w^0, \gamma^0, \alpha^0)$ of program

     $$\begin{cases} u^T(-Cx^0 + Cd^-) + w^T(b - Ad^-) + \gamma^T(d^+ - d^-) \longrightarrow \min, \\ u^T C - w^T A - \gamma^T + \alpha^T = -e^T C, \\ u, \alpha, \gamma \geq 0, \end{cases}$$

     by using the direct support method.
   - Obtain an optimal extreme point solution of the L.P

     $$\begin{cases} \lambda^T Cx \longrightarrow \max, \\ Ax = b, \\ d^- \leq x \leq d^+, \end{cases}$$

     with $\lambda = (u^0 + e)$ by using the direct support method for the resolution of a L.P with bounded variables, let x^1 be the obtained solution.
3. Set $s = 2$.
4. Calculate x^s, s^{th} efficient extreme point.
5. Test if there exists at least i such as $\beta_i(x^s, J_B^s) \leq \epsilon_i$.
   - If so, go to 6.
   - If not, go to 7.
6. Can we improve another objective?
   - If so, go to 7.
   - If not, go to 12.
7. Is there $j \in J_N$ such that the relations (16) are not satisfied?
   - If so, go to 8.

- If not, go to 11.

8. Consider all $j \in J_N$.

9. Test if the introduction of the j^{th} corresponding column leads to an unprocessed basis?

   - If so, set $s = s + 1$ and go to 4.
   - If not, go to 10.

10. Is there an already stored non explored basis?

    - If so, set $s = s + 1$ and go to 4.
    - If not, stop, all the vertices are determined.

11. Consider the following program:

$$\begin{cases} g = e^T s \longrightarrow \max, \\ -Cx + Is = -Cx^s, \\ Ax = b, \\ d^- \leq x \leq d^+, \\ s \geq 0. \end{cases}$$

    - If $\max g = 0$, go to 12.
    - If not, go to 13.

12. The solution x^s is efficient, go to 13.

13. Test if there exists $j \in J_N^s$ such that the relations (16) are satisfied?

    - If so, go to 14.
    - If not, go to 10.

14. Test if $s \leq n - m$.

    - If so, go to 15.
    - If not, go to 10.

15. Store the corresponding basic indices which lead to an unprocessed basis and go to 10.

In addition to the application of this algorithm to the planning of production on the level of a dairy [9], we present a numerical example.

5 Numerical Example

Let's consider the bicriterion linear problem with bounded variables:

$$\begin{cases} z_1(x) = 2x_1 - 3x_2 - x_3 \ \rightarrow \max, \\ z_2(x) = 3x_1 + x_2 \ \rightarrow \max, \\ x_1 - x_2 + 3x_3 = 3, \\ -7x_1 + x_2 + 2x_3 = 2, \\ -2 \leq x_1 \leq 2, \\ -4 \leq x_2 \leq 4, \\ -6 \leq x_3 \leq 6. \end{cases}$$

Let $x^0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ be an initial feasible solution of this problem.
The application of the algorithm has led to find two efficient extreme points:

$$x^1 = \begin{pmatrix} -0.8696 \\ -4 \\ -0.0435 \end{pmatrix}, x^2 = \begin{pmatrix} 0.8792 \\ 4 \\ 2.0472 \end{pmatrix}.$$

References

[1] H.P. Benson. Finding an initial efficient extreme point for a linear multiple objective program. *Journal of Operational Research Society*, 32(6):495–498, 1981.

[2] M.O. Bibi. Support method for solving a linear-quadratic problem with polyhedral constraints on control. *Optimization*, 37:139–147, 1996.

[3] J.G. Ecker and I.A. Kouada. Finding all efficient extreme points for multiple objective linear programs. *Mathematical Programming*, 14:249–261, 1978.

[4] J.P. Evans and R.E. Steuer. *Generating efficient extreme points in linear multiple objective programming: Two algorithms and computing experience, in Multiple Criteria decision making, 349-365.* J.L. Cochrane and M. Zeleny, 1973.

[5] J.P. Evans and R.E. Steuer. A revised simplex method for linear multiple objective programs. *Mathematical* P*rogramming*, 5(1):54–72, August 1973.

[6] R. Gabasov et al. *Constructive methods of optimization.* P.I.-University Press, Minsk, 1984.

[7] S. RADJEF. Sur la programmation linéaire multiobjectifs. Master's thesis, Université A. Mira de Béjaïa, 2001.

[8] S. RADJEF, M.O. BIBI, and M.S. RADJEF. Sur les procédures de recherche des points extrêmes efficaces dans les problèmes linéaires multi-objectifs. *Preprint, Rapport Interne LAMOS N° 03/02*, 2002.

[9] K. Ait Yahia and F. Benkerrou. *Méthodes d'aide à la planification de la production au niveau de la laiterie du Djurdjura, Mémoire d'ingéniorat.* Département de Recherche Opérationnelle, Université de Béjaïa, 2000.

[10] P.L. Yu and M. Zeleny. The set of all non-dominated solutions in linear cases and a multicriteria simplex method. *Journal of mathematical analysis and applications*, 49(2):430–468, 1975.