From CVaR to Uncertainty Set: Implications in Joint Chance Constrained Optimization *

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Abstract

In this paper we review the different tractable approximations of individual chance constraint problems using robust optimization on a varieties of uncertainty set, and show their interesting connections with bounds on the condition-value-at-risk CVaR measure popularized by Rockafellar and Uryasev. We also propose a new formulation for approximating joint chance constrained problems that improves upon the standard approach. The standard approach decomposes the joint chance constraint into a problem with m individual chance constraints and then applies safe robust optimization approximation on each one of them. Our approach builds on a classical worst case bound for order statistics problem, and is applicable even if the constraints are correlated. We provide an application of the model on a network resource allocation network with uncertain demand. The new chance constrained formulation led to more than 8-12% improvement over the standard approach.

*The research is supported by Singapore MIT Alliance, NUS Risk Management Institute, NUS academic research grants R-314-000-066-122 and R-314-000-068-122.

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1 Introduction

Data uncertainties prevail in many real world linear optimization models. If ignored, the so called "optimal solution" obtained by solving a model using the "nominal data" or point estimates can become infeasible in the model when the true data differs from the nominal one. We consider a linear optimization model as follows

$$\begin{aligned} Z(\tilde{\boldsymbol{z}}) = \quad & \min \quad \boldsymbol{c}'\boldsymbol{x} \\ & \text{s.t.} \quad \boldsymbol{A}(\tilde{\boldsymbol{z}})\boldsymbol{x} \geq \boldsymbol{b}(\tilde{\boldsymbol{z}}) \\ & \qquad\quad \boldsymbol{x} \in X, \end{aligned} \tag{1}$$

where $X \subseteq \Re^n$ is a polyhedron and the data entries, $\boldsymbol{A}(\tilde{\boldsymbol{z}}) \in \Re^{m \times n}$ and $\boldsymbol{b}(\tilde{\boldsymbol{z}}) \in \Re^m$ are uncertain and affinely dependent on a vector of primitive uncertainties, $\tilde{\boldsymbol{z}}$,

$$\begin{aligned} \boldsymbol{A}(\tilde{\boldsymbol{z}}) &= \boldsymbol{A}^0 + \sum_{k=1}^{N} \boldsymbol{A}^k \tilde{z}_k \\ \boldsymbol{b}(\tilde{\boldsymbol{z}}) &= \boldsymbol{b}^0 + \sum_{k=1}^{N} \boldsymbol{b}^k \tilde{z}_k. \end{aligned}$$

To overcome such infeasibility, Soyster [24] introduced a worst case model that ensures that its solutions remains feasible for all possible realization of the uncertain data. Soyster proposed the following model [24],

$$\begin{aligned} & \min \quad \boldsymbol{c}'\boldsymbol{x} \\ & \text{s.t.} \quad \boldsymbol{A}(\boldsymbol{z})\boldsymbol{x} \geq \boldsymbol{b}(\boldsymbol{z}) \qquad \forall \boldsymbol{z} \in \mathcal{W}, \\ & \qquad\quad \boldsymbol{x} \in X, \end{aligned} \tag{2}$$

where

$$\mathcal{W} = \{\boldsymbol{z} : -\underline{\boldsymbol{z}} \leq \boldsymbol{z} \leq \bar{\boldsymbol{z}}\} \qquad \underline{\boldsymbol{z}}, \bar{\boldsymbol{z}} > \boldsymbol{0}$$

is the interval support of the primitive uncertainties $\tilde{\boldsymbol{z}}$. Soyster [24] showed that the model can be represented as a polynomially sized linear optimization model. However, Soyster's model can be extremely conservative in addressing model where the violation of constraints may be tolerated as a tradeoff for better attainment in objective.

Perhaps the most natural way of safeguarding a constraint against violation is to control its violation probability. Such a constraint is known as a probabilistic or a chance constraint, which was first introduced by Charnes, Cooper, and Symonds [11]. A chance constrained model is defined as follows

$$\begin{aligned} Z_\epsilon = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{P}(\boldsymbol{A}(\tilde{\boldsymbol{z}})\boldsymbol{x} \geq \boldsymbol{b}(\tilde{\boldsymbol{z}})) \geq 1 - \epsilon \\ & \boldsymbol{x} \in X, \end{aligned} \tag{3}$$

where the chance constraint requires all the m linear constraints to be jointly feasible with probability at least $1-\epsilon$, where $\epsilon \in (0,1)$ is the desired safety factor.

Chance constrained problems can be classified as individual chance constrained problem when $m = 1$, and joint chance constrained problem when $m > 1$. It is well known that under multivariate normal distribution, an individual chance constrained problem is second-order cone representable. In other words, the optimization model becomes a second-order cone optimization problem (SOCP), which is computationally tractable, both in theory and practice (see for instance Alizadeh and Goldfarb [1]). However, for general distributions, chance constrained problems are computationally intractable. For instance, Nemirovski and Shapiro [20] noted that evaluating the distribution of a weighted sum of uniformly distributed independent random variables is already NP-hard.

Needless to say, joint chance constrained problems are clearly harder than individual chance constraint problems. For instance, with only right hand side disturbances, we can transform an individual chance constrained problem to an equivalent linearly constrained problem. However, this property does not necessarily apply in a joint chance constrained problem. In fact, convex joint chance constrained problems are hard to come by. For instance, with only right hand side disturbances, a joint chance constrained problem is convex only when the distributions is logconcave. Solution techniques for solving such problems includes supporting hyperplane, central cutting plane and reduced gradient methods (see for instance Prekopa [22] and Mayer [17].)

The intractability of chance constrained problem using exact probability distributions has spurred recent interests in robust optimization in which data uncertainties are described using uncertainty sets. Moreover, robust optimization often requires only a mild assumption on probability distributions such as known supports, $\mathcal{W}$, covariances and other forms of deviation measures, notably the forward and backward deviations derived from moment generating functions proposed by Chen, Sim and Sun [13]. For some practitioners, this could be viewed as an advantage over having to obtain the entire joint probability distributions of the uncertain data. One of the goals of robust optimization is to provide a tractable approach for obtaining a solution that remains feasible in the chance constrained model (3) for all distributions that conform to the mild distributional assumption. Hence, such solutions are viewed as "safe" approximations of the chance constrained problems.

Robust optimization has been rather successful in constructing safe approximation of individual chance constrained problems. Given an uncertainty set, $\mathcal{U}$, the robust counterpart of an individual linear constrained problem with affinely dependent primitive uncertainties $\boldsymbol{z}$ is defined as

$$\boldsymbol{a}(\boldsymbol{z})\boldsymbol{x} \geq b(\boldsymbol{z}) \qquad \forall \boldsymbol{z} \in \mathcal{U}.$$

Clearly, the Soyster's model (2) is a special case of robust counterpart in which the uncertainty set $\mathcal{U}$ is chosen to be the worst case support set, $\mathcal{W}$. For computational tractability, the chosen uncertainty set $\mathcal{U}$ is usually in the form of tractable convex representable sets such as second-order cone representable ones and even polytopes. Variants of uncertainty sets include symmetrical ones such as a simple ellipsoid proposed by Ben-Tal and Nemirovski [3, 4] and independently by El-Ghaoui et al. [15] and a normed constrained type polytope proposed by Bertsimas and Sim [8]. More recently, Chen, Sim and Sun [13] proposed an asymmetrical uncertainty set that generalizes the symmetric ones. All these models are computationally attractive in the form of SOCPs or even in the form linear optimization problems. In the recent work of Nemirovski and Shapiro [20], they incorporated moment generating functions for providing safe and tractable approximations of an individual chance constrained problem. Despite the improved approximation, it is not readily second-order cone representable, and hence computationally more expensive. Other forms of deterministic approximation of an individual chance constrained problem includes using Chebyshev's inequality, Bernstein's inequality, Hoefding's inequality to bound the probability of violating individual constraints. See, for example, Pintér [21].

Besides deterministic approximations of chance constrained problems, there is a body of works on approximating chance constrained problems using Monte Carlo sampling (see for instance Calafiore and Campi [10] and Ergodan and Iyengar [14]). However, the solutions obtained via sampling appears to be conservative compared to those obtained using deterministic approximation. See computation studies in Nemirovski and Shapiro [20] and also Chen, Sim and Sun [13].

While robust optimization has been pretty successful in approximating individual chance constrained problems, it is rather unsatisfactory in approximating joint chance constrained problems. The "standard method" for approximating a joint constrained problem is to decompose a joint chance constrained problem into a problem with m individual chance constraints. Clearly, by Bonferroni's inequality, a sufficient condition for ensuring feasibility in the joint chance constrained problem is to ensure that the total sum of violation probabilities of the individual chance constraints is less than ϵ. The natural choice proposed in Nemirovski and Shapiro [20] and also Chen, Sim and Sun [13] is to divide the violation probability equality among the m individual chance constraints. To the best of our knowledge, prior to this work, we do not know of any systematic approach for selecting better allocation of the safety factors among the individual chance constraints. Unfortunately, this approach ignores the fact that the individual chance constraints could be correlated, and hence the approximation obtained, using the best allocation of the safety factors, could be extremely conservative. This motivates our research to achieve better approximations of joint chance constrained problems. We build instead on a classical result on order statistics (cf. Meijilson and Nadas [18]) to bound the probability of violation for the joint chance constraint. We show that by choosing the right multiplers which can be used in conjunction with this classical inequality, we can derived an improved approximation to the above method (using Bonferroni's inequality) for the joint chance constraint problem.

Our specific contributions in this paper include the followings:

1. We review the different tractable approximations of individual chance constraint problems using robust optimization and show their interesting connections with bounds on the condition-value-at-risk CVaR measure popularized by Rockafellar and Uryasev [23].

2. We propose a new formulation for approximating joint chance constrained problems that improves upon the standard approach using Bonferroni's inequality.

3. We provide an application of the model on a network resource allocation problem with uncertain demand and study the performance of the new chance constrained formulation over the approach using Bonferroni's inequality.

The rest of the paper is organized as follows. In section 2, we focus on robust optimization approximation of individual chance constrained problems. In section 3, we propose a new approximation of joint chance constrained problem. In Section 4, We anlalyze the efficacy of joint chance constrained problem through a computational study of emergency supply allocation network. Finally, we conclude this paper in Section 5.

Notations We denote random variables with tilde sign, such as $\tilde{x}$. Bold face lower case letters represent vectors, such as $\boldsymbol{x}$ and bold face upper case letters represent matrices, such as $\boldsymbol{A}$. In addition, we denote $x^+ = \max\{x, 0\}$.

2 Individual Chance Constrained Problems

In this section, we will establish the relation between bounds on the condition-value-at-risk (CVaR) measure popularized by Rockafellar and Uryasev [23] and the different tractable approximations of individual chance constrained problems using robust optimization. For simplicity, we consider a linear individual chance constraint as follows

$$\mathrm{P}\Big(y(\tilde{\boldsymbol{z}}) \le 0)\Big) \ge 1-\epsilon, \tag{4}$$

where $y(\tilde{\boldsymbol{z}})$ are affinely dependent of $\tilde{\boldsymbol{z}}$,

$$y(\tilde{\boldsymbol{z}}) = y^0 + \sum_{k=1}^{N} y^k \tilde{z}_k,$$

and $(y^0, y^1, \ldots, y^N)$ are the decision variables. To illustrate the generality, we can represent the following chance constrained problem

$$\mathrm{P}\Big(\boldsymbol{a}(\tilde{\boldsymbol{z}})'\boldsymbol{x} \ge b(\tilde{\boldsymbol{z}}))\Big) \ge 1-\epsilon,$$

where

$$\begin{aligned} \boldsymbol{a}(\tilde{\boldsymbol{z}}) &= \boldsymbol{a}^0 + \sum_{k=1}^{N} \boldsymbol{a}^k \tilde{z}_k \\ b(\tilde{\boldsymbol{z}}) &= b^0 + \sum_{k=1}^{N} b^k \tilde{z}_k, \end{aligned}$$

by enforcing the following affine relations

$$y^k = -\boldsymbol{a}^{k\prime}\boldsymbol{x} + b^k \qquad \forall k = 0, \ldots, N.$$

The chance constrained problem (4) is not necessarily convex in its decision variables, $(y^0, y^1, \ldots, y^N)$. A step towards tractability is by convexifying the chance constrained problem (4) using the conditional-value-at-risk (CVaR) measure, $\rho_{1-\epsilon}(\tilde{v})$, which is a functional on a random variable $\tilde{v}$ defined as follows

$$\rho_{1-\epsilon}(\tilde{v}) \triangleq \min_{\beta} \left\{ \beta + \frac{1}{\epsilon}\mathrm{E}\left((\tilde{v}-\beta)^+\right) \right\}.$$

The CVaR measure is a special class of optimized certainty equivalent (OCE) risk measure introduced by Ben-Tal and Teboulle [6] and is popularized by Rockafellar and Uryasev [23] as a tractable alternative for solving value-at-risk problems in financial applications. Recent works of Bertsimas and Brown [7] and Natarajan et al. [19] have uncovered the relation between financial risk measures and uncertainty sets in robust optimization. Our interest in CVaR measure is due to Nemirovski and Shapiro [20], who have established that the following CVaR constrained problem,

$$\rho_{1-\epsilon}(y(\tilde{\boldsymbol{z}})) \le 0 \tag{5}$$

is the tightest convex approximation of an individual chance constrained problem. However, despite its convexity, it remains unclear how we can evaluate the CVaR measure precisely. The key difficulty lies in the evaluation of the expectation, $\mathrm{E}((\cdot)^+)$, which involves multi-dimension integration. Such evaluation is typically analytically prohibitive above the forth dimension. Although it is possible to approximate CVaR using sampling average approximation, its solution may not be a safe approximation of the chance constrained problem (4). Furthermore, sampling average approximation of the CVaR measure relies on full knowledge of the underlying distributions, $\tilde{\boldsymbol{z}}$, which may become a practical concern due to the limited availability of independent stationary historical data.

2.1 Bounding $\mathrm{E}((\cdot)^+)$

Providing bounds on $\mathrm{E}((\cdot)^+)$ is pivotal in developing tractable approximations of individual and joint chance constrained problems. We show next different ways of bounding $\mathrm{E}((\cdot)^+)$ using mild distributional information of $\tilde{\boldsymbol{z}}$, such as supports, covariances and deviation measures. The results in bounding $\mathrm{E}((\cdot)^+)$ has also been presented in Chen and Sim [12]. For completeness, we list some of the known bounds on $\mathrm{E}((\cdot)^+)$.

The primitive uncertainties, $\tilde{\boldsymbol{z}}$ may be partially characterized using the forward and backward deviations, which are recently introduced by Chen, Sim and Sun [13].

Definition 1 *Given a random variable $\tilde{z}$ with zero mean, the forward deviation is defined as*

$$\sigma_f(\tilde{z}) \triangleq \sup_{\theta>0}\left\{\sqrt{2\ln(\mathrm{E}(\exp(\theta\tilde{z})))/\theta^2}\right\} \tag{6}$$

and backward deviation is defined as

$$\sigma_b(\tilde{z}) \triangleq \sup_{\theta>0}\left\{\sqrt{2\ln(\mathrm{E}(\exp(-\theta\tilde{z})))/\theta^2}\right\}. \tag{7}$$

The forward and backward deviations are derived from the moment generating functions of $\tilde{z}$ and can be bounded from the support of $\tilde{z}$.

Theorem 1 *(Chen, Sim and Sun [13]) If $\tilde{z}$ has zero mean and distributed in $[-\underline{z}, \bar{z}]$, $\underline{z}, \bar{z} > 0$, then*

$$\sigma_f(\tilde{z}) \leq \bar{\sigma}_f(\tilde{z}) = \frac{\underline{z}+\bar{z}}{2}\sqrt{g\left(\frac{\underline{z}-\bar{z}}{\underline{z}+\bar{z}}\right)}$$

and

$$\sigma_b(\tilde{z}) \leq \bar{\sigma}_b(\tilde{z}) = \frac{\underline{z}+\bar{z}}{2}\sqrt{g\left(\frac{\bar{z}-\underline{z}}{\underline{z}+\bar{z}}\right)},$$

where

$$g(\mu) = 2\max_{s>0}\left\{\frac{\phi_\mu(s)-\mu s}{s^2}\right\},$$

and

$$\phi_\mu(s) = \ln\left(\frac{e^s + e^{-s}}{2} + \frac{e^s - e^{-s}}{2}\mu\right).$$

Moreover the bounds are tight.

Assumption U: We assume that the uncertainties $\{\tilde{z}_j\}_{j=1:N}$ are zero mean random variables, with positive definite covariance matrix, $\boldsymbol{\Sigma}$. Let $\mathcal{W} = \{\boldsymbol{z} : -\underline{\boldsymbol{z}} \leq \boldsymbol{z} \leq \bar{\boldsymbol{z}}\}$ denote the smallest compact convex set containing the support of $\tilde{\boldsymbol{z}}$. Of the N primitive uncertainties, the first I random variables, that is, $\tilde{z}_j$, $j = 1, \ldots, I$ are stochastically independent. Moreover, the corresponding forward and backward deviations (or their bounds used in Theorem 1) are given by $p_j = \sigma_f(\tilde{z}_j) > 0$ and $q_j = \sigma_b(\tilde{z}_j) > 0$ respectively for $j = 1, \ldots, I$, and we denote $\boldsymbol{P} = \operatorname{diag}(p_1, \ldots, p_I)$ and $\boldsymbol{Q} = \operatorname{diag}(q_1, \ldots, q_I)$.

Theorem 2 *(Chen and Sim [12]) Suppose the primitive $\tilde{\boldsymbol{z}}$ satisfies Assumption U. The following functions are upper bounds of* $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right)$.

(a)

$$\begin{aligned}\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \leq \pi^1(y_0, \boldsymbol{y}) &\triangleq \left(y_0 + \max_{\boldsymbol{z}\in\mathcal{W}} \boldsymbol{z}'\boldsymbol{y}\right)^+ \\ &= \min_{r,\boldsymbol{s},\boldsymbol{t}} \left\{r \mid r \geq y_0 + \boldsymbol{s}'\overline{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}}, \boldsymbol{s} - \boldsymbol{t} = \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t} \geq \boldsymbol{0}, r \geq 0\right\}.\end{aligned}$$

The bound is tight whenever $y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0$ for all $\boldsymbol{z} \in \mathcal{W}$.

(b)

$$\begin{aligned}\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) &= y_0 + E\left((-y_0 - \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \\ &\leq \pi^2(y_0, \boldsymbol{y}) \\ &\triangleq y_0 + \left(-y_0 + \max_{\boldsymbol{z}\in\mathcal{W}} (-\boldsymbol{y})'\boldsymbol{z}\right)^+ \\ &= \min_{r,\boldsymbol{s},\boldsymbol{t}} \left\{r \mid r \geq \boldsymbol{s}'\overline{\boldsymbol{z}} + \boldsymbol{t}'\underline{\boldsymbol{z}}, \boldsymbol{s} - \boldsymbol{t} = -\boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t} \geq \boldsymbol{0}, r \geq y_0\right\}.\end{aligned}$$

The bound is tight whenever $y_0 + \boldsymbol{y}'\boldsymbol{z} \geq 0$ for all $\boldsymbol{z} \in \mathcal{W}$.

(c)

$$\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) = \tfrac{1}{2}\left(y_0 + E|y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}|\right) \leq \pi^3(y_0, \boldsymbol{y}) \triangleq \tfrac{1}{2}y_0 + \tfrac{1}{2}\sqrt{{y_0}^2 + \boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}.$$

(d)

$$\begin{aligned}&\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \leq \inf_{\mu>0} \tfrac{\mu}{e} E\left(\exp\left(\tfrac{y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}}{\mu}\right)\right) \\ &\leq \pi^4(y_0, \boldsymbol{y}) \triangleq \begin{cases} \inf_{\mu>0}\left\{\frac{\mu}{e}\exp\left(\frac{y_0}{\mu} + \frac{\|\boldsymbol{u}\|_2^2}{2\mu^2}\right)\right\} & \text{if } y_j = 0 \ \forall j = I+1, \ldots, N \\ +\infty & \text{otherwise}\end{cases}\end{aligned}$$

where $u_j = \max\{p_j y_j, -q_j y_j\}$.

(e)

$$\begin{aligned}
&\mathrm{E}\left((y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \leq y_0 + \inf_{\mu>0} \tfrac{\mu}{e} E\left(\exp(\tfrac{-y_0-\boldsymbol{y}'\tilde{\boldsymbol{z}}}{\mu})\right)\\
&\leq \pi^5(y_0,\boldsymbol{y}) \triangleq \begin{cases} y_0 + \inf_{\mu>0}\left\{\frac{\mu}{e}\exp\left(-\frac{y_0}{\mu}+\frac{\|\boldsymbol{v}\|_2^2}{2\mu^2}\right)\right\} & \textit{if } y_j = 0\ \forall j = I+1,\ldots,N \\ +\infty & \textit{otherwise} \end{cases}
\end{aligned}$$

where $v_j = \max\{-p_j y_j, q_j y_j\}$.

Remark : Observe that $\pi^i(y_0,\boldsymbol{y})$, $i = 1,\ldots,5$ are convex and positive homogenous functions, that is,

$$\pi^i(ky_0, k\boldsymbol{y}) = k\pi^i(y_0,\boldsymbol{y}) \qquad \forall k \geq 0. \tag{8}$$

Furthermore,

$$\pi^i(y_0,\boldsymbol{0}) = y_0^+. \tag{9}$$

More importantly, Chen and Sim [12] shows that the bound can be strengthened further by suitably decomposing $(y_0,\boldsymbol{y})$ into $(y_0^i,\boldsymbol{y^i})$, and using a linear combination of the bounds $\pi^i(y_0^i,\boldsymbol{y^i})$ to obtain a stronger bound.

Theorem 3 *(Chen and Sim [12]) Suppose* $\pi^i(y_0,\boldsymbol{y})$*, for all* $i \in \mathcal{L}$*, is an upperbound to* $E(y_0+\boldsymbol{y}'z)^+$*,* $\pi^i(y_0,\boldsymbol{y})$ *is convex and positive homogenous. Define*

$$\begin{aligned}
\pi^{\mathcal{L}}(y_0,\boldsymbol{y}) \triangleq \min_{y_{l0},\boldsymbol{y}_l} \quad & \sum_{l\in\mathcal{L}} \pi^l(y_{l0},\boldsymbol{y}_l)\\
\text{s.t.} \quad & \sum_{l\in\mathcal{L}} y_{l0} = y_0\\
& \sum_{l\in\mathcal{L}} \boldsymbol{y}_l = \boldsymbol{y}.
\end{aligned}$$

Then

$$0 \leq \mathrm{E}\left((y_0+\boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \leq \pi^{\mathcal{L}}(y_0,\boldsymbol{y}) \leq \min_{l\in\mathcal{L}} \pi^l(y_0,\boldsymbol{y}). \tag{10}$$

Moreover, $\pi^{\mathcal{L}}(y_0,\boldsymbol{y})$ *inherits the convexity and positive homogenous properties of the individual functions* $\pi^i(y_0,\boldsymbol{y})$*,* $i \in \mathcal{L}$*.*

For details, the interested reader may refer to Chen and Sim [12].

Proposition 1 *Under Assumption U and suppose* $\pi(y_0,\boldsymbol{y})$ *is an upperbound to* $E(y_0+\boldsymbol{y}'z)^+$*, then*

$$\pi(y_0,\boldsymbol{y}) = 0 \tag{11}$$

only if

$$y_0 + \max_{\boldsymbol{z}\in\mathcal{W}} \boldsymbol{y}'\boldsymbol{z} \leq 0. \tag{12}$$

Proof : Note that

$$0 = \pi(y_0, \boldsymbol{y}) \geq \mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) \geq 0.$$

Suppose

$$y_0 + \boldsymbol{y}'\boldsymbol{z}^* = y_0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}'\boldsymbol{z} > 0$$

for some $\boldsymbol{z}^* \in \mathcal{W}$. Since the objective function is linear, we can assume WLOG that $\boldsymbol{z}^*$ is an extreme point in $\mathcal{W}$.

Let $B_\epsilon(\boldsymbol{z}^*)$ denote an open ball with radius ϵ around $\boldsymbol{z}^*$, with

$$y_0 + \boldsymbol{y}'\boldsymbol{z} > 0 \text{ for all } z \in B_\epsilon(\boldsymbol{z^*}).$$

Since $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right) = 0$, we must have

$$P\left(B_\epsilon(\boldsymbol{z}^*)\right) = 0.$$

Thus the support for $\tilde{\boldsymbol{z}}$ lies in the convex hull $\mathcal{W}'$ of the (closed) set $\mathcal{W} \setminus B_\epsilon(\boldsymbol{z}^*)$. $\boldsymbol{z}^* \notin \mathcal{W}'$ since it is an extreme point in $\mathcal{W}$. This contradicts our earlier assumption that $\mathcal{W}$ denote the smallest convex set containing the support for $\tilde{\boldsymbol{z}}$. ■

2.2 Bounds on CVaR and Robust Optimization

There are several attractive proposals of robust optimization that approximate individual chance constrained problems which we have mentioned. In such a proposal, the solution, $(y_0, \boldsymbol{y})$ to the following robust counterpart

$$y_0 + \max_{\boldsymbol{z} \in \mathcal{U}} \boldsymbol{y}'\boldsymbol{z} \leq 0$$

guarantees that

$$\mathrm{P}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} \leq 0) \geq 1 - \epsilon. \tag{13}$$

Clearly, the choice of uncertainty set depends on the underlying assumption of primitive uncertainty.

Another approach of approximating the chance constraint problem is to provide an upper bound of $\rho_{1-\epsilon}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})$, so that if the bound is nonnegative, the chance constraint (13) will also be satisfied. For a given upperbound $\pi(y_0, \boldsymbol{y})$ to $E(\cdot)^+$, we define

$$\eta_{1-\epsilon}(y_0, \boldsymbol{y}) \triangleq \min_\beta \left\{ \beta + \frac{1}{\epsilon}\pi(y_0 - \beta, \boldsymbol{y}) \right\}.$$

Clearly,

$$\rho_{1-\epsilon}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}) = \min_\beta \left\{ \beta + \frac{1}{\epsilon}\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} - \beta)^+\right) \right\} \leq \eta_{1-\epsilon}(y_0, \boldsymbol{y})$$

and a sufficient condition for satisfying (13) is

$$\eta_{1-\epsilon}(y_0, \boldsymbol{y}) \leq 0. \tag{14}$$

Note that if the epigraph of $\pi(\cdot,\cdot)$ can be approximated by a second-order cone, the constraint (14) is also approximately second-order cone representable.

We show next that the two appraoches are esentially equivalent.

Theorem 4 *Under Assumption U and suppose $\pi(y_0, \boldsymbol{y})$ is an upperbound to $\mathrm{E}(y_0 + \boldsymbol{y}'z)^+$, $\pi(y_0, \boldsymbol{y})$ is convex and positive homogenous, with $\pi(y_0, \boldsymbol{0}) = y_0^+$, then*

$$\eta_{1-\epsilon}(y_0, \boldsymbol{y}) = y_0 + \max_{\boldsymbol{z} \in \mathcal{U}(\epsilon)} \boldsymbol{y}'\boldsymbol{z}.$$

for some convex uncertainty set $\mathcal{U}(\epsilon)$.

Proof : The set $\{(u, y_0, \boldsymbol{y}) : u \geq \pi(y_0, \boldsymbol{y})\}$ is a convex cone as it is the epigraph of a convex positive homogeneous function. $\mathcal{K} \stackrel{\Delta}{=} cl\,\{(u, y_0, \boldsymbol{y}) : u \geq \pi(y_0, \boldsymbol{y})\}$ is thus a closed convex cone. We show next that $\mathcal{K}$ is pointed.

Note that $\mathrm{E}(y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+ = 0$ ensures that $y_0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}'\boldsymbol{z} \leq 0$ under Assumption U. Suppose $(u, y_0, \boldsymbol{y}) \in \mathcal{K}$ and $(-u, -y_0, -\boldsymbol{y}) \in \mathcal{K}$. Since $\pi(y_0, \boldsymbol{y}) \geq 0$ and $\pi(-y_0, -\boldsymbol{y}) \geq 0$, we must have $u \geq 0, -u \geq 0$, i.e., $u = 0$. This forces $\pi(y_0, \boldsymbol{y}) = \pi(-y_0, -\boldsymbol{y}) = 0$. Hence by Proposition 1,

$$y_0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}'\boldsymbol{z} \leq 0, \quad -y_0 + \max_{\boldsymbol{z} \in \mathcal{W}} (-\boldsymbol{y})'\boldsymbol{z} \leq 0.$$

This is possible only when $y_0 = 0$ and $\boldsymbol{y} = \boldsymbol{0}$. Hence $\mathcal{K}$ is a pointed cone.

The dual cone by $\mathcal{K}^* \stackrel{\Delta}{=} \{(v, z_0, \boldsymbol{z}) : (v, z_0, \boldsymbol{z}) \cdot (u, y_0, \boldsymbol{y}) \geq 0 \ \forall \ (u, y_0, \boldsymbol{y}) \in \mathcal{K}\}$ is thus also a closed convex pointed cone. Thus both $\mathcal{K}$ and $\mathcal{K}^*$ have non-empty interior.

Suppose

$$\eta_{1-\epsilon}(y_0, \boldsymbol{y}) = \min_{\beta} \{\beta + \pi(y_0 - \beta, \boldsymbol{y})/\epsilon\}$$

is unbounded. Let $\{\beta_n\}$ be a sequence of numbers with $\lim_{n\to\infty} \beta_n = -\infty$, and $\lim_{n\to\infty} \{\beta_n + \pi(y_0 - \beta_n, \boldsymbol{y})/\epsilon\} = -\infty$. We may assume $\beta_n < 0$ for all n.

$$\lim_{n\to\infty} \{\beta_n + \pi(y_0 - \beta_n, \boldsymbol{y})/\epsilon\} = \lim_{n\to\infty} \left\{\beta_n + (-\beta_n)\pi\left(\frac{y_0}{-\beta_n} + 1, \frac{\boldsymbol{y}}{-\beta_n}\right)/\epsilon\right\} = \infty,$$

since

$$\lim_{n\to\infty} \pi(\frac{y_0}{-\beta_n} + 1, \frac{\boldsymbol{y}}{-\beta_n}) = \pi(1, \boldsymbol{0}) = 1,$$

and $\epsilon < 1$. This is a contradiction. Thus $\eta_{1-\epsilon}(y_0, \boldsymbol{y})$ must be bounded.

We have, using strong duality theorem, that

$$\begin{aligned}
\eta_{1-\epsilon}(y_0, \boldsymbol{y}) &= \min_{\beta,u} \{\beta + u/\epsilon : (u, y_0 - \beta, \boldsymbol{y}) \in \mathcal{K}\} \\
&= \min_{\beta,u} \{\beta + u/\epsilon : (u, -\beta, \boldsymbol{0}) \succ_{\mathcal{K}} (0, -y_0, -\boldsymbol{y})\} \\
&= \max \{y_0 z_0 + \boldsymbol{y}'\boldsymbol{z} : (v, -z_0, -\boldsymbol{z}) \in \mathcal{K}^*, v = 1/\epsilon, z_0 = 1\} \\
&= \max \{y_0 + \boldsymbol{y}'\boldsymbol{z} : (1/\epsilon, -1, -\boldsymbol{z}) \in \mathcal{K}^*\}
\end{aligned}$$

Hence

$$\eta_{1-\epsilon}(y_0, \boldsymbol{y}) = y_0 + \max_{\boldsymbol{z} \in \mathcal{U}(\epsilon)} \boldsymbol{y}'\boldsymbol{z},$$

with

$$\mathcal{U}(\epsilon) \triangleq \{\boldsymbol{z} : (1/\epsilon, -1, -\boldsymbol{z}) \in \mathcal{K}^*\}.$$

$\blacksquare$

For the functions $\pi^i(y_0, \boldsymbol{y})$, $i = 1, \ldots, 5$, the corresponding uncertainty sets can be computed explicitly. Consider the following uncertainty sets:

$$\begin{aligned}
&\mathcal{U}_1(\epsilon) \triangleq \mathcal{W}, \\
&\mathcal{U}_2(\epsilon) \triangleq \{\boldsymbol{z} \mid \boldsymbol{z} = (1 - 1/\epsilon)\boldsymbol{\zeta}, \text{for some } \boldsymbol{\zeta} \in \mathcal{W}\}, \\
&\mathcal{U}_3(\epsilon) \triangleq \left\{\boldsymbol{z} \mid \|\boldsymbol{z}\|_2 \le \sqrt{\frac{1-\epsilon}{\epsilon}}\right\}, \\
&\mathcal{U}_4(\epsilon) \triangleq \left\{\boldsymbol{z} \mid \exists \boldsymbol{s}, \boldsymbol{t} \in \Re^I, (z_1, \ldots, z_I) = \boldsymbol{s} - \boldsymbol{t}, \|\boldsymbol{P}^{-1}\boldsymbol{s} + \boldsymbol{Q}^{-1}\boldsymbol{t}\| \le \sqrt{-2\ln\epsilon}\right\}, \\
&\mathcal{U}_5(\epsilon) \triangleq \left\{\boldsymbol{z} \mid \exists \boldsymbol{s}, \boldsymbol{t} \in \Re^I, (z_1, \ldots, z_I) = \boldsymbol{s} - \boldsymbol{t}, \|\boldsymbol{Q}^{-1}\boldsymbol{s} + \boldsymbol{P}^{-1}\boldsymbol{t}\| \le \frac{1-\epsilon}{\epsilon}\sqrt{-2\ln(1-\epsilon)}\right\}.
\end{aligned}$$

Corollary 1

$$\eta^i_{1-\epsilon}(y_0, \boldsymbol{y}) \triangleq \min_{\beta} \left\{\beta + \frac{1}{\epsilon}\pi^i(y_0 - \beta, \boldsymbol{y})\right\} = y_0 + \max_{\boldsymbol{z} \in \mathcal{U}_i(\epsilon)} \boldsymbol{y}'\boldsymbol{z}.$$

Proof :

Uncertainty Set $\mathcal{U}_1$:

$$\begin{aligned}
\eta^1_{1-\epsilon}(y_0, \boldsymbol{y}) &= \min_{\beta} \left(\beta + \frac{\pi^1(y_0 - \beta, \boldsymbol{y})}{\epsilon}\right) \\
&= \min_{\beta} \left(\beta + \frac{1}{\epsilon}(y_0 - \beta + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}'\boldsymbol{z})^+\right) \\
&= y_0 + \max_{\boldsymbol{z} \in \mathcal{U}_1} \boldsymbol{y}'\boldsymbol{z}.
\end{aligned}$$

Uncertainty Set $\mathcal{U}_2$:

$$\begin{aligned}
\eta^2_{1-\epsilon}(y_0,\boldsymbol{y}) &= \min_\beta \left(\beta + \frac{\pi^2(y_0-\beta,\boldsymbol{y})}{\epsilon}\right)\\
&= y_0 + \min_\beta \left(\beta + \frac{\pi^2(-\beta,\boldsymbol{y})}{\epsilon}\right)\\
&= y_0 + \min_\beta \left\{\beta + \frac{1}{\epsilon}\left(\left(\max_{\boldsymbol{z}\in\mathcal{W}}(-\boldsymbol{y})'\boldsymbol{z}+\beta\right)^+ - \beta\right)\right\}\\
&= y_0 + \min_\beta \left\{\beta(1-1/\epsilon) + \frac{1}{\epsilon}\left((\max_{\boldsymbol{z}\in\mathcal{W}}(-\boldsymbol{y})'\boldsymbol{z}+\beta)^+\right)\right\}\\
&= y_0 + (1/\epsilon - 1)\min_\beta \left\{-\beta + \frac{1}{1-\epsilon}\left((\max_{\boldsymbol{z}\in\mathcal{W}}(-\boldsymbol{y})'\boldsymbol{z}+\beta)^+\right)\right\}\\
&= y_0 + (1/\epsilon-1)\max_{\boldsymbol{z}\in\mathcal{W}}\boldsymbol{y}'(-\boldsymbol{z}) + (1/\epsilon-1)\min_\beta\left(-\beta+\frac{1}{1-\epsilon}(\beta)^+\right)\\
&= y_0 + \max_{\boldsymbol{z}\in\mathcal{U}_2}\boldsymbol{y}'\boldsymbol{z}.
\end{aligned}$$

Uncertainty Set $\mathcal{U}_3$:

$$\begin{aligned}
\eta^3_{1-\epsilon}(y_0,\boldsymbol{y}) &= \min_\beta \left(\beta + \frac{\pi^3(y_0-\beta,\boldsymbol{y})}{\epsilon}\right)\\
&= \min_\beta \left(\beta + \frac{y_0-\beta+\sqrt{(y_0-\beta)^2+\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}}{2\epsilon}\right)\\
&= y_0 + \sqrt{\frac{1-\epsilon}{\epsilon}}\sqrt{\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}\\
&= y_0 + \max_{\boldsymbol{z}\in\mathcal{U}_3}\boldsymbol{y}'\boldsymbol{z},
\end{aligned}$$

where the second equality follows from choosing the optimum β,

$$\beta^* = y_0 + \frac{\sqrt{\boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}(1-2\epsilon)}{2\sqrt{\epsilon(1-\epsilon)}}.$$

Uncertainty Set $\mathcal{U}_4$:

For notational convenience, we denote

$$\begin{aligned}
\boldsymbol{y}_{\mathcal{I}} &= (y_1,\ldots,y_I)\\
\boldsymbol{y}_{\bar{\mathcal{I}}} &= (y_{I+1},\ldots,y_N).
\end{aligned}$$

$$\begin{aligned}
\eta^4_{1-\epsilon}(y_0,\boldsymbol{y}) &= \min_\beta \left(\beta + \frac{\pi^4(y_0-\beta,\boldsymbol{y})}{\epsilon}\right)\\
&= \min_{\beta,\mu,\boldsymbol{u}} \left(\beta + \frac{\frac{\mu}{e}\exp(\frac{y_0-\beta}{\mu}+\frac{\|\boldsymbol{u}\|_2^2}{2\mu^2})}{2\epsilon} \mid \boldsymbol{u}\geq \boldsymbol{P}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{u}\geq -\boldsymbol{Q}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{y}_{\bar{\mathcal{I}}}=\boldsymbol{0}\right)\\
&= \min_{\mu,\boldsymbol{u}}\left(y_0 + \frac{\|\boldsymbol{u}\|_2^2}{2\mu^2} - \mu\ln\epsilon \mid \boldsymbol{u}\geq \boldsymbol{P}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{u}\geq -\boldsymbol{Q}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{y}_{\bar{\mathcal{I}}}=\boldsymbol{0}\right)\\
&= \min_{\boldsymbol{u}}\left(y_0 + \sqrt{-2\ln\epsilon}\,u_0 \mid \boldsymbol{P}^{-1}\boldsymbol{u}\geq \boldsymbol{y}_{\mathcal{I}}, \boldsymbol{Q}^{-1}\boldsymbol{u}\geq -\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{y}_{\bar{\mathcal{I}}}=\boldsymbol{0}, \|\boldsymbol{u}\|_2\leq u_0\right)\\
&= y_0 + \max_{\boldsymbol{z}\in\mathcal{U}_4}\boldsymbol{y}'\boldsymbol{z},
\end{aligned}$$

where the second and third equalities follow from choosing the tightest β^* and μ^*, that is

$$\beta^* = y_0 + \frac{\|\boldsymbol{u}\|_2^2}{2\mu^2} - \mu \ln \epsilon - \mu,$$

$$\mu^* = \frac{\|\boldsymbol{u}\|_2}{\sqrt{-2\ln \epsilon}}.$$

The last equality is the result of strong conic duality and has been derived in Chen, Sim and Sun [13].

Uncertainty Set $\mathcal{U}_5$:

Following from the above exposition,

$$\begin{aligned}
\eta^5_{1-\epsilon}(y_0, \boldsymbol{y}) &= \min_{\beta} \left(\beta + \frac{\pi^5(y_0 - \beta, \boldsymbol{y})}{\epsilon}\right) \\
&= \min_{\beta,\mu,\boldsymbol{v}} \left(\beta + \frac{y_0 - \beta + \frac{\mu}{e}\exp(-\frac{y_0-\beta}{\mu} + \frac{\|\boldsymbol{v}\|_2^2}{2\mu^2})}{2\epsilon} \mid \boldsymbol{v} \ge -\boldsymbol{P}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{v} \ge \boldsymbol{Q}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{y}_{\bar{\mathcal{I}}} = \boldsymbol{0}\right) \\
&= \min_{\mu,\boldsymbol{v}} \left(y_0 + (\frac{1}{\epsilon} - 1)(\frac{\|\boldsymbol{v}\|_2^2}{2\mu^2} - \mu \ln(1-\epsilon)) \mid \boldsymbol{v} \ge -\boldsymbol{P}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{v} \ge \boldsymbol{Q}\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{y}_{\bar{\mathcal{I}}} = \boldsymbol{0}\right) \\
&= \min_{\boldsymbol{v}} \left(y_0 + \frac{1-\epsilon}{\epsilon}\sqrt{-2\ln(1-\epsilon)}\|\boldsymbol{v}\| \mid \mid \boldsymbol{P}^{-1}\boldsymbol{v} \ge -\boldsymbol{y}_{\mathcal{I}}, \boldsymbol{Q}^{-1}\boldsymbol{v} \ge \boldsymbol{y}_{\mathcal{I}}, \boldsymbol{y}_{\bar{\mathcal{I}}} = \boldsymbol{0}\right) \\
&= y_0 + \max_{\boldsymbol{z}\in\mathcal{U}_5} \boldsymbol{y}'\boldsymbol{z}.
\end{aligned}$$

■

We show next that the uncertainty set corresponding to the stronger bound $\pi^{\mathcal{L}}(y_0, \boldsymbol{y})$ can also be obtained in similar way.

Proposition 2 *Let $\mathcal{U}_i$, $i \in \mathcal{L}$, be compact uncertainty sets such that their intersections*

$$\mathcal{U}_{\mathcal{L}} = \bigcap_{i\in\mathcal{L}} \mathcal{U}_i,$$

has a non-empty interior. Then

$$\max_{\boldsymbol{z}\in\mathcal{U}_{\mathcal{L}}} \boldsymbol{y}'\boldsymbol{z} = \min_{\boldsymbol{y}_i, i\in\mathcal{L}} \left(\sum_{i\in\mathcal{L}} \max_{\boldsymbol{z}_i\in\mathcal{U}_i} \boldsymbol{y}_i'\boldsymbol{z}_i \mid \sum_{i\in\mathcal{L}} \boldsymbol{y}_i = \boldsymbol{y}\right).$$

Proof : We observe that the problem

$$\begin{aligned}
\max \quad & \boldsymbol{y}'\boldsymbol{z} \\
\text{s.t.} \quad & \boldsymbol{z} \in \mathcal{U}_{\mathcal{L}}
\end{aligned}$$

is equivalently

$$\begin{aligned}
\max \quad & \boldsymbol{y}'\boldsymbol{z} \\
\text{s.t.} \quad & \boldsymbol{z}_i = \boldsymbol{z} \\
& \boldsymbol{z}_i \in \mathcal{U}_i \qquad \forall i \in \mathcal{L}.
\end{aligned} \tag{15}$$

By strong duality, we have

$$\begin{aligned}&\max_{\boldsymbol{z}}\{\boldsymbol{y}'\boldsymbol{z} \ : \ \boldsymbol{z} = \boldsymbol{z}_i, i \in \mathcal{L}\}\\ =\ &\min_{\boldsymbol{y}_i, i\in\mathcal{L}} \left\{ \sum_{i\in\mathcal{L}} \boldsymbol{y}_i'\boldsymbol{z}_i \ : \ \sum_{i\in\mathcal{L}} \boldsymbol{y}_i = \boldsymbol{y} \right\}.\end{aligned}$$

Hence, the problem (15) is equivalent to

$$\max_{\boldsymbol{z}\in\mathcal{U}_{\mathcal{L}}} \boldsymbol{y}'\boldsymbol{z} = \max_{\boldsymbol{z}_i\in\mathcal{U}_i, i\in\mathcal{L}} \left\{ \min_{\boldsymbol{y}_i, i\in\mathcal{L}} \left\{ \sum_{i\in\mathcal{L}} \boldsymbol{y}_i'\boldsymbol{z}_i \mid \sum_{i\in\mathcal{L}} \boldsymbol{y}_i = \boldsymbol{y} \right\} \right\}.$$

Observe the set $\mathcal{U}_{\mathcal{L}}$ is a compact set with nonempty interior. Hence, $\max_{\boldsymbol{z}\in\mathcal{U}} \boldsymbol{y}'\boldsymbol{z}$ is therefore finite. Furthermore, there exists finite optimal primal and dual solutions $\boldsymbol{z}_i$ and $\boldsymbol{y}_i$, $i \in \mathcal{L}$ that satisfy strong duality. Hence, we can exchange "max" with "min", so that

$$\begin{aligned}\max_{\boldsymbol{z}\in\mathcal{U}} \boldsymbol{y}'\boldsymbol{z} &= \min_{\boldsymbol{y}_i, i\in\mathcal{L}} \left\{ \max_{\boldsymbol{z}_i\in\mathcal{U}_i, i\in\mathcal{L}} \sum_{i\in\mathcal{L}} \boldsymbol{y}_i'\boldsymbol{z}_i \mid \sum_{i\in\mathcal{L}} \boldsymbol{y}_i = \boldsymbol{y} \right\}\\ &= \min_{\boldsymbol{y}_i, i\in\mathcal{L}} \left\{ \sum_{i\in\mathcal{L}} \max_{\boldsymbol{z}_i\in\mathcal{U}_i} \boldsymbol{y}_i'\boldsymbol{z}_i \mid \sum_{i\in\mathcal{L}} \boldsymbol{y}_i = \boldsymbol{y} \right\}.\end{aligned}$$

■

Theorem 5 *Suppose $\tilde{\boldsymbol{z}}$ satisfies Assumption U. Let*

$$\mathcal{U}_{\mathcal{L}}(\epsilon) \triangleq \bigcap_{l\in\mathcal{L}} \mathcal{U}_l(\epsilon).$$

and suppose $\mathcal{U}_{\mathcal{L}}(\epsilon)$ has an non-empty interior. Then

$$\eta_{1-\epsilon}^{\mathcal{L}}(y_0, \boldsymbol{y}) = y_0 + \max_{\boldsymbol{z}\in\mathcal{U}_{\mathcal{L}}(\epsilon)} \boldsymbol{y}'\boldsymbol{z}.$$

Proof :

For notational convenience, we ignore the representation of uncertainty sets as functions of ϵ. Observe that for any $\epsilon \in (0, 1)$, the sets $\mathcal{U}_i(\epsilon)$ are compact and contain $\mathbf{0}$ in their interiors.

$$\begin{aligned}\eta^{\mathcal{L}}(y_0, \boldsymbol{y}) &= \min_{\beta} \left(\beta + \frac{\pi^{\mathcal{L}}(y_0 - \beta, \boldsymbol{y})}{\epsilon} \right)\\ &= \min_{\boldsymbol{\beta}, y_{l0}, \boldsymbol{y}_l, l\in\mathcal{L}} \left(\beta + \sum_{l\in\mathcal{L}} \left(\frac{\pi^l(y_{l0} - \beta_l, \boldsymbol{y}_l)}{\epsilon} \right) \mid \sum_{l\in\mathcal{L}} \boldsymbol{y}_l = \boldsymbol{y}, \sum_{l\in\mathcal{L}} y_{l0} = y_0, \sum_{l\in\mathcal{L}} \beta_l = \beta \right)\\ &= \min_{y_{l0}, \boldsymbol{y}_l, l\in\mathcal{L}} \left(\sum_{l\in\mathcal{L}} \min_{\beta_l} \left(\beta_l + \frac{\pi^l(y_{l0} - \beta_l, \boldsymbol{y}_l)}{\epsilon} \right) \mid \sum_{l\in\mathcal{L}} \boldsymbol{y}_l = \boldsymbol{y}, \sum_{l\in\mathcal{L}} y_{l0} = y_0 \right)\\ &= \min_{y_{l0}, \boldsymbol{y}_l, l\in\mathcal{L}} \left(\sum_{l\in\mathcal{L}} \left(y_{l0} + \max_{\boldsymbol{z}\in\mathcal{U}_l(\epsilon)} \boldsymbol{y}_l'\boldsymbol{z} \right) \mid \sum_{l\in\mathcal{L}} \boldsymbol{y}_l = \boldsymbol{y}, \sum_{l\in\mathcal{L}} y_{l0} = y_0 \right)\\ &= y_0 + \min_{\boldsymbol{y}_l, l\in\mathcal{L}} \left(\sum_{l\in\mathcal{L}} \left(\max_{\boldsymbol{z}\in\mathcal{U}_l(\epsilon)} \boldsymbol{y}_l'\boldsymbol{z} \right) \mid \sum_{l\in\mathcal{L}} \boldsymbol{y}_l = \boldsymbol{y} \right)\\ &= y_0 + \max_{\boldsymbol{z}\in\mathcal{U}_{\mathcal{L}}(\epsilon)} \boldsymbol{y}'\boldsymbol{z},\end{aligned}$$

where the last inequality is due to Proposition 2. ■

Hence, the different approximations of individual chance constrained problems using robust optimization are the consequences of applying different bounds on $\mathrm{E}((\cdot)^+)$. Notably, when the primitive uncertainties are characterized only by their means and covariance, the corresponding uncertainty set is an ellipsoid of the form $\mathcal{U}_3$. See, for instance, Bertsimas et al. [9] and El-Ghaoui et al. [16]. When $I = N$, that is all the primitive uncertainties are independently distributed, Chen, Sim and Sun [13] proposed the asymmetrical uncertainty set

$$\mathcal{U}_A(\epsilon) = \underbrace{\mathcal{W}}_{=\mathcal{U}_1(\epsilon)} \bigcap \mathcal{U}_4(\epsilon),$$

which generalizes the uncertainty set proposed by Ben-Tal and Nemirovski [5]. Noting that $\mathcal{U}_A(\epsilon) \subseteq \mathcal{U}_{\{1,2,4,5\}}(\epsilon)$, we can therefore improve upon the approximation using the uncertainty set $\mathcal{U}_{\{1,2,4,5\}}(\epsilon)$. However, in most application of chance constrained problems, the safety factor, ϵ is relatively small. In which case, the uncertainty sets of $\mathcal{U}_2(\epsilon)$ and $\mathcal{U}_5(\epsilon)$ are usually exploded to engulf the uncertainty sets of $\mathcal{W}$ and $\mathcal{U}_4(\epsilon)$, respectively . For instance, under symmetric distributions, that is $\boldsymbol{P} = \boldsymbol{Q}$ and $\bar{\boldsymbol{z}} = \underline{\boldsymbol{z}}$, it is easy to establish that for $\epsilon < 0.5$, we have

$$\mathcal{U}_{\{1,2,4,5\}}(\epsilon) = \underbrace{\mathcal{U}_1(\epsilon)}_{=\mathcal{W}} \bigcap \underbrace{\mathcal{U}_2(\epsilon)}_{\supseteq\mathcal{W}} \bigcap \mathcal{U}_4 \bigcap \underbrace{\mathcal{U}_5}_{\supseteq\mathcal{U}_4} = \mathcal{U}_A(\epsilon).$$

3 Joint Chance Constrained Problem

Unfortunately, the notion of uncertainty set in classical robust optimization does not carry forward as well in addressing joint chance constrained problems. We consider a linear joint chance constraint as follows,

$$\mathrm{P}\Big(y_j(\tilde{\boldsymbol{z}}) \leq 0, j \in \mathcal{M}\Big) \geq 1 - \epsilon, \tag{16}$$

where $\mathcal{M} = \{1, \ldots, m\}$, $y_j(\tilde{\boldsymbol{z}})$ are affinely dependent of $\tilde{\boldsymbol{z}}$,

$$y_j(\tilde{\boldsymbol{z}}) = y_j^0 + \sum_{k=1}^{N} y_j^k \tilde{z}_k \qquad j \in \mathcal{M}.$$

$(y_1^0, \ldots, y_1^N, \ldots, y_m^0, \ldots, y_m^N)$ being the decision variables. For notational convenience, we represent

$$\boldsymbol{y}_j = (y_j^1, \ldots, y_j^N),$$

so that $y_i(\tilde{\boldsymbol{z}}) = y_i^0 + \boldsymbol{y}_i' \tilde{\boldsymbol{z}}$ and denote

$$\boldsymbol{Y} = (y_1^0, \ldots, y_1^N, \ldots, y_m^0, \ldots, y_m^N),$$

as the collection of decision variables in the joint chance constrained problem. By suitable affine constraints imposed on the decision variables $\boldsymbol{Y}$ and $\boldsymbol{x}$, we can represent the joint chance constraint in Model (3) in the form of constraint (16).

It is not surprising that a joint chance constraint is more difficult to solve than an individual one. For computational tractability, the common approach is to decompose the joint constrained problem into a problem with m individual constraints of the form

$$\mathrm{P}\Big(y_i(\tilde{\boldsymbol{z}}) \leq 0\Big) \geq 1 - \epsilon_i,\ i \in \mathcal{M}. \tag{17}$$

By enforcing Bonferroni's inequality on their safety factors,

$$\sum_{i\in\mathcal{M}} \epsilon_i \leq \epsilon. \tag{18}$$

any feasible solution that satisfies the set of individual chance constrained problem will also satisfy the corresponding joint chance constrained problem. See for instance, Chen, Sim and Sun [13] and Nemirovski and Shapiro [20]. Consequently, using the techniques discussed in the previous section, we can then build tractable safe approximations as follows

$$\eta_{1-\epsilon_i}(y_i^0, \boldsymbol{y}_i) \leq 0,\ i \in \mathcal{M}. \tag{19}$$

The main issue with using Bonferroni's inequality is the choice of ϵ_i. Unfortunately, the problem becomes non-convex and possibly intractable if ϵ_i are made variables and enforcing the constraint (18) as part of the optimization model. As such, it is natural to choose, $\epsilon_i = \epsilon/m$ as proposed in Chen, Sim and Sun [13] and Nemirovski and Shapiro [20].

In some instances, Bonferroni's inequality may be rather conservative even for an optimal choice of ϵ_i. For instance, suppose $y_i(\tilde{\boldsymbol{z}})$ are completely correlated, such as

$$y_i(\tilde{\boldsymbol{z}}) = \delta_i(a^0 + \boldsymbol{a}'\tilde{\boldsymbol{z}}), \qquad i \in \mathcal{M} \tag{20}$$

for some $\delta_i > 0$, the least conservative choice of ϵ_i is $\epsilon_i = \epsilon$ for all $i \in \mathcal{M}$, which would violate the condition (18) imposed by Bonferroni's inequality. As a matter of fact, it is easy to see that the least conservative choice of ϵ_i while satisfying Bonferroni's inequality is $\epsilon_i = \epsilon/m$ for all $i = 1, \ldots, m$. Hence, if $y_i(\tilde{\boldsymbol{z}})$ are correlated, the efficacy of Bonferroni's inequality will possibly diminish.

We propose a new tractable way for approximating the joint chance constraint problem. Given a vector of positive constants, $\boldsymbol{\alpha} \in \Re^N$, $\boldsymbol{\alpha} > \boldsymbol{0}$, an index set $\mathcal{J} \subseteq \mathcal{M}$, an upperbound $\pi(y_0, \boldsymbol{y})$ for

$\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right)$, we define the following function,

$$\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \triangleq \min_{w_0, \boldsymbol{w}} \left(\underbrace{\min_{\beta} \left(\beta + \frac{1}{\epsilon}\pi(w_0 - \beta, \boldsymbol{w}) \right)}_{=\eta_{1-\epsilon}(w_0, \boldsymbol{w})} + \frac{1}{\epsilon} \left\{ \sum_{i \in \mathcal{J}} \pi(\alpha_i y_i^0 - w_0, \alpha_i \boldsymbol{y}_i - \boldsymbol{w}) \right\} \right).$$

The next result shows we can use the above function to approximate a joint chance constrained problem.

Theorem 6 **(a)** *Suppose $\tilde{\boldsymbol{z}}$ satisfies Assumption U, then*

$$\rho_{1-\epsilon}\left(\max_{i \in \mathcal{J}} \{\alpha_i y_i(\tilde{\boldsymbol{z}})\} \right) \leq \gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}).$$

Consequently, the joint chance constraint (16) is satisfied if

$$\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \leq 0 \tag{21}$$

and

$$y_i^0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}_i' \boldsymbol{z} \leq 0 \qquad \forall i \in \mathcal{M} \backslash \mathcal{J}. \tag{22}$$

(b) *For fixed $\boldsymbol{\alpha}$, the epigraph of the function $\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J})$ with respect to $\boldsymbol{Y}$ is second-order cone representable and positive homogenous. Similarly, for a fixed $\boldsymbol{Y}$, the epigraph of the function $\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J})$ with respect to $\boldsymbol{\alpha}$ is second-order cone representable and positive homogenous.*

Proof : **(a)** Under Assumption U, the set $\mathcal{W}$ is the support of the primitive uncertainty, $\tilde{\boldsymbol{z}}$, hence, the robust counterpart (22) implies

$$\mathrm{P}(y_i^0 + \boldsymbol{y}_i' \tilde{\boldsymbol{z}} > 0) = 0, \qquad \forall i \in \mathcal{M} \backslash \mathcal{J}.$$

Hence, with $\boldsymbol{\alpha} > 0$, we have

$$\mathrm{P}\Big(y_i^0 + \boldsymbol{y}_i' \tilde{\boldsymbol{z}} \leq 0, i \in \mathcal{M}\Big) = \mathrm{P}\Big(y_i^0 + \boldsymbol{y}_i' \tilde{\boldsymbol{z}} \leq 0, i \in \mathcal{J}\Big) = \mathrm{P}\Big(\max_{i \in \mathcal{J}} \{\alpha_i y_i^0 + \alpha_i \boldsymbol{y}_i' \tilde{\boldsymbol{z}}\} \leq 0\Big).$$

Therefore, it suffices to show that if $\boldsymbol{Y}$ is feasible in the constraint (21), then the CVaR measure,

$$\rho_{1-\epsilon}\left(\max_{i \in \mathcal{J}} \{\alpha_i y_i(\tilde{\boldsymbol{z}})\} \right) \leq 0.$$

Using the classical inequality (cf. Meijison and Nadas [18]) that

$$\mathrm{E}\left(\max_{i=1,\ldots,n} X_i - \beta \right)^+ \leq \mathrm{E}\left(Y - \beta\right)^+ + \sum_{i=1}^n \mathrm{E}\left(X_i - Y\right)^+, \text{ for any r.v. } Y, \tag{23}$$

we have

$$\begin{aligned}
& \rho_{1-\epsilon}\Big(\max_{i\in\mathcal{J}}\{\alpha_i(y_i^0+\boldsymbol{y}_i'\tilde{\boldsymbol{z}})\}\Big) \\
= & \min_{\beta}\left\{\beta+\frac{1}{\epsilon}\mathrm{E}\left(\Big(\max_{i\in\mathcal{J}}\{\alpha_i(y_i^0+\boldsymbol{y}_i'\tilde{\boldsymbol{z}})\}-\beta\Big)^+\right)\right\} \\
\le & \min_{\beta,w_0,\boldsymbol{w}}\left\{\beta+\frac{1}{\epsilon}\left(\mathrm{E}\left((w_0-\beta+\boldsymbol{w}'\tilde{\boldsymbol{z}})^+\right)+\sum_{i\in\mathcal{J}}\mathrm{E}\left((\alpha y_i^0-w_0+(\alpha_i\boldsymbol{y}_i-\boldsymbol{w})'\tilde{\boldsymbol{z}})^+\right)\right)\right\} \\
\le & \min_{\beta,w_0,\boldsymbol{w}}\left\{\beta+\frac{1}{\epsilon}\left(\pi(w_0-\beta,\boldsymbol{w})+\sum_{i\in\mathcal{J}}\pi(\alpha y_i^0-w_0,\alpha_i\boldsymbol{y}_i-\boldsymbol{w})\right)\right\} \\
= & \gamma_{1-\epsilon}(\boldsymbol{Y},\boldsymbol{\alpha},\mathcal{J})\le 0.
\end{aligned}$$

(b) For a fixed $\boldsymbol{\alpha}$, the corresponding epigraph can be expressed as

$$Y_1=\{(\boldsymbol{Y},t):\gamma_{1-\epsilon}(\boldsymbol{Y},\boldsymbol{\alpha},\mathcal{J})\le t\}=\left\{(\boldsymbol{Y},t):\begin{array}{l}\exists w_0,r_0,\ldots,r_m\in\Re,\boldsymbol{w}\in\Re^N\\ r_0+\frac{1}{\epsilon}\sum_{i\in\mathcal{J}}r_i\le t\\ \eta_{1-\epsilon}(w_0,\boldsymbol{w})\le r_0\\ \pi(\alpha_i y_i^0-w_0,\alpha_i\boldsymbol{y}_i-\boldsymbol{w})\le r_i\quad\forall i\in\mathcal{J}\end{array}\right\}.$$

Since the epigraphs of $\eta_{1-\epsilon}(\cdot,\cdot)$ and $\pi(\cdot,\cdot)$ are second-order cone representable, the set Y_1 is also second-order cone representable. For positive homogeneity, we observe that since $\pi(\cdot,\cdot)$ is positive homogenous, we have that for all $k\ge 0$,

$$\begin{aligned}
& \gamma_{1-\epsilon}(k\boldsymbol{Y},\boldsymbol{\alpha},\mathcal{J}) \\
= & \min_{\beta,w_0,\boldsymbol{w}}\left(\beta+\frac{1}{\epsilon}\left\{\pi(w_0-\beta,\boldsymbol{w})+\sum_{i\in\mathcal{J}}\pi(k\alpha_i y_i^0-w_0,k\alpha_i\boldsymbol{y}_i-\boldsymbol{w})\right\}\right) \\
= & k\min_{\beta,w_0,\boldsymbol{w}}\left(\frac{1}{k}\beta+\frac{1}{\epsilon}\left\{\pi\left(\frac{1}{k}w_0-\frac{1}{k}\beta,\frac{1}{k}\boldsymbol{w}\right)+\sum_{i\in\mathcal{J}}\pi\left(\alpha_i y_i^0-\frac{1}{k}w_0,\alpha_i\boldsymbol{y}_i-\frac{1}{k}\boldsymbol{w}\right)\right\}\right) \\
= & k\min_{\beta,w_0,\boldsymbol{w}}\left(\beta+\frac{1}{\epsilon}\left\{\pi(w_0-\beta,\boldsymbol{w})+\sum_{i\in\mathcal{J}}\pi(\alpha_i y_i^0-w_0,\alpha_i\boldsymbol{y}_i-\boldsymbol{w})\right\}\right) \\
= & k\gamma_{1-\epsilon}(\boldsymbol{Y},\boldsymbol{\alpha},\mathcal{J}).
\end{aligned}$$

Similarly, the same exposition applies when $\boldsymbol{Y}$ is fixed and $\boldsymbol{\alpha}$ being the decision variable. ■

Remark : Note that the constraints (22) do not depend on the values of α_j for all $j\in\mathcal{M}\backslash\mathcal{J}$. Speaking intuitively, we can perceive $\alpha_j=\infty$ for all $j\in\mathcal{M}\backslash\mathcal{J}$. However, to avoid dealing with infinite entities, we define the set $\mathcal{J}$ as part of the input to the function $\gamma_{1-\epsilon}(\cdot,\cdot,\cdot)$. Throughout this paper, we will restrict the focus of $\boldsymbol{\alpha}$ to only elements corresponding to the indices in the set $\mathcal{J}$. Unfortunately, the function $\gamma_{1-\epsilon}(\boldsymbol{Y},\boldsymbol{\alpha},\mathcal{J})$ is not jointly convex in both $\boldsymbol{Y}$ and $\boldsymbol{\alpha}$. Nevertheless, for a given $\boldsymbol{Y}$, it is a tractable convex function with respect to $\boldsymbol{\alpha}$ and in the attractive form of SOCP. We will later exploit this property for improving the choice of $\boldsymbol{\alpha}$.

The inequality obtained using (23) is tight when the variables $y_i(\tilde{\boldsymbol{z}})$, $i = 1, \ldots, n$ are negatively correlated. More specifically, if the sets

$$\mathcal{S}_i \triangleq \{\boldsymbol{z} : y_i(\tilde{\boldsymbol{z}}) \geq \beta\}, \quad i = 1, \ldots, n$$

are mutually disjoint, then

$$\mathrm{E}\left(\max_i y_i(\tilde{\boldsymbol{z}}) - \beta\right)^+ = \sum_{i=1}^n \mathrm{E}\left(y_i(\tilde{\boldsymbol{z}}) - \beta\right)^+,$$

and hence the inequality (23) cannot be tighten further substantially. Interestingly, by introducing the parameters $\boldsymbol{\alpha}$ and random variable $w_0 + \boldsymbol{w}'\boldsymbol{z}$, our approach is also able to handle the situation when the variables are positively correlated. In the example (20) where $y_i(\tilde{\boldsymbol{z}})$, $i \in \mathcal{M}$ are completely positively correlated, the following condition

$$\eta_{1-\epsilon}(a^0, \boldsymbol{a}) \leq 0$$

is also sufficient to guarantee feasibility in the joint chance constrained problem. Choosing $\alpha_i = 1/\delta_i > 0$, we see that

$$\begin{aligned}
&\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{M}) \\
= &\min_{w_0, \boldsymbol{w}} \left(\eta_{1-\epsilon}(w_0, \boldsymbol{w}) + \frac{1}{\epsilon}\left\{\sum_{i \in \mathcal{M}} \pi(\alpha_i y_i^0 - w_0, \alpha_i \boldsymbol{y}_i - \boldsymbol{w})\right\}\right) \\
= &\min_{w_0, \boldsymbol{w}} \left(\eta_{1-\epsilon}(w_0, \boldsymbol{w}) + \frac{1}{\epsilon}\left\{\sum_{i \in \mathcal{M}} \pi(\alpha_i \delta_i a^0 - w_0, \alpha_i \delta_i \boldsymbol{a} - \boldsymbol{w})\right\}\right) \\
\leq &\ \eta_{1-\epsilon}(a^0, \boldsymbol{a}) + \frac{1}{\epsilon}\left\{\sum_{i \in \mathcal{M}} \pi(a^0 - a_0, \boldsymbol{a} - \boldsymbol{a})\right\} \\
= &\ \eta_{1-\epsilon}(a^0, \boldsymbol{a}) \leq 0.
\end{aligned}$$

Therefore, we see that the new bound is potentially better than the application of Bonferroni's inequality on individual chance constraints. By choosing the right combination of $(\boldsymbol{\alpha}, \mathcal{J})$, we can prove a stronger result as follows.

Theorem 7 *Let $\epsilon_i \in (0,1), i \in \mathcal{M}$ and $\sum_{i \in \mathcal{M}} \epsilon_i \leq \epsilon$. Suppose $\boldsymbol{Y}$ satisfies*

$$\eta_{1-\epsilon_i}(y_i^0, \boldsymbol{y}_i) \leq 0 \qquad \forall i \in \mathcal{M},$$

then there exists $\boldsymbol{\alpha} > 0$, and a set $\mathcal{J} \subseteq \mathcal{M}$ such that $(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J})$ are feasible in the constraints (21) and (22).

Proof : Let β_i be the optimal solution to

$$\underbrace{\min_\beta \left(\beta + \frac{1}{\epsilon_i}\left(\pi(y_i^0 - \beta, \boldsymbol{y}_i)\right)\right)}_{=\eta_{1-\epsilon_i}(y_i^0, \boldsymbol{y}_i)}.$$

Since $\eta_{1-\epsilon_i}(y_i^0, \boldsymbol{y}_i) \leq 0$ and that

$$\pi(y_i^0 - \beta_i, \boldsymbol{y}_i) \geq \mathrm{E}\left((y_i^0 - \beta_i + \boldsymbol{y}_i'\tilde{\boldsymbol{z}})^+\right) \geq 0,$$

we must have $\beta_i \leq 0$. Let $\mathcal{J} = \{i | \beta_i < 0\}$,

$$\alpha_j = -\frac{1}{\beta_j} \qquad \forall j \in \mathcal{J}.$$

Since $\beta_j = 0$ for all $j \in \mathcal{M}\backslash\mathcal{J}$, we have

$$0 \leq \pi(y_i^0, \boldsymbol{y}_i) \leq 0 \qquad \forall i \in \mathcal{M}\backslash\mathcal{J}$$

From Proposition 1, it follows that

$$y_i^0 + \boldsymbol{y}_i'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}, \qquad \forall i \in \mathcal{M}\backslash\mathcal{J}$$

which satisfies the set of inequalities in (22).

For $i \in \mathcal{J}$, the constraint $\eta_{1-\epsilon_i}(y_i^0, \boldsymbol{y}_i) \leq 0$ is equivalent to

$$\frac{1}{-\beta_i}\pi(y_i^0 - \beta_i, y_i) \leq \epsilon_i$$

Since the function $\pi(\cdot,\cdot)$ is positive homogenous, we have

$$\begin{aligned}
\frac{1}{-\beta_i}\pi(y_i^0 - \beta_i, y_i) &= \pi\left(\frac{1}{-\beta_i}y_i^0 + 1, \frac{1}{-\beta_i}y_i\right) \\
&= \pi\left(\alpha_i y_i^0 + 1, \alpha_i y_i\right) \\
&\leq \epsilon_i \qquad \forall i \in \mathcal{J}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \\
= \; &\min_{\beta, w_0, \boldsymbol{w}} \left(\beta + \frac{1}{\epsilon}\left\{\pi(w_0 - \beta, \boldsymbol{w}) + \sum_{i \in \mathcal{J}} \pi(\alpha_i y_i^0 - w_0, \alpha_i \boldsymbol{y}_i - \boldsymbol{w})\right\}\right) \\
\leq \; &-1 + \frac{1}{\epsilon}\left\{\pi(-1+1, \boldsymbol{0}) + \sum_{i \in \mathcal{J}} \pi(\alpha_i y^0 + 1, \alpha_i \boldsymbol{y} - \boldsymbol{0})\right\} \\
= \; &-1 + \frac{1}{\epsilon}\sum_{i \in \mathcal{J}} \pi(\alpha_i y^0 + 1, \alpha_i \boldsymbol{y}) \\
\leq \; &-1 + \frac{1}{\epsilon}\sum_{i \in \mathcal{J}} \epsilon_i \leq 0,
\end{aligned}$$

where the first inequality is due to the choice of $\beta = -1$, $w_0 = -1$, $\boldsymbol{w} = \boldsymbol{0}$ and the last inequality follows from $\sum_{i \in \mathcal{M}} \epsilon_i \leq \epsilon$. ■

3.1 Optimizing over $\boldsymbol{\alpha}$

Consider a joint chance constrained model as follows

$$\begin{aligned} Z_\epsilon = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{P}(y_i(\tilde{\boldsymbol{z}}) \leq 0, \quad i \in \mathcal{M}) \geq 1-\epsilon \\ & (\boldsymbol{x}, \boldsymbol{Y}) \in X, \end{aligned} \tag{24}$$

in which X is efficiently computable convex set, such as a polyhedron or a second-order cone representable set. Given a set of constant, $\boldsymbol{\alpha} > 0$ and a set $\mathcal{J}$, we consider the following optimization model.

$$\begin{aligned} Z_\epsilon^1(\boldsymbol{\alpha}, \mathcal{J}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \leq 0 \\ & y_i^0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}_i'\boldsymbol{z} \leq 0 \qquad \forall i \in \mathcal{M} \backslash \mathcal{J} \\ & (\boldsymbol{x}, \boldsymbol{Y}) \in X. \end{aligned} \tag{25}$$

Under Assumption U, suppose Model (25) is feasible, the solution $(\boldsymbol{x}, \boldsymbol{Y})$ is also feasible in Model (24), albeit more conservatively.

The main concern here is how to choose $\boldsymbol{\alpha}$ and $\mathcal{J}$. A likely choice, is say $\alpha_j = 1/m$, for all $j \in \mathcal{M}$ and $\mathcal{J} = \mathcal{M}$. Alternatively, we may use the classical approach by decomposing into m individual chance constraint problem with $\epsilon_i = \epsilon/m$. Base on Theorem 7, we can find a feasible $\boldsymbol{\alpha} > \boldsymbol{0}$ and set $\mathcal{J}$ such that Model (25) is also feasible.

Our aim is to improve upon the objective by minimizing $\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J})$ over α_j, $j \in \mathcal{J}$, resulting in greater slack in the model (25). Hence, this approach will lead to improvement in the objective, or at least will not increase the value.

Given a feasible solution, $\boldsymbol{Y}$ in Model (25), our aim is to improve upon the objective by readjusting the set $\mathcal{J}$ and the weights α_j, $j \in \mathcal{J}$, that will result in greater slack in the model (25) over the solution, $\boldsymbol{Y}$. We define the following set,

$$\mathcal{K}(\boldsymbol{Y}) \triangleq \left\{ i \ : \ y_i^0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}_i'\boldsymbol{z} > 0 \right\}.$$

Note that we can obtain the set $\mathcal{K}(\boldsymbol{Y})$ by solving the following linear optimization problem

$$\begin{aligned} \min \quad & \sum_{i=1}^m s_i \\ \text{s.t.} \quad & y_i^0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}_i'\boldsymbol{z} \leq s_i, \end{aligned} \tag{26}$$

so that $\mathcal{K}(\boldsymbol{Y}) = \{i \ : \ s_i^* > 0\}$, $\boldsymbol{s}^*$ being its optimal solution.

Since $\boldsymbol{Y}$ is feasible in Model (25), we must have $\mathcal{K}(\boldsymbol{Y}) \subseteq \mathcal{J}$. If the set $\mathcal{K}(\boldsymbol{Y})$ is nonempty, we consider the following optimization problem over α_j, $j \in \mathcal{K}(\boldsymbol{Y})$,

$$\begin{aligned} Z^1_\alpha(\boldsymbol{Y}) = \min \quad & \gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{K}(\boldsymbol{Y})) \\ \text{s.t.} \quad & \sum_{j \in \mathcal{K}(\boldsymbol{Y})} \alpha_j = 1 \\ & \alpha_j \geq 0 \qquad \forall j \in \mathcal{K}(\boldsymbol{Y}). \end{aligned} \tag{27}$$

By choosing $\pi(y_0, \boldsymbol{y}) \leq \pi^1(y_0, \boldsymbol{y})$, we can ensure that the objective function of Problem (27) is finite. Moreover, since the feasible region of Problem (27) is compact, the optimal solution for α_j, $j \in \mathcal{K}(\boldsymbol{Y})$ is therefore achievable.

Proposition 3 *Assume there exists* $(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J})$*,* $\boldsymbol{\alpha} > 0$*, such that* $\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \leq 0$*. Let* $\boldsymbol{\alpha}^*$ *be the optimum solution of Problem (27).*

(a)

$$Z^1_\alpha(\boldsymbol{Y}) \leq 0.$$

(b) *Moreover, the solution* $\boldsymbol{\alpha}^*$ *satisfies,*

$$\alpha^*_i > 0 \qquad \forall i \in \mathcal{K}(\boldsymbol{Y}).$$

Proof : (a) Since $\mathcal{K}(\boldsymbol{Y}) \subset \mathcal{J}$, and under the assumption that there exists $(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J})$, $\boldsymbol{\alpha} > 0$, such that $\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \leq 0$, by using the same $\boldsymbol{\alpha}$, we observe that

$$\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{K}) \leq \gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \leq 0.$$

Due to the positive homogenous property of Theorem 6(b), we scale $\boldsymbol{\alpha}$ by a positive constant so that it is feasible in Problem (27). Hence, the result follows.

(b) Suppose there exists a nonempty set $\mathcal{G} \subset \mathcal{K}(\boldsymbol{Y})$ such that $\alpha^*_i = 0, \forall i \in \mathcal{G}$, we will show that the following holds,

$$y^0_i + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{y}'_i \boldsymbol{z} \leq 0 \quad \forall i \in \mathcal{K}(\boldsymbol{Y}) \backslash \mathcal{G},$$

which is a contradiction. We have argued that $Z^1_\alpha(\boldsymbol{Y}) \leq 0$. Let $k \in \mathcal{G}$, that is, $\alpha^*_k = 0$. Observe that for some suitably chosen $(\beta, w_0, \boldsymbol{w})$,

$$
\begin{aligned}
0 \geq\ & \gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}^*, \mathcal{K}(\boldsymbol{Y})) \\
=\ & \beta + \frac{1}{\epsilon}\left\{\pi(w_0-\beta, \boldsymbol{w}) + \sum_{i\in\mathcal{K}(\boldsymbol{Y})} \pi(\alpha_i^* y_i^0 - w_0, \alpha_i^* \boldsymbol{y}_i - \boldsymbol{w})\right\} \\
=\ & \beta + \frac{1}{\epsilon}\left\{\pi(w_0-\beta, \boldsymbol{w}) + \pi(-w_0, -\boldsymbol{w})\right\} + \frac{1}{\epsilon}\sum_{i\in\mathcal{K}(\boldsymbol{Y})\backslash\{k\}} \pi(\alpha_i^* y_i^0 - w_0, \alpha_i^* \boldsymbol{y}_i - \boldsymbol{w}) \\
\geq\ & \beta + \frac{1}{\epsilon}\left\{E\left(w_0 + \boldsymbol{w}'\boldsymbol{z} - \beta\right)^+ + E\left(-w_0 - \boldsymbol{w}'\boldsymbol{z}\right)^+\right\} \\
\geq\ & \beta + \tfrac{1}{\epsilon}(-\beta)^+,
\end{aligned}
$$

where the second equality is due to $\alpha_k^* = 0$. Since, $\epsilon \in (0,1)$, the inequality $\beta + \frac{1}{\epsilon}(-\beta)^+ \leq 0$ is satisfied if and only if $\beta = 0$. We now argue that

$$
\pi(y_i^0, \boldsymbol{y}_i) = 0 \quad \forall i \in \mathcal{K}(\boldsymbol{Y})\backslash\mathcal{G} \tag{28}
$$

which, from Proposition 1, implies

$$
y_i^0 + \max_{\boldsymbol{z}\in\mathcal{W}} \boldsymbol{y}_i'\boldsymbol{z} \leq 0 \quad \forall i \in \mathcal{K}(\boldsymbol{Y})\backslash\mathcal{G}.
$$

Indeed, for any $l \in \mathcal{K}(\boldsymbol{Y})\backslash\mathcal{G}$, we observe that

$$
\begin{aligned}
0 \geq\ & \beta + \frac{1}{\epsilon}\left\{\pi(w_0-\beta, \boldsymbol{w}) + \sum_{i\in\mathcal{K}(\boldsymbol{Y})} \pi(\alpha_i^* y_i^0 - w_0, \alpha_i^* \boldsymbol{y}_i - \boldsymbol{w})\right\} \\
=\ & \frac{1}{\epsilon}\left\{\pi(w_0, \boldsymbol{w}) + \sum_{i\in\mathcal{K}(\boldsymbol{Y})} \pi(\alpha_i^* y_i^0 - w_0, \alpha_i^* \boldsymbol{y}_i - \boldsymbol{w})\right\} \qquad \text{Substituting } \beta = 0, \\
\geq\ & \frac{1}{\epsilon}\left\{\pi(w_0, \boldsymbol{w}) + \pi(\alpha_l^* y_l^0 - w_0, \alpha_l^* \boldsymbol{y}_l - \boldsymbol{w})\right\} \\
\geq\ & \frac{1}{\epsilon}\left\{\pi(\alpha_l^* y_l^0, \alpha_l^* \boldsymbol{y}_l)\right\} \\
=\ & \frac{\alpha_l^*}{\epsilon}\pi(y_l^0, \boldsymbol{y}_l) \geq 0.
\end{aligned}
$$

Hence, the equality (28) is achieved by noting that $\alpha_l^* > 0$.

■

We propose an algorithm for improving the choice of $\boldsymbol{\alpha}$ and the set $\mathcal{J}$. Again, we assume that we can find an initial feasible solution of Model (25).

Algorithm 1 .

Input: $\boldsymbol{Y}$

1. *Solve Problem (26) with Input $\boldsymbol{Y}$. Obtain optimal solution $\boldsymbol{s}^*$.*

2. *Set* $\mathcal{K}(\boldsymbol{Y}) := \{i | s^*_j > 0, j \in \mathcal{M}\}$.

3. *Solve Problem (27) with Input* $\boldsymbol{Y}$. *Obtain optimal solution* $\boldsymbol{\alpha}^*$. *Set* $\mathcal{J} := \mathcal{K}(\boldsymbol{Y})$.

4. *Solve Model (25) with Input* $(\boldsymbol{\alpha}, \mathcal{J})$. *Obtain optimal solution* $(\boldsymbol{x}^*, \boldsymbol{Y}^*)$. *Set* $\boldsymbol{Y} := \boldsymbol{Y}^*$.

5. *Repeat Step 1 until a termination criterion is met or until* $\mathcal{J} = \emptyset$.

6. *Output solution* $(\boldsymbol{x}^*, \boldsymbol{Y}^*)$.

Theorem 8 *In Algorithm 1, the sequence of objectives obtained by solving Model (25) is non-increasing.*

Proof : Starting with a feasible solution of Model (25), we are assured that there exists $(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J})$, $\boldsymbol{\alpha} > 0$, such that $\gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{J}) \leq 0$. With Proposition 3(b), the condition in Step 3 ensures that $\alpha^*_j > 0$ for all $j \in \mathcal{J}$. Moreover, Proposition 3(a) ensures that the updates on $\boldsymbol{\alpha}$ and $\mathcal{J}$ do not affect the feasibility of its previous solution $(\boldsymbol{x}, \boldsymbol{Y})$ in the Model (25). Hence, its objective value will not increase. ■

The implementation of Algorithm 1 may involve perpetual updates of the set $\mathcal{J}$ and result in reformulating Problem 25. A practical solution is to ignore the set $\mathcal{J}$ and solve the following model,

$$\begin{aligned} Z^2_\epsilon(\boldsymbol{\alpha}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{M}) \leq 0 \\ & (\boldsymbol{x}, \boldsymbol{Y}) \in X, \end{aligned} \tag{29}$$

for a given $\boldsymbol{\alpha} \geq \mathbf{1}$ such that $\mathbf{1}'\boldsymbol{\alpha} = M$, where M is a large number. The updates of $\boldsymbol{\alpha}$ is done by solving

$$\begin{aligned} Z^2_\alpha(\boldsymbol{Y}) = \min \quad & \gamma_{1-\epsilon}(\boldsymbol{Y}, \boldsymbol{\alpha}, \mathcal{M}) \\ \text{s.t.} \quad & \textstyle\sum_{j \in \mathcal{J}} \alpha_j = M \\ & \boldsymbol{\alpha} \geq 1. \end{aligned} \tag{30}$$

The algorithm is also simplified as follows,

Algorithm 2 .

Input: $\boldsymbol{Y}$

1. *Solve Problem (30) with Input* $\boldsymbol{Y}$. *Obtain optimal solution* $\boldsymbol{\alpha}^*$. *Set* $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$

2. *Solve Model (29) with Input* $\boldsymbol{\alpha}$. *Obtain optimal solution* $(\boldsymbol{x}^*, \boldsymbol{Y}^*)$. *Set* $\boldsymbol{Y} = \boldsymbol{Y}^*$.

3. *Repeat Step 1 until a termination criterion is met.*

4. Output solution $(\boldsymbol{x}^*, \boldsymbol{Y}^*)$.

The following result is straightforward.

Theorem 9 *Assume $\boldsymbol{Y}$ is feasible in Model (29) for some $\boldsymbol{\alpha} \geq \mathbf{1}$ and $\mathbf{1}'\boldsymbol{\alpha} = M$. Then, the sequence of objectives obtained by solving Model (25) in Algorithm 2 is non-increasing.*

Like most "Big M approaches", the quality of the solution improves with larger values of M. However, M cannot be too large that it results in numerical instability of optimization problem. Although, the Big M approach does not provide the theoretical guaranteed improvement over classical approach using Bonferroni's inequality, it seems to perform very well from our numerical studies.

4 Computations studies

We consider a resource allocation problem on a network in which the demands are uncertain. The network we consider is an directed graph with node set $\mathcal{V}$, $|\mathcal{V}| = n$ and arc set $\mathcal{E}$, $|\mathcal{E}| = r$. At each node, i, $i \in \mathcal{V}$, we decide on the quantity of resource x_i to allocate, which will incur a cost of c_i per unit resource. When the demands $\tilde{d}_i$, $i \in \mathcal{V}$ are realized, resources at the nodes or from neighboring nodes are used to meet the demands. The goal is to minimize the total allocation cost subjected to a service level constraint of meeting all demands with probability at least $1 - \epsilon$. We assume that the resource at each node i can only be transshipped across to its outgoing neighboring nodes defined as

$$\mathcal{N}^-(i) \triangleq \{j : (i,j) \in \mathcal{E}\},$$

and received from its incoming neighboring nodes defined as

$$\mathcal{N}^+(i) \triangleq \{j : (j,i) \in \mathcal{E}\}.$$

Transhipment of resources received from other nodes is prohibited.

In our model, we ignore operating costs such as the transhipment costs. One of such applications is with regards to allocation of equipment such as ambulances or time critical medical supplies for emergency response to local or neighboring demands. The costs associated with their procurement is more significant than the operating cost of transhipment, which may occur rather infrequently. We list the notations of the model as follows

c_i	: Unit cost of having one resourse at node i, $i \in \mathcal{V}$;
$d_i(\tilde{\boldsymbol{z}})$	: Demand at node i, $i \in \mathcal{V}$ as a function of the primitive uncertainties $\tilde{\boldsymbol{z}}$;
x_i	: Quantity at resource at node i, $i \in \mathcal{V}$;
$w_{ij}(\tilde{\boldsymbol{z}})$	: Transshipment quantity from node i to node j, $(i,j) \in \mathcal{E}$ in respond to realization of $\tilde{\boldsymbol{z}}$.

The problem can be formulated as a joint chance constrained problem as follows,

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \mathrm{P}\left(\begin{array}{ll} x_i + \sum\limits_{j\in\mathcal{N}^+(i)} w_{ji}(\tilde{\boldsymbol{z}}) - \sum\limits_{j\in\mathcal{N}^-(i)} w_{ij}(\tilde{\boldsymbol{z}}) \geq d_i(\tilde{\boldsymbol{z}}) & i=1,\ldots,n \\ x_i \geq \sum\limits_{j\in\mathcal{N}^-(i)} w_{ij}(\tilde{\boldsymbol{z}}) & i=1,\ldots,n \\ \boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0} & \end{array}\right) \geq 1-\epsilon \\ & \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{w}(\tilde{\boldsymbol{z}}) \end{array} \tag{31}$$

We assume that the demand at each node are independently distributed and represented as

$$d_j(\tilde{\boldsymbol{z}}) = d_j^0 + \tilde{z}_j,$$

where $\tilde{z}_j$ are independent zero mean random variables with unknown distribution.

By introducing new variables, we can transform the model (31) to the "standard form" model as follows

$$\begin{array}{lll} \min & \boldsymbol{c}'\boldsymbol{x} & \\ \text{s.t.} & x_i + \sum\limits_{j\in\mathcal{N}^+(i)} w_{ji}(\tilde{\boldsymbol{z}}) - \sum\limits_{\in\mathcal{N}^-(i)} w_{ij}(\tilde{\boldsymbol{z}}) + \boldsymbol{r}(\tilde{\boldsymbol{z}}) = d_i(\tilde{\boldsymbol{z}}) & i=1,\ldots,n \\ & x_i + s_i(\tilde{\boldsymbol{z}}) = \sum\limits_{j\in\mathcal{N}^-(i)} w_{ij}(\tilde{\boldsymbol{z}}) & i=1,\ldots,n \\ & \boldsymbol{w}(\tilde{\boldsymbol{z}}) + \boldsymbol{t}(\tilde{\boldsymbol{z}}) = \boldsymbol{0} & \\ & \boldsymbol{y}(\tilde{\boldsymbol{z}}) = \begin{pmatrix} \boldsymbol{r}(\tilde{\boldsymbol{z}}) \\ \boldsymbol{s}(\tilde{\boldsymbol{z}}) \\ \boldsymbol{t}(\tilde{\boldsymbol{z}}) \end{pmatrix} & \\ & \mathrm{P}(\boldsymbol{y}(\tilde{\boldsymbol{z}}) \leq \boldsymbol{0}) \geq 1-\epsilon & \\ & \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{r}(\tilde{\boldsymbol{z}}), \boldsymbol{s}(\tilde{\boldsymbol{z}}), \boldsymbol{t}(\tilde{\boldsymbol{z}}), \boldsymbol{y}(\tilde{\boldsymbol{z}}), \boldsymbol{w}(\tilde{\boldsymbol{z}}). & \end{array} \tag{32}$$

Note that the dimension of $\boldsymbol{y}(\tilde{\boldsymbol{z}})$ is $m = 2n + r$.

The transhipment variables $\boldsymbol{w}(\tilde{\boldsymbol{z}})$ is an arbitrary function of $\tilde{\boldsymbol{z}}$. In order to obtain a bound on Problem 31, we apply the linear decision rule on the transhipment variables $\boldsymbol{w}(\tilde{\boldsymbol{z}})$ advocated in Ben-Tal eta al. [2] and Chen, Sim and Sun [13] as follows,

$$\boldsymbol{w}(\tilde{\boldsymbol{z}}) = \boldsymbol{w}^0 + \sum_{j=1}^{n} \boldsymbol{w}^j \tilde{z}_j.$$

Under the assumption of linear decision on $\boldsymbol{w}(\tilde{\boldsymbol{z}})$ and with suitable affine mapping, we have

$$\begin{aligned}
\boldsymbol{r}(\tilde{\boldsymbol{z}}) &= \boldsymbol{r}^0 + \textstyle\sum_{j=1}^{n} \boldsymbol{r}^j \tilde{z}_j \\
\boldsymbol{s}(\tilde{\boldsymbol{z}}) &= \boldsymbol{s}^0 + \textstyle\sum_{j=1}^{n} \boldsymbol{s}^j \tilde{z}_j \\
\boldsymbol{t}(\tilde{\boldsymbol{z}}) &= \boldsymbol{t}^0 + \textstyle\sum_{j=1}^{n} \boldsymbol{t}^j \tilde{z}_j \\
\boldsymbol{y}(\tilde{\boldsymbol{z}}) &= \boldsymbol{y}^0 + \textstyle\sum_{j=1}^{n} \boldsymbol{y}^j \tilde{z}_j,
\end{aligned}$$

which are affine functions with respect to the primitive uncertainty, $\tilde{\boldsymbol{z}}$. Hence, we transform the problem from one with infinite variables (optimizing over functional) to a restricted one with polynomial number of variables. Therefore, we can apply our proposed framework to obtain an approximate solutions to Problem (32).

In our test problem, we generate 15 nodes randomly positioned on a square grid and restrict to the r shortest arcs on the grid in terms of Euclidean distances. We assume $c_i = 1$. For the demand uncertainty, we assume that $d_j^0 = 10$ and the demand at each node, $d_j(\tilde{\boldsymbol{z}})$ takes value from zero to 100. Therefore, we have $\tilde{z}_j \in [-10, 90]$. Using Theorem 1, we can determine the bounds on the forward and backward deviations, which yields $p_j = 42.67$ and $q_j = 30$.

For the evaluation of bounds, we use $\mathcal{L} = \{1, 2, 4, 5\}$. We formulate Model using an in-house developed software, *PROF* (Platform for Robust Optimization Formulation). The Matlab based software is essentially an SOCP modeling environment that contains reusable functions for modeling multiperiod robust optimization using decision rules. We have implemented bounds for the CVaR measure and expected positivity of a weighted sum of random variables. The software calls upon CPLEX 10.1 to solve the underlying SOCP.

We first solve the problem using the classical approach by decomposing the joint chance constrained problem into m constraints of the form (19), with $\epsilon_i = \epsilon/(2n + r)$. We denote the optimal solution as $\boldsymbol{x}^B$ and its objective as Z^B. Subsequently, we use Algorithm 2, the big M approach, with $M = 10^6$, to improve upon the solution. We report results at the end of twenty iterations. Here, we denote the optimal solution as $\boldsymbol{x}^N$ and its objective as Z^N. We also benchmark against the worst case solution, which corresponds to all the demands at its maximum value. Hence the worse case solution is $x_i^W = 100$ for all $i \in \mathcal{W}$ and $Z^W = 1500$.

Figure 1 shows an illustration of the solution. The size of the hexagon on each location, i reflects upon the quantity x_i. Each link refers to two directed arcs in opposite directions. We present solutions in Table 1. It is interesting to note that the solutions obtained using the classical approach has significant resources needed at nodes $5, 10, 12$ and 13, which forms a complete graph with node 15. After several iterations of optimization, the new solutions centrally locates the resources at node 15, diminishing the resources needed at nodes $5, 10, 12$ and 13.

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x^B	14	61	73	100	13	213	136	112	7	161	27	8	9	61	161
x^N	18	41	77	100	1	257	82	59	15	2	11	0	0	41	337

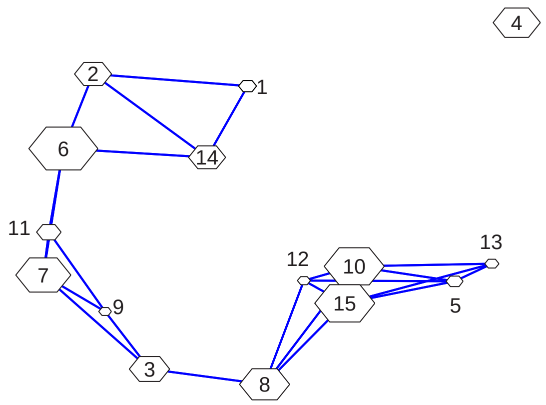
Table 1: Resource allocation: 15 nodes, 50 arcs (rounded to nearest integer)

In Table 2, we compare the relative improvement of Z^N against Z^B and Z^N against Z^W. The new method has $8 - 12\%$ improvement compared with classical approach of applying Bonferroni's inequality and has $30 - 42\%$ improvement compared with the worst case solution. We also note that the improvement generally increases over the classical approach when the number of connectivity increases. This is probably due to the increases correlation among the constraints as connectivity increases. Even though minimum distributional information are provided, this experiment shows that the new method solves the joint chance constrained problem more efficiently.

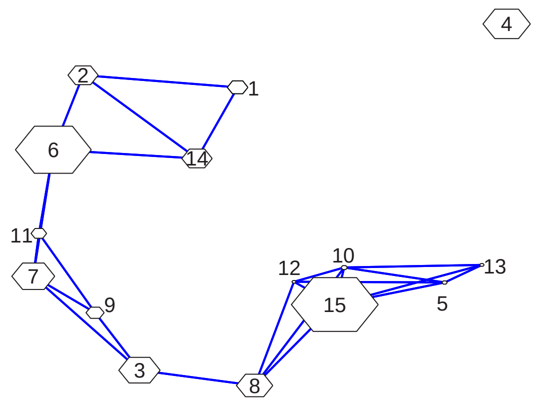
In addition, we find that the improvement increases as the number of facilities increases. Moreover, we tested the convergence rate of Algorithm 2. Figure 2 shows that the improvement is made mostly in the first several steps.

# of Nodes	# of Arcs	Z^W	Z^B	Z^N	$(Z^W - Z^N)/Z^W$	$(Z^B - Z^N)/Z^B$
15	50	1500	1158.1	1043.3	30.45%	9.91%
15	60	1500	1059.7	968.1	35.46%	8.64%
15	70	1500	1027.3	929.5	38.03%	9.52%
15	80	1500	1009.3	890.1	40.66%	11.81%
15	90	1500	989.1	865.7	42.29%	12.48%

Table 2: Comparisons among Worst case solution Z^W, Solution using Bonferroni's inequality Z^B and Solution using new approximation Z^N.



Solution using Bonferroni's inequality



Solution using New Method

Figure 1: Inventory allocation: 15 nodes, 50 arcs

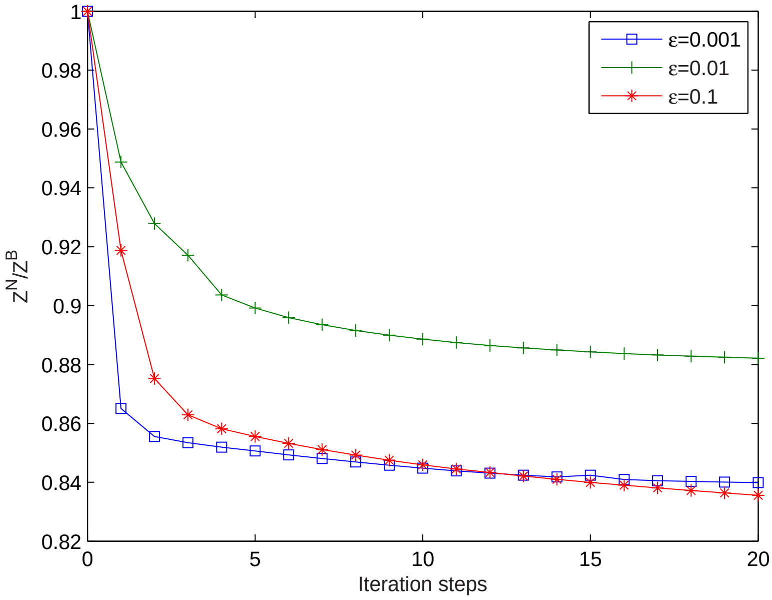


Figure 2: A sample convergence plot

5 Conclusion

In this paper, we propose a general technique to deal with joint chance constrained optimization problems. The standard approach decomposes the joint chance constraint into a problem with m individual chance constraints and then applies safe robust optimization approximation on each one of them. Our approach builds on a classical worst case bound for order statistics problem, where the bound is tight when the random variables are negatively correlated. By introducing new parameters $(\boldsymbol{\alpha}, w_0, \boldsymbol{w}, \mathcal{J})$ into the worst case bound, we enlarge the search space so that our approach can also deals with positively correlated variables, and improves upon the solution obtained using the standard approach via Bonferroni's inequality.

The quality of solution obtained using this approach depends largely on the availability of good upperbound $\pi(y_0, \boldsymbol{y})$ for the function $\mathrm{E}\left((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\right)$. As a by product of this study, we show that any such bound satisfying convexity, positively homogeneity, and with $\pi(y_0, \mathbf{0}) = y_0^+$, can be used to construct an uncertainty set to develop a robust optimization framework for (single) chance constrained problems. This provides a unified perspective on the choice of uncertainty set in the development of robust optimization methodology.

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