Recursive Construction of Optimal Self-Concordant Barriers for Homogeneous Cones *

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November 20, 2006

Abstract. In this paper, we give a recursive formula for optimal dual barrier functions on homogeneous cones. This is done in a way similar to the primal construction of Güler and Tunçel [1] by means of the dual Siegel cone construction of Rothaus [2]. We use invariance of the primal barrier function with respect to a transitive subgroup of automorphisms and the properties of the duality mapping, which is a bijection between the primal and the dual cones. We give simple direct proofs of self-concordance of the primal optimal barrier and provide an alternative expression for the dual universal barrier function.

Key words. Optimal self-concordant barrier, duality, homogeneous cone, Siegel cone, Legendre-Fenchel transformation, universal barrier.

1. Introduction

This paper provides a new expression for an optimal self-concordant barrier F_K^* for the dual cone K^* of a homogeneous cone K. The formula for F_K^* is obtained recursively using the dual Siegel cone construction of Rothaus [2]. This is similar to the ideas of Güler and Tunçel [1] who use the primal Siegel cone construction of Gindikin [3]. We use the properties of the duality mapping, Fenchel dual and invariance properties of the primal optimal barrier of Güler and Tunçel in our proofs. We provide an alternative formula for the universal barrier function as well.

*Research partially supported by the National Science Foundation under grant DMS-0411955. The author is thankful to Osman Güler for suggesting to investigate the invariance properties of the optimal barriers, which were used in the proof of self-concordance in the primal case, and, eventually, in the derivation of the dual optimal barrier. His valuable comments helped to improve the paper.

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Consider the primal-dual pair of convex programming problems in conic form

$$\begin{aligned} \min\ & \langle c, x\rangle \\ \text{s.t. } & Ax = b \qquad\qquad \text{(P)} \\ & x \in \overline{K} \end{aligned} \qquad\qquad \begin{aligned} \max\ & \langle b, y\rangle \\ \text{s.t. } & A^T y + s = c \qquad\qquad \text{(D)} \\ & s \in \overline{K^*} \end{aligned}$$

where $A \in \mathbb{R}^{n\times m}$, $x, c, s \in \mathbb{R}^n$, $y, b \in \mathbb{R}^m$, $K \subset \mathbb{R}^n$ is a regular open convex cone, that is, K is non-empty and does not contain straight lines. We will refer to an open regular cone simply as a *cone*. The cone dual to K is defined as

$$K^* = \{y \in \mathbb{R}^n : \langle x, y\rangle > 0,\ \forall\, x \in \overline{K},\ x \neq 0\}.$$

The theory of interior-point methods provides polynomial time algorithms for convex programming problems. These methods are based on generating iterates that follow a path in the interior of the feasible region that converges to the solution of (P). This path is called the *central path* and it is formed of the minimizers of a family of auxiliary problems. For instance,

$$x^*(t) := \underset{\text{s.t. } Ax=b}{\arg\min} \{\langle c, x\rangle + tF(x)\}$$

is the *primal* central path. The auxiliary function $F : K \to \mathbb{R}$ is convex and satisfies the barrier property $F(x) \to \infty$ as $x \to \partial K$, which ensures that the minimizers $x^*(t)$ stay strictly inside of the feasible region. The iterations in the interior-point methods are generated by means of a variant of the Newton method. The self-concordance property of F defined below is crucial for obtaining convergence results and polynomial time complexity.

Definition 1.1. A convex, three-times differentiable, function $F : K \to \mathbb{R}$ is called a *θ-self-concordant barrier* for a cone K if it satisfies

(a) $F(x) \to \infty$ as $x \to \partial K$ (the barrier property);

(b) $|D^3F(x)[h, h, h]| \leq 2\left(D^2F(x)[h, h]\right)^{3/2}$ (the self-concordance property);

(c) $(DF(x)[h])^2 \leq \theta D^2F(x)[h, h]$ (θ is the *parameter* of the barrier function),

where $D^kF(x)[h, \ldots, h]$ denotes the kth directional derivative of F at the point x in the direction $h \in \mathbb{R}^n$.

If a barrier function satisfies property (b) and is θ-logarithmically homogeneous, that is,

$$F(tx) = F(x) - \theta \log t, \qquad \forall\, t > 0, \tag{1}$$

then we say that F is *θ-normal.* It is well known [4] that equation (1) implies property (c). The absolute value in property (b) can be omitted.

The value θ in Definition (1.1) is called the *parameter* of F. Since interior-point methods have $O(\theta)$ complexity, it is important to use a barrier with the smallest possible value of θ. Such a barrier and the corresponding parameter are said to be *optimal*.

This paper discusses optimal barriers for *homogeneous programming* problems, in which the cone K in (P) is homogeneous, more precisely,

Definition 1.2. A cone $K \subset \mathbb{R}^n$ is said to be *homogeneous* if for any $x, y \in K$ there exists a non-degenerate linear transformation A of $\mathbb{R}^n$ such that $AK = K$ and $Ax = y$. In other words, the group $Aut(K)$ of all automorphisms of K acts *transitively* on K.

Homogeneous cones form a subclass of convex cones that has attractive properties from the standpoint of developing efficient interior-point algorithms. The availability of self-concordant barriers, which preferably have some additional features that improve the performance of these methods, is essential for this purpose.

Nesterov and Nemirovskii [4] show that any cone K in $\mathbb{R}^n$ admits a logarithmically homogeneous *universal* barrier function u_K with $\theta = O(n)$. The universal barrier is defined as a volume integral and is hard to compute even for simple sets of low dimensions. This gives the motivation to look for efficient ways to evaluate the universal barrier function and/or to find computable barriers preferably with the best possible parameter value.

The geometry of the underlying cone is an important factor that influences the values of the parameters of associated barrier functions. Nesterov and Nemirovskii [4] show that some convex polytopes in $\mathbb{R}^n$ cannot admit barriers with parameters less then n (Proposition 2.3.6, p. 42 in [4]). Güler and Tunçel [1] use the special structure of homogeneous cones to discern that one cannot associate a self-concordant barrier with a homogeneous cone having θ less than the Carathéodory number of the cone, that is the minimum number of extreme directions needed to represent any point in the cone as a nonnegative linear combination of these directions. They also show that another geometric characteristic of a homogeneous cone K, the rank of K (discussed below), is equal to the optimal barrier parameter. They use a recursive scheme to construct an optimal barrier by means of Siegel domains. To prove the self-concordance property of this barrier, they refer to a more general construction of Nesterov and Nemirovskii [4]. In Section 2 of this paper, we use invariance properties of homogeneous cones (described below) to prove the self-concordance property of F_K.

It is well known [5] that if K is homogeneous then so is its dual K^*. In Section 3, we utilize the work of Rothaus [6] on the dual Siegel cone construction to introduce a recursive formula for an optimal dual barrier function F_K^*. The dual Siegel cone construction is a generalization of the idea of representing the cone S_{++}^{n+1} of symmetric $(n+1) \times (n+1)$ positive definite matrices in terms of S_{++}^n by means of the Schur complement technique:

$$S^{n+1}_{++} = \left\{ \begin{pmatrix} s & v^T \\ v & U \end{pmatrix} : \ s > \langle v, U^{-1} v \rangle, \ s \in \mathbb{R}, \ v \in \mathbb{R}^n, \ U \in S^n_{++} \right\}.$$

We also present an alternative expression for the universal barrier function $u_{SC(K,B)^*}$ for the dual Siegel cone $SC(K,B)^*$. For this purpose, we use the observation of Güler [7] that the universal barrier $u_K(x)$ for a cone K is equal to $\log \phi_K(x)$ up to an additive constant, where ϕ_K is a characteristic function of K defined by

$$\phi_K(x) := \int_{K^*} e^{-\langle x, y \rangle} dy.$$

One of the important properties of the characteristic function [5] is that for any automorphism $A \in Aut(K)$,

$$\phi_K(Ax) = \phi_K(x)/|\det A|. \tag{2}$$

Therefore, the kth directional derivative of the function $F(x) := \log \phi_K(x)$ at a point $Ax \in K$ along the direction Ah is equal to

$$D^k F(Ax)[Ah, \ldots, Ah] = D^k F(x)[h, \ldots, h].$$

Consequenly, statements, like the self-concordance property (b), need only be verified *at a single point* [7]. This idea will be used to prove the self-concordance of the primal and dual optimal barriers for Siegel cones.

2. Self-Concordance of the Primal Optimal Barrier for a Homogeneous Cone

Homogeneous cones possess the remarkable property that any such cone can be recursively built from lower-dimensional cones starting from the positive half-line [5, 8, 3]. This construction is based on the Siegel domain theory. Güler and Tunçel [1] utilize this theory to recursively construct an optimal barrier F_K for a homogeneous cone K and show that the optimal parameter θ is equal to the rank of K defined as the number of recursive steps necessary to build K. To prove the self-concordance of F_K, they use the fact that the Siegel cone construction is a particular case of the more general construction described by Nesterov and Nemirovskii ([4], Theorem 5.1.8). In this section, we give a simpler self-concordance proof using the invariance of F_K with respect to a transitive subset of automorphisms of K, which implies that it is enough to prove the self-concordance *at a single point*.

We start by giving all the necessary definitions and auxiliary results related to the Siegel cone construction, and then proceed with the proof.

Definition 2.1. Let K be a cone in $\mathbb{R}^k$. A mapping $B : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^k$ is called a *K-bilinear symmetric form* if it satisfies the following conditions

(1) $B(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v)$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$,

(2) $B(u,v) = B(v,u)$,

(3) $B(u,u) \in \overline{K}$,

(4) $B(u,u) = 0$ implies $u = 0$.

Definition 2.2. The Siegel domain corresponding to $K \subset \mathbb{R}^k$ and a K-bilinear symmetric form B is the set

$$S(K,B) = \{(x,u) \in \mathbb{R}^k \times \mathbb{R}^p : \, x - B(u,u) \in K\}.$$

Definition 2.3. A K-bilinear symmetric form B is called *homogeneous* if K is homogeneous and there exists a transitive subgroup $G \subset Aut(K)$ such that for every $g \in G$ there exists a linear transformation $\overline{g}$ of $\mathbb{R}^p$ such that

$$gB(u,v) = B(\overline{g}u, \overline{g}v). \tag{3}$$

The Siegel domain $S(K,B)$ corresponding to a homogeneous K-bilinear symmetric form B is affine homogeneous (see [7]). Moreover, any affine homogeneous domain is affine equivalent to a Siegel domain (Vinberg [5]). The cone fitted to a Siegel domain $S(K,B)$ given by

$$SC(K,B) = \{(x,u,t) \in \mathbb{R}^k \times \mathbb{R}^p \times \mathbb{R} : \, tx - B(u,u) \in K, \, t > 0\} \tag{4}$$

is called the *Siegel cone* corresponding to K and B. This cone is homogeneous ([5], see also [3, 7]). This can be shown by checking that the linear maps T_1, T_2, T_3 defined below generate a transitive subgroup (which will be referred later as G_{SC}) of $Aut(SC(K,B))$.

$$T_1(x,u,t) = (x, \sqrt{\alpha}u, \alpha t), \alpha > 0, \tag{5}$$

$$T_2(x,u,t) = (x + 2B(u,a) + tB(a,a), u + ta, t), \; a \in \mathbb{R}^p, \tag{6}$$

$$T_3(x,u,t) = (gx, \overline{g}u, t), g \in G \subseteq Aut(K), \tag{7}$$

where G is a transitive subgroup of $Aut(K)$. The above construction shows that we can build homogeneous cones out of lower dimensional ones. Remarkably, the converse is also true [5], that is, for every homogeneous cone $\overline{K}$ of dimension at least 2, there exists a lower dimensional homogeneous cone K and a homogeneous K-bilinear symmetric form B such that $\overline{K}$ is linearly isomorphic to $SC(K,B)$ (see [5, 3]).

The following well-known result was obtained by Nesterov and Nemirovskii ([4], see also [9] for an easy proof in case of hyperbolic barriers) and will be used in self-concordance proofs. It is included here for easy reference.

Lemma 2.1. *Let F be a θ-self-concordant barrier for a closed convex domain $G \subset E$ and let h be an element of the recession cone of G, i.e., $x + th \in G$ for all $x \in G$ and all $t > 0$. Then, for all $x \in int\, G$, we have*

$$\left(D^2F(x)[h,h]\right)^{1/2} \leq -DF(x)[h].$$

To present the proof of self-concordance of the optimal primal barrier, we repeat the statement of Theorem 4.1 in [1].

Theorem 2.1. *(Güler, Tunçel, [1]) If a θ-normal barrier for a homogeneous cone K has an optimal parameter θ, then*

$$F_{SC(K,B)}(x,u,t) = F_K\left(x - B(u,u)/t\right) - \log t. \tag{8}$$

is an optimal $(\theta+1)$-normal barrier for the Siegel cone $SC(K,B)$.

Proof. The function $F_{SC(K,B)}$ is invariant under linear transformations from G_{SC}, i.e. for every $T \in G_{SC}$,

$$F_{SC(K,B)}(T(x,u,t)) = F_{SC(K,B)}(x,u,t) + const.$$

This can be easily proved by checking the above property for the generators T_1, T_2, and T_3 of G_{SC} using the homogeneity property (3) of B and the induction hypothesis that F_K is invariant with respect to $g \in G$. Therefore,

$$D^k F_{SC(K,B)}(T(x,u,t))[Th,\ldots,Th] = D^k F_{SC(K,B)}(x,u,t)[h,\ldots,h]$$

and it is enough to prove the self-concordance property just at the point $(x,0,1) \in SC(K,B)$. To compute the directional derivatives of $F_{SC(K,B)}$ at this point in the direction $h := (\eta,\xi,\tau)$ write

$$\begin{aligned} F_{SC(K,B)}((x,0,1)+\alpha h) &= F_{SC(K,B)}((x+\alpha\eta, \alpha\xi, 1+\alpha\tau)) \\ &= F_K\left(x+\alpha\eta - B(\alpha\xi,\alpha\xi)/(1+\alpha\tau)\right) - \log(1+\alpha\tau) \\ &= F_K\left(x+\alpha\left(\eta - \alpha B(\xi,\xi)/(1+\alpha\tau)\right)\right) - \log(1+\alpha\tau). \end{aligned}$$

Substitution of the Taylor's expansions for $1/(1+\alpha\tau)$ and $\log(1+\alpha\tau)$ into the above expression gives

$$\begin{aligned} F_{SC(K,B)}((x+\alpha\eta,\alpha\xi,1+\alpha\tau)) &= F_K(x) + \alpha DF_K(x)[\Delta] \\ &\quad + (\alpha^2/2!)D^2F_K(x)[\Delta,\Delta] + (\alpha^3/3!)D^2F_K(x)[\Delta,\Delta,\Delta] \\ &\quad - (\alpha\tau - (\alpha^2\tau^2)/2 + (\alpha^3\tau^3)/3 - \ldots), \end{aligned}$$

where $\Delta = \eta - \alpha B(\xi,\xi)(1-\alpha\tau+\alpha^2\tau^2-\ldots)$. By collecting terms of order α, α^2, α^3 we obtain the first, the second, and the third directional derivatives of $F_{SC(K,B)}$:

$$\begin{aligned} DF_{SC(K,B)}(x,0,1)[h] &= DF_K(x)[\eta] - \tau, \\ D^2F_{SC(K,B)}(x,0,1)[h,h] &= D^2F_K(x)[\eta,\eta] - 2DF_K(x)[B(\xi,\xi)] + \tau^2 \\ D^3F_{SC(K,B)}(x,0,1)[h,h,h] &= D^3F_K(x)[\eta,\eta,\eta] - 6D^2F_K(x)[\eta,B(\xi,\xi)] \\ &\quad + 6\tau DF_K(x)[B(\xi,\xi)] - 2\tau^3. \end{aligned}$$

The self-concordance property of F_K and the Cauchy-Schwarz inequality imply

$$
\begin{aligned}
|D^3F_{SC(K,B)}(x,0,1)[h,h,h]| \le 2|D^2F_K(x)[\eta,\eta]|^{3/2} \\
&+ 3\left(D^2F_K(x)[\eta,\eta]\right)^{1/2}\left(D^2F(x)[2B(\xi,\xi),2B(\xi,\xi)]\right)^{1/2} \\
&+ 3|\tau||DF_K(x)[2B(\xi,\xi)]| + 2|\tau|^3.
\end{aligned}
$$

Using Lemma 2.1, we can conclude that $DF_K(x)[-2B(\xi,\xi)] \ge 0$ and

$$
\begin{aligned}
|D^3F_{SC(K,B)}| \le 2|D^3F_K(x)[\eta,\eta]|^{3/2} \\
&+ 3\left(D^2F_K(x)[\eta,\eta]\right)^{1/2} DF_K(x)[-2B(\xi,\xi)] \\
&+ 3|\tau|DF_K(x)[-2B(\xi,\xi)] + 2|\tau|^3.
\end{aligned}
$$

Introducing the following notation

$$
a := \left(D^2F_K(x)[\eta,\eta]\right)^{1/2}, \quad b := \left(DF_K(x)[-2B(\xi,\xi)]\right)^{1/2}, \quad c := |\tau|,
$$

the self-concordance property of $F_{SC(K,B)}$ reduces to

$$
a^3 + (3/2)ab^2 + (3/2)cb^2 + c^3 \le (a^2+b^2+c^2)^{3/2}. \tag{9}
$$

Without loss of generality, we can assume that $a^2+b^2+c^2 = 1$, since the polynomial on the left-hand side is homogeneous. Therefore, we need to show that

$$
a^3 + (3/2)ab^2 + (3/2)cb^2 + c^3 \le 1.
$$

It is easy to see that

$$
\begin{aligned}
a^3 + (3/2)ab^2 + (3/2)cb^2 + c^3 &= (a+c)(a^2+c^2+(3/2)b^2 - ac) \\
&= (a+c)(3-(a+c)^2)/2.
\end{aligned}
$$

The function $g(t) = t(3-t^2)/2$ attains its maximum value 1 on the set of all nonnegative t at the point $t=1$, and inequality (9) follows.

Since $F_{SC(K,B)}$ is $(\theta+1)$-logarithmically homogeneous, its barrier parameter is equal to $(\theta+1)$, which is the best possible since it is equal to the rank of the Siegel cone $SC(K,B)$ (see [1]). □

Remark 2.1. To construct an optimal barrier for a homogeneous cone, we always start with an optimal 1-normal barrier $F_{\mathbb{R}_{++}}$ for the positive half line $\mathbb{R}_{++}$. Due to 1-logarithmic homogeneity it should satisfy $DF(x)[x] = 1$, hence the function $F_{\mathbb{R}_{++}} = -\log x + C$ is the only possible choice. Therefore, we always start the construction of the optimal barrier from $-\log x$.

It is interesting that at the second step of the construction we always obtain a cone isomorphic to a second-order cone (a particular case of a symmetric cone) [10], since $\mathbb{R}_{++}$-bilinear symmetric form B can be written as symmetric positive definite $p \times p$ matrix, which, in turn, can be reduced to a positive definite diagonal matrix Σ in the appropriate basis. Therefore,

$$SC(\mathbb{R}_{++}, B) = \{(x, u, t) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R} : \ tx - \langle u, \Sigma u\rangle > 0\}.$$

Since by the rotation of variables, the term tx can be transformed to $(t^2 - x^2)/2$, the isomorphism of $SC(\mathbb{R}_{++}, B)$ to the interior of the second order cone $SOC = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{p+1} : \ t^2 - x_1^2 - x_2^2 - \ldots - x_{p+1}^2 \geq 0, \ t \geq 0\}$ is evident.

3. Optimal and Universal Barriers for the Dual Homogeneous Cones

This section contains main results of the paper. We provide recursive formulas for an optimal and the universal barriers for dual homogeneous cones.

We start with describing the dual Siegel cone construction of Rothaus [6]. Let B be a K-bilinear symmetric homogeneous form. For each $y \in \mathbb{R}^k$ define a symmetric linear mapping $U(y) : \mathbb{R}^p \to \mathbb{R}^p$ by

$$\langle U(y)u, v\rangle := \langle B(u, v), y\rangle. \tag{10}$$

Property (c) in Definition (2.1) implies that if $y \in K^*$, then $U(y)$ is positive definite.

Recall that Definition 2.3 requires the existence of a transitive subgroup $G \subseteq Aut(K)$ such that for all $g \in G$ there exists a linear transformation $\overline{g} : \mathbb{R}^p \to \mathbb{R}^p$ such that

$$gB(u, v) = B(\overline{g}u, \overline{g}v).$$

It can be shown (see Proposition 3.2 in [1]) that for any $g \in G$ there exists a linear operator $T_{\overline{g}} : S^p \to S^p$ defined by $T_{\overline{g}}(U(y)) := (\overline{g})^* U(y)(\overline{g})$ such that

$$T_{\overline{g}}(U(y)) = U(g^* y). \tag{11}$$

Theorem 3.1. *The (open) dual Siegel cone is given by*

$$SC(K, B)^* = \{(y, v, s) \in K^* \times \mathbb{R}^p \times \mathbb{R} : s > \langle U(y)^{-1} v, v\rangle\}, \tag{12}$$

with respect to the inner product

$$\langle (x, u, t), (y, v, s)\rangle = \langle x, y\rangle + 2\langle u, v\rangle + st. \tag{13}$$

Proof. The dual cone $SC(K, B)^*$ is defined as

$$SC(K, B)^* = \{(y, v, s) : \ \langle x, y\rangle + 2\langle u, v\rangle + ts > 0, \ \forall \, (x, u, t) \in \overline{SC(K, B)}, \ (x, u, t) \neq 0\}$$

If $t = 0$, then $-B(u, u) \in \overline{K}$, and, consequently, $u = 0$. A point $(x, 0, 0) \neq 0$ belongs to $\overline{SC(K, B)}$ if and only if $x \in \overline{K}$, $x \neq 0$. Therefore,

$$SC(K, B)^* = \{(y, v, s) : \ y \in int\, K^*, \ \langle x, y\rangle + 2\langle u, v\rangle + s > 0, \ \forall \, (x, u, 1) \in \overline{SC(K, B)}\}.$$

A point $(x, u, 1) \in \overline{SC(K,B)}$ if and only if $x - B(u,u) \in \overline{K}$. If we write

$$\begin{aligned}\langle x, y\rangle + 2\langle u, v\rangle + s = &\langle x - B(u,u), y\rangle + \\ &\langle (u - U(y)^{-1}v), U(y)(u - U(y)^{-1}v)\rangle + \\ &(s - \langle v, U(y)^{-1}v\rangle),\end{aligned}$$

then it is easy to see that the first two inner products on the right-hand side are nonnegative, and are equal to 0 if we pick $u = U(y)^{-1}v$ and $x = B(u,u)$. □

The cone $SC(K,B)^*$ is homogeneous. We can construct a transitive subset of $SC(K,B)^*$ by computing adjoint transformations T_i^* corresponding to T_i from (5)-(7) (see [5]). It can be easily checked that T_i^* are defined as follows

$$\begin{aligned}&T_1^*(y, v, s) = (y, \sqrt{\alpha}v, \alpha s),\ \alpha > 0,\\ &T_2^*(y, v, s) = (y, U(y)a + v, s + 2\langle a, v\rangle + \langle a, U(y)a\rangle),\ a \in \mathbb{R}^p,\\ &T_3^*(y, v, s) = (g^*y, (\overline{g})^*v, s),\ g \in G \subseteq Aut(K)\end{aligned}$$

generate a transitive subgroup of $Aut(SC(K,B)^*)$ that will be denoted by G_{SC}^*.

The explicit form of the generators of a transitive subgroup of $Aut(SC)$ and their triangular structure give us an opportunity to provide an alternative formula for the universal barrier for $SC(K,B)^*$.

Theorem 3.2. *Let K be a homogeneous cone. The characteristic function $\phi_{SC(K,B)^*}$ and the universal barrier $u_{SC(K,B)^*}$ for $SC(K,B)^*$ are given by*

$$\begin{aligned}&\phi_{SC(K,B)^*}(y, v, s) = const \cdot \phi_{K^*}(y) \cdot |\det(\overline{g})^*| \cdot (s - \langle U(y)^{-1}v, v\rangle)^{-(p/2+1)},\\ &u_{SC(K,B)^*}(y, v, s) = u_{K^*}(y) + \log|\det(\overline{g})^*| - (p/2+1)\log(s - \langle U(y)^{-1}v, v\rangle) + const,\end{aligned}$$

respectively.

Here, the mapping $\overline{g}$ is like in Definition 2.3, it depends on the choice of y.

Proof. Fix $(y_0, 0, 1) \in SC(K,B)^*$. For any $(y, v, s) \in SC(K,B)$ there exists an automorphism T^* of $SC(K,B)^*$ such that $T^*(y, v, s) = (y_0, 0, 1)$. If we pick $\alpha = (s - \langle v, U(y)^{-1}v\rangle)$, $a = -U(y)^{-1}v$, and $g \in G$ such that $g^*y = y_0$, then $T^* = T_3^*T_1^*T_2^*$ with transformations T_i, $i = 1, 2, 3$ defined in the above theorem. The equality (2) gives

$$\begin{aligned}\phi_{SC(K,B)^*}(y, v, s) = &\ \phi_{SC(K,B)^*}(T^{-*}(y_0, 0, 1))\\ &= \phi_{SC(K,B)^*}(y_0, 0, 1)/|\det(T^{-*})|\\ &= const \cdot |\det T^*|.\end{aligned}$$

Therefore, to compute the value of the characteristic function of $SC(K,B)^*$ at (y,v,s), it is enough to calculate $\det T^* = \det T_1^* \det T_2^* \det T_3^*$. It is easy to see that $\det T_1^* = \alpha^{p/2+1} = (s - \langle U(y)^{-1}v, v\rangle)^{-(p/2+1)}$, $\det T_2^* = 1$, $\det T_3^* = \det g^* \det(\overline{g})^*$. From the invariance property (2) of the characteristic function ϕ_{K^*} we obtain $\phi_{K^*}(y) = \phi_{K^*}(y_0)/|\det(g)^{-*}| = const \cdot |\det g^*|$, and, consequently, $|\det T_3^*| = const \cdot \phi_{K^*}(y) \cdot |\det(\overline{g})^*|$. The expression for the characteristic function follows. □

It was shown by Nesterov and Nemirovskii [4] that if F is a θ-normal barrier for K, then the (modified) Fenchel transformation

$$(F)^*(y) = \sup_{x \in K^o} \{-\langle x, y\rangle - F(x)\}$$

is a θ-normal barrier for K^*. In case of homogeneous cones the optimal parameter is equal to the rank of the cone, and the ranks of K and K^* coincide. This implies that the Fenchel dual of the optimal barrier F_K is an optimal barrier for K^*. In general, it is hard or even impossible to give a symbolic expression for the Fenchel dual (see, for instance, [11],[12]). However, we are able to calculate it for optimal barriers on homogeneous cones as shown in the following

Theorem 3.3. *Let K be a homogeneous cone. If F_K is an optimal θ-normal barrier for K, then*

$$F^*_{SC(K,B)}(y,v,s) = F^*_K(y) - \log(s - \langle U(y)^{-1}v, v\rangle) + const. \tag{14}$$

is $(\theta+1)$-normal barrier for $SC(K,B)^*$, and the parameter $(\theta+1)$ is the best possible.

We will formulate and prove two lemmas that will be used in the proof of Theorem 3.3.

Lemma 3.1. *The primal optimal barrier $F_{SC(K,B)}$ is invariant with respect to transformations from G_{SC}, i.e.*

$$F_{SC(K,B)}(T(x,u,t)) = F_{SC(K,B)}(x,u,t) + \eta(T), \quad \forall\, T \in G_{SC} \tag{15}$$

(the function η does not depend on the choice of the point $(x,u,t) \in SC(K,B)$).

Consequently, η has the following properties

(i) $\eta(I) = 0$,

(ii) $\eta(T_1T_2) = \eta(T_1) + \eta(T_2)$, $\forall\, T_1, T_2 \in G_{SC}$,

(iii) $\eta(T^{-1}) = -\eta(T)$, $\forall\, T \in G_{SC}$

Proof. The proof of this lemma consists of simple verification of (15) for the generators T_1, T_2, T_3 of G_{SC}. □

Lemma 3.2. *Let F be a convex function defined on a homogeneous cone K. If $F(gx) = F(x) + \eta(g)$ for all g from a transitive subset G of $Aut(K)$, then*

$$F^*(g^*y) = F^*(y) + \eta(g), \qquad \forall\, g \in G. \tag{16}$$

Proof. This lemma is a direct consequence of the definition of the Fenchel conjugate and the fact that K is homogeneous: $F^*(g^*y) = \sup_{x\in K^o} \{-\langle g^*y, x\rangle - F(x)\} = \sup_{x\in K^o} \{-\langle y, gx\rangle - F(gx)\} + \eta(g) = \sup_{z\in K^o} \{-\langle y, z\rangle - F(z)\} + \eta(g) = F^*(y) + \eta(g)$. □

Now we present the proof of Theorem (3.3).

Proof. Let (y, v, s) be an arbitrary point of $SC(K, B)^*$. It can be mapped to the point $(y, 0, 1)$ by means of transformations T_1^* and T_2^* with $\alpha = (s - \langle v, U(y)^{-1}v\rangle)^{-1}$ and $a = -U(y)^{-1}v$, more precisely, $T_1^* T_2^*(y, v, s) = (y, 0, 1)$. Therefore, it follows from Lemmas 3.1 and 3.2 that

$$F^*_{SC}(y, v, s) = F^*_{SC}(y, 0, 1) - \eta(T_1) - \eta(T_2).$$

Consequently, to compute the value of F^*_{SC} at (y, v, s), we need to compute $F^*_{SC}(y, 0, 1)$ (which is easier, and can be done using the properties of the duality mapping) and $\eta(T_1)$, $\eta(T_2)$. It can be easily checked that $\eta(T_1) = -\log\alpha = \log(s - \langle v, U(y)^{-1}v\rangle)$ and $\eta(T_2) = 0$. Hence,

$$F^*_{SC}(y, v, s) = F^*_{SC}(y, 0, 1) - \log(s - \langle v, U(y)^{-1}v\rangle). \tag{17}$$

To compute $F^*_{SC}(y, 0, 1)$ we will use logarithmic homogeneity of the barriers and the properties of the duality mapping * ($x^* := -\nabla F_K(x) \in K^*$ if $x \in K$, $(x, u, t)^* := -\nabla F_{SC(x,u,t)} \in SC(K, B)^*$ if $(x, u, t) \in SC(K, B)$). In order to find $(x, u, t)^*$, we expand $F_{SC(K,B)}(x, u, t) = F_K(tx - B(u, u)) + (\theta - 1)\log t$ into the Taylor series to get

$$\begin{aligned} DF_{SC(K,B)}(x, u, t)[\Delta x, \Delta u, \Delta t] &= DF_K(tx - B(u, u))[t\Delta x + x\Delta t - 2B(u, \Delta u)] \\ &\quad + (\theta - 1)\Delta t/t \\ &= \quad \langle\, t\nabla F_K(tx - B(u, u)), \quad \Delta x\rangle + \\ &\quad + 2\langle -\nabla F_K(tx - B(u, u)), B(u, \Delta u)\rangle + \\ &\quad + (\langle \nabla F_K(tx - B(u, u)), x\rangle + (\theta - 1)/t)\, \Delta t. \end{aligned}$$

It follows from (10) that $\langle -\nabla F_K(tx - B(u, u)), B(u, \Delta u)\rangle = \langle U(-\nabla F_K(tx - B(u, u))u, \Delta u)\rangle$, therefore,

$$\begin{aligned} (x, u, t)^* &= -\begin{pmatrix} t\,\nabla F_K(tx - B(u, u)) \\ U(-\nabla F_K(tx - B(u, u)))u \\ \langle \nabla F_K(tx - B(u, u)), x\rangle + (\theta - 1)/t \end{pmatrix} \\ &= \begin{pmatrix} (x - B(u, u)/t)^* \\ -U((tx - B(u, u))^*)u \\ \langle (tx - B(u, u))^*, x\rangle + (1 - \theta)/t \end{pmatrix} \end{aligned} \tag{18}$$

with respect to the inner product (13).

Equality (18) implies $(x^*, 0, 1) = (x, 0, 1)^*$ for any $x \in K$. Since the duality mapping is onto ([5], Proposition 7), there exists $x \in K$ such that $x^* = y$. Hence, $F^*_{SC}(y, 0, 1) = F^*_{SC}(x^*, 0, 1) = F^*_{SC}((x, 0, 1)^*)$. Logarithmic homogeneity of F_K and F_{SC} implies $F_K(x) + F^*_K(x^*) = -\theta$, $F_{SC}(x, 0, 1) + F^*_{SC}((x, 0, 1)^*) = -(\theta + 1)$, therefore, $F^*_{SC}(y, 0, 1) = -(\theta + 1) - F_{SC}(x, 0, 1) = -\theta - 1 - F_K(x) = F^*_K(x^*) - 1$, Consequently,

$$F^*_{SC}(y, 0, 1) = F_K(y) - 1. \tag{19}$$

Equation (14) now follows from (17) and (19). □

4. Some Examples

To illustrate how the constructions of optimal primal and dual barriers works, consider the well-known case of the cone of symmetric positive definite matrices.

Example 4.1. *(Symmetric positive definite matrices)*
It is well known that $F_{S^n_{++}}(X_n) = -\log\det X_n$. The cone S^{n+1}_{++} can be obtained from S^n_{++}, if in (4), we use S^n_{++}-bilinear symmetric homogeneous form $B_n(u, v) := (uv^T + vu^T)/2$, $u, v \in \mathbb{R}^n$.

Consider $X_{n+1} = \begin{pmatrix} X_n & u \\ u^T & t \end{pmatrix} \in S^{n+1}_{++}$, where X_n is $n \times n$ principal minor of X_{n+1}, $u \in \mathbb{R}^n$, and $t \in \mathbb{R}$. It is easy to see that

$$\det X_{n+1} = \det \begin{pmatrix} X_n & u \\ u^T & t \end{pmatrix} = \det \begin{pmatrix} I & u \\ 0 & t \end{pmatrix} \det \begin{pmatrix} X_n - uu^T/t & 0 \\ u^T/t & I \end{pmatrix} = t\det(X_n - uu^T/t). \tag{20}$$

Hence, by applying $-\log$ to both sides of the above equation, we obtain

$$-\log\det X_{n+1} = -\log\det(X_n - uu^T/t) - \log t,$$

which is exactly (8). The constructions of the primal barrier corresponds to application of (20) at each step.

The construction of the dual barrier is similar. Since S^{n+1}_{++} is symmetric, the optimal primal and dual barriers are the same. For $Y_n \in S^n$ the transformation $U_n(Y_n)$ corresponding to U in (10) is identity, i.e. $U_n(Y_n) = Y_n$, since

$$\langle B(u, v), Y_n \rangle = \langle (uv^T + vu^T)/2, Y_n \rangle = \mathrm{Tr}\,(Y_n uv^T) = \mathrm{Tr}\,(v^T Y_n u) = \langle Y_n u, v \rangle.$$

If $Y_{n+1} = \begin{pmatrix} Y_n & v \\ v^T & s \end{pmatrix} \in S^{n+1}_{++}$, where $v \in \mathbb{R}^n$, and $s \in \mathbb{R}$. It is easy to see that

$$\begin{aligned} \det Y_{n+1} = \det \begin{pmatrix} Y_n & v \\ v^T & s \end{pmatrix} &= \det \begin{pmatrix} Y_n & 0 \\ v^T & I \end{pmatrix} \det \begin{pmatrix} I & Y_n^{-1} v \\ 0 & s - v^T Y_n^{-1} v \end{pmatrix} \\ &= (s - \langle v, Y_n^{-1} v \rangle) \det Y_n. \end{aligned}$$

Hence, we again obtain the formula for

$$-\log\det Y_{n+1} = -\log\det Y_n - \log(s - \langle v, Y_n^{-1} v\rangle)$$

similar to (14).

Example 4.2. *(Epigraph of matrix norms)*

As shown in [1], the optimal primal barrier F_K for

$$K = \{(x,u,t) \in S^m \times \mathbb{R}^{n\times m} \times \mathbb{R} :\ tx - u^T u \in S^m_{++}, t > 0\}$$

is equal to

$$F_K(x,u,t) = -\log\det\left(x - u^T u/t\right) - \log t.$$

Compute the optimal dual barrier F_{K^*}. The S^n_{++}-bilinear symmetric form in this case is equal to $B(u,v) = (u^T v + v^T u)/2$. If $y \in S^m$, then

$$\langle B(u,v), y\rangle = \mathrm{Tr}\ ((u^T v + v^T u)y)/2 = \mathrm{Tr}\ (yuv^T) = \langle uy, v\rangle.$$

Hence, $U(y)u = uy$, and if $y \in S^m_{++}$, then $U(y)^{-1}u = uy^{-1}$. Therefore,

$$K^* = \{(y,v,s) \in S^m_{++} \times \mathbb{R}^{n\times m} \times \mathbb{R} :\ s > \langle vy^{-1}, v\rangle\}.$$

and the optimal dual barrier is equal to

$$\begin{aligned} F_{K^*} &= -\log\det y - \log(s - \langle vy^{-1}, v\rangle) \\ &= -\log\det y - \log(s - \langle v^T v, y^{-1}\rangle). \end{aligned}$$

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