Sensitivity analysis in linear semi-infinite programming via partitions

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Abstract

This paper provides sufficient conditions for the optimal value function of a given linear semi-infinite programming problem to depend linearly on the size of the perturbations, when these perturbations are directional, involve either the cost coefficients or the right-hand-side function or both, and they are sufficiently small. Two kinds of partitions are considered. The first one concerns the effective domain of the optimal value as a function of the cost coefficients, and consists of maximal regions on which this value function is linear. The second class of partitions considered in the paper concern the index set of the constraints through a suitable extension of the concept of optimal partition from ordinary to semi-infinte linear programming. These partitions provide convex sets, in particular segments, on which the optimal value is a linear function of the size of the perturbations, for the three types of perturbations considered in this paper.

Key words Sensitivity analysis, linear semi-infinite programming, linear programming, optimal value function.

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1 Introduction

Given a linear semi-infinite programming (LSIP) problem and a perturbation direction of the cost vector and/or the right-hand-side (RHS) function, we give conditions guaranteeing the linearity of the optimal value function with respect to the size of the perturbation provided this size is sufficiently small. The preceding works are, first, a stream of papers on sensitivity analysis in ordinary and parametric linear programming (LP) from an optimal partition perspective ([1], [2], [4], [10], [6], [7], [11], [12], [13], [14], [15], [16], [17], [18]) and, second, the recent paper [8], where conditions are given for the linearity (not only on segments) of the optimal value function of a LSIP problem with respect to (non-simultaneous) perturbations of the cost vector or the RHS function from a duality perspective.

We consider given a vector $c \in \mathbb{R}^n$, two (possibly infinite) sets of indices, U and V, such that $U \cap V = \emptyset$ and $U \neq \emptyset$, and two functions $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$, where $T := U \cup V$. We associate with the triple $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ (the data) a primal nominal problem in $\mathbb{R}^n$,

$$\begin{array}{lll} P: & \text{Inf} & c'x \\ & \text{s.t.} & a_t'x \geq b_t,\ t \in U, \\ & & a_t'x = b_t,\ t \in V, \end{array}$$

which is assumed to be consistent, and its corresponding dual nominal problem in $\mathbb{R}^{(T)}$ (the linear space of *generalized finite sequences*, i.e., the functions $\lambda : T \to \mathbb{R}$ such that $\lambda_t = 0$ for all $t \in T$ except maybe for a finite number of indices),

$$\begin{array}{lll} D: & \text{Sup} & \sum_{t\in T} \lambda_t b_t \\ & \text{s.t.} & \sum_{t\in T} \lambda_t a_t = c, \\ & & \lambda_t \geq 0,\ t \in U. \end{array}$$

These problems are called *bounded* when their optimal values, denoted by v^P and v^D, are finite. In contrast with LP, in LSIP the boundedness of both problems does not imply their solvability and $v^P = v^D$. We denote by F and F^* (by Λ and Λ^*) the *feasible* and the *optimal sets* of P (of D, respectively). We assume throughout that $\emptyset \neq F \neq \mathbb{R}^n$.

If we replace c by $z \in \mathbb{R}^n$ in P and D we get parametric LSIP problems whose optimal value depends on z. These optimal value functions, from $\mathbb{R}^n$

to $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, are concave, proper and (positively) homogeneous (a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *homogeneous* if $f(\lambda x) = \lambda f(z)$ for all $z \in \mathbb{R}^n$ and $\lambda > 0$).

The size of the perturbations of c can be measured through the Euclidean norm in $\mathbb{R}^n$, $\|\cdot\|$, with associated distance d. Concerning the perturbations of $b : T \to \mathbb{R}$, we consider the linear space $\mathbb{R}^T$ equipped with the pseudometric $\delta(f, g) := \sup_{t \in T} |f(t) - g(t)|$, for f, $g \in \mathbb{R}^T$ (we may have $\delta(f, g) = +\infty$). The zero-vector in $\mathbb{R}^T$ is denoted by 0_T.

The canonical basis, the zero-vector and the open unit ball in $\mathbb{R}^n$ will be denoted by $\{e_1, ..., e_n\}$, 0_n and $B(0_n; 1)$, respectively. For any set $X \neq \emptyset$, we denote by $|X|$, $\operatorname{cl} X$, $\operatorname{int} X$, $\operatorname{rint} X$, $\operatorname{conv} X$, $\operatorname{cone} X$, $\operatorname{aff} X$, $\operatorname{span} X$ and X^0 the *cardinality*, the *closure*, the *interior*, the *relative interior*, the *convex hull*, the *convex conical hull* (of $X \cup \{0_n\}$), the *affine hull*, the *linear hull*, and the *positive polar* of X, respectively. The dimension of a convex set $X \subseteq \mathbb{R}^n$ will be denoted by $\dim X$. A vector $y \in \mathbb{R}^n$ is a *feasible direction* at $x \in X$ if there exists $\varepsilon > 0$ such that $x + \varepsilon y \in X$. The *cone of feasible directions* at x will be denoted by $D(X; x)$.

Now we summarize some basic concepts and results of LSIP theory that will be used throughout (all these results can be found in [9]).

Let problem P be defined by the triple (a, b, c). Its *characteristic cone* is

$$K := \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; -\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in V; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

The Farkas lemma establishes that $u'x \geq \alpha$ for all $x \in F$ if and only if $(u, \alpha) \in \operatorname{cl} K$. Thus $\operatorname{cl} K$ only depends on F whereas Λ depends on K (and so on the constraint system of P). Given $x \in F$, the *set of active indices* at x is $T(x) := \{t \in T \mid a_t' x = b_t\}$. Obviously, $V \subseteq T(x)$. The *active cone* at x is

$$A(x) := \operatorname{cone}\{a_t, \ t \in T(x); -a_t, \ t \in V\}.$$

It is easy to see that $x \in F^*$ if and only if $c \in D(F; x)^0$ and also that $A(x) \subseteq D(F; x)^0$ for all $x \in F$. Consequently, if $c \in A(x)$ (the KKT condition) then $x \in F^*$, and the converse statement holds if K is closed.

A point $x^* \in F$ is a *strongly unique* optimal solution if there exists $\alpha > 0$ such that $c'x \geq c'x^* + \alpha \|x - x^*\|$ for all $x \in F$ (in which case $F^* = \{x^*\}$). This happens if and only if $c \in \operatorname{int} D(F; x^*)^0$.

The weak duality theorem establishes that $v^D \leq v^P$. The equality holds if either K is closed or $c \in \operatorname{rint} M$, where $M := \operatorname{cone}\{a_t, t \in T; -a_t, t \in V\}$ is the so-called *first moment cone*. Moreover the first condition entails $\Lambda^* \neq \emptyset$ if $\Lambda \neq \emptyset$ and the second one $F^* \neq \emptyset$.

F is bounded if and only if $M = \mathbb{R}^n$ and F^* is bounded if and only if $c \in \operatorname{int} M$. Since M is invariant through the perturbations considered in this paper, if the primal feasible set is bounded, the same is true for the perturbed problems. The strong Slater condition (existence of $\overline{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $a_t'\overline{x} \geq b_t + \varepsilon$ for all $t \in U$, and $a_t'\overline{x} = b_t$ for all $t \in V$), together with the linear independence of $\{a_t, t \in V\}$ if $V \neq \emptyset$, guarantees the solvability of the problem obtained by replacing b with $w \in \mathbb{R}^T$ provided $\delta(w, b)$ is sufficiently small. Under both assumptions, the perturbed problems have zero duality gap for sufficiently small perturbations of the data.

This paper is structured as follows. Section 2 shows that the effective domain of any convex homogeneous function can be partitioned into maximal relatively open convex cones where the function is *linear* (i.e., finite, convex and concave) which are called *linearity cones* of the given function. Section 3 extends and analyzes the concepts of complementary solution and optimal partition from LP to LSIP. Section 4 examines the linearity of the optimal value functions associated with perturbations of c on convex sets (e.g, on segments emanating from c and on relatively open convex cones) by means of the theory developed in Section 2 (as both optimal value functions are concave in the case of perturbations of c) and Section 3. Sections 5 and 6 give sufficient conditions for the optimal value function to depend linearly on the size of the perturbations when the perturbed data are the RHS function b or both parameters, c and b, respectively. These conditions are expressed in terms of optimal partitions. Finally, Section 7 contains the conclusions.

2 Linearity cones of convex homogeneous functions

The effective domain of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is denoted by $\operatorname{dom} f$. In this section we prove that, if f is convex and homogeneous, then there exists a partition of $(\operatorname{dom} f) \setminus \{0_n\}$ into maximal relatively open convex cones on which f is linear.

Lemma 1 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex function. Then the following statements hold:*

(i) *If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping, then $f \circ A$ is also convex. Moreover, if f is homogeneous (linear), then $f \circ A$ is also homogeneous (linear, respectively).*

(ii) *If $C \subset \mathbb{R}^n$ is convex and h is a linear function on $\mathbb{R}^n$ such that $f(x) \leq h(x)$ for all $x \in C$ and $f(\overline{x}) = h(\overline{x})$ for a certain $\overline{x} \in \operatorname{rint} C$, then $f(x) = h(x)$ for all $x \in C$.*

(iii) *If C and D are convex sets such that $(\operatorname{rint} C) \cap (\operatorname{rint} D) \neq \emptyset$ and $D \subset \operatorname{aff} C$, f is linear on D and $f(x) = d'x + \delta$ for all $x \in C$, with $d \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$, then $f(x) = d'x + \delta$ for all $x \in D$.*

Proof: (i) It is immediate.

(ii) Since $f - h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is also convex, we can assume $f(x) \leq 0$ for all $x \in C$ and $f(\overline{x}) = 0$.

Take an arbitrary $x \in C$. Since $\overline{x} \in \operatorname{rint} C$, there exists $\mu > 1$ such that $z := (1-\mu)x + \mu\overline{x} \in C$. Then we have $\overline{x} = \mu^{-1}z + (1-\mu^{-1})x$, with $0 < \mu^{-1} < 1$ and, by convexity of f we get

$$0 = f(\overline{x}) \leq \mu^{-1} f(z) + \left(1 - \mu^{-1}\right) f(x) \leq \left(1 - \mu^{-1}\right) f(x).$$

Consequently, $0 \leq f(x)$. Since we are assuming $f(x) \leq 0$, we have $f(x) = 0$.

(iii) Take an arbitrary $x \in D$. Select a point $\overline{x} \in (\operatorname{rint} C) \cap (\operatorname{rint} D)$. Since $x \in D \subset (\operatorname{aff} C) \cap (\operatorname{aff} D)$, based on the same arguments as in part (ii), there exists an element $z \in C \cap D$ and $\varepsilon > 0$, $\varepsilon < 1$, such that $\overline{x} = \varepsilon z + (1-\varepsilon)x$.

Taking into account that $\overline{x}, z \in C$ and the linearity of f on $[x, z] \subset D$, we have

$$d'\overline{x} + \delta = f(\overline{x}) = \varepsilon f(z) + (1-\varepsilon) f(x) = \varepsilon (d'z + \delta) + (1-\varepsilon) f(x),$$

from which we get $f(x) = d'x + \delta$. □

Lemma 2 *Let C and D be two cones in $\mathbb{R}^n$ such that C is convex, relatively open and $C \cap D \neq \emptyset$. Then $C \subset C + D$.*

Proof: Let $c \in C \cap D$. Given $x \in C$, since $c, x \in C$ and this is relatively open, there exists $\mu > 1$ such that $y := (1-\mu)c + \mu x \in C$. Then $x = \mu^{-1}y + (1-\mu^{-1})c \in C + D$. Hence $C \subset C + D$. □

Proposition 1 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex homogeneous function. Let $\{C_i,\ i \in I\}$ be a finite family of relatively open convex cones containing $c \in \mathbb{R}^n \setminus \{0_n\}$ on which f is linear. Then f is linear on $\sum_{i\in I} C_i$.*

Proof: Given $J \subset I$, $J \neq \emptyset$, we denote $C_J = \sum_{i\in J} C_i$, which is also a relatively open convex cone (the three properties are preserved by the sum) containing c (because $c = \sum_{i\in J} |J|^{-1} c \in C_J$). If $\emptyset \neq H \subsetneq J \subset I$, then by Lemma 2, $C_H \subset C_H + C_{J\setminus H} = C_J$.

Let $\dim C_I = m \leq n$. The case when $m = 1$ is trivial, so we suppose that $m \geq 2$. Let k be the minimum cardinality of the sets $J \subset I$ such that $\dim C_J = m$. We can assume without loss of generality that $\dim C_K = m$, where $K = \{1, ..., k\} \subset I$. Obviously, $C_K \subset C_I$.

First we show that

$$2 \leq \dim C_1 < \dim (C_1 + C_2) < ... < \dim C_K = m. \tag{1}$$

If $\dim C_1 = 1$, then $\operatorname{span} C_1 = \operatorname{span} \{c\} \subset \operatorname{span} C_2$ and $\dim \sum\_i = 2^k C_i = m$, contradicting the definition of k.

Analogously, if there exists $j \in \{1, ..., k-1\}$ such that $\dim \sum_{i=1}^{j+1} C_i = \dim \sum_{i=1}^{j} C_i$, then $\operatorname{span} C_{j+1} \subset \operatorname{span} \sum_{i=1}^{j} C_i$ and we have $\dim \sum_{i\in K\setminus\{j\}} C_i = m$, contradicting again the definition of k.

Observe that (1) entails that $k + 1 \leq \dim C_K = m$, $\operatorname{span} C_i \neq \operatorname{span} C_j$ if $i \neq j = 1, 2, ..., k$ (since $C_1, .., C_k$ can be re-ordered arbitrarily) and $\dim \sum_{i\in K\setminus\{j\}} C_i < m$. where $j = 1, 2, ..., k$.

Now we select m vectors of $\mathbb{R}^n$ as follows. Let $m_0 = 1$. Let $m_1 = \dim C_1 \geq 2$ and let us select in C_1 a set of m_1 linearly independent vectors, $\{v_1, ..., v_{m_1}\}$, where $v_1 = c$. Now, let $m_2 = \dim (C_1 + C_2) > m_1$. Since $C_1 \subset C_1 + C_2$ (by Lemma 2), there exist $w_i \in C_1$ and $v_i \in C_2$, $i = m_1 + 1, ..., m_2$, such that $\{v_1, ..., v_{m_1}, w_{m_1+1} + v_{m_1+1}, ..., w_{m_2} + v_{m_2}\}$ form a basis of $\operatorname{span} \{C_1 + C_2\}$. Since $w_i \in \operatorname{span} C_1$, $i = m_1 + 1, ..., m_2$, the system of m_2 vectors $\{v_1, ..., v_{m_2}\}$ is also a basis of $\operatorname{span} \{C_1 + C_2\}$. By induction, considering all the k cones, we obtain $m_k = m$ linearly independent vectors

$$v_1, ..., v_{m_1}, v_{m_1+1}, ..., v_{m_2}, ..., v_{m_{k-1}+1}, ..., v_{m_k} \in C_K,$$

such that $m_{i-1} < m_i$, $i = 1, 2, ..., k$, and

$$C_K \subset C_I \subset \operatorname{span} \left\{v_1, ..., v_{m_1}, v_{m_1+1}, ..., v_{m_2}, ..., v_{m_{k-1}+1}, ..., v_{m_k}\right\}.$$

Now, we define a new family of relatively open cones $\{B_1, ..., B_k\}$, containing c, as follows: $B_1 = C_1$ and $B_i = C_i \cap \operatorname{span}\left\{v_1, v_{m_{i-1}+1}, ..., v_{m_i}\right\}$, $i = 2, ..., k$ (recall that any linear subspace is relatively open). Obviously, $B_K := \sum_{i \in K} B_i$ is a relatively open convex cone such that $c \in B_K$, $\dim B_K = m$ and $B_K \subset C_K \subset C_I$.

Let A be a (non-singular) linear transformation on $\mathbb{R}^n$ such that $Av_i = e_i$, $i = 1, 2, ..., m$. Therefore $e_1 \in AB_i \subset \operatorname{span}\left\{e_1, e_{m_{i-1}+1}, ..., e_{m_i}\right\}$, $i = 1, 2, ..., k$, and all the sets

$$AB_K \subset AC_K \subset AC_I \subset A \operatorname{span}\{v_1, ..., v_m\} = \operatorname{span}\{e_1, ..., e_m\} = \mathbb{R}^m \times \{0_{n-m}\}$$

are relatively open, have the same dimension m and contain e_1.

The function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that $g = f \circ A^{-1}$ is convex and homogeneous (by Lemma 1, part (i)), so that $g(0_n) = 0$. We denote $d = (d_1, ..., d_m, 0..., 0) \in \mathbb{R}^n$, where $d_j := g(e_j)$, $j = 1, ..., m$. Observe that

$$d_1 = g(e_1) = f(A^{-1}e_1) = f(v_1) = f(c). \tag{2}$$

Given $i = 1, 2, ..., k$, since f is linear on $B_i \subset C_i$, g is also linear on

$$AB_i \subset \operatorname{span}\left\{e_1, e_{m_{i-1}+1}, ..., e_{m_i}\right\},$$

and we can express

$$g(x) = x_1 d_1 + \sum_{j=m_{i-1}+1}^{m_i} x_j d_j$$

for all $x \in AB_i$ and for all $i = 1, 2, ..., k$.

Now we prove that

$$g(y) \leq d'y \text{ for all } y \in AB_K. \tag{3}$$

Take an arbitrary $y \in AB_K$. Then we can write $y = \sum_{i \in K} y^i$, with $y^i \in AB_i$, $i = 1, 2, ..., k$. Let $y^i = y_1^i e_1 + \sum_{j=m_{i-1}+1}^{m_i} y_j^i e_j$, $i = 1, 2, ..., k$. Then we have

$$\begin{aligned} g(y) &= g(k \sum_{i \in K} k^{-1} y^i) = k g(\sum_{i \in K} k^{-1} y^i) \\ &\leq \sum_{i \in K} g(y^i) = \sum_{i \in K} \left(y_1^i d_1 + \sum_{j=m_{i-1}+1}^{m_i} y_j^i d_j \right) \\ &= \left(\sum_{i \in K} y_1^i \right) d_1 + \sum_{i \in K} \sum_{j=m_{i-1}+1}^{m_i} y_j^i d_j = d'y. \end{aligned}$$

From (2), (3) and item (ii) of Lemma 1, recalling that $e_1 \in \operatorname{rint} AB_K = AB_K$, we get

$$g(y) = d'y \text{ for all } y \in AB_K. \tag{4}$$

In order to extend (4) to the whole cone AC_I, let us fix $i \in I$. Since we have $e_1 \in (\operatorname{rint} AC_i) \cap (\operatorname{rint} AB_K) = AC_i \cap AB_K$ and $AC_i \subset \mathbb{R}^m \times \{0_{n-m}\} = \operatorname{aff}(AB_K)$, according to item (iii) of Lemma 1, formula (4) entails $g(y) = d'y$ for all $y \in AC_i$.

Now, let us take an arbitrary point $y \in AC_I$, whereby $y = \sum_{i \in I} y^i$, with $y^i \in AC_i$, $i \in I$. Since g is a convex homogeneous function we have

$$g(y) \leq \sum_{i \in I} g(y^i) = \sum_{i \in I} d'y^i = dy.$$

Applying again item (ii) of Lemma 1, we conclude that g is linear on AC_I. Therefore $f = g \circ A$ is linear on $A^{-1}(AC_I) = C_I$. □

Let us illustrate Proposition 1 with two simple examples.

Example 1 *Consider the convex cones* $C_1 = \{x \in \mathbb{R}^3 \mid x_1 = 0,\ x_3 > 0\}$ *and* $C_2 = \{x \in \mathbb{R}^3 \mid x_2 = 0,\ x_3 > 0\}$. *They are relatively open and* $e_3 \in C_1 \cap C_2$. *Thus, any convex homogeneous function* $f : \mathbb{R}^3 \to \overline{\mathbb{R}}$ *which is linear on both cones,* C_1 *and* C_2, *is also linear on* $C_I = C_1 + C_2 = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$. *Concerning the objects used in the above proof,* $m = 3$, $k = 2$, *i.e.,* $K = I = \{1, 2\}$, *and we could choose* $v_1 = e_3$, $v_2 = e_2$ *and* $v_3 = e_1$, *so that* A *is the symmetry in* $\mathbb{R}^3$ *with respect to the plane* $x_3 = x_2$. *Then* $B_i = C_i$, $i = 1, 2$, $B_K = C_I$ *and so* $AB_K = AC_I = \{y \in \mathbb{R}^3 \mid y_1 > 0\}$.

Example 2 *The function* $f(x) = \max\{-x_1, -x_2\}$ *is convex and homogeneous on* $\mathbb{R}^2$ *and it vanishes on the relatively open convex cones* $C_1 = \mathbb{R}_{++} \times \{0\}$ *and* $C_2 = \{0\} \times \mathbb{R}_{++}$, *but it is not even linear on its sum* $C_1 + C_2 = \mathbb{R}^2_{++}$. *Observe also that* $C_i \cap (C_1 + C_2) = \emptyset$ *although* $C_i \subset \operatorname{cl}(C_1 + C_2)$, $i = 1, 2$. *This example shows that the assumptions on the intersection of the relatively open convex cones in Lemma 2 and Proposition 1 are not superfluous.*
Consider also the convex cone $C_3 = \mathbb{R}^2_+$. *Obviously,* $C_1 \cap C_3 \neq \emptyset$ *but* $C_3 \nsubseteq C_1 + C_3$, *so that Lemma 2 only guarantees that the relatively open convex cone is contained in the sum of the two cones.*

Proposition 2 *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a convex homogeneous function and let* $c \in \mathbb{R}^n \backslash \{0_n\}$. *Then there exists a largest relatively open convex cone containing* c *on which* f *is linear.*

Proof: Let $\mathcal{C} := \{C_i, i \in I\}$ be the class of all relatively open convex cones containing c on which f is linear. We shall prove that $C := \cup_{i\in I} C_i \in \mathcal{C}$ (i.e., C is the maximum of $\mathcal{C}$ for the inclusion).

Since f is linear on cone $\{c\} \setminus \{0_n\}$, this is an element of $\mathcal{C}$ so that $I \neq \emptyset$.

Let us denote with $\mathcal{J}$ the family of all nonempty finite subsets of I. For each $J \in \mathcal{J}$, the sum $C_J := \sum_{i\in J} C_i$ is a relatively open convex cone containing c and so $C_J \in \mathcal{C}$ by Proposition 1. Since $\mathcal{C} \subset \{C_J, J \in \mathcal{J}\} \subset \mathcal{C}$, we have $C = \cup_{J\in\mathcal{J}} C_J$. On the other hand, given $\{J, H\} \subset \mathcal{J}$ such that $J \subset H$, we have shown in Proposition 1 that

$$C_J \subset C_H. \tag{5}$$

Now we show that C satisfies all the requirements.

<u>C is a convex cone</u>: The union of cones is a cone. On the other hand, given $x^1, x^2 \in C$, if $x^i \in C_{J_i}$, $i = 1, 2$, taking $J = J_1 \cup J_2 \in \mathcal{J}$, (5) yields $x^i \in C_J$, $i = 1, 2$. Since C_J is convex, we have $[x^1, x^2] \subset C_J \subset C$.

<u>C is relatively open</u>: Let $x \in C$ and let $y \in \operatorname{aff} C$. Then we can write

$$y = \sum_{i=1}^{m} \lambda_i y_i, \; m \in \mathbb{N}, \; \sum_{i=1}^{m} \lambda_i = 1, \text{ and } y_i \in C, i = 1, ..., m.$$

By (5) there exists $J \in \mathcal{J}$ such that $x, y_i \in C_J$, $i = 1, ..., m$. Since C_J is relatively open, there exists $\mu > 1$ such that $\mu x + (1 - \mu) y \in C_J \subset C$. Then $x \in \operatorname{rint} C$.

<u>f is linear on C</u>: Let $x^1, x^2 \in C$. Let $J \in \mathcal{J}$ such that $x^1, x^2 \in C_J$. Since f is linear on C_J, we have $f((1 - \lambda) x^1 + \lambda x^2) = (1 - \lambda) f(x^1) + \lambda f(x^2)$ for all $\lambda \in [0, 1]$. □

Given a convex (concave) homogeneous function f, we define the *linearity cone* of f at $z \in (\operatorname{dom} f) \setminus \{0_n\}$ as the largest relatively open convex cone containing z on which f is linear (this definition is correct by Proposition 2). We denote it by C_z.

Proposition 3 *The linearity cones of a convex (concave) homogeneous function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ constitute a partition of* $(\operatorname{dom} f) \setminus \{0_n\}$.

Proof: We denote by $\mathcal{C}_z$ be the family of all the relatively open convex cones containing $z \in (\operatorname{dom} f) \setminus \{0_n\}$ on which f is linear. Obviously, C_z is the maximum of $\mathcal{C}_z$ for the inclusion.

Let us assume that the statement is not true. Let $z^1, z^2 \in (\operatorname{dom} f) \setminus \{0_n\}$ such that $C_{z^1} \cap C_{z^2} \neq \emptyset$ and $C_{z^1} \neq C_{z^2}$. Take an arbitrary $z \in C_{z^1} \cap C_{z^2}$. Since $C_{z^1}, C_{z^2} \in \mathcal{C}_z$, we have $C_{z^1}, C_{z^2} \subset C_z$, with $C_{z^i} \subsetneq C_z$ for some $i = 1, 2$. Then, C_{z^i} cannot be the linearity cone of f at z^i. □

3 Optimal partitions

Let us consider the primal LSIP problem P introduced in Section 1 and its dual problem D. We associate with each primal-dual feasible solution, $(x, \lambda) \in F \times \Lambda$, the *supporting sets* $\sigma(x) := \{t \in U \mid a_t' x > b_t\}$ and $\sigma(\lambda) := \{t \in U \mid \lambda_t > 0\}$. The couple $(x, \lambda) \in F \times \Lambda$ is called a *complementary solution of the pair* $P - D$ if $\sigma(x) \cap \sigma(\lambda) = \emptyset$.

The next two results clarify the relationship between optimality and complementary solutions in LSIP (which is more complex than in LP).

Proposition 4 *The pair* $(x, \lambda) \in F \times \Lambda$ *is a complementary solution of* $P - D$ *if and only if it is a primal-dual optimal solution and* $v^D = v^P$. *In that case, the following statements are true:*
(i) If $\overline{x} \in F$ *satisfies* $a_t' \overline{x} = b_t$ *for all* $t \in \sigma(\lambda)$, *then* $\overline{x} \in F^*$.
(ii) If $\overline{\lambda} \in \Lambda$ *satisfies* $\overline{\lambda}_t = 0$ *for all* $t \in \sigma(x)$, *then* $\overline{\lambda} \in \Lambda^*$.

Proof: Let (x, λ) be a complementary solution of $P - D$. Then $\sigma(x) \cap \sigma(\lambda) = \emptyset$, i.e., $\lambda_t (a_t' x - b_t) = 0$ for all $t \in U$. Since $a_t' x = b_t$ for all $t \in V$, we have $\sum\limits_{t \in T} \lambda_t (a_t' x - b_t) = 0$, so that

$$\sum_{t \in T} \lambda_t b_t = \left(\sum_{t \in T} \lambda_t a_t \right)' x = c' x,$$

and the weak duality theorem yields the coincidence of optimal values (i.e., $v^D = v^P$), $x \in F^*$ and $\lambda \in \Lambda^*$. The converse statement is trivial.

Now we assume that (x, λ) is a complementary solution of $P - D$.

(i) Let $\overline{x} \in F$ be such that $a_t' \overline{x} = b_t$ for all $t \in \sigma(\lambda)$. Then we have

$$\begin{aligned} v^D = \sum_{t \in T} \lambda_t b_t &= \sum_{t \in V \cup \sigma(\lambda)} \lambda_t b_t = \sum_{t \in V \cup \sigma(\lambda)} \lambda_t a_t' \overline{x} \\ &= \sum_{t \in T} \lambda_t a_t' \overline{x} = \left(\sum_{t \in T} \lambda_t a_t \right)' \overline{x} = c' \overline{x} \geq v^P, \end{aligned}$$

and the conclusion is consequence of the weak duality theorem.

(ii) Let $\overline{\lambda} \in \Lambda$ be such that $\overline{\lambda}_t = 0$ for all $t \in \sigma(x)$. Then

$$v^P = c'x = \left(\sum_{t\in T} \overline{\lambda}_t a_t\right)' x = \left(\sum_{t\in V\cup\sigma(\lambda)} \overline{\lambda}_t a_t\right)' x$$
$$= \sum_{t\in V\cup\sigma(\lambda)} \overline{\lambda}_t b_t = \sum_{t\in T} \overline{\lambda}_t b_t \leq v^D,$$

so that $\overline{\lambda} \in \Lambda^*$ again by the weak duality theorem. □

Corollary 1 *Given a point $\overline{x} \in F$, there exists $\overline{\lambda} \in \Lambda$ such that $(\overline{x}, \overline{\lambda})$ is a complementary solution of $P - D$ if and only if $\overline{x}$ is an optimal solution for some finite subproblem of P.*

Proof: If $(\overline{x}, \overline{\lambda})$ is a complementary solution of $P - D$, by Proposition 4, $\left(\sum_{t\in T} \overline{\lambda}_t a_t\right)' \overline{x} = c'\overline{x} = \sum_{t\in T} \overline{\lambda}_t b_t$, so that $\sum_{t\in T} \overline{\lambda}_t (a_t'\overline{x} - b_t) = 0$, i.e., $c \in A(\overline{x})$. Thus $\overline{x}$ is an optimal solution of the problem resulting of replacing U by $\sigma(\overline{\lambda})$ in P. Replacing in that problem $\{a_t'x = b_t,\ t \in V\}$ by an equivalent finite subsystem, we obtain an equivalent finite subproblem with optimal solution $\overline{x}$.

Conversely, assume that $\overline{x}$ is an optimal solution of the finite subproblem of P obtained substituting U and V with the finite subsets $\overline{U}$ and $\overline{V}$. Since the KKT condition characterizes optimality in LP, there exists $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ such that $\overline{\lambda}_t = 0$ for all $t \in T \diagdown (\overline{U} \cup \overline{V})$, $\overline{\lambda}_t \geq 0$ for all $t \in U$, $\sum_{t\in T} \overline{\lambda}_t (a_t'\overline{x} - b_t) = 0$, and $c \in \sum_{t\in T} \overline{\lambda}_t a_t$. Then it is easy to show that $(\overline{x}, \overline{\lambda})$ is a complementary solution of $P - D$, again by Proposition 4. □

The constraint system of P is called *locally Farkas-Minkowski* (see [9, Chapter 5] and references therein) if $u'x \geq \alpha$ for all $x \in F$, with $u'x = \alpha$ for some $x \in F$, implies that $u'x \geq \alpha$ for every x solution of some finite subsystem. This property is equivalent to assert that, for every $z \in \mathbb{R}^n$, if $\overline{x}$ is an optimal solution of $P(z)$, then it is also optimal solution for some finite subproblem of $P(z)$. Thus Corolary 1 gives two new characterizations of this class of linear semi-infinite systems.

A triple $(B, N, Z) \in (2^U)^3$ is called an *optimal partition* if there exists a complementary solution (x, λ) such that $B = \sigma(x)$, $N = \sigma(\lambda)$ and

$Z = U \diagdown (B \cup N)$ (for the sake of brevity we omit problems and couples of problems when they are implicit in the context). Obviously, the non-empty elements of the *tripartition* (B, N, Z) give a partition of U (similar tripartitions have been used in [2] and [7] in order to extend the optimal partition approach from LP to quadratic programming). We say that a tripartition $\left(\overline{B}, \overline{N}, \overline{Z}\right)$ is *maximal* if

$$\overline{B} = \bigcup_{x \in F^*} \sigma(x), \qquad \overline{N} = \bigcup_{\lambda \in \Lambda^*} \sigma(\lambda) \quad \text{and} \quad \overline{Z} = U \setminus (\overline{B} \cup \overline{N}).$$

Note that the definition of the maximal partition imply that $B \subset \overline{B}$ and $N \subset \overline{N}$ for every optimal partition (B, N, Z). The uniqueness of the maximal partition is straightforward consequence of the definition. If there exist an optimal solution pair $\overline{x} \in F^*$ and $\overline{\lambda} \in \Lambda^*$ such that $\sigma(\overline{x}) = \overline{B}$ and $\sigma(\overline{\lambda}) = \overline{N}$, then the maximal partition is called the *maximal optimal partition.*

Proposition 5 *The maximal optimal partition exists if and only if* $v^D = v^P$ *and there exist* $\overline{x} \in F^*$ *and* $\overline{\lambda} \in \Lambda^*$ *such that* $\sigma(x) \subset \sigma(\overline{x})$ *and* $\sigma(\lambda) \subset \sigma\left(\overline{\lambda}\right)$ *for all* $(x, \lambda) \in F^* \times \Lambda^*$*. In particular, if* $\left(\overline{B}, \overline{N}, \overline{Z}\right)$ *is an optimal partition such that* $\overline{Z} = \emptyset$*, then it is a maximal optimal partition.*

Proof: The first statement is straightforward consequence of Proposition 4.

Now, let $\left(\overline{x}, \overline{\lambda}\right)$ be a complementary solution such that $\overline{B} = \sigma(\overline{x})$, $\overline{N} = \sigma\left(\overline{\lambda}\right)$, and $\overline{B} \cup \overline{N} = U$ (in which case $\left(\overline{x}, \overline{\lambda}\right)$ is called *strictly complementary solution of* $P - D$). Let (B, N, Z) be an arbitrary optimal partition and let (x, λ) be a complementary solution such that $\sigma(x) = B$ and $\sigma(\lambda) = N$. Again by Proposition 4, the pairs $(\overline{x}, \lambda)$ and $\left(x, \overline{\lambda}\right)$ are also complementary solutions, so that $\overline{B} \cap N = \emptyset$ and $B \cap \overline{N} = \emptyset$, i.e., $N \subset U \diagdown \overline{B} = \overline{N}$ and $B \subset U \diagdown \overline{N} = \overline{B}$. □

The next example illustrates the existence of maximal optimal partitions $\left(\overline{B}, \overline{N}, \overline{Z}\right)$ such that $\overline{Z} \neq \emptyset$.

Example 3 *Consider the problem* P *in* $\mathbb{R}^2$ *such that* $T = \{-2, -1-, 0, 1, ...\}$, *the objective function is the null one, and the constraints are* $tx_1 \geq -1$, *for* $t = 1, 2, .., -x_1 \geq 0$ *(*$t = 0$*),* $x_2 \geq 0$ *(*$t = -1$*), and* $-x_2 \geq -1$ *(*$t = -2$*). We have* $F^* = \{0\} \times [0, 1]$ *and* $\Lambda^* = \{0_T\}$. *It is easy to show that* $(T \diagdown \{0\}, \emptyset, \{0\})$ *is the maximal optimal partition.*

The solvability of P guarantees the existence of a point $\overline{x}$ such that $\sigma(x) \subset \sigma(\overline{x})$ for all $x \in F^*$ due to the finite dimension of the space of variables (take $\overline{x} \in \operatorname{rint} F^*$). Concerning D, if Λ^* is the convex hull of a finite set, then its arithmetic mean, $\overline{\lambda}$, satisfies $\sigma(\lambda) \subset \sigma\left(\overline{\lambda}\right)$ for all $\lambda \in \Lambda^*$. Nevertheless, $v^D = v^P$ and primal-dual solvability do not guarantee the existence of the maximal optimal partition, as the following example shows.

Example 4 *Consider the following LSIP problem in* $\mathbb{R}^2$:

$$\begin{array}{lll} P: & Inf & x_2 \\ & s.t. & -x_1 + x_2 \geq 0 \quad (t=1) \\ & & x_1 + x_2 \geq 0 \quad\;\; (t=2) \\ & & x_2 \geq 0. \qquad\quad\;\; t = 3, 4, ... \end{array}$$

Obviously, $v^D = v^P = 0$, *with* $F^* = \{0_2\}$. *For* $r \in \mathbb{N}$ *we denote by* λ^r *the indicator function of* $\{r\}$. *Since* $\Lambda^* = \Lambda = \operatorname{conv}\left\{\frac{\lambda^1+\lambda^2}{2}, \lambda^3, \lambda^4, ...\right\}$, $\bigcup_{\lambda\in\Lambda^*} \sigma(\lambda) = T$ *and so the maximal partition* $(\emptyset, T, \emptyset)$ *cannot be optimal.*

From Proposition 4, if (B, N, Z) is an optimal partition of P, a sufficient optimality condition for $\overline{x} \in F$ ($\overline{\lambda} \in \Lambda$) is that $\sigma(\overline{x}) \cap N = \emptyset$ ($\sigma\left(\overline{\lambda}\right) \cap B = \emptyset$, respectively). When the maximal optimal partition exists, it provides the weakest optimality criterion based on optimal partitions.

4 Perturbing c

The perturbed problems of P and D to be considered in this section are

$$\begin{array}{lll} P(z): & \text{Inf} & z'x \\ & \text{s.t.} & a_t'x \geq b_t,\ t \in U, \\ & & a_t'x = b_t,\ t \in V, \end{array}$$

and

$$\begin{array}{lll} D(z): & \text{Sup} & \sum\limits_{t\in T} \lambda_t b_t \\ & \text{s.t.} & \sum\limits_{t\varepsilon T} \lambda_t a_t = z, \\ & & \lambda_t \geq 0,\ t \in U, \end{array}$$

where the parameter z ranges on $\mathbb{R}^n$. We denote the optimal values of $P(z)$ and $D(z)$ as $v^P(z)$ and $v^D(z)$, respectively (since Sections 4-6 deal with optimal value functions of different parameters, in order to avoid confusion, our notation makes explicit the corresponding argument, e.g., we write $v^P(z)$ and $v^D(z)$ instead of just v^P and v^D). With this notation, the effective domain of $v^D(z)$ is the first moment cone M and the optimal values of the nominal problem P and its dual D are $v^P(c)$ and $v^D(c)$, respectively. In [8, Section 2] we have shown that $v^P(z)$ is linear on a certain neighborhood of c (or on an open convex cone containing c) if and only if $c \in \operatorname{int} D(F; x^*)^0$ or, equivalently, if and only if P has a strongly unique solution. Moreover, $v^P(z)$ is linear on a segment emanating from c in the direction of $d \in \mathbb{R}^n \setminus \{0_n\}$ if P and D are solvable, with $v^D = v^P$, and the following problem is also solvable and has zero duality gap:

$$\begin{array}{lll} D_d: & \text{Sup} & \sum_{t \in T} \lambda_t b_t + \mu v^P(c) \\ & \text{s.t.} & \sum_{t \varepsilon T} \lambda_t a_t + \mu c = d, \\ & & \lambda_t \geq 0,\ t \in U. \end{array}$$

This is the case, in particular, if P is a bounded LP problem and d satisfies $\inf \{d'x \mid x \in F^*\} \neq -\infty$.

Lemma 3 *Let* $\left\{ \left(c^i, \lambda^i\right), i \in I \right\} \subset \mathbb{R}^n \times \mathbb{R}^{(T)}$ *and* $\overline{x} \in \mathbb{R}^n$ *be such that* $\left(\overline{x}, \lambda^i\right)$ *is a complementary solution of* $P(c^i) - D(c^i)$ *for all* $i \in I$. *Then* $P(z)$ *and* $D(z)$ *are solvable and*

$$v^P(z) = v^D(z) = \overline{x}'z \text{ for all } z \in \operatorname{conv}\left\{c^i, , i \in I\right\}. \tag{6}$$

Proof: Let $z \in \operatorname{conv}\{c^i, , i \in I\}$. Then there exists $\mu \in \mathbb{R}_+^{(I)}$ such that

$$z = \sum_{i \in I} \mu_i c^i \text{ and } \sum_{i \in I} \mu_i = 1.$$

Since the feasible set is the same for $P(z)$ and for all $P(c^i)$, $i \in I$, $\overline{x}$ is a feasible solution of $P(z)$.

Consider the element $\lambda^z := \sum_{i \in I} \mu_i \lambda^i \in \mathbb{R}^{(T)}$. We shall prove that λ^z is a feasible solution of $D(z)$. In fact, since λ^i is a feasible solution of $D(c^i)$, we have $\lambda_t^i \geq 0$ for all $t \in U$ and $\sum_{t \in T} \lambda_t^i a_t = c^i$ for all $i \in I$. Thus, $\lambda_t^z = \sum_{i \in I} \mu_i \lambda_t^i \geq 0$ for all $t \in U$ and

$$\sum_{t \in T} \lambda_t^z a_t = \sum_{i \in I} \mu_i \sum_{t \in T} \lambda_t^i a_t = \sum_{i \in I} \mu_i c^i = z.$$

Since $\sigma(\lambda^z) \subset \cup_{i\in I}\sigma\left(\lambda^i\right)$ and $\sigma(\overline{x}) \cap \sigma\left(\lambda^i\right) = \emptyset$ for all $i \in I$, we have $\sigma(\overline{x}) \cap \sigma(\lambda^z) = \emptyset$, i.e., $(\overline{x}, \lambda^z)$ is a complementary solution of $P(z)$. Then, applying Proposition 4 to $P(z)$, we conclude that $v^P(z) = v^D(z) = z'\overline{x}$. $\square$

Proposition 6 *Let* $\{c^i, i \in I\} \subset \mathbb{R}^n$ *be such that there exists a common optimal partition for the family of problems* $\{P(c^i), i \in I\}$. *Then* $v^P(z) = v^D(z)$ *is linear on* $\operatorname{conv}\{c^i, i \in I\}$.

Proof: Let (B, N, Z) be optimal partition for $P(c^i)$, for all $i \in I$. Select $j \in I$ arbitrarily and let $\overline{x} = x^j$. According to the final remark in Section 2, $\overline{x}$ is an optimal solution for $P(c^i)$, for all $i \in I$. Then, by Proposition 4, $\left(\overline{x}, \lambda^i\right)$ is a complementary solution of $P(c^i) - D(c^i)$, for all $i \in I$. Applying Lemma 3, $P(z)$ and $D(z)$ are solvable and $v^P(z) = v^D(z) = z'\overline{x}$ for all $z \in \operatorname{conv}\{c^i, i \in I\}$.

Under the additional assumption, since $v^P(z)$ is linear on $\operatorname{conv}\{c^i, , i \in I\}$ and this is a neighborhood of c, P has a strongly unique solution. $\square$

Under the assumption of Proposition 6, if $c \in \operatorname{int}\operatorname{conv}\{c^i, i \in I\}$ (e.g., if all the problems $P(c^i)$ have the same maximal optimal partition), then P has a strongly unique optimal solution. This is the case if there exists a common optimal partition for all the problems $P(z)$ such that z belongs to a certain neighborhood of c. In fact, the next example shows that the linearity of $v^P(z) = v^D(z)$ on a neighborhood of c does not entail the existence of a set $\{c^i, i \in I\}$ as in Proposition 6.

Example 5 *Let us consider the LSIP problem with index set* $\mathbb{Z}$

$$
\begin{array}{lll}
P: & Inf & x_1 + x_2 \\
 & s.t. & \quad tx_1 \geq -1,\ t = 1, 2, 3, ..., \\
 & & -tx_2 \geq -1,\ t = 0, -1, -2, ...
\end{array}
$$

Since the characteristic cone is $K = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 < 0\} \cup \{0_3\}$, $F = \mathbb{R}^2_+$, 0_2 *is the strongly unique solution of* P *and* $v^P(z) = 0$ *for all* $z \in \mathbb{R}^2_+$ *(the effective domain of* $v^P(z)$*). Given* $z \in \mathbb{R}^2_+$, *since* $v^D(z) \leq v^P(z) = 0$ *and the sequence* $\{\lambda^r\} \subset \mathbb{R}^{(\mathbb{Z})}_+$ *such that*

$$
\lambda^r_t = \begin{cases} \frac{z_1}{r}, & t = r, \\ \frac{z_2}{r}, & t = -r, \\ 0, & otherwise, \end{cases}
$$

is feasible for $D(z)$ *and satisfies* $\sum_{t\in\mathbb{Z}}\lambda_t^r b_t = -\frac{z_1+z_2}{r} \to 0$ *as* $r\to\infty$*, we have also* $v^D(z)=0$ *for all* $z\in\mathbb{R}_+^2$ *although* $D(z)$ *is only solvable when* $z=0_2$*. Thus no complementary solution exists for* $D(z)$ *if* $z\neq 0_2$*. It is easy to see that the maximal optimal partition of* $P(0_2)$ *is* $(\mathbb{Z},\emptyset,\emptyset)$*.*

Corollary 2 *Given* $d\in\mathbb{R}^n$*, if there exists* $\varepsilon>0$ *such that* $P(c+\varepsilon d)$ *and* P *have a common optimal partition, then* $v^P(z)=v^D(z)$ *is linear on* $[c,c+\varepsilon d]$*.*

Proof: Apply Proposition 6 to $\{c^1,c^2\}$, where $c^1:=c$ and $c^2:=c+\varepsilon d$. □

Example 6 *Consider the primal LSIP problem*

$$\begin{array}{lll} P: & Inf & c'x \\ & s.t. & -(\cos t)\,x_1-(\sin t)\,x_2\geq -1,\ t\in\left[0,\frac{\pi}{2}\right], \\ & & x_1\geq 0\ (t=2),\ x_2\geq 0\ (t=3). \end{array}$$

for three different cost vectors:

(a) $c=(1,1)'$*. If* $z\in\mathbb{R}_{++}^2$*, there exists a unique complementary solution of* $P(z)-D(z)$: $\left(0_2,\overline{\lambda}\right)$*, where*

$$\overline{\lambda}_t=\begin{cases} z_1, & t=2,\\ z_2, & t=3,\\ 0, & otherwise.\end{cases}$$

Since $\left(\left[0,\frac{\pi}{2}\right],\{2,3\},\emptyset\right)$ *is a common optimal (actually maximal) partition for* $\left\{P(z), z\in\mathbb{R}_{++}^2\right\}$*,* $v^P(z)=v^D(z)$ *is linear on* $\mathbb{R}_{++}^2$ *by Proposition 6. In fact,* $v^P(z)=v^D(z)=0$ *for all* $z\in\mathbb{R}_{++}^2$ *(Figure 1 represents the graph of* $v^P(z)=v^D(z)$*).*

(b) $c=(1,0)'$*.* $P(c)$ *has a maximal optimal partition,* $\left(\left[0,\frac{\pi}{2}\right]\cup\{3\},\{2\},\emptyset\right)$ *(and two other optimal partitions). If* $d\notin\operatorname{cone}\{c\}$ *and* $\varepsilon>0$ *is sufficiently small,* $z:=c+\varepsilon d$ *satisfies* $z_1>0$ *and either* $z_2>0$ *(in which case the maximal partition of* $P(z)$ *is* $\left(\left[0,\frac{\pi}{2}\right],\{2,3\},\emptyset\right)$*, as in (a)) or or* $z_2<0$*. In this case the unique complementary solution is* $\left((0,1),\overline{\lambda}\right)$*, where*

$$\overline{\lambda}_t=\begin{cases} -z_2, & t=\frac{\pi}{2},\\ z_1, & t=2,\\ 0, & otherwise.\end{cases}$$

Thus the maximal optimal partition of $P(z)$ *is* $\left(\left[0,\frac{\pi}{2}\right[\cup\{3\},\left\{\frac{\pi}{2},2\right\},\emptyset\right)$*. This implies that, for any* $d\in\mathbb{R}^2$*, there exists* $\varepsilon>0$ *such that* $v^P(z)=v^D(z)$ *is linear on* $[c,c+\varepsilon d]$*.*

(c) $c = (-1,-1)'$. *The unique complementary solution is* $\left(x^0, \lambda^0\right)$ *such that* $x^0 = \frac{1}{\sqrt{2}}(1,1)'$ *and*

$$\lambda_t^0 = \begin{cases} \sqrt{2}, & t = \frac{\pi}{4}, \\ 0, & \textit{otherwise}, \end{cases}$$

so that the maximal optimal partition of $P(-1,-1)$ *is* $(B, N, \emptyset)$ *where* $B = \left\{\left[0, \frac{\pi}{2}\right] \setminus \left\{\frac{\pi}{4}\right\}\right\} \cup \{2,3\}$ *and*$N = \left\{\frac{\pi}{4}\right\}$. *Given an arbitrary* $d \in \mathbb{R}^2$, $c + \rho d \in \mathbb{R}^2_{--}$ *if* ρ *is sufficiently small. For such a* ρ, *the optimal set of* $P(c+\rho d)$ *is* $F^*(c+\rho d) = \{x^\rho\}$, *where* $x^\rho = -\frac{c+\rho d}{\|c+\rho d\|} \in \mathbb{R}^2_{++}$. *There exists a unique* $\alpha \in \left]0, \frac{\pi}{2}\right[$ *(depending on* ρ*) such that* $x^\rho = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$. *Obviously,* $\sigma(x^\rho) = \left\{\left[0, \frac{\pi}{2}\right] \setminus \{\alpha\}\right\} \cup \{2,3\}$. *Similarly, the optimal set of* $D(c+\rho d)$ *is* $\Lambda^*(c+\rho d) = \{\lambda^\rho\}$, *where*

$$\lambda_t^\rho = \begin{cases} \|c+\rho d\|, & t = \alpha, \\ 0, & \textit{otherwise}. \end{cases}$$

Thus $\sigma(x^\rho) = B$ *and* $\sigma(x^\rho) = N$ *if and only if* $d \in \operatorname{span}\{c\}$. *Observe that, given* $d \in \mathbb{R}^2$, *there exists* $\varepsilon > 0$ *such that* $v^P(z) = v^D(z)$ *is linear on* $[c, c+\varepsilon d]$ *if and only if* $d \in \operatorname{span}\{c\}$.

Figure 1 shows the existence of a partition of $\left(\operatorname{dom} v^P(z)\right) \setminus \{0_2\} = \mathbb{R}^2 \setminus \{0_2\}$ in relatively open convex cones on which $v^P(z)$ is linear. In fact, since the hypograph of $v^P(z)$ is the convex cone $\operatorname{cl} K$ ([9, Theorem 8.1]), $v^P(z)$ is a concave, proper, upper semi-continuous homogeneous function and, according to Proposition 3, $\left\{C_z^P, z \in \left(\operatorname{dom} v^P(z)\right) \setminus \{0_n\}\right\}$, where C_z^P denotes the linearity cone of $v^P(z)$ at z, is a partition of $\left(\operatorname{dom} v^P(z)\right) \setminus \{0_n\}$ in maximal regions of linearity.

In the particular case of Example 6, the partition associated with $v^P(z)$ has infinitely many elements, e.g., $C^P_{(1,1)} = \mathbb{R}^2_{++}$, $C^P_{(-1,-1)} = \operatorname{cone}\{(-1,-1)\} \setminus \{0_2\}$, and $C^P_{(1,0)} = \operatorname{cone}\{(1,0)\} \setminus \{0_2\}$. Observe that $\left\{C_z^P, z \in \mathbb{R}^2 \setminus \{0_2\}\right\}$ is a partition of $\mathbb{R}^2 \setminus \{0_2\}$ such that

$$\dim C_z^P = \begin{cases} 1, & z \in \left[\mathbb{R}^2_- \cup (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)\right] \setminus 0_2, \\ 2, & \text{otherwise}. \end{cases}$$

Concerning $v^D(z)$, it is also concave, proper and homogeneous. We denote by $\left\{C_z^D, z \in M \setminus \{0_n\}\right\}$ the corresponding partition. In Example 6, $v^D(z) = v^P(z)$, so that both functions have the same partition. This is not true in general, as the following example shows.

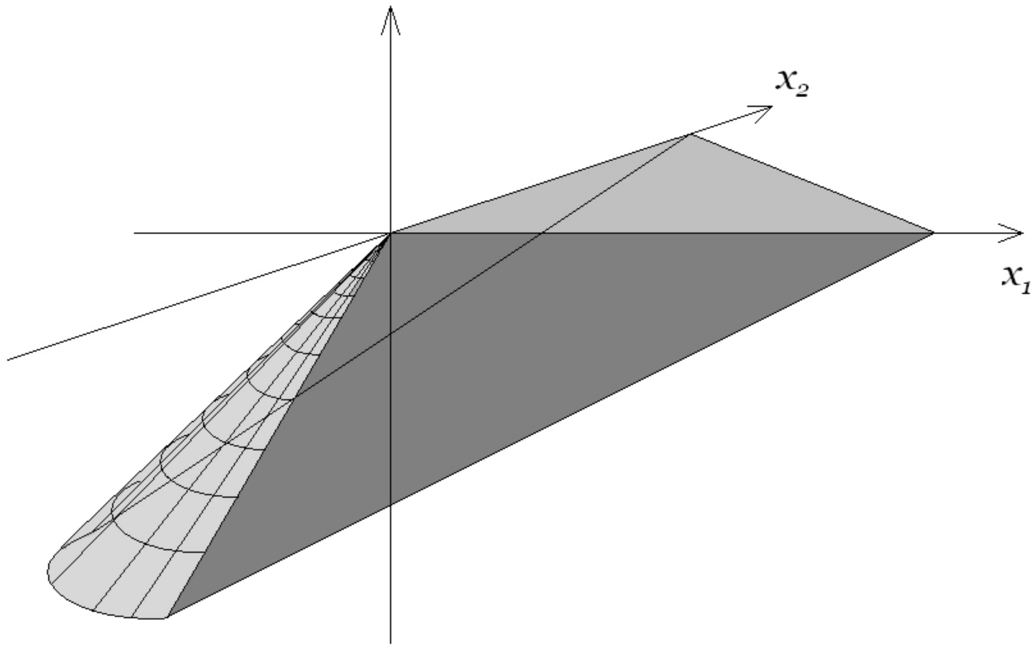

Figure 1

Example 7 *Take* $n = 3$, $T = \{t \in \mathbb{R}^3 \mid t_1 + t_2 + t_3 = 1, t_i > 0, i = 1, 2, 3\} \cup \{(1, 1, 0)\}$, *and the constraints* $t_1 x_1 + t_2 x_2 + t_3 x_3 \geq 0$ *for all* $t \neq (1, 1, 0)$ *and* $x_1 + x_2 \geq -1$ *otherwise. Then the linearity cones of* $v^P(z)$ *are the seven faces of* $\operatorname{dom} v^P(z) = \mathbb{R}^3_+$ *different from* $\{0_3\}$ *whereas* $v^D(z)$ *has only two linearity cones,* $\mathbb{R}^3_{++}$ *and* $\operatorname{cone}\{(1, 1, 0)\} \setminus \{0_3\}$.

Proposition 7 *Let* $c \neq 0_n$. *If* $d \in \operatorname{span} C^P_c$ ($d \in \operatorname{span} C^D_c$), *then there exists* $\varepsilon > 0$ *such that* $v^P(z)$ ($v^D(z)$, *respectively) is linear on* $[c, c + \varepsilon d]$.

Proof: If $d \in \operatorname{span} C^P_c$, then there exists $\varepsilon > 0$ such that $[c, c + \varepsilon d] \subset C^P_c$. Since $v^P(z)$ is linear on C^P_c the conclusion is immediate (the proof is the same for $v^D(z)$). □

5 Perturbing b

The perturbed problems in this section are

$$\begin{array}{lll} P(w): & \text{Inf} & c'x \\ & \text{s.t.} & a_t'x \geq w_t,\ t \in U, \\ & & a_t'x = w_t,\ t \in V, \end{array}$$

and

$$\begin{array}{lll} D(w): & \text{Sup} & \sum_{t\in T} \lambda_t w_t \\ & \text{s.t.} & \sum_{t \varepsilon T} \lambda_t a_t = c, \\ & & \lambda_t \geq 0,\ t \in U, \end{array}$$

with respective optimal values $v^P(w)$ and $v^D(w)$. With this notation, the optimal values of the nominal problem P and its dual D are $v^P(b)$ and $v^D(b)$, respectively. Observe that now $v^P(w), v^D(w) : \mathbb{R}^T \to \overline{\mathbb{R}}$, so that we cannot expect simple counterparts for the results in Section 4 unless $|T| < \infty$. In fact, in LP, $v^P(w), v^D(w) : \mathbb{R}^{|T|} \to \overline{\mathbb{R}}$ are ordinary homogeneous convex functions, so that Proposition 7 applies (observe that the parameter is now the gradient of the objective function of D, as in Section 4 but exchanging the roles of the problems). In such a case, if there exists $x^* \in F^*$ such that $\{a_t, t \in T(x^*)\}$ is a basis of $\mathbb{R}^n$, then $v^P(w) = c'x(w)$ on a certain neighborhood of b, where $x(w)$ is the unique solution of the system $\{a_t'x = w_t, t \in T(x^*)\}$ (by Cramer's rule). Then $\dim C_b^P = |T|$ and $v^P(w)$ is linear on a certain neighborhood of b.

If T is infinite, the first difficulty comes from the fact that the perturbations of w affect the feasible set of the primal problem and possibly its consistency and the second from the infinite dimension of $\mathbb{R}^T$ which does not allows us to use Proposition 3. In [8, Section 2] it is shown that, if $v^P(w)$ is linear on a certain neighborhood of b (in the pseudometric space $\left(\mathbb{R}^T, \delta\right)$, then D has at most one optimal solution (the converse is true under strong assumptions). Moreover, $v^P(w)$ is linear on a segment emanating from b in the direction of a bounded function $d \in \mathbb{R}^T \setminus \{0_T\}$ if P and D are solvable with the same optimal value, the problem

$$\begin{array}{lll} P_d: & \text{Inf} & c'x + v^P(b)\,y \\ & \text{s.t.} & a_t'x + b_t y \geq d_t,\ t \in U, \\ & & a_t'x + b_t y = d_t,\ t \in V \end{array}$$

is also solvable and has zero duality gap, and either there exists an optimal solution of P_d, (z^*, y^*), such that $y^* \geq 0$ or there exists an optimal solution of P, x^*, such that either $T(x^*) = T$ or there exist two scalars μ and η such that $0 < \mu \leq a_t' x^* - b_t \leq \eta$ for all $t \notin T(x^*)$. This is the case, in particular, if $|T| < \infty$ and P and D_d are both bounded.

Lemma 4 *Let* $\{(b^i, x^i), i \in I\} \subset \mathbb{R}^T \times \mathbb{R}^n$ *and* $\overline{\lambda} \in \mathbb{R}^{(T)}$ *such that* $(x^i, \overline{\lambda})$ *is a complementary solution of* $P(b^i) - D(b^i)$ *for all* $i \in I$*. Then* $P(w)$ *and* $D(w)$ *are solvable and*

$$v^P(w) = v^D(w) = \sum_{t \in T} \overline{\lambda}_t w_t \ \text{ for all } w \in \operatorname{conv}\left\{b^i, i \in I\right\}. \tag{7}$$

Proof: Let $w = \sum_{i \in I} \mu_i b^i$, with $\sum_{i \in I} \mu_i = 1$ and $\mu \in \mathbb{R}_+^{(I)}$.

We shall prove that $x^w := \sum_{i \in I} \mu_i x^i$ is a feasible solution of $P(w)$. In fact, given $i \in I$, we have $a_t' x^i \geq b_t^i$ for all $t \in U$ and $a_t' x^i = b_t^i$ for all $t \in V$, so that $a_t' x^w \geq w_t$ for all $t \in U$ and $a_t' x^w = w_t$ for all $t \in V$.

On the other hand, if $t \in U$ satisfies $a_t' x^w > w_t$, i.e., $\sum_{i \in I} \mu_i (a_t' x^i - b_t^i) > 0$, then there exists $j \in I$ such that $\mu_j \left(a_t' x^j - b_t^j\right) > 0$ so that $a_t' x^j - b_t^j > 0$. Since $(x^j, \overline{\lambda})$ is a complementary solution of $P(b^j)$, we must have $\overline{\lambda}_t = 0$. We have shown that the primal-dual feasible solution $(x^w, \overline{\lambda})$ of $P(w)$ is a complementary solution of that problem. Applying Proposition 4 we get the aimed conclusion. □

Proposition 8 *Let* $\operatorname{conv}\{b^i, i \in I\}$ *be such that all the problems* $P(b^i)$, $i \in I$*, have the same optimal partition. Then* $v^P(w) = v^D(w)$ *is linear on* $\operatorname{conv}\{b^i, i \in I\}$.

Proof: It is a straightforward consequence of Lemma 4. □

In particular, if $b \in \operatorname{int} \operatorname{conv}\{b^i, i \in I\}$ (e.g., the maximal partition is the same for all the problems $P(w)$ such that w belongs to a certain neighborhood of b), then D has a unique optimal solution. We can have $v^P(w) = v^D(w)$ linear (or even constant) on a certain neighborhood of b such that no optimal partition exists on that neighborhood.

Example 8 *(Example 5 revisited) Let* $w \in \mathbb{R}^T$ *be such that*

$$\delta(w,b) = \sup_{t\in T} |w(t) + 1| < 1.$$

It is easy to see that $-2 < w(t) < 0$ *for all* $t \in T$*. Thus* $P(w)$ *and* P *have the same characteristic cone*

$$K = \left\{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 < 0\right\} \cup \{0_3\},$$

in which case

$$v^P(w) = \sup\{\gamma \in \mathbb{R} \mid (1,1,\gamma) \in \operatorname{cl} K\} = 0$$

and

$$v^D(w) = \sup\{\gamma \in \mathbb{R} \mid (1,1,\gamma) \in K\} = 0.$$

Since $0 \notin \{\gamma \in \mathbb{R} \mid (1,1,\gamma) \in K\}$*,* $D(w)$ *is not solvable and so* $P(w)$ *has no complementary solution.*

Corollary 3 *Given* $d \in \mathbb{R}^T$*, if there exists* $\varepsilon > 0$ *such that* $P(b + \varepsilon d)$ *has the same optimal partition as* P*, then* $v^P(w) = v^D(w)$ *is linear on* $[b, b + \varepsilon d]$*.*

Proof: It follows from Lemma 4. □

Let us mention that the recent paper [3] provides an upper bound for $v^D(b) - v^D(w)$ when $D(b)$ is consistent and $P(w)$ is also consistent in some neighborhood of b.

6 Perturbing c and b

The main advantage of the optimal partition approach is that it allows to study the simultaneous perturbation of cost and RHS coefficients. We denote by (z,w) the result of perturbing the vector (c,b) (called *rim data* in the LP literature). To do this we consider the parametric problem

$$\begin{array}{lll} P(z,w): & \text{Inf} & z'x \\ & \text{s.t.} & a_t'x \geq w_t,\ t \in U, \\ & & a_t'x = w_t,\ t \in V, \end{array}$$

and its corresponding dual

$$
\begin{array}{lll}
D(z,w): & \text{Sup} & \sum_{t\in T} \lambda_t w_t \\
& \text{s.t.} & \sum_{t\in T} \lambda_t a_t = z, \\
& & \lambda_t \geq 0, \; t \in U.
\end{array}
$$

In order to describe the behavior of the value functions of these problems we define a class of functions after giving a brief motivation. Let L be a linear space and let $\varphi : L^2 \to \mathbb{R}$ be a bilinear form on L. Let $C = \operatorname{conv}\{v_i, i \in I\} \subset L$ and let $q_{ij} := \varphi(v_i, v_j)$, $(i,j) \in I^2$. Then any $v \in C$ can be expressed as

$$
v = \sum_{i\in I} \mu_i v_i, \;\; \sum_{i\in I} \mu_i = 1, \;\; \mu \in \mathbb{R}_+^{(I)}. \tag{8}
$$

Then we have

$$
\varphi(v,v) = \sum_{i,j\in I} \mu_i \mu_j q_{ij}. \tag{9}
$$

Accordingly, given $q : C \to \mathbb{R}$, where $C = \operatorname{conv}\{v_i, i \in I\} \subset L$, we say that q is quadratic on C if there exist real numbers q_{ij}, $i,j \in I$, such that (9) holds for all $v \in C$ satisfying (8).

Proposition 9 *Let $\{(c^i, b^i), i \in I\} \subset \mathbb{R}^n \times \mathbb{R}^T$ be such that there exists a common optimal partition for the family of problems $P(c^i, b^i)$, $i \in I$. Then $P(z,w)$ and $D(z,w)$ are solvable and $v^P(z,w) = v^D(z,w)$ on* $\operatorname{conv}\{c^i, i \in I\} \times \operatorname{conv}\{b^i, i \in I\}$ *and $v^P(z,w)$ is quadratic on* $\operatorname{conv}\{(c^i, b^i), i \in I\}$. *Moreover, if $(c,b) \in$* $\operatorname{conv}\{c^i, i \in I\} \times \operatorname{conv}\{b^i, i \in I\}$*, then $v^P(z,b)$ and $v^P(c,w)$ are linear on* $\operatorname{conv}\{c^i, i \in I\}$ *and* $\operatorname{conv}\{b^i, i \in I\}$*, respectively.*

Proof: Let (B,N,Z) be a common optimal partition of $P(c^i, b^i)$ for all $i \in I$. Let $(z,w) \in \operatorname{conv}\{c^i, i \in I\} \times \operatorname{conv}\{b^i, i \in I\}$. Then we can write

$$
z = \sum_{i\in I} \delta_i c^i, \;\; w = \sum_{i\in I} \gamma_i b^i, \;\; \sum_{i\in I} \delta_i = \sum_{i\in I} \gamma_i = 1, \;\; \delta, \gamma \in \mathbb{R}_+^{(T)}. \tag{10}
$$

Let $\left(x^i, \lambda^i\right) \in \mathbb{R}^n \times \mathbb{R}^{(T)}$ be a complementary solution of $P(c^i, b^i) - D(c^i, b^i)$, $i \in I$, corresponding to (B,N,Z). We shall prove that $\overline{x} := \sum_{i\in I} \gamma_i x^i$ and $\overline{\lambda} := \sum_{i\in I} \delta_i \lambda^i$ constitute a complementary solution of $P(z,w)$.

Since $a_t' x^i \geq b_t^i$ for all $t \in U$ and $a_t' x^i = b_t^i$ for all $t \in V$, we have $a_t' \overline{x} \geq w_t$ for all $t \in U$ and $a_t' \overline{x} = w_t$ for all $t \in V$, i.e., $\overline{x}$ is a feasible solution of $P(z,w)$.

On the other hand, $\lambda_t^i \geq 0$ for all $t \in U$ and all $i \in I$ entails $\overline{\lambda}_t \geq 0$ for all $t \in U$, whereas $\sum\limits_{t \varepsilon T} \lambda_t^i a_t = c^i$ for all $i \in I$ implies $\sum\limits_{t \varepsilon T} \overline{\lambda}_t a_t = z$.

We have shown that $(\overline{x}, \overline{\lambda})$ is a primal-dual feasible solution. Moreover, if $t \in U$ satisfies $a_t' \overline{x} > w_t$, i.e., $\sum_{i \in I} \gamma_i (a_t' x^i - b_t^i) > 0$, the there exists $j \in I$ such that $a_t' x^j > b_t^j$. Thus, by the assumption on the optimal partition of the family of problems, $t \in B$ and so $\lambda_t^i = 0$ for all $i \in I$. Hence $\overline{\lambda}_t = 0$ and $(\overline{x}, \overline{\lambda})$ turns out to be complementary solution of $P(z, w)$. Then, according to Proposition 4, applied to $P(z, w)$, we have that $P(z, w)$ and $D(z, w)$ are solvable and $v^P(z, w) = v^D(z, w)$. Since $(\overline{x}, \overline{\lambda})$ is a primal-dual optimal solution, we have

$$v^P(z, w) = \overline{x}' z = \sum_{t \varepsilon T} \overline{\lambda}_t w_t = v^D(z, w). \tag{11}$$

Let $q_{ij} = (c^i)' x^j$, $i, j \in I$ and let $C := \operatorname{conv}\{(c^i, b^i), i \in I\}$. Let $(z, w) = \sum_{i \in I} \mu_i (c^i, b^i)$, $\sum_{i \in I} \mu_i = 1$ and $\mu \in \mathbb{R}_+^{(T)}$. Then, since we can take $\delta_i = \gamma_i = \mu_i$ in (10), (11) yields

$$v^P(z, w) = \left(\sum_{j \in I} \mu_j x^j \right)' \left(\sum_{i \in I} \mu_i c^i \right) = \sum_{i,j \in I} \mu_i \mu_j q_{ij}.$$

Now assume that $(c, b) \in \operatorname{conv}\{c^i, i \in I\} \times \operatorname{conv}\{b^i, i \in I\}$.

Let $b = \sum\_i \in I \gamma_i b^i$, with $\sum_{i \in I} \gamma_i = 1$, $\gamma \in \mathbb{R}_+^{(T)}$. Then $\overline{x} := \sum_{i \in I} \gamma_i x^i$ is constant and (11) yields $v^P(z, b) = z' \overline{x}$ for all $z \in \operatorname{conv}\{c^i, i \in I\}$. Similarly, $v^P(c, w) = \sum\limits_{t \varepsilon T} \overline{\lambda}_t w_t$ if $w \in \operatorname{conv}\{b^i, i \in I\}$, with $\overline{\lambda}$ fixed, and this is a linear function of w. □

Obviously, if $(c, b) \in \operatorname{int} \operatorname{conv}\{(c^i, b^i), i \in I\}$, then $v^P(z, w) = v^D(z, w)$ is quadratic on a neighborhood of (c, b). In particular, if the problems $P(z, w)$ have a common optimal partition when (z, w) ranges on a certain neighborhood of (c, b), then we can assert that P has strongly unique solution and D has unique solution). In Example 5, $v^P(c, w) = v^D(c, w) = 0$ for all (c, w) such that $\delta(w, b) < 1$ and $\|z - c\| < 1$. Nevertheless, the only perturbed problems which have optimal partition are of the form $P(0_n, w)$, so that the condition in Proposition 9 fails.

Corollary 4 *Given $(d, f) \in \mathbb{R}^n \times \mathbb{R}^T$, if there exists $\varepsilon > 0$ such that the problem $P((c, b) + \varepsilon (d, f))$ has the same maximal optimal partition as P,* *then* $v^P(z,w) = v^D(z,w)$ *is quadratic on the interval* $[(c,b),(c,b)+\varepsilon(d,f)]$. *Moreover,* $v^P(z,b)$ *(*$v^P(c,w)$*) is linear function of* z *on* $[c, c+\varepsilon d]$ *(of* w *on* $[b, b+\varepsilon f]$*, respectively).*

Proof: It is an immediate consequence of Proposition 9. □

7 Conclusions

In this paper we examine the linearity of the primal and the dual optimal value functions (which can be different in LSIP) relative to the size of perturbations of the cost vector, the RHS vector or both, on convex subsets of their effective domain. The new results on sensitivity analysis in LSIP in Sections 4-6 have been obtained by means of two different partition approaches whose fundamentals are developed in Sections 2 and 3:

1. Partition of the domain of the optimal value functions in maximal relatively open convex cones where they are linear (the so-called linearity cones). The partition corresponding to the primal value function only depends on the primal feasible set whereas the corresponding to the dual optimal value function depends on the constraints. The advantage of this approach is that it provides a significant insight into the optimal value functions. The inconveniences are, first, that this approach only applies to perturbations of c and, second, that computing linearity cones may be a difficult task in practice.

2. Optimal partitions of the index set of the inequality constraints. The advantage of this approach is that it yields sufficient conditions for the linearity of the optimal value functions for a variety of convex sets for the three types of perturbations considered in this paper. The multiplicity of optimal partitions and the possible lack of a maximal partition in LSIP is the main difficulty when checking these sufficient conditions in practice (at least in comparison with LP).

A third approach to sensitivity analysis in LSIP, valid for perturbation of b or c (but not both) has been sketched at the beginning of Sections 4 and 5, where we recall the corresponding extensions of Gauvin's formulae [5]. The main inconvenience of this approach is that it only provides linearity tests for the optimal value functions on segments, and its main advantage consists of the fact that these tests also provide directional derivatives in the direction of the corresponding segment.

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