Preprint ANL/MCS-P1379-1006

SPECTRAL STOCHASTIC FINITE-ELEMENT METHODS FOR PARAMETRIC CONSTRAINED OPTIMIZATION PROBLEMS

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Abstract. We present a method to approximate the solution mapping of parametric constrained optimization problems. The approximation, which is of the spectral stochastic finite element type, is represented as a linear combination of orthogonal polynomials. Its coefficients are determined by solving an appropriate finite-dimensional constrained optimization problem. We show that, under certain conditions, the latter problem is solvable because it is feasible for a sufficiently large degree of the polynomial approximation and has an objective function with bounded level sets. In addition, the solutions of the finite dimensional problems converge for an increasing degree of the polynomials considered, provided that the solutions exhibit a sufficiently large and uniform degree of smoothness. We demonstrate that our framework is applicable to one-dimensional parametric eigenvalue problems and that the resulting method is superior in both accuracy and speed to black-box approaches.

Key words. spectral approximations, orthogonal polynomials, parametric problems, stochastic finite element, constrained optimization.

AMS subject classifications. 65K05, 42C05.

1. Introduction.

This paper is concerned with the application of stochastic finite-element methods to the determination of the parametric variation of the solution of parametric constrained optimization. Parametric problems appear in a variety of circumstances, and, relevant to this work, when the parameters of the problem are uncertain [14]. Applications of parametric problems include elastoplasticity [1], radioactive waste disposal [16], elasticity problems [14], disease transmission [3], and nuclear reactor safety assessment [20].

In parametric uncertainty analysis of nonlinear equations, the problem is to characterize the dependence with respect to parameters of the solution of a nonlinear equation $F(x, \omega) = 0$, $x \in R^n$, $\omega \in \Omega \subset R^m$, $F : R^n \times R^m \to R^n$. In addition, the function $F(\cdot, \cdot)$ is smooth in both its arguments. Under the assumption of non singularity of $\nabla_x F(x, \omega)$ in a sufficiently large open set that contains (x_0, ω_0), one can determine a smooth mapping $x(\omega)$ that satisfies $x(\omega_0) = x_0$ and $F(x(\omega_0), \omega_0)$. The essence of parametric uncertainty analysis is to characterize the mapping $x(\omega)$ either by approximating it to an acceptable degree or by computing some of its integral characteristics, such as averages with appropriate weighting functions.

Perhaps the most widespread approach in carrying out this endeavor is the use of some form of the Monte Carlo method [18, 24]. In this approach, the parameter ω is interpreted as a random variable with an appropriate probability density function, and either the probability density function of $x(\omega)$ is approximated or computed, or appropriate averages $E_\omega\left[g(x(\omega))\right]$ are computed for suitable expressions of the multidimensional merit function g. Here E_ω is the expectation operator with respect with the probability density function of ω. In the Monte Carlo approach, values for $x(\omega)$ are produced for an appropriate set of sample points ω_i, in which case for each sample point the original nonlinear problem must be solved for its argument x.

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Recently, there has been substantial interest in carrying out the analytical computation as far as possible in characterizing the mapping $x(\omega)$. The component of this endeavor that is relevant to this work is the spectral stochastic finite element (SFEM) method [14, 13]. In this method, the mapping $x(\omega)$ is approximated by a Fourier-type expansion with respect to a basis of polynomials $P_0(\omega),\ P_1(\omega),\ \ldots, P_{M_K}(\omega)$ that are orthogonal with respect to the probability density function of ω, that is $E_\omega\left(P_i\left(\omega\right)P_j\left(\omega\right)\right) = \delta_{ij}, \quad 0 \leq i,j \leq M_K$. For $x_0, x_1 \ldots x_{M_K} \in R^n$, one defines the spectral approximation $\tilde{x}(\omega) = \tilde{x}(\omega; x_0, x_1, \ldots x_{M_K}) = \sum_{i=0}^{M_K} x_i P_i(\omega)$, and the SFEM formulation is obtained by determining the vectors $x_0^*, x_1^* \ldots x_{M_K}^*$ that satisfy the Galerkin projection conditions

$$E_\omega\left(F\left(\tilde{x}(\omega), \omega\right) P_k(\omega)\right) = 0_n, \quad k = 0, 1, \ldots, M_K.$$

This procedure results in a nonlinear system of equations that is $M_K + 1$ times larger than the original nonlinear system of equations for a given choice of the parameter ω. The advantage over the Monte Carlo method is that once this nonlinear system of equations is solved, the original nonlinear problem no longer needs to be solved. In the SFEM approach one generates directly an approximation of the mapping $x(\omega)$, and if either several of its momentum or its probability density function need to be evaluated, then a Monte Carlo method can be used on the explicit approximation $\tilde{x}(\omega) = \sum_{k=0}^{M_K} x_k^* P_k(\omega)$, *without the need to solve any further system of nonlinear equations.* Because the polynomials are used as the generators of the space over which approximation is carried out and the parameter ω has a stochastic interpretation the expansion defined by this approximation is called the *chaos polynomial expansion* [13].

Of course, the success of this method resides in the ability to suitably choose the set of polynomials P_i so that the residual decreases rapidly for relatively small values of M_K, before the size of the Galerkin projected problem explodes, a situation that occurs if one considers in the approximating set all the polynomials of degree up to K and if m is large. Nonetheless, for cases where n is huge (as are the cases originating in the discretization of partial differential equations) and m is relatively moderate, the SFEM has shown substantially more efficiency compared to the Monte Carlo approach, even when all polynomials of degree up to K were considered as generators of the approximating subspace [1]. In this work, we choose as the basis for the approximation the set of all polynomials of degree up to K, and we will defer the investigation of choosing a smaller subset to future research.

Stochastic finite element approaches have been applied primarily to the problem of parametric nonlinear equations [14, 13, 1, 6]. The object of this paper is to analyze the properties of SFEM and its extensions when the original problem is a *constrained optimization problem.* In this work we are not be interested in the stochastic aspect of the method proper, but in possible ways of generating the approximation $\tilde{x}(\omega)$ and in the properties of the resulting optimization problems. In this case, our work is perhaps better described as *spectral approximations for parametric constrained optimization problems.* Nonetheless, we will still refer to our method as SFEM, since generating the approximation $\tilde{x}(\omega)$ is by far the most conceptually involved part of SFEM.

2. Background on Spectral Methods. In this section, we use the framework from [9]. The choice of orthogonal polynomials is based on the scalar product

$$\langle g, h \rangle_W = \int_\Omega W(\omega) g(\omega) h(\omega) d\omega,$$

where g, h are continuous functions from $\mathbb{R}^m$ to $\mathbb{R}$. Here $\Omega \in \mathbb{R}^m$ is a compact set with a nonempty interior, and $W(\omega)$ is a weight function that satisfies the following.

1. $W(\omega) \geq 0$, $\forall \omega \in \Omega$.
2. Any multivariable polynomial function $P(\omega)$ is integrable, that is,

$$\int_\Omega W(\omega)|P(\omega)|d\omega < \infty.$$

We define the semi norm

$$||g||_W = \sqrt{\langle g, g\rangle_W}$$

on the space of continuous functions. If, in addition, $||g||_W = 0 \Rightarrow g = 0$, then $||\cdot||_W$ is a norm. We will concern ourselves only with this case, in which we denote by $L^2_W = L^2_W(\Omega)$ the completion of the space of continuous functions whose norm $||\cdot||_W$ is finite.

With respect to the scalar product $\langle\cdot,\cdot\rangle_W$, we can orthonormalize the set of polynomials in the variable ω. We obtain the orthogonal polynomials $P_i(\omega)$ that satisfy the following.

- $\langle P_i, P_j\rangle_W = \delta_{ij}$, $0 \leq i, j$. By convention, we always take P_0 to be the constant polynomial.
- The set $\{P_i\}_{i=0,1,2,\ldots}$ forms the basis of the complete space L^2_W.
- If $k_1 \leq k_2$, then $\deg(P_{k_1}) \leq \deg(P_{k_2})$. To simplify our notation, we introduce the definition

$$M_K = \max\{k|\deg(P_k) \leq K\}.$$

We define $L^2_{p,W} = \underbrace{L^2_W \otimes L^2_W \otimes \ldots L^2_W}_{p}$. We use the notation $L^2_W = L^2_{p,W}$ when the value of p can be inferred from the context. The Fourier coefficients of a function $f : \Omega \to \mathbb{R}^p$ are defined as

$$c_k(f) = \int_\Omega f P_k(\omega)W(\omega)d\omega \in \mathbb{R}^p, \quad f \in L^2_W,\ k = 0, 1, \ldots,$$

and they satisfy Bessel's identity [9]

$$f \in \mathcal{L}^2_W \Rightarrow \sum_{k=0}^{\infty} ||c_k(f)||^2 = \int_\Omega ||f(\omega)||^2 = ||f||^2_W. \tag{2.1}$$

The projection of a function $f \in L^2_W$ onto the space of the polynomials of degree at most K can be calculated as [9]

$$\Pi^K_W(f) = \sum_{k=0}^{M_K} c_k(f)P_k(\omega).$$

The most common type multidimensional weight function is probably the one of the separable type, that is, $W(\omega_1, \omega_2, \ldots, \omega_m) = \prod_{i=1}^m w_i(\omega_i)$. In this case, the orthogonal polynomials can be chosen to be products of orthogonal polynomials in each individual variable [5, 9]. We refer to such orthogonal polynomials as *tensor products*. The case $\Omega = [-1, 1]^m$, with $w_i(x) = \frac{1}{2}$, $i = 1, 2, \ldots, m$ is the one of tensor Legendre polynomials, whereas the one with $w_i(x) = \frac{1}{\pi\sqrt{1-x^2}}$, $i = 1, 2, \ldots, m$ is the one of Chebyshev polynomials [5].

Following the multidimensional Jackson theorem [12, Theorem 2], there exists a parameter C that depends only on the function f such that

$$D^\alpha f(\omega) \text{ are Lipschitz } \forall \alpha \in \mathbb{N}^m, ||\alpha||_1 = q-1 \Longrightarrow \left|\left|f - \Pi_W^K(f)\right|\right|_W \le C\frac{1}{K^q}. \quad (2.2)$$

Here, we denote by D^α the derivative of multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$,

$$D^\alpha(f) = \frac{\partial^{\sum_{i=1}^m \alpha_i} f}{\partial\omega_1^{\alpha_1}\partial\omega_2^{\alpha_1} \ \ldots \partial\omega_m^{\alpha_m}}.$$

If $m = 1$, then the polynomial functions are polynomials of only one variable, and we can obtain an orthonormal family that satisfies $\deg P_k = k$, and $M_K = K + 1$.

In addition, a reciprocal of (2.2) holds in certain circumstances. There exists a parameter t that depends only on $W(x)$ and on m such that

$$\max\left\{||f||_\infty, \left|\left|\frac{\partial f}{\partial\omega_1}\right|\right|_\infty, \left|\left|\frac{\partial f}{\partial\omega_2}\right|\right|_\infty, \ldots, \left|\left|\frac{\partial f}{\partial\omega_m}\right|\right|_\infty\right\} \le C_S \sum_{k=0}^\infty ||c_k(f)|| \deg(P_k)^t < \infty. \quad (2.3)$$

Indeed, for tensor Legendre and Chebyshev polynomials such a conclusion follows by techniques described in [5] from choosing an appropriate t, computing the Sobolev norm of weak derivatives of f whose projection can be explicitly computed for either case followed by an application of Sobolev's theorem.

Finally, for some orthogonal polynomial families, the following holds.

$$\Lambda^K = \sup_{\omega\in\Omega} \sqrt{\sum_{k=0}^{M_K} \left(P_k(\omega)\right)^2} \le C_\Lambda {M_K}^d, \quad (2.4)$$

where d and C_Λ are parameters, depending on m, Ω, but not on K. For tensor-product Chebyshev polynomials, and $\Omega = [-1, 1]^m$, it is immediate that $C_\Lambda = 1$ and $d = \frac{m}{2}$. For tensor-product Legendre polynomials, one can choose $d = m$, following the properties of separable weight functions [9, Proposition 7.1.5] as well as the asymptotic properties of Λ^K for the Legendre case when $m = 1$ [21, Lemma 21].

In addition, for the case where

$$\int_\Omega W(\omega) d(\omega) = 1,$$

we can interpret $W(\omega)$ as a probability density function (this case can be achieved for any weight function after rescaling with a constant). In that case, we may refer to ω as a random variable, and it is the case we treat in this work.

Notations The expectation of a function $f(\omega)$ of the random variable is

$$E_\omega\left[f(\omega)\right] = \int_\Omega f(\omega)W(\omega)d\omega \stackrel{\Delta}{=} \langle f(\omega)\rangle.$$

The last notation is useful to compact mathematical formulas. Note that the symbol of the scalar product includes a comma ($<\cdot,\cdot>$). We use $||u||$ to denote the Euclidean norm of a vector $u \in \mathbb{R}^p$. For $f : \Omega \to \mathbb{R}^p$, the quantity $||f||_W = ||f(\omega)||_W$ is the L_W^2 norm, defined in (2.1), whereas $||f(\omega)||_\infty = ||||f(\omega)||||_\infty$.

When proving an inequality or equality, we will display on top of the respective sign the equation that justifies it. For example $\stackrel{(2.1)}{=}$ is an identity justified by Bessel's identity (2.1).

3. Constrained Optimization Problems. Consider the following constrained optimization (O) problem

$$
(O)\quad \begin{array}{rll} x(\omega) = \arg\min_x & f(x,\omega) & \\ \text{subject to} & g(x,\omega) & = 0_p. \end{array}
$$

Here, the function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. We are interested in approximating the mapping $x(\omega)$, where $\omega \in \Omega$.

3.1. SFEM formulations. An SFEM formulation can be obtained by writing the optimality conditions for the problem (O) after we introduce the Lagrange multiplier mapping $\lambda(\omega) : \Omega \to \mathbb{R}^m$, followed by the procedure outlined in [14]. The optimality conditions result in

$$
\begin{array}{rcl} \nabla_x f(x(\omega),\omega) + \lambda^T(\omega)\nabla_x g(x(\omega),\omega) & = & 0_n \\ g(x(\omega),\omega) & = & 0_p. \end{array} \tag{3.1}
$$

We introduce the SFEM parametrization of the approximation

$$
\tilde{x}^K(\omega) = \sum_{k=0}^{M_K} x_k P_k(\omega), \qquad \tilde{\lambda}^K(\omega) = \sum_{k=0}^{M_K} \lambda_k P_k(\omega).
$$

Here, we have that the coefficients of the expansion satisfy $x_k \in \mathbb{R}^n$ and $\lambda_k \in \mathbb{R}^p$, $k = 0, 1, 2, \ldots$. The procedure outlined in [14] results in the following system of nonlinear equations

$$
\left.\begin{array}{rcl} \left\langle P_k(\omega)\left(\nabla_x f(\tilde{x}^K(\omega),\omega) + \left(\tilde{\lambda}^K(\omega)\right)^T \nabla_x g(\tilde{x}^K(\omega),\omega)\right)\right\rangle & = & 0_n, \\ \left\langle P_k(\omega) g(\tilde{x}^K(\omega),\omega)\right\rangle & = & 0_p, \end{array}\right\} 0 \le k \le M_K. \tag{3.2}
$$

We could try to solve the equations (3.2) in order to obtain the SFEM approximation. Once we do that, however, we face the problem of determining whether the resulting system of nonlinear equations has a solution, and how we can determine it. One could imagine that certain results can be proved under the assumption that the solution of (O), $\tilde{x}^*(\omega)$, has sufficiently small variation. A result of this type will be shown in Subsection 3.3 though for weaker assumptions than the small variation of the solution. But more important from a practical perspective, we started with an optimization structure to our original problem (O) and, at first sight, the equations (3.2) do not have an optimization problem structure. This situation restricts the type of algorithms that we could use to solve the problem. Nonetheless, this difficulty is only superficial, as shown by the following theorem, which relates the solution of the nonlinear equations (3.2) to the solution of the following stochastic optimization problem:

$$
(SO(K))\quad \begin{array}{lll} \min_{\{x_k\}_{k=0,1,\ldots,M_K}} & \langle f(\tilde{x}(\omega),\omega)\rangle & \\ & \langle g(\tilde{x}(\omega),\omega) P_k(\omega)\rangle & = 0_p,\ k = 0, 1, \ldots, M_K. \end{array}
$$

Theorem 3.1. *Consider the coefficients $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{M_K}$ that are a solution of the minimization problem (SO(K)) and assume that they satisfy the KKT conditions* *with the Lagrange multipliers $\hat\lambda_0, \hat\lambda_1, \dots, \hat\lambda_{M_K}$. With these coefficients and multipliers we define the functions*

$$\hat{x}^K(\omega) = \sum_{k=0,1,\dots,M_K} \hat{x}_k P_k(\omega), \quad \hat\lambda^K(\omega) = \sum_{k=0,1,\dots,M_K} \hat\lambda_k P_k(\omega).$$

Then, $\hat{x}(\omega)$ and $\hat\lambda(\omega)$ satisfy the equations (3.2), assuming that f and g have Lipschitz first derivatives.

Proof The optimality conditions for (SO(K)), that are satisfied by the solution, since the constraint qualifications holds [19], result, for a fixed $k \in \{0, 1, \dots, M_K\}$, in

$$\begin{aligned}
0_n &= \nabla_{x_k} \left\langle f(\hat{x}^K(\omega), \omega) \right\rangle + \nabla_{x_k} \sum_{k'=0,1,\dots,M_K} \hat\lambda_{k'}^T \left\langle g(\hat{x}^K(\omega), \omega) P_{k'}(\omega) \right\rangle \\
&= \nabla_{x_k} \left\langle f(\hat{x}^K(\omega), \omega) \right\rangle + \nabla_{x_k} \left\langle \left(\sum_{k'=0,1,\dots,M_K} P_{k'}(\omega) \hat\lambda_{k'} \right) g(\hat{x}^K(\omega), \omega) \right\rangle \\
&= \nabla_{x_k} \left\langle f(\hat{x}^K(\omega), \omega) + \left(\hat\lambda^K(\omega)\right)^T g(\hat{x}^K(\omega), \omega) \right\rangle \\
&= \left\langle P_k(\omega) \left(\nabla_x f(\hat{x}^K(\omega), \omega) + \left(\hat\lambda^K(\omega)\right)^T \nabla_x g(\hat{x}^K(\omega), \omega) \right) \right\rangle,
\end{aligned}$$

where we have used the fact that the expectation operator commutes with multiplication with a parameter. We have also used the fact that f and g have Lipschitz continuous derivatives, which allows us to interchange the derivative and the expectation operator.

The last equation represents the first set of equations in (3.2) for $\hat{x}^K(\omega)$ and $\hat\lambda^K(\omega)$. Since the second set of equations must be satisfied from the feasibility conditions, the proof of the theorem is complete. □

The preceding theorem represents the main practical advance brought by this work, because it provides an alternative way of formulating the stochastic finite-element approximation when the original problem has an optimization structure. The computational advantage of the formulation (SO(K)) over the nonlinear equation formulation (3.2) is that it preserves the optimization structure and allows one to use optimization software that is guaranteed to obtain a solution of (3.2) under milder conditions than solving the nonlinear equation directly.

3.2. Assumptions. Our goal is to show that, under certain assumptions, a solution of (SO(K)) approximates a solution of (O). A key step is to ensure that the problem (SO(K)) has a feasible point whose Jacobian of the constraints is well conditioned in the neighborhood of $\tilde{x}^*(\omega)$, the solution of the problem (O). As we will later see, this result, in addition to a bounded level set condition, will be the key to ensuring that (SO(K)) is feasible and, in turn, that (SO(K)) has a solution.

An important result is the following.

Theorem 3.2 (Kantorovich's theorem for nonsquare systems of equations, [7, 23]). *Assume that $f : X \to Y$ is defined and differentiable on a ball $\mathcal{B} = \{x | \, ||x - x_0|| \leq r\}$, and assume that its derivative $F(x)$ satisfies the Lipschitz condition on $\mathcal{B}$:*

$$||F(x) - F(z)|| \leq L \, ||x - z|| \, , \forall x, z \in \mathcal{B}.$$

Here, X and Y are Banach spaces, $F(x)$ maps X onto Y, and the following estimate holds:

$$||F(x_0)^* y|| \geq \mu \,||y|| \;\; \textit{for any } y \in Y \tag{3.3}$$

with $\mu > 0$ (the star denotes conjugation). Introduce the function $H(t) = \sum_{k=1}^{\infty} t^{2^k}$, and suppose that $h = \frac{L\mu^2 ||F(x_0)||}{2} < 1$, $\rho = \frac{2H(t)}{L\mu} \leq r$. Then the equation $F(x) = 0$ has a solution that satisfies $||x - x_0|| \leq \rho$. **Note** This result is stated slightly differently in [23], where (3.3) is required for all $x \in \mathcal{B}$. However, the purpose in that reference is to prove a rate of convergence result for an iterative process. From Graves' theorem [7, Theorem 1.2] the Theorem 3.2 follows as stated. Note that for Kantorovich's Theorem for square systems [23](where the spaces X and Y are the same), the condition corresponding to (3.3) is also stated only at x_0.

We can immediately see that the nature of the constraints in (SO(K)) is quite a bit different from the one of (O). It is clear how to assume well-posedness of the constraints at the solution $\tilde{x}^*(\omega)$.

$$\textbf{[A3]} \quad \sigma_{\min}\left(\nabla_x g(\tilde{x}^*(\omega), \omega)\right) \geq \sigma_m, \forall \omega \in \Omega.$$

Here $\sigma_{\min}$ is the smallest singular value of a given matrix. It is not clear how to immediately translate [A3] into a proof of well-conditioning for the constraints of (SO(K)):

$$\langle g(\tilde{x}(\omega), \omega) P_k(\omega) \rangle = 0_p, \quad k = 0, 1, 2 \ldots, M_K,$$

which we investigate in this section. We have that $\nabla_{x_i} \left\langle g(\tilde{x}^K(\omega), \omega) P_k \right\rangle$ are the blocks of the Jacobian matrix at an SFEM approximation $\tilde{x}^K(\omega)$. Since $g(x, \omega)$ has Lipschitz continuous derivatives from Assumption [A2] below and Ω is compact, we can interchange the average and the differentiation and use the chain rule to obtain that the blocks are $\left\langle \nabla_x g(\tilde{x}^k(\omega), \omega) P_i P_k \right\rangle$.

Therefore, for fixed K, the Jacobian has dimension $p(K+1) \times n(K+1)$.

$$J^K(\tilde{x}^K) = \begin{bmatrix} J_{00}(\tilde{x}^K) & J_{01}(\tilde{x}^K) & \cdots & J_{0K}(\tilde{x}^K) \\ J_{10}(\tilde{x}^K) & J_{11}(\tilde{x}^K) & \cdots & J_{1K}(\tilde{x}^K) \\ \vdots & \vdots & \vdots & \vdots \\ J_{K0}(\tilde{x}^K) & J_{K1}(\tilde{x}^K) & \cdots & J_{KK}(\tilde{x}^K) \end{bmatrix},$$

where

$$J_{ij}(\tilde{x}^K) = \left\langle \nabla_x g(\tilde{x}^k(\omega), \omega) P_i(\omega) P_j(\omega) \right\rangle \in \mathbb{R}^{p \times n}, \quad i, j = 0, 1, \ldots, K.$$

We want to show that the matrix J^K is uniformly well-conditioned with respect to K, for K sufficiently large, at $\tilde{x}^{*,K} = \Pi_W^K(\tilde{x}^*)$. In that sense, we need to prove that its smallest singular value is bounded below. To obtain such a bound, we need a more workable expression for the minimum singular value. The minimum singular value of a matrix B of dimension $p \times n$ is the following $\inf - \sup$ condition [4]:

$$\sigma_{\min} = \inf_{\lambda \in \mathbb{R}^p} \sup_{u \in \mathbb{R}^n} \frac{\lambda^T B u}{||\lambda|| \, ||u||} = \inf_{\lambda \in \mathbb{R}^p, ||\lambda|| = 1} \; \sup_{u \in \mathbb{R}^n, ||u|| = 1} \lambda^T B u.$$

To prove our results, we need to invoke several assumptions. One of the assumptions will involve a statement about functions that have bounded level sets. We say that a function $\chi : \mathbb{R} \to \mathbb{R}$ has bounded level sets if the sets $\mathcal{L}_M^\chi = \chi^{-1}\left((-\infty, M]\right)$ are bounded for any $M \in \mathbb{R}$.

A1 Uniformly bounded level sets assumption: There exist a function $\chi(\cdot)$ that is convex and nondecreasing, and that has bounded level sets and a parameter $\gamma > 0$ such that

$$\chi\left(||x||^{\gamma}\right) \leq f(x,\omega),\ \forall \omega \in \Omega. \tag{3.4}$$

A2 Smoothness assumption: The functions $f(x,\omega)$ and $g(x,\omega)$ have Lipschitz continuous derivatives.

A4 The solution of the problem (O), $\tilde{x}^*(\omega)$ has Lipschitz continuous derivatives.

A5 There exists $L > 0$ such that $||\nabla_x g(x_1,\omega) - \nabla_x g(x_2,\omega)|| \leq L\left(||x_1 - x_2||\right)$, $\forall x_1, x_2 \in \mathbb{R}^n$.

A6 There exists a c such that

$$\left|\left|\left(\bar{J}^{K,Q}(\tilde{x}^*)\right)\right|\right| \leq c,\ \forall K, Q \in \mathbb{N}.$$

Here,

$$\bar{J}^{K,Q}(\tilde{x}) = \begin{bmatrix} \bar{J}_{0,K+1}(\tilde{x}) & \bar{J}_{0,K+2}(\tilde{x}) & \cdots & \bar{J}_{0,K+Q}(\tilde{x}) \\ \bar{J}_{1,K+1}(\tilde{x}) & \bar{J}_{1,K+2}(\tilde{x}) & \cdots & \bar{J}_{1,K+Q}(\tilde{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \bar{J}_{K,K+1}(\tilde{x}) & \bar{J}_{K,K+2}(\tilde{x}) & \cdots & \bar{J}_{K,K+Q}(\tilde{x}) \end{bmatrix},$$

where

$$\bar{J}_{i,j}(\tilde{x}) = \left\langle \nabla_x G^{\dagger}(\omega) P_i(\omega) P_j(\omega) \right\rangle \in \mathbb{R}^{p\times n}, \quad i,j = 0,1,2,\ldots,$$

and $G^{\dagger}(\omega) \in \mathbb{R}^{n\times p}$ (the pseudoinverse) is a matrix-valued Lipschitz mapping such that

$$\nabla_x g(\tilde{x}^*(\omega),\omega) G^{\dagger}(\omega) = I_p,\ \left|\left|G^{\dagger}(\omega)\right|\right| \leq \frac{1}{\sigma_m},\ \forall \omega \in \Omega. \tag{3.5}$$

The pseudoinverse exists following Assumptions [A2],[A3], and [A4].

A7 $cC_G < \frac{1}{4}$. Here $C_G > 0$ is the smallest value that satisfies

$$||\nabla_x g(\tilde{x}^*(\omega),\omega)|| \leq C_G, \quad \forall \omega \in \Omega,$$

which exists following assumptions [A2] and [A4].

Discussion All the assumptions invoked here are standard fare except for [A7]. If the constraint function $g(x,\omega)$ is linear in x and does not depend on ω, then it immediately follows that $c = 0$. Therefore the condition $cC_G \leq \frac{1}{4}$ represents a small variation assumption.

Notation We denote by $\tilde{x}^{*,K}(\omega) = \Pi_W^K \tilde{x}^*(\omega)$.

3.3. Solvability and convergence results. Define now

$$\begin{aligned} G_2^K(\lambda,u) &= \sum_{i,j=0}^{M_K} \lambda_i^T \left\langle P_i \nabla_x g(\tilde{x}^{*,K}(\omega),\omega) P_j \right\rangle u_j, \\ G^K &= \inf_{\substack{\lambda_k \in \mathbb{R}^p, \\ k=0,1,\ldots,K \\ \sum_{i=0}^{M_K} ||\lambda_k||^2 = 1}} \ \sup_{\substack{u_k \in \mathbb{R}^n, \\ k=0,1,\ldots,K \\ \sum_{j=0}^{M_K} ||u_k||^2 = 1}} G_2^K(\lambda,u). \end{aligned}$$

LEMMA 3.3. *Define* $\Gamma^K = A_0 - A_1 \left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_\infty - A_2 \left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_\infty^2$, *where*

$$A_0 = 1 - 4c^2 C_G^2,\ A_1 = 8c^2 C_G L,\ \text{and}\ A_2 = 4c^2 L^2 + 2\frac{L^2}{\sigma_m^2}. \tag{3.6}$$

Then, if $\Gamma^K > 0$, *it follows that* $c\sigma_m < 1$ *and*

$$G^K \geq \sqrt{\frac{4\Gamma^K}{\frac{1}{\sigma_m^2} - c^2}}.$$

Notation For simplicity, we use the notation $\tilde{x} = \tilde{x}^*$ and $\tilde{x}^K = \tilde{x}^{*,K}$.

Proof Define

$$\Theta^K = \left\{ \tilde{\lambda}(\omega) = \sum_{i=0}^{M_K} \lambda_i P_i(\omega) \middle| \lambda \in \mathbb{R}^p, i = 0, 1, \ldots, M_K, \sum_{i=0}^{M_K} \left|\left|\lambda_i\right|\right|^2 = 1 \right\} \tag{3.7}$$

$$\Upsilon^K = \left\{ \tilde{u}(\omega) = \sum_{i=0}^{M_K} u_i P_i(\omega) \middle| u_i \in \mathbb{R}^p, i = 0, 1, \ldots, M_K, \sum_{i=0}^{M_K} \left|\left|u_i\right|\right|^2 = 1 \right\} \tag{3.8}$$

It immediately follows from (2.1) that $\tilde{\lambda} \in \Theta^K$ implies that $\left|\left|\tilde{\lambda}(\omega)\right|\right|_W = 1$, and $\tilde{u} \in \Upsilon^K$ implies that $\left|\left|\tilde{u}(\omega)\right|\right|_W = 1$. We will use $\tilde{u}$ and $\tilde{\lambda}$ as the functional image of $\{\lambda_k\}_{k=0,1,\ldots,M_K}$, and, respectively, $\{u_k\}_{k=0,1,\ldots,M_K}$. We have that

$$G_2^K(\lambda, u) = \left\langle \tilde{\lambda}(\omega)^T \nabla_x g(\tilde{x}^K(\omega), \omega) \tilde{u}(\omega) \right\rangle.$$

We define $\mathbb{R}^n \ni e_{nK} = \frac{1}{\sqrt{n(M_K+1)}} (1, 1, \ldots, 1)^T$ and

$$0 \leq H^K(\tilde{\lambda}) = \sum_{i=0}^{M_K} \left|\left| \left\langle \tilde{\lambda}^T(\omega) \nabla_x g(\tilde{x}^K(\omega), \omega) P_i(\omega) \right\rangle \right|\right|^2. \tag{3.9}$$

We now define

$$u_k = \begin{cases} \frac{1}{\sqrt{H^K}} \left\langle \tilde{\lambda}^T(\omega) \nabla_x g(\tilde{x}^K(\omega), \omega) P_i(\omega) \right\rangle & H^K \neq 0 \\ e_{nK} & H^k = 0. \end{cases},$$

which results in $\tilde{u} \in \Upsilon^K$. With this choice we get that $G(\tilde{\lambda}, \tilde{u}) = H^K(\tilde{\lambda})$, and, using the expression of G^K, we obtain that

$$G^K \geq \inf_{\tilde{\lambda} \in \Theta^K} \sqrt{H^K(\tilde{\lambda})}.$$

So we now proceed to bound below $H^K(\tilde{\lambda})$.

From the definition of $G^\dagger(\omega)$, we have that

$$\tilde{\lambda}(\omega)^T = \tilde{\lambda}(\omega)^T \nabla_x g(\tilde{x}(\omega), \omega) G^\dagger(\omega), \quad \forall \omega \in \Omega. \tag{3.10}$$

Define

$$\mathbb{R}^n \ni \theta_i(\tilde{\lambda}) = \left\langle \tilde{\lambda}(\omega)^T \nabla_x g(\tilde{x}^K(\omega), \omega) P_i(\omega) \right\rangle$$

$$\mathbb{R}^{n \times p} \ni \mu_{ik}(G^\dagger) = \left\langle P_i(\omega) G^\dagger(\omega) P_k(\omega) \right\rangle.$$

We now discuss the well-posedness of the preceding quantities. Since $\tilde{\lambda}$ is a polynomial and, from assumptions [A2] and [A4] the function $\tilde{\lambda}^T(\omega)\nabla_x g(\tilde{x}^K(\omega),\omega)$ is continuous and Ω is compact, it follows that $\left|\left|\tilde{\lambda}(\omega)\nabla_x g(\tilde{x}^K(\omega),\omega)\right|\right|_W^2 < \infty$. This, in turn, means that $\theta_i(\tilde{\lambda})$ is well defined (the integral that defines it is absolutely convergent), and

$$\begin{array}{rcl}
\sum_{i=0}^{\infty}\left|\left|\theta_i(\tilde{\lambda})\right|\right|^2 & \stackrel{(2.1)}{=} & \left|\left|\tilde{\lambda}(\omega)\nabla_x g(\tilde{x}^K(\omega),\omega)\right|\right|_W^2 = \left\langle \left|\left|\tilde{\lambda}(\omega)\nabla_x g(\tilde{x}^K(\omega),\omega)\right|\right|^2\right\rangle \\
& \leq & \left\langle \left|\left|\tilde{\lambda}(\omega)\right|\right|^2 \left|\left|\nabla_x g(\tilde{x}^K(\omega),\omega)\right|\right|^2\right\rangle \\
& \leq & \left|\left|\tilde{\lambda}(\omega)\right|\right|_W^2 \left|\left|\nabla_x g(\tilde{x}^K(\omega),\omega)\right|\right|_\infty^2 \\
& = & \left|\left|\nabla_x g(\tilde{x}^K(\omega),\omega)\right|\right|_\infty^2 \leq \left(C_G + L\left|\left|\tilde{x}-\tilde{x}^K\right|\right|_\infty\right)^2,
\end{array} \tag{3.11}$$

where the last inequality follows from $\left|\left|\tilde{\lambda}(\omega)\right|\right|_W^2 = 1$ (since $\tilde{\lambda}\in\Theta^K$), the triangle inequality

$$\begin{array}{rcl}
\left|\left|\nabla_x g(\tilde{x}^K(\omega),\omega)\right|\right|_\infty & \leq & \left|\left|\nabla_x g(\tilde{x}(\omega),\omega)\right|\right|_\infty + \left|\left|\nabla_x g(\tilde{x}^K(\omega),\omega) - \nabla_x g(\tilde{x}(\omega),\omega)\right|\right|_\infty \\
& \leq & C_G + L\left|\left|\tilde{x}-\tilde{x}^K\right|\right|_\infty,
\end{array}$$

and the notations from Assumptions [A5], and [A7].

From (3.10), using the extension of $< h,l >_W = \sum_{i=0}^{\infty} c_i(h)c_i(l)$, that holds for $h,l\in L_W^2$, to matrix-valued mappings, we obtain that, for $\forall k\leq M_K$, we have that

$$\begin{array}{l}
\left\langle P_k\tilde{\lambda}(\omega)^T\right\rangle = \sum_{i=1}^{\infty}\left\langle \tilde{\lambda}(\omega)^T\nabla_x g(\tilde{x}(\omega),\omega)P_i(\omega)\right\rangle \mu_{ik}(G^\dagger) = \sum_{i=1}^{\infty}\theta_i(\tilde{\lambda})^T\mu_{ik}(G^\dagger) + \\
\qquad\qquad \left\langle P_k(\omega)\tilde{\lambda}(\omega)^T\left(\nabla_x g(\tilde{x}(\omega),\omega) - \nabla_x g(\tilde{x}^K(\omega),\omega)\right)G^\dagger(\omega)\right\rangle.
\end{array}$$

Since $\tilde{\lambda}\in\Theta^K$ and from the preceding equation, we have that

$$1 = \sum_{k=0}^{M_K}\left|\left|\left\langle P_k,\tilde{\lambda}(\omega)^T\right\rangle\right|\right|^2 \leq 2\sum_{k=0}^{M_K}\left|\left|\sum_{i=0}^{\infty}\theta_i(\tilde{\lambda})^T\mu_{ik}(G^\dagger)\right|\right|^2 + 2T_3 \leq 4(T_1+T_2)+2T_3, \tag{3.12}$$

where the last two inequalities follow from the inequality $||a+b||^2 \leq 2(||a||^2+||b||^2)$ applied twice and from Bessel's identity (2.1) where

$$\begin{array}{l}
T_1 = \sum_{k=0}^{M_K}\left|\left|\sum_{i=0}^{M_K}\theta_i(\tilde{\lambda})^T\mu_{ik}(G^\dagger)\right|\right|^2 \quad T_2 = \sum_{k=0}^{M_K}\left|\left|\sum_{i=M_K+1}^{\infty}\theta_i(\tilde{\lambda})^T\mu_{ik}(G^\dagger)\right|\right|^2 \\
T_3 = \sum_{k=0}^{M_K}\left|\left|\left\langle P_k(\omega)\tilde{\lambda}(\omega)^T\left(\nabla_x g(\tilde{x}(\omega),\omega) - \nabla_x g(\tilde{x}^K(\omega),\omega)\right)G^\dagger(\omega)\right\rangle\right|\right|^2.
\end{array}$$

We now find upper bounds on T_1, T_2, and T_3. We define $\tilde{\theta}(\omega) = \sum_{i=0}^{M_K}\theta_i(\tilde{\lambda})P_i(\omega)$. We obtain that

$$\begin{array}{rcl}
T_1 & = & \sum_{k=0}^{M_K}\left|\left|\left\langle\tilde{\theta}(\omega)^T G^\dagger(\omega)P_k\right\rangle\right|\right|^2 \stackrel{(2.1)}{\leq} \left\langle\left|\left|\tilde{\theta}(\omega)^T G^\dagger(\omega)\right|\right|^2\right\rangle \leq \left\langle\left|\left|\tilde{\theta}(\omega)\right|\right|^2\left|\left|G^\dagger(\omega)\right|\right|^2\right\rangle \\
& \stackrel{\text{by }(3.5)}{\leq} & \frac{1}{\sigma_m^2}\left\langle\left|\left|\tilde{\theta}(\omega)\right|\right|^2\right\rangle \stackrel{(2.1)}{=} \frac{1}{\sigma_m^2}\sum_{k=0}^{M_K}\left|\left|\theta_k(\tilde{\lambda})\right|\right|^2 \stackrel{(3.9)}{=} \frac{H^K(\tilde{\lambda})}{\sigma_m^2}.
\end{array}$$

Using [A6], we obtain that

$$\begin{aligned} T_2 &\leq c^2 \sum_{k=M_K+1}^{\infty} \left|\left|\theta_k(\tilde{\lambda})\right|\right|^2 = c^2 \sum_{k=0}^{\infty} \left|\left|\theta_k(\tilde{\lambda})\right|\right|^2 - c^2 \sum_{k=0}^{M_K} \left|\left|\theta_k(\tilde{\lambda})\right|\right|^2 \\ &\overset{\text{by (3.11), (3.9)}}{\leq} c^2 \left(C_G + L\left|\left|\tilde{x} - \tilde{x}^K\right|\right|\right)^2 - c^2 H^K(\tilde{\lambda}). \end{aligned}$$

Finally, using Bessel's identity (2.1), Cauchy-Schwarz, [A5], and that $\left|\left|\lambda(\tilde{\omega})\right|\right|_W = 1$, which follows from $\tilde{\lambda} \in \Theta^K$, we obtain that

$$\begin{aligned} T_3 &\overset{(2.1)}{\leq} \left|\left|\tilde{\lambda}(\omega)^T \left(\nabla_x g(\tilde{x}(\omega), \omega) - \nabla_x g(\tilde{x}^K(\omega), \omega)\right) G^{\dagger}(\omega)\right|\right|_W^2 \\ &\leq \left\langle \left|\left|\tilde{\lambda}(\omega)\right|\right|^2 \left|\left|\left(\nabla_x g(\tilde{x}(\omega), \omega) - \nabla_x g(\tilde{x}^K(\omega), \omega)\right)\right|\right|^2 \left|\left|G^{\dagger}(\omega)\right|\right|^2 \right\rangle \\ &\overset{\text{by [A5],(3.5)}}{\leq} \left(\frac{L}{\sigma_m}\right)^2 \left|\left|\tilde{x} - \tilde{x}^K\right|\right|_\infty^2. \end{aligned}$$

Replacing the bounds obtained for T_1, T_2, and T_3 in (3.12), we obtain that

$$4H^K(\tilde{\lambda}) \left(\frac{1}{\sigma_m^2} - c^2\right) \geq \Gamma^K = A_0 - A_1 \left|\left|\tilde{x} - \tilde{x}^K\right|\right|_\infty - A_2 \left|\left|\tilde{x} - \tilde{x}^K\right|\right|_\infty^2,$$

where A_0, A_1, A_2 are defined in (3.6). Since $\Gamma^K > 0$ implies that $A_0 > 0$, which in turn implies that $cC_G < \frac{1}{4}$, we get, from [A3], that $c\sigma_m < \frac{1}{4}$. The conclusion follows from the preceding displayed inequality. □

A key point of our analysis consists of obtaining bounds between $\left|\left|\tilde{x}^K\right|\right|_\infty$ and $\left|\left|\tilde{x}^K\right|\right|_W$.

Lemma 3.4.

$$\left|\left|\tilde{x}^K\right|\right|_W \leq \left|\left|\tilde{x}^K\right|\right|_\infty \leq \Lambda^K \left|\left|\tilde{x}^K\right|\right|_W$$

Proof It is immediate that $\left|\left|\tilde{x}^K\right|\right|_W \leq \left|\left|\tilde{x}^K\right|\right|_\infty$. We have, using Cauchy-Schwarz, that

$$\left|\left|\tilde{x}^K(\omega)\right|\right| = \left|\left|\sum_{k=0}^{M_K} x_k P_k(\omega)\right|\right| \leq \sum_{k=0}^{M_K} \left|\left|x_k\right|\right| \left|P_k(\omega)\right| \leq \sqrt{\sum_{k=0}^{M_K} \left|\left|x_k\right|\right|^2} \sqrt{\sum_{k=0}^{M_K} \left|P_k(\omega)\right|^2}.$$

From the definition of Λ^K, (2.4), we obtain that

$$\sup_{\omega \in \Omega} \left|\left|\tilde{x}^K(\omega)\right|\right| \leq \Lambda^K \left|\left|\tilde{x}^K(\omega)\right|\right|_W,$$

which proves the claim. □

Lemma 3.5.

$$\left|\left|J^K\left(\tilde{x}_1(\omega)\right) - J^K\left(\tilde{x}_2(\omega)\right)\right|\right| \leq L\Lambda^K \left|\left|\tilde{x}_1(\omega) - \tilde{x}_2(\omega)\right|\right|_W$$

Notation We will denote $\tilde{x}_1 = \tilde{x}_1(\omega)$, $\tilde{x}_2 = \tilde{x}_2(\omega)$.

Proof By algebraic manipulations and notations similar to the ones in Lemma 3.3, we obtain that

$$\begin{aligned}
\left|\left|J^K\left(\tilde{x}_1\right)-J^K\left(\tilde{x}_2\right)\right|\right| &= \\
\sup_{\substack{\lambda_k \in \mathbb{R}^p,\ u_k \in \mathbb{R}^n \\ k=0,1,\ldots,K \\ \sum_{k=0}^{M_K}||\lambda_k||^2=1 \\ \sum_{k=0}^{M_K}||u_k||^2=1}} \sum_{i,j=0}^{M_K} \lambda_i^T \left\langle P_i\left(\nabla_x g(\tilde{x}_1,\omega)-\nabla_x g(\tilde{x}_2,\omega)\right)P_j\right\rangle u_j &= \\
\sup_{\tilde{\lambda}\in\Theta^K,\tilde{u}\in\Upsilon^K}\left\langle \tilde{\lambda}(\omega)^T\left(g(\tilde{x}_1,\omega)-\nabla_x g(\tilde{x}_2,\omega)\right)\tilde{u}(\omega)\right\rangle &\overset{\text{by [A5]}}{\leq} \\
L\left|\left|\tilde{x}_2-\tilde{x}_1\right|\right|_\infty \sup_{\tilde{\lambda}\in\Theta^K,\tilde{u}\in\Upsilon^K}\left\langle\left|\left|\tilde{\lambda}(\omega)\right|\right|\left|\left|\tilde{u}(\omega)\right|\right|\right\rangle \overset{\text{Cauchy-Schwarz}}{\leq} L\left|\left|\tilde{x}_2-\tilde{x}_1\right|\right|_\infty &\overset{\text{by Lemma 3.4}}{\leq} \\
L\Lambda^K\left|\left|\tilde{x}_2-\tilde{x}_1\right|\right|,
\end{aligned}$$

which completes the claim. □

LEMMA 3.6. *The objective function of the problem (SO(K)) has bounded level sets.*

Proof Take $\tilde{x}^K(\omega)=\sum_{k=0}^{M_K}x_kP_k(\omega)$. Consider the level set of height M of $\tilde{f}^K(\tilde{x}^K)=\left\langle f(\tilde{x}^K(\omega),\omega)\right\rangle$,

$$\mathcal{L}_K(M)=\left\{(x_0,x_1,\ldots,x_{M_K})\in\mathbb{R}^{mM_K}\Big|\quad \tilde{f}^K\left(\tilde{x}\right)\leq M\right\}.$$

Using Assumption [A1], we obtain that, if $(x_0,x_1,\ldots,x_{M_K})\in\mathcal{L}_K(M)$, then

$$\begin{aligned}
M \quad &> \quad \tilde{f}\left(\tilde{x}^K\right)=\left\langle f\left(\tilde{x}^K(\omega),\omega\right)\right\rangle \overset{\text{[A1]}}{\geq}\left\langle\chi\left(\left|\left|\tilde{x}^K(\omega)\right|\right|^\gamma\right)\right\rangle \\
&\overset{\text{by Jensen's inequality}}{\geq} \chi\left(\left\langle\left|\left|\tilde{x}^K(\omega)\right|\right|^\gamma\right\rangle\right)\Longrightarrow\left\langle\left|\left|\tilde{x}^K(\omega)\right|\right|^\gamma\right\rangle\in\mathcal{L}_M^\chi. \qquad (3.13)
\end{aligned}$$

We denote

$$L_K^\gamma=\min_{\sum_{k=0}^{M_K}||x_k||^2=1}\left\langle\left|\left|\tilde{x}^K(\omega)\right|\right|^\gamma\right\rangle.$$

Since the unit ball $\mathcal{B}^K\in\mathbb{R}^{n(M_K+1)}$, defined as

$$\mathcal{B}^K=\left\{(x_0,x_1,\ldots,x_{M_K})\in\mathbb{R}^{n(M_K+1)}\Big|\sum_{k=0}^{M_K}||x_k||^2=1\right\},$$

is a compact set, the quantity L_K^γ is well defined.

It also immediately follows that $L_K^\gamma>0$, $\forall K>0$. Indeed, if there existed a K for which $L_K^\gamma=0$, it would follow that for some choice of $x_0,x_1,\ldots,x_K$, such that $\sum_{k=0}^{M_K}||x_k||^2=1$, we have that $\tilde{x}^K(\omega)=0$, $\forall\omega\in\Omega$, which contradicts the fact that $P_k(\omega)$ are linearly independent because they are a subset of a basis.

In return (3.13) results in $\chi\left(L_K^\gamma\left(\sum_{k=0}^{M_K}||x_k||^2\right)\right)\leq M$, which, in turn, results in $\left(\sum_{k=0}^{M_K}||x_k||^2\right)\leq\frac{\chi^{-1}(M)}{L_K^\gamma}$. Since we assumed that the function χ has bounded level sets, the conclusion follows. □

Note Lemma 3.6 ensures that the solution of the systems of nonlinear equations that defines the spectral stochastic finite element method [14, 13] does exist, at least for the case where the system of nonlinear equations is derived from the optimality conditions of an unconstrained optimization problem. The same result can be obtained for constrained problems from Lemma 3.3 for all K when $g(x,\omega)$ is linear in x and does not depend on ω, since [A7] is satisfied with $c=0$. To our knowledge, this is a new result for the case where the the variation of $\tilde{x}^*(\omega)$ is not necessarily small.

LEMMA 3.7. *Assume that*

(a) $lim_{K\to\infty} \left|\left|\tilde{x}^* - \Pi_W^K \tilde{x}^*\right|\right|_\infty = 0$ *and*

(b) that $lim_{K\to\infty}\Lambda^K \left|\left|\tilde{x}^* - \Pi_W^K \tilde{x}^*\right|\right|_W = 0$.

For any $r>0$*, there exists* K_0 *such that (SO(K)) has a feasible point* $\bar{x}^K(\omega)$ *that satisfies* $\left|\left|\bar{x}^K - \Pi_W^K \tilde{x}^*\right|\right|_W \le r$, $\forall K \ge K_0$.

Proof We seek to apply Kantorovich's Theorem 3.2. With the notations in the assumptions [A1]–[A7], and from the definition of G^K preceding Lemma 3.3 and from the definition of Λ^K (2.4), it follows, using Lemma 3.5, that the conditions of the theorem are satisfied at $\tilde{x}^{*,K} = \Pi_W^K(\tilde{x}^*)$ provided the following two conditions hold:

$$(i)\ h = \frac{L\Lambda^K \left(G^K\right)^2 g^K}{2} < 1, \quad (ii')\ \rho = \frac{2H(h)}{L\Lambda^K G^K} < r,$$

where $g^K = \sqrt{\sum_{k=0}^{M_K} ||\langle g(\tilde{x}^{*,K}(\omega),\omega)P_k\rangle||^2}$. Note that if $h<\frac{1}{2}$, we have that $h \le H(h) \le 2h$. Therefore a sufficient condition for the condition (ii) to hold is

$$(ii)\ 4G^K g^K < r.$$

We have that

$$\left(g^K\right)^2 = \sum_{k=0}^{M_K} \left|\left|\langle g(\tilde{x}^{*,K}(\omega),\omega)P_k\rangle\right|\right|^2 \overset{(2.1)}{\le} \left|\left|g(\tilde{x}^{*,K}(\omega),\omega)\right|\right|_W^2.$$

From Leibnitz-Newton and assumptions [A2] and [A4] we get

$$g(\tilde{x}^{*,K}(\omega),\omega) = g(\tilde{x}^*(\omega),\omega) + \int_0^1 \nabla_x g(\bar{x}(t,\omega),\omega)\left(\tilde{x}^{*,K}(\omega) - \tilde{x}^*(\omega)\right)dt,$$

where $\bar{x}(t,\omega) = t\tilde{x}^{*,K}(\omega) + (1-t)\tilde{x}^*(\omega)$. Since $\tilde{x}^*(\omega)$ is a solution of (O), for any $\omega \in \Omega$, we get $g(\tilde{x}^*(\omega),\omega) = 0$, $\forall \omega \in \Omega$. Using Assumptions [A7] and [A5], we obtain the following

$$\begin{aligned}\left|\left|g(\tilde{x}^{*,K}(\omega),\omega)\right|\right| &\le \left|\left|\tilde{x}^{*,K}(\omega) - \tilde{x}^*(\omega)\right|\right| \int_0^1 (C_G + L\,||\bar{x}(t,\omega) - \tilde{x}^*(\omega)||)dt \\ &\le \left(C_G + \frac{L}{2}\left|\left|\tilde{x}^{*,K}(\omega) - \tilde{x}^*(\omega)\right|\right|\right)\left|\left|\tilde{x}^{*,K}(\omega) - \tilde{x}^*(\omega)\right|\right|.\end{aligned}$$

In turn, this implies that

$$g^K \le \left|\left|g(\tilde{x}^{*,K}(\omega),\omega)\right|\right|_W \le \left(C_G + \frac{L}{2}\left|\left|\tilde{x}^{*,K} - \tilde{x}\right|\right|_\infty\right)\left|\left|\tilde{x}^{*,K} - \tilde{x}^*\right|\right|_W. \tag{3.14}$$

From assumption (a) of this theorem, we have that $\exists K_0$ such that, $\forall K \ge K_0$,

$$L\left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_\infty \le C_G, \quad A_1\left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_\infty + A_2\left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_\infty^2 \le \frac{A0}{2},$$

where A_0, A_1, and A_2 are defined in (3.6). With the notations of Lemma 3.3, $\Gamma^K \geq \frac{A_0}{2}$, and thus from Assumption [A7], which ensures that $A_0 > 0$, we get that $G^K \geq A_3 \stackrel{\Delta}{=} \sqrt{\frac{2A_0}{\frac{1}{\sigma_m^2} - c}} > 0$. Therefore, for $K \geq K_0$ and from (i),(ii), and (3.14) we get that the conditions of Kantorovich's theorem 3.2 are satisfied provided that

$$2LA_3^2 C_G \Lambda^K \left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_W \leq 1, \quad 8A_3 C_G \left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_W \leq r.$$

From assumptions (a) and (b), it follows that these conditions are satisfied, by eventually choosing a larger K_0, for all $K \geq K_0$. Therefore, Kantorovich's theorem 3.2 applies to give the conclusion. The proof is complete. □

THEOREM 3.8. *Assume that $\tilde{x}^*(\omega)$ is smooth (infinitely differentiable). Then there exists K_0 such that (SO(K)) has a solution, $\forall K \geq K_0$.*

Proof The key of the proof is that we are able to choose q as large as necessary in (2.2) for $f = \tilde{x}^*$. Choose $q = t + m + 2$. We obtain from (2.3) that

$$\left|\left|\tilde{x}^* - \Pi_W^K \tilde{x}^*\right|\right|_\infty \leq mC_S \sum_{k=M_{K+1}+1}^{\infty} ||c_k(f)|| \deg(P_k)^t.$$

Since the number of polynomials of degree at most K is $\begin{pmatrix} m+K \\ m \end{pmatrix}$ [9] we obtain from (2.2) and (2.1) that $||c_k(f)|| \leq CQ^{-q}$, and from the preceding displayed equation and (2.3), that

$$\begin{aligned} \left|\left|\tilde{x}^* - \Pi_W^K \tilde{x}^*\right|\right|_\infty &\leq mCC_S \sum_{Q=K+1}^{\infty} \begin{pmatrix} m+Q \\ m \end{pmatrix} Q^{-(t+m+2)} Q^t \\ &\leq \frac{CC_S}{(m-1)!} \sum_{Q=K+1}^{\infty} Q^{-2} \left(\frac{m+Q}{Q}\right)^m \stackrel{K\to\infty}{\longrightarrow} 0 \end{aligned}$$

and thus

$$\lim_{K\to\infty} \left|\left|\tilde{x}^* - \Pi_W^K \tilde{x}^*\right|\right|_\infty = 0. \tag{3.15}$$

In addition, from (2.4) and (2.2) we obtain that

$$\Lambda^K \left|\left|\tilde{x}^* - \Pi_W^K \tilde{x}^*\right|\right|_W \leq C \frac{1}{K^q} \begin{pmatrix} m+K \\ m \end{pmatrix}^d \leq C \frac{1}{m! K^{q-dm}} \left(\frac{m+K}{K}\right)^{md}.$$

Therefore, if we choose $q \geq md + 1$, we get that $\Lambda^K \left|\left|\tilde{x}^* - \tilde{x}^{*,K}\right|\right|_W \stackrel{K\to\infty}{\longrightarrow} 0$. From (3.15), conditions (a) and (b) of Lemma 3.7 are satisfied. We apply Lemmas 3.7 and 3.6 to obtain that problem (SO(K)) is feasible for $K \geq K_0$ and has bounded level sets and thus has a solution [19]. The proof is complete. □

THEOREM 3.9. *Let $m = 1$ and $W(x) = \sqrt{1-x^2}^{-1}$ (the Chebyshev polynomials case). Then $SO(K)$ has a solution for all $K \geq K_0$.*

Proof From [A4], $\tilde{x}^*(\omega)$ is continuous and has bounded variation; therefore $\left|\left|\tilde{x}^*(\omega) - \Pi_W^K \tilde{x}^*(\omega)\right|\right|_\infty \to 0$ as $K \to \infty$ [17, Theorem 1]. Also, from [A4], (2.4), and (2.2) we obtain that

$$\Lambda^K \left|\left|\tilde{x}^*(\omega) - \Pi_W^K \tilde{x}^*(\omega)\right|\right|_W \leq K^{\frac{1}{2}} C \frac{1}{K} \stackrel{K\to\infty}{\longrightarrow} 0.$$

Conditions (a) and (b) of Lemma 3.7 therefore are satisfied. Therefore Lemmas 3.7 and 3.6 apply to give that the problem (SO(K)) is feasible for $K \geq K_0$ and its objective function has bounded level sets. Therefore (SO(K)) is solvable [19], and the proof is complete. $\square$

Discussion Theorem 3.8 completely addresses the issue of solvability of (SO(K)) in the case of smooth solution functions, independent of the dimension of the problem. The result for nonsmooth solution functions Theorem 3.9 is restrictive in terms of both dimensions and polynomial type, and its extension is deferred to future research.

Finally, we approach the issue of limits of solutions of (SO(K)) for increasing K. For convergence as $K \to \infty$ we need to invoke stronger assumptions, that allow us to guarantee the existence of convergent subsequences.

THEOREM 3.10. *Assume that the conditions of Theorem 3.8 are satisfied, that the sequence of solutions of the problem (SO(K)) satisfies the Kuhn-Tucker conditions, and that there exists a $C_X > 0$ such that the solution and multiplier sequences (λ_k^K, x_k^K) satisfy*

$$\sum_{k=0}^{M_K} \left|\left|\lambda_k^K\right|\right| \deg(P_k)^t < C_X; \quad \sum_{k=0}^{M_K} \left|\left|x_k^K\right|\right| \deg(P_k)^t < C_X,$$

where t is the parameter from (2.3). Define $\tilde{\lambda}^K(\omega) = \sum_{k=0}^{M_K} \lambda_k^K P_k(\omega)$ and $\tilde{x}^K(\omega) = \sum_{k=0}^{M_K} x^K P_k(\omega)$. Then the sequence $(\tilde{x}^K(\omega), \tilde{\lambda}^K(\omega))$ has a uniformly convergent subsequence. Any limit $(\hat{x}(\omega), \hat{\lambda}(\omega))$ of such a subsequence satisfies the nonlinear system of equations (3.1).

Proof From (2.3) it follows that the sequence $(\tilde{\lambda}^K(\omega), \tilde{x}^K(\omega))$ satisfies

$$\forall \omega_1,\, \omega_2 \in \Omega \left\{ \begin{array}{l} \left|\left|\tilde{\lambda}^K(\omega_1)\right|\right|,\ \left|\left|\tilde{x}^K(\omega_1)\right|\right| \leq C_S C_X, \\ \left|\left|\tilde{\lambda}^K(\omega_1) - \tilde{\lambda}^K(\omega_2)\right|\right|,\ \left|\left|\tilde{x}^K(\omega_1) - \tilde{x}^K(\omega_2)\right|\right| \leq C_S C_X \left|\left|\omega_1 - \omega_2\right|\right|. \end{array} \right.$$

Therefore the families $\tilde{\lambda}^K(\omega)$, $\tilde{x}^K(\omega)$ are equicontinuous and equibounded. We can apply the Arzela-Ascoli theorem [15, Theorem 6.41] to determine that there exists a uniformly convergent subsequence, $\tilde{x}^{K_l}, \tilde{\lambda}^{K_l}$ with a corresponding limit pair. Let $(\hat{x}, \hat{\lambda})$ be such a limit function pair, which must also be continuous because the convergence of the subsequence of Lipschitz functions is continuous. Using Theorem (3.1), we get that $\tilde{x}^{K_l}, \tilde{\lambda}^{K_l}$ satisfies the equation (3.2), for $l \geq 0$. Using assumptions [A2] and [A4], we can take the limit in that equation and obtain that

$$\begin{aligned} \left\langle P_k(\omega) \left(\nabla_x f(\hat{x}(\omega), \omega) + \left(\hat{\lambda}(\omega)\right)^T \nabla_x g(\hat{x}(\omega), \omega) \right) \right\rangle &= 0_n,\ k \geq 0, \\ \left\langle P_k(\omega) g(\hat{x}(\omega), \omega) \right\rangle &= 0_p,\ k \geq 0. \end{aligned}$$

From Bessel's identity (2.1), we get that

$$\left|\left| \left(\nabla_x f(\hat{x}(\omega), \omega) + \hat{\lambda}(\omega)^T \nabla_x g(\hat{x}(\omega), \omega) \right) \right|\right|_W^2 + \left|\left| g(\hat{x}(\omega), \omega) \right|\right|_W^2 = 0,$$

which, in turn, proves our claim. The proof is complete. $\square$

Discussion Of course, it would be important to prove the convergence of the approximating sequences $\tilde{x}^K(\omega)$ and $\tilde{\lambda}^K(\omega)$ without assuming that they exhibit sufficient smoothness in the limit. For this initial investigation, we provide this limited result, and we defer the issue of extending it to further research. A promising approach seems to be to quantify the uniform validity with ω of the second order sufficient conditions for problem (O) and infer the smoothness in the limit from it.

4. Applications and Numerical Examples. Motivating our investigation was the study of parametric eigenvalue problems as they appear in neutron diffusion problems in nuclear reactor criticality analysis [8]. We thus investigate how our developments apply to eigenvalue problems.

4.1. Parametric Eigenvalue Problems. In the following, we study our formulation for two parametric eigenvalue problems, of sizes $n = 2$ and $n = 1000$. In the formulation of the problem for both cases is $(Q + \omega D_Q)x(\omega) = \lambda(\omega)\, x(\omega)$, where Q and D_Q are matrices of size n, $\lambda(\omega)$ and $x(\omega)$ are the smallest eigenvalue and the corresponding eigenvector of the matrix $(Q + \omega D_Q)$. Our theory is applied via the interpretation of the problem as $x(\omega) = \arg\min_{x(\omega)^T x(\omega)=1} x(\omega)^T (Q + \omega D_Q)\, x(\omega)$, where $\lambda(\omega)$ is the Lagrange multiplier of the constraint, all for a fixed value of ω. Here, $\omega \in [-1, 1]$, and the stochastic finite element problem is constructed by using either Legendre or Chebyshev polynomials [9].

As in our theoretical developments, the problem to be solved has $n \times (M_K + 1)$ unknowns and $M_K + 1$ constraints, and, with the notation $\Phi^K = \{0, 1, \ldots, M_K\}$, can be stated as

$$\begin{array}{ll} \min_{\{x_i\}_{i\in\Phi^K}} & \left\langle \left(\sum_{i=0}^{M_K} x_i P_i(\omega)\right)^T (Q + \omega D_Q) \left(\sum_{i=0}^{M_K} x_i P_i(\omega)\right) \right\rangle \\ \text{s.t.} \forall k \in \Phi^K & \\ & \left\langle \left(\sum_{i=0}^{M_K} x_i P_i(\omega)\right)^T \left(\sum_{i=0}^{M_K} x_i P_i(\omega)\right) P_k(\omega) \right\rangle = \langle P_k(\omega) \rangle. \end{array}$$

The problem is set up by computing the terms involved after breaking up the parentheses, computing the terms $\langle \omega P_i(\omega)\, P_j(\omega) \rangle = L_{i,j}$ and $\langle P_i(\omega)\, P_j(\omega)\, P_k(\omega) \rangle = \hat{L}_{i,j,k}$. This procedure was carried out by numerical quadrature in MATLAB, after which, the resulting problem became

$$\begin{array}{ll} \min & \sum_{i=1}^{K} x_i Q x_i + \sum_{i,j=1}^{K} L_{ij} x_i D_Q x_j \\ \text{s.t.} & \sum_{i,j=1}^{K} \hat{L}_{ijk} x_i x_j = E_\omega [P_k(\omega)] \quad k = 1, 2, \ldots, K. \end{array}$$

The problem was coded in AMPL [11] and solved by using the KNITRO interior point solver [22]. Once the problem was solved, the parametric approximation of the solution and of the multiplier were constructed as $\tilde{x}(\omega) = \sum_{i=1}^{K} x_k P_k(\omega), \quad \lambda(\omega) = \sum_{i=1}^{K} \lambda_k P_k(\omega)$.

It is immediate that the problem satisfies assumptions [A2]–[A6]. Assumption [$A1$] is satisfied only if the resulting matrix is positive definite for any value of ω, which can be ensured if one adds a suitable fixed multiple of the identity to the matrix. In that case [A1] is satisfied with $\gamma = 1$ and $\chi(r) = r^2$. Since the effect of that is only to shift the λ values, we can assume without loss of generality that [A1] is satisfied. [A7] is a difficult assumption to verify numerically, and like any small variation assumptions, it is bound to be too conservative.

4.2. Problems with Inequality Constraints. It is well known that we can transform an inequality constraint $g_1(x, \omega) \leq 0$ into an equality constraint by using a slack s_1 and representing the inequality as $g_1(x, \omega) + s_1^2 = 0$ [2]. The resulting problem can be represented as (O), and our approach can be used to solve it. To generate the problem (SO(K)), we use a parameterization for the slacks $s_1(\omega) =$

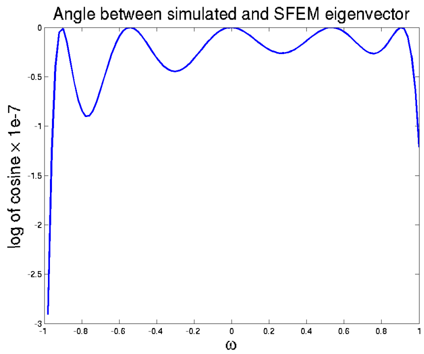

FIG. 4.1. *The case $n = 2$. The eigenvector angle error*

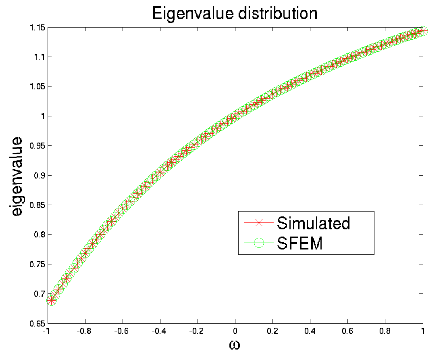

FIG. 4.2. *The case $n = 2$. The eigenvalue.*

$\sum_{k=0}^{K} P_k(\omega)s_{1,k}$. When we enforce the constraints of (SO(K)), we get expressions similar to the constraints of the eigenvalue problem in the preceding section, which means that the effect of s_1 on the constraints can be represented finitely in the spectral basis. Therefore, if the functions of the inequality constraints can be represented finitely in the spectral basis, the introduction of slacks will not destroy that. This means our approach and Theorem 3.1 applies to inequality constraints as well, once we have formulated them as slacks. Theorem 3.8 cannot be expected to apply because the solution $\tilde{x}^*(\omega)$ is not smooth in general when inequality constraints are present. Theorem 3.9 may apply but it is limited to the case $m = 1$. For the convergence analysis of problems with inequality constraints, further analysis is necessary.

4.3. The $n = 2$ Problem. For this problem, we chose $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $D_Q = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}$, and we use only Legendre polynomials. We have computed the minimum eigenvalue and the corresponding eigenvector as a function of the parameter ω, both by solving the eigenvalue problem at 100 equally spaced points in the interval between minus 1 and 1, and by using our constrained optimization formulation the stochastic finite element method. The results of the two approaches have been plotted in Figure 4.1 for the angle between the eigenvectors obtained by the two approaches and in Figure 4.2 for the eigenvalue. We call here, in Figures 4.1-4.2, and subsequently, the first approach simulation and the second approach "SFEM". It can be seen that the error for the cosine of the angle between eigenvalues is in the seventh decimal place, and the eigenvalue results are virtually indistinguishable. Note that the size of the variation for which we were computing the eigenvector reaches half the size of the maximum element in the original matrix, so the variation is far from being considered small. The results show the soundness of our approach and provide good evidence for convergence. In addition, the solution of the problem seems to be smooth, so the conditions for both Theorems 3.8 and 3.10, as well as their conclusions, appear to be satisfied.

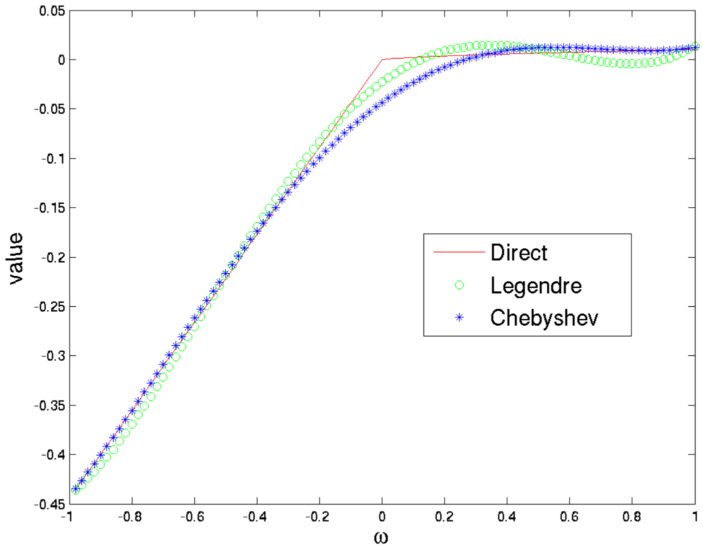

FIG. 4.3. *The case $n = 1000$. The eigenvalue.*

4.4. The $n = 1000$ Problem. For this problem, we chose

$$Q_{i,j} = \begin{cases} 1 & i=j=1 \\ 1 & i=j=n \\ -1 & |i-j|=1 \\ 2 & 1<i=j<n \\ 0 & \text{otherwise} \end{cases}, \quad D_{Q\ i,j} = \begin{cases} 2\frac{i}{n}\frac{n-i}{n}\cos\left(\frac{i}{n}\right) & i=j \\ 0 & \text{otherwise} \end{cases}.$$

This problem mimics a one-dimensional criticality analysis of the neutron flux in a nuclear reactor [8]. Again, we computed a minimum eigenvalue and the corresponding eigenvector as a function of the parameter ω both by solving the eigenvalue problem at 100 equally spaced points in the interval between minus 1 and 1, as well as by using our constrained optimization formulation of the stochastic finite element method. In this case the calculation was done with both Chebyshev and Legendre polynomials.

The results are displayed in the Figures 4.3-4.4 for the match between the eigenvalues, as well as the angle between the eigenvectors. We see that the eigenvalues match very well, with a relative error below 5% (with respect to the vector infinity norm). Note that the optimization problem was solved in 8 seconds for the Legendre polynomials and 13 seconds for the Chebyshev polynomials, whereas the simulated eigenvalues took more than 1,300 seconds to compute on the same machine. Also note that the possible objection that MATLAB is much slower for the simulation approach does not apply here, since we have timed only the call to the eigenvalue function in MATLAB, which is an external call to a compiled function. Of course, the numbers are relatively difficult to compare, given that we did not truly try to find the minimum number of simulated values with which to evaluate the parametric dependence of the eigenvalue to the same tolerance level. Nonetheless the results show the tremendous advantage that our approach has for the efficient evaluation of the parametric dependence of the minimum eigenvalue.

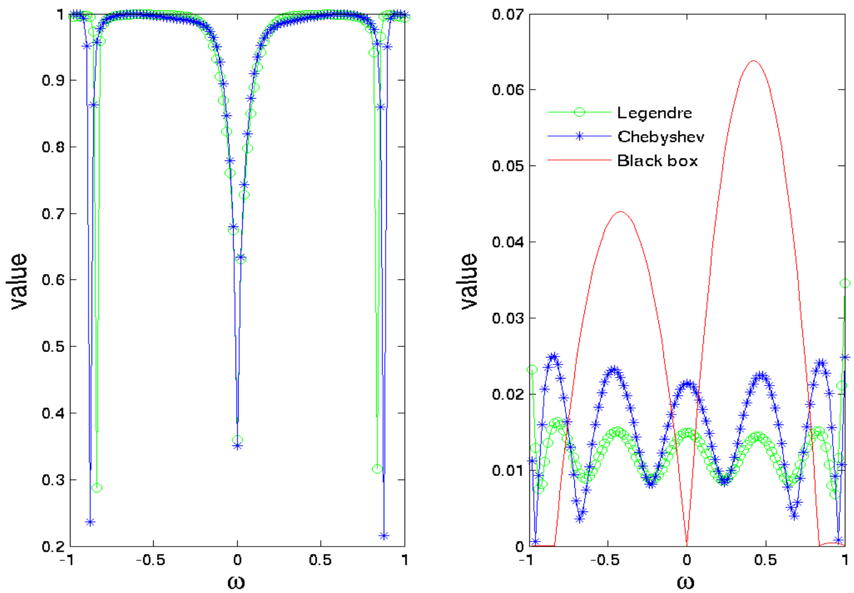

FIG. 4.4. *The case $n = 1000$: eigenvector angle and residual.*

At a first glance to the left panel of Figure 4.4, our approach did much worse in calculating the behavior of the eigenvector, in effect, the variable of our optimization problem. The figure seems to show errors in the cosine as large as 60%. A deeper investigation revealed that the cusps in the figure have to do with the degeneracy of the eigenvalue problem at those ω values. Indeed, if instead we are evaluating the residual error $\left\|(Q + \omega D_Q)\, \tilde{x}(\omega) - \tilde{\lambda}(\omega)\, \tilde{x}(\omega)\right\|$, we see in the right panel of Figure 4.4 that that residual is always below 0.035 and 97% of the time below 0.02 for the Legendre case and is always below 0.025 for the Chebyshev case. By comparison, if one would compute the exact minimum eigenvalue at the points minus 1, 1 and at the coordinates of the three cusps and used a linear interpolation with these nodes and the minimum eigenvectors obtained by simulation (denoted by the "Black box" in Figure 4.4) we see that the error would actually be quite a bit worse, by about a factor of two, and on average by a factor of four. While such comparisons must be carried out on much larger classes of problems, we find here evidence *that the optimization based SFEM approach may be much more robust than black-box algorithms*, at least for parametric eigenvalue problems. We call a "black-box" algorithm for parametric analysis a non-intrusive algorithm that uses only input-output information of the non-parametric problem (in our example, an eigenvalue solver), in order to generate the parametric approximation. Such algorithms are perhaps the easiest to implement for parametric analysis and uncertainty quantification [10]. Our example shows that such algorithms may encounter difficulties for a small dimension of the parameter space for problems of the type presented here, in addition to the well-documented difficulties for a large dimension of a parameter space [10].

Concerning the validation of the theoretical results, we note that the conditions of Theorem 3.9 are satisfied for the Chebyshev polynomials case, though the Legendre polynomials also seems to provide good approximating properties. The latter is relevant since the Legendre polynomials are the choice in the widespread case of uniform distribution.

5. Conclusions. We have shown that, in the study of the parametric dependence of problems that originate in optimization problems, the stochastic finite element (SFEM) method can be formulated as an optimization problem. The major advantage of our approach is that the resulting nonlinear problem has a solution that can be found by optimization algorithms.

The formulation will include constraints if the original problem had any, and the stochastic finite element approximation to the parametric dependence of the Lagrange multipliers is obtained implicitly from the solution, rather than explicitly as one would expect from the typical stochastic finite element formulation. We have shown that, under certain assumptions, the SFEM problem is well-posed and that the sequence of SFEM approximations of increasing degree converges to a solution of the parametric problem. In particular, if the constraints are linear, a solution of the SFEM approach exists without a small variation assumption of the solution $\tilde{x}^*(\omega)$ of the parametric problem (O).

In the case where our approach is used for studying the parametric dependence of the solution of minimum eigenvalue problems, we have shown that our method can be orders of magnitude faster compared to the simulation-based exploration of the parameter space. In addition, we have evidence that the method may be quite a bit more accurate than worst-case choices of simulation based on black-box exploration of the parameter space. The resulting problem is not convex, and it is difficult to guarantee that the global minimum can be actually found by the software. Nonetheless the software that we used KNITRO showed no difficulty in actually determining the minimum value.

Several issues remain to be analyzed. These include being able to guarantee that the minimum found is actually a global minimum, determining efficient ways of choosing the polynomial basis functions for a large number of dimensions of the parameter space, efficiently solving the larger coupled optimization problem, showing that the limit of solutions of (SO(K)) is sufficiently smooth rather than assuming it in Theorem 3.10, and providing convergence results for inequality-constrained problems and problems without smooth solutions.

Acknowledgments. We are grateful to Michel Deville, Paul Fischer, and Yvon Maday, for discussions on orthogonal polynomials and to Paul Hovland, Pino Palmotti, and Wong-Sik Yang for discussions about stochastic finite elements, uncertainty quantification, and their applications. Mihai Anitescu was supported by the Mathematical, Information, and Computational Sciences Division subprogram of the Office of Advanced Scientific Computing Research, Office of Science, U.S. Dept. of Energy, under Contract DE-AC02-06CH11357 and by the Argonne National Laboratory Directed Research and Development grant AFCI-168 (PI: Andrew Siegel).

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