OPTIMAL DISTORTION EMBEDDINGS OF DISTANCE REGULAR GRAPHS INTO EUCLIDEAN SPACES

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ABSTRACT. In this paper we give a lower bound for the least distortion embedding of a distance regular graph into Euclidean space. We use the lower bound for finding the least distortion for Hamming graphs, Johnson graphs, and all strongly regular graphs. Our technique involves semidefinite programming and exploiting the algebra structure of the optimization problem so that the question of finding a lower bound of the least distortion is reduced to an analytic question about orthogonal polynomials.

1. INTRODUCTION

By $\mathbb{R}^n$ we denote the Euclidean space of column vectors $x = (x_1, \ldots, x_n)^t$ with standard inner product $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ and corresponding norm $\|x\| = \sqrt{x \cdot x}$. Let (X, d) be a finite metric space with n elements. We say that an embedding $\varrho : X \to \mathbb{R}^n$ into Euclidean space has *distortion* D if for all $x, y \in X$ the inequalities

$$d(x, y) \leq \|\varrho(x) - \varrho(y)\| \leq D d(x, y)$$

hold.

By $c_2(X, d)$ we denote the *least distortion* for which (X, d) can be embedded into $\mathbb{R}^n$ and say that an embedding of (X, d) is *optimal* if it has distortion $c_2(X, d)$.

In [3] Bourgain showed that $c_2(X, d) = O(\log n)$ and in [7] Linial, London and Rabinovich proved that this bound is tight. In the last years embeddability questions, especially of finite graphs where the metric is given by the shortest path metric, were studied by theoretical computer scientists. For example they were used to design approximation algorithms (see e.g. [9], [6] and [10], Chapter 15).

Despite this interest for only very few graphs the exact least distortion and a least distortion embedding is explicitly known. The list only includes unit cubes (due to Enflo, see [4]), cycles, and strong graph product of cycles (due to Linial and Magen, see [8]). Extending work of Linial and Magen we give a lower bound for the least distortion of distance regular graphs. It turns out that the bound is tight in many examples and *we conjecture that it is always tight*. We compute least

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distortions for the following important examples: Hamming graphs (which include the cube), Johnson graphs, and all strongly regular graphs.

This paper is organized as follows. In Section 2 we give the necessary definitions and state our results. In Section 3 we prove the lower bound and in Section 4 we work out the three cases.

2. Statement of Results

Before we formulate our results we recall some definitions and results of the theory of distance regular graphs. For a comprehensive treatment we refer to [1] and [2].

Let $G = (V, E)$ be an *undirected graph* given by a finite set V of *vertices* and a subset $E \subseteq \binom{V}{2}$ of two-element subsets of V called *edges*. By $d : V \times V \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ we denote the length of a shortest path connecting two vertices x and y in G where we set $d(x, y) = \infty$ whenever there is no connection at all. The *diameter* of G is $\operatorname{diam} G = \max_{x,y \in V} d(x, y)$. A *connected* graph G, that is a graph with finite diameter, gives a finite metric space (V, d). In this situation we write for the least distortion $c_2(G)$ instead of $c_2(V, d)$.

A connected graph G is called *distance regular* if there are constants a_i, b_i, c_i where $i \in \{0, \ldots, \operatorname{diam} G\}$ so that the following holds: For every pair of vertices $x, y \in V$ with $d(x, y) = i$ we have

$$
\begin{aligned}
a_i &= \operatorname{card}(\{z \in V : d(x, z) = 1 \text{ and } d(z, y) = i\}), \\
b_i &= \operatorname{card}(\{z \in V : d(x, z) = 1 \text{ and } d(z, y) = i + 1\}), \\
c_i &= \operatorname{card}(\{z \in V : d(x, z) = 1 \text{ and } d(z, y) = i - 1\}).
\end{aligned} \tag{1}
$$

The number

$$k_i = \operatorname{card}(\{y \in V : d(x, y) = i\}), \tag{2}$$

is called the i*-th degree* of G. It is independent of x.

The following three families are important examples of distance regular graphs. We will find their least distortions in Section 4.

Example 2.1 (Hamming Graphs)**.** *Let X be a finite set of cardinality $q \geq 2$. The vertex set of the* Hamming graph $H(q, n)$ *is X^n, the set of all vectors of length n. Two vertices $x, y \in X^n$ are adjacent if x and y differ in exactly one coordinate. The shortest path metric of $H(q, n)$ coincides with the Hamming distance. The diameter of $H(q, n)$ is n.*

Example 2.2 (Johnson Graphs)**.** *Let V be a set of size v and n be an integer with $v \geq 2n$. The vertex set of the* Johnson graph $J(v, n)$ *is the set $\binom{V}{n}$ of all n-element subsets of V. Two vertices x, y of $J(v, n)$ are adjacent if the intersection $x \cap y$ has cardinality $n - 1$. The diameter of $J(v, n)$ is n.*

Example 2.3 (Strongly Regular Graphs)**.** *A* strongly regular graph *with parameters (ν, k, λ, μ) is a graph with ν vertices where every vertex is adjacent to k vertices, where every pair of adjacent vertices has precisely λ common neighbors, and where every pair of nonadjacent vertices has precisely μ common neighbors. If a strongly regular graph has diameter 2, then it is a distance regular graph with*

$k_1 = k$, $a_1 = \lambda$, $c_2 = \mu$. *Otherwise it is a disjoint union of equal-sized complete graphs.*

For $i \in \{0, \ldots, \operatorname{diam} G\}$ we define the *i-th adjacency matrix* $A_i \in \{0,1\}^{V \times V}$ component wise by $(A_i)_{xy} = 1$ whenever $d(x,y) = i$ and $(A_i)_{xy} = 0$ otherwise. We have the following relation between the adjacency matrices

$$A_1 A_i = c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1}. \tag{3}$$

Hence we can write $A_i = v_i(A_1)$ for univariate polynomials v_i of degree i. By $\theta_0 > \ldots > \theta_{\operatorname{diam} G}$ we denote the different eigenvalues of the matrix A_1. Notice that $v_i(\theta_0) = k_i$ and that $v_i(\theta_0)$ is the largest eigenvalue of A_i.

Now we can state our principal theorem.

Theorem 2.4. *Let G be a distance regular graph with $n = \operatorname{diam} G$. Then,*

$$c_2(G)^2 \geq \frac{n^2 v_n(\theta_0)}{v_1(\theta_0)} \min_{j \in \{1,\ldots,n\}} \left\{ \frac{v_1(\theta_0) - v_1(\theta_j)}{v_n(\theta_0) - v_n(\theta_j)} \right\}. \tag{4}$$

We prove this theorem in Section 3. The proof is based on the following observations. In general one can compute a least distortion embedding by solving a semidefinite programming problem. Using the commutativity of the algebra spanned by the adjacency matrices one can simplify the semidefinite programming program considerably (even to a linear program, see e.g. [5]). Then, using duality theory of semidefinite programming one gets a lower bound for the least distortion.

In Section 4 we apply this theorem to the distance regular graphs we introduced above to get their least distortions. The following theorem summarizes the results.

Theorem 2.5.

(a) *For the Hamming graph $H(q,n)$ we have*
$$c_2(H(q,n)) = \sqrt{n}.$$

(b) *For the Johnson graph $J(v,n)$ we have*
$$c_2(J(v,n)) = \sqrt{n}.$$

(c) *For a strongly regular graph G of diameter 2 with parameters (ν, k, λ, μ) we have*
$$c_2(G) = \sqrt{\frac{4(\nu - k - 1)(k - r)}{k(\nu - k + r)}},$$
where $r = \frac{1}{2}\big(\lambda - \mu + \sqrt{\nu}\big)$.

3. Proof of Theorem 2.4

Linial, London and Rabinovich [7] were the first who noticed that finding a least distortion embedding of a finite metric space (X,d) into Euclidean space can be expressed as a semidefinite programming problem:

$$\begin{aligned} &\textbf{minimize} && C \\ &\textbf{subject to} && Q = (q_{xy}) \in \mathbb{R}^{X \times X} \text{ is positive semidefinite}, \\ & && d(x,y)^2 \leq q_{xx} - 2q_{xy} + q_{yy} \leq C d(x,y)^2 \text{ for all } x, y \in X. \end{aligned} \tag{5}$$

Here Q is the *Gram matrix* of an embedding $\varrho : X \to \mathbb{R}^n$ defined entry wise by $q_{xy} = \varrho(x) \cdot \varrho(y)$. Note that Q defines the embedding ϱ uniquely up to orthogonal transformations. The minimum C of the semidefinite programming problem (5) equals $c_2(X, d)^2$.

Semidefinite programming problems are convex minimization problems and they can be solved efficiently in polynomial time in the sense that one can approximate an optimal solution to any fixed precision (see the survey [11]). Furthermore, semidefinite programming problems respect the symmetries of the instances. Hence, there is a least distortion embedding of a distance regular graph which inherits the symmetries of the graph. Now we make this statement precise. For this we start with a definition.

Definition 3.1. *Let (X, d) be a finite metric space. We say that an embedding $\varrho : X \to \mathbb{R}^n$ into Euclidean space is* faithful *if for every two pairs (x, y) and $(x', y') \in X \times X$ we have*

$$d(x, y) = d(x', y') \Longrightarrow \|\varrho(x) - \varrho(y)\| = \|\varrho(x') - \varrho(y')\|. \tag{6}$$

Lemma 3.2. *Let $G = (V, E)$ be a distance regular graph. Then, there exists a faithful embedding of G into Euclidean space with minimal distortion.*

Proof. Let $Q \in \mathbb{R}^{V \times V}$ be the Gram matrix of an embedding $\varrho : V \to \mathbb{R}^n$. We denote the entries of Q by $q_{xy} = \varrho(x) \cdot \varrho(y)$. Suppose that ϱ has distortion D so that we have the inequality

$$d(x, y)^2 \leq q_{xx} - 2q_{xy} + q_{yy} \leq D^2 d(x, y)^2 \tag{7}$$

for all $x, y \in V$.

Because of (3) the algebra $\mathcal{A}$ generated by the adjacency matrices A_i is commutative. The algebra $\mathcal{A}$ is called the *Bose-Mesner algebra* of G and it has basis A_i with $i = 0, \ldots, \operatorname{diam} G$. It is equipped with the inner product $\langle A, B \rangle = \operatorname{trace}(A^t B)$.

Now we show that the orthogonal projection $\bar{Q}$ of Q onto $\mathcal{A}$ is a Gram matrix of a faithful embedding having distortion D.

First we argue that $\bar{Q}$ is positive semidefinite. Because $\mathcal{A}$ is commutative the adjacency matrices A_i have a common basis of eigenvectors. Decompose the space $\mathbb{R}^V$ into an orthogonal direct sum of maximal common eigenspaces:

$$\mathbb{R}^V = V_0 \perp V_1 \perp \ldots \perp V_{\operatorname{diam} G}. \tag{8}$$

Then, the matrices of the orthogonal projection $E_i : \mathbb{R}^V \to V_i$ form a basis of $\mathcal{A}$. Since they are positive semidefinite we have $\langle Q, E_i \rangle \geq 0$. Hence the orthogonal projection

$$\bar{Q} = \sum_{i=0}^{\operatorname{diam} G} \frac{\langle Q, E_i \rangle}{\langle E_i, E_i \rangle} E_i \tag{9}$$

is positive semidefinite.

To show that $\bar{Q}$ is faithful and satisfies the desired inequalities we use the representation

$$\bar{Q} = \sum_{i=0}^{\operatorname{diam} G} \frac{\langle Q, A_i \rangle}{\langle A_i, A_i \rangle} A_i. \tag{10}$$

Notice here that the adjacency matrices form an orthogonal basis of $\mathcal{A}$. Let $x, y \in V$ be two vertices at distance $d = d(x, y)$. For the entry $\bar{q}_{xy}$ of $\bar{Q}$ we have

$$\bar{q}_{xy} = \sum_{i=0}^{\operatorname{diam} G} \frac{\langle Q, A_i \rangle}{\langle A_i, A_i \rangle} (A_i)_{xy} = \frac{\langle Q, A_d \rangle}{\langle A_d, A_d \rangle} = \frac{1}{\operatorname{card}(M_d)} \sum_{(x',y') \in M_d} q_{x'y'}, \tag{11}$$

where $M_d = \{(x, y) \in V \times V : d(x, y) = d\}$. From (11) it follows immediately that the embedding $\bar{\varrho}$ given by $\bar{Q}$ is faithful. Furthermore we obviously have

$$d(x, y)^2 = \frac{1}{\operatorname{card}(M_d)} \sum_{(x',y') \in M_d} d(x', y')^2. \tag{12}$$

Applying this to (7) and using the definition of $\bar{Q}$ gives

$$\begin{aligned} d(x, y)^2 &\leq \frac{1}{\operatorname{card}(M_d)} \sum_{(x',y') \in M_d} \left(q_{x'x'} - 2q_{x'y'} + q_{y'y'} \right) \\ &= \bar{q}_{xx} - 2\bar{q}_{xy} + \bar{q}_{yy} \\ &\leq \frac{D^2}{\operatorname{card}(M_d)} \sum_{(x',y') \in M_d} d(x', y')^2 \\ &= D^2 d(x, y)^2, \end{aligned} \tag{13}$$

hence the embedding given by $\bar{Q}$ has distortion D. □

Remark 3.3. *If the graph G is distance transitive, then one can partially simplify the proof of Lemma 3.2: The* automorphism group $\operatorname{Aut}(G)$ *is the set of permutations $\sigma \in \operatorname{Sym}(V)$ with $\{x, y\} \in E$ if and only if $\{\sigma(x), \sigma(y)\} \in E$, and we say that G is* distance transitive *if for every pair of vertex pairs $(x, y), (x', y')$ with $d(x, y) = d(x', y')$ there exists $\sigma \in \operatorname{Aut}(G)$ so that $(\sigma(x), \sigma(y)) = (x', y')$. Then, the orthogonal projection $\bar{Q}$ is simply the symmetrization of Q, i.e.*

$$\bar{Q} = \frac{1}{|\operatorname{Aut}(G)|} \sum_{\sigma \in \operatorname{Aut}(G)} (q_{\sigma(x),\sigma(y)}), \tag{14}$$

and $\bar{Q}$ is positive semidefinite because it is the sum of positive semidefinite matrices.

Using duality theory of semidefinite programming Linial, London and Rabinovich [7] and Linial and Magen [8] gave the following characterization of the least possible distortion for a finite metric space.

Lemma 3.4. *Let (X, d) be a finite metric space.*

(a) *The least distortion of an embedding of* (X, d) *into Euclidean space is given by*

$$c_2(X,d)^2 = \max_Q \frac{\sum_{\{(x,y):q_{xy}>0\}} d(x,y)^2 q_{xy}}{\sum_{\{(x,y):q_{xy}<0\}} d(x,y)^2(-q_{xy})}, \tag{15}$$

where the maximum is taken among all positive semidefinite matrices Q *in which all row sums vanish. (Note that the quotient is invariant under scaling of* Q *with positive reals.)*

(b) *Let* ϱ *be an embedding of* (X, d) *into Euclidean space having minimal distortion* $c_2(X, d)$*. For a matrix attaining the maximum in (15) and for a pair* $(x, y) \in X \times X$ *we have* $q_{xy} > 0$ *only for the* most contracted pairs (x, y)*, that is for* (x, y) *the fraction* $\|\varrho(x) - \varrho(y)\|/d(x, y)$ *is minimal among all pairs in* $X \times X$*, we have* $q_{xy} < 0$ *only for the* most expanded pairs (x, y)*, that is for* (x, y) *the fraction* $\|\varrho(x)-\varrho(y)\|/d(x, y)$ *is maximal among all pairs in* $X \times X$*, and* $q_{xy} = 0$ *for all other pairs.*

Proof. See [7, Corollary 3.5] and [8, Claim 1.4]. □

Remark 3.5. *For the embedding of finite metric spaces given by the shortest path metric of a graph, Linial and Magen showed (*[8, Claim 2.2]*) that most expanded pairs are always adjacent vertices.*

Now we finish the proof of Theorem 2.4. Let G be a distance regular graph and let ϱ be an embedding of G into Euclidean space with minimal distortion $c_2(G)$. By Lemma 3.2 we can assume that ϱ is faithful. Hence, by the previous remark, all pairs (x, y) with $d(x, y) = 1$ are most expanded, and there is an index $i \in \{2, \ldots, \operatorname{diam} G\}$ so that all pairs (x, y) with $d(x, y) = i$ are most contracted.

For proving a lower bound on the distortion of ϱ we suppose that $i = n$, where $n = \operatorname{diam} G$. So the lower bound can only be tight when the most contracted pairs are at distance n.

We define

$$Q_\alpha = (k_1 - \alpha k_n)A_0 - A_1 + \alpha A_n, \quad \alpha \in \mathbb{R}. \tag{16}$$

When Q_α is positive semidefinite, then Q_α satisfies the assumption of Lemma 3.4 (a). Hence,

$$c_2(G)^2 \geq \left\{\frac{k_n n^2 \alpha}{k_1} : Q_\alpha \text{ is positive semidefinite}\right\}. \tag{17}$$

In order to maximize $\frac{k_n n^2 \alpha}{k_1}$ we have to maximize α so that Q_α is positive semidefinite. Recall that the adjacency matrices have a common system of eigenvectors. Let x_j be a common eigenvector of the adjacency matrices which is an eigenvector of the eigenvalue θ_j of A_1. Then, $A_1 x_j = \theta_j x_j$, and

$$Q_\alpha x_j = (k_1 - \alpha k_n - \theta_j + \alpha v_n(\theta_j))x_j, \tag{18}$$

and the matrix Q_α is positive semidefinite if and only if

$$k_1 - \alpha k_n - \theta_j + \alpha v_n(\theta_j) \geq 0, \quad \text{for all } j \in \{0, \ldots, n\}. \tag{19}$$

The largest eigenvalue of the adjacency matrix of a k-regular graph is exactly k. So, $k_n - v_n(\theta_j)$ is positive for $j \in \{1, \ldots, n\}$ and $k_n - v_n(\theta_j) = 0$ for $j = 0$. Hence,

$$\begin{aligned} \alpha &= \min_{j \in \{1,\ldots,n\}} \frac{k_1 - \theta_j}{k_n - v_n(\theta_j)} \\ &= \min_{j \in \{1,\ldots,n\}} \frac{v_1(\theta_0) - v_1(\theta_j)}{v_n(\theta_0) - v_n(\theta_j)}, \end{aligned} \tag{20}$$

which yields the statement of the theorem.

4. Examples

4.1. **Hamming Graphs.** Now we show using Theorem 2.4 that the optimal distortion of the Hamming graph $H(q,n)$ is $\sqrt{n}$ and we give an embedding of $H(q,n)$ into Euclidean space having this distortion.

We use the notation we introduced in Section 2. The eigenvalues of the i-th adjacency matrix of $H(q,n)$ are well-known (see for example [1, Chapter 3.2]). They are $v_i(\theta_j) = K_i(j)$ where $j \in \{0, \ldots, n\}$ and where K_i is the *i-th Krawtchouk polynomial*

$$K_i(u) = \sum_{t=0}^{i} (-q)^t (q-1)^{(i-t)} \binom{n-t}{i-t} \cdot \binom{u}{t}. \tag{21}$$

In particular we have

$$k_i = K_i(0) = \binom{n}{i} (q-1)^i, \tag{22}$$

$$\theta_j = K_1(j) = n(q-1) - qj, \tag{23}$$

$$v_n(\theta_j) = (-1)^j (q-1)^{n-j}. \tag{24}$$

Let us determine the value of $\alpha = \min_{j \in \{1,\ldots,n\}} \frac{k_1 - \theta_j}{k_n - v_n(\theta_j)}$. The minimum is attained for $j = 1$ so that we have

$$\alpha = \frac{k_1 - \theta_1}{k_n - v_n(\theta_1)} = \frac{1}{(q-1)^{n-1}}, \tag{25}$$

since for $j = 2, \ldots, n$ the inequality

$$\frac{k_1 - \theta_j}{k_n - v_n(\theta_j)} = \frac{qj}{(q-1)^n - (-1)^n (q-1)^{n-j}} \geq \frac{1}{(q-1)^{n-1}} \tag{26}$$

holds true. Hence by Theorem 2.4 the distortion of an optimal embedding is bounded by

$$c_2(H(q,n))^2 \geq \frac{n^2 \alpha k_n}{k_1} = n. \tag{27}$$

We have equality since the embedding ϱ we define below has distortion $\sqrt{n}$. Let X^n be the vertex set of $H(q,n)$. With $e_x \in \mathbb{R}^X$ denote the standard unit vector defined component wise by $(e_x)_y = 1$ if $x = y$ and $(e_x)_y = 0$ otherwise. For a vertex $(x_1, \ldots, x_n) \in X^n$ in $H(q, n)$ set

$$\varrho(x_1, \ldots, x_n) = \sqrt{n/2}(e_{x_1}, \ldots, e_{x_n})^t \in (\mathbb{R}^X)^n. \tag{28}$$

If $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = i$, then $\|\varrho(x_1, \ldots, x_n) - \varrho(y_1, \ldots, y_n)\| = \sqrt{ni}$ and we have the desired inequalities

$$d(x, y)^2 = i^2 \leq \|\varrho(x) - \varrho(y)\|^2 = ni \leq nd(x, y)^2 = ni^2, \tag{29}$$

where we abbreviate $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ by x and y. The image of this embedding forms the vertex set of the direct product, taken n times, of a regular simplex with q vertices.

Remark 4.1. *In particular this implies the classical result of Enflo* [4] *that the least distortion embedding of the n-dimensional unit cube $H(2, n)$ is $\sqrt{n}$. Enflo's proof uses inductive and combinatorial arguments and does not easily generalize to different finite metric spaces. Linial and Magen* [8, Theorem 2.4] *give another proof of Enflo's theorem which is in a sense an ad-hoc variant of our proof.*

4.2. **Johnson Graphs.** Here we show that the optimal distortion of the Johnson graph $J(v, n)$ is $\sqrt{n}$ and we give an embedding of $J(v, n)$ into Euclidean space having this distortion.

The eigenvalues of the i-th adjacency matrix of $J(v, n)$ are well-known (see for example [1, Chapter 3.2]). They are $v_i(\theta_j) = E_i(j)$ where E_i is the *i-th Eberlein polynomial* (or *dual Hahn polynomial*)

$$E_i(u) = \sum_{t=0}^{i} (-1)^t \binom{u}{t}\binom{n-u}{i-t}\binom{v-n-u}{i-t}. \tag{30}$$

In particular we have

$$k_i = E_i(0) = \binom{n}{i}\binom{v-n}{i}, \tag{31}$$

$$\theta_j = E_1(j) = j^2 - (v + 1)j + n(v - n). \tag{32}$$

Let us determine the value of α. We have

$$\alpha = \min_{j=1,\ldots,n} \frac{k_1 - \theta_j}{k_n - v_n(\theta_j)} = \min_{j=1,\ldots,n} \frac{(v + 1)j - j^2}{\binom{v-n}{n} - (-1)^j\binom{v-n-j}{n-j}}. \tag{33}$$

We shall show that the minimum is attained for $j = 1$ so that

$$\alpha = \frac{v}{\binom{v-n}{n} + \binom{v-n-1}{n-1}}. \tag{34}$$

We compare the numerator of the right hand side of (34) with the one of (33). This gives the following inequality which holds true for all j in the interval $[1, m]$

$$v \leq (v + 1)j - j^2. \tag{35}$$

We compare the denominators getting the inequality

$$\binom{v-n}{n} - (-1)^j \binom{v-n-j}{n-j} \leq \binom{v-n}{n} + \binom{v-n-1}{n-1}, \tag{36}$$

which holds because $\binom{v-n-1}{n-1} = \binom{v-n-j}{n-j} \prod_{t=1}^{j-1} \frac{v-n-t}{n-t}$ and $v-n-t \geq n-t$ since $v \geq 2n$. Altogether this shows that the value α is the one stated in (34). Hence the squared distortion of an embedding is at least n. We have equality since the embedding ϱ described below has distortion $\sqrt{n}$.

Let $\binom{V}{n}$ be the vertex set of $J(v,n)$. With $e_v \in \mathbb{R}^V$ denote the standard unit vector as in the last section. For a n-element subset $X \subseteq V$ define the embedding $\varrho(X) = \sqrt{n} \sum_{x \in X} e_x$. If two n-element subsets X, Y have distance i in $J(v,n)$, then $\|\varrho(X) - \varrho(Y)\| = \sqrt{ni}$. Hence, the distortion of ϱ is $\sqrt{n}$. The image of this embedding forms the vertex set of the n-hypersimplex in dimension v.

4.3. **Strongly Regular Graphs.** In this section we will show that the optimal distortion of a strongly regular graph $G = (V,E)$ of diameter 2 with parameters (ν, k, λ, μ) is $\left(\frac{4(\nu-k-1)(k-r)}{k(\nu-k+r)}\right)^{1/2}$, where $r = \frac{1}{2}\left(\lambda - \mu + \sqrt{\nu}\right)$. In the following we shall make use of [2, Theorem 1.3.1] where fundamental facts about the parameters ν, k, λ, μ are provided.

The eigenvalues of the first adjacency matrix A_1 are

$$k, \quad r = \frac{1}{2}\left(\lambda - \mu + \sqrt{\nu}\right), \quad s = \frac{1}{2}\left(\lambda - \mu - \sqrt{\nu}\right). \tag{37}$$

We have

$$A_1^2 = kA_0 + \lambda A_1 + \mu A_2, \tag{38}$$

and hence

$$v_2(u) = \frac{1}{\mu}\left(u^2 - \lambda u - k\right). \tag{39}$$

Using the identities $\lambda = \mu + r + s$ and $rs = \mu - k$ we compute $v_2(r) = -r-1$ and $v_2(s) = -s-1$. Because $r \geq 0$ and $s \leq -1$ we have the inequality

$$\frac{k-r}{(\nu-k-1)-(-1-r)} \leq \frac{k-s}{(\nu-k-1)-(-1-s)} \tag{40}$$

Now Theorem 2.4 gives the lower bound

$$c_2(G)^2 \geq \frac{4(\nu-k-1)(k-r)}{k(\nu-k+r)}. \tag{41}$$

By reviewing the proof of Theorem 2.4 for the case of distance regular graphs with diameter 2, i.e. for connected strongly regular graphs, one sees that Theorem 2.4 is tight in these cases. The reason for this is that in a faithful embedding all the most contracted pairs are pairs of vertices which are not adjacent. So this case is especially convenient since we do not have to construct an embedding to upper bound the least distortion.

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