Constrained linear system with disturbance: stability under disturbance feedback ⋆

C. Wang [a], C.J. Ong [b,∗], Melvyn Sim [c]

[a] *Department of Mechanical Engineering, National University of Singapore*

[b] *Department of Mechanical Engineering, National University of Singapore and Singapore-MIT Alliance*

[c] *NUS Business School, National University of Singapore and Singapore-MIT Alliance*

Abstract

This paper proposes a control parametrization under Model Predictive Controller (MPC) framework for constrained linear discrete time systems with bounded additive disturbances. The proposed approach has the same feasible domain as that obtained from parametrization over the family of time-varying state feedback policies. In addition, the closed-loop system is stable in the sense that the state converges to a bounded set that has a characterization determined by a feedback gain.

Key words: Model Predictive Control; Disturbance Feedback, Constrained Systems with disturbance

1 Introduction

This paper considers constrained linear discrete-time system with additive disturbances:

$$x_{t+1} = Ax_t + Bu_t + Dw_t, \quad (1)$$

$$(x_t, u_t) \in Y,\ w_t \in W \ \text{ for all } t \geq 0 \quad (2)$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ are the state and control of the system at time t respectively, $w_t \in W \subset \mathbb{R}^p$ is the disturbance on the system at time t and Y represents the joint state and control constraints imposed on the system.

The control of such systems has been addressed in the literature. One popular approach is the Model Predictive Controller (MPC) where a finite-horizon (FH) optimization problem is solved at every time t. It is quite well known (Bemporad, 1998; Chisci, Rossiter & Zappa, 2001; Löfberg, 2003; Mayne, Rawlings, Rao & Scokaert, 2000) that the optimization is to be over families of feedback policies for stability and conservatism considera-

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∗ Corresponding author. Tel. +65-6516-2217. Fax +65-6779-1459.

Email addresses: wangchen@nus.edu.sg (C. Wang), mpeongcj@nus.edu.sg (C.J. Ong), dscsimm@nus.edu.sg (Melvyn Sim).

tions. One popular parametrization (Bemporad, 1998; Chisci et al., 2001; Rossiter, Kouvaritakis & Rice, 1998; Lee & Kouvaritakis, 1999; Mayne, Seron & Raković, 2005) of the feedback policies is $u_t = Kx_t + c_t$ where K is fixed and c_t is the new variable. An advantage of this fixed-gain parametrization is the available characterization of the asymptotic behavior of the closed-loop system. Specifically, it can be shown that the state of the closed-loop system converges to the minimal invariant set of $x_{t+1} = (A + BK)x_t + Dw_t$ (Chisci et al., 2001). However, the domain of initial states for which the FH problem admits a feasible solution is limited in size because of the somewhat restricted family of feedback policies considered.

More recent works (Löfberg, 2003; Goulart, Kerrigan & Maciejowski, 2006) consider larger families of feedback policies in an effort to reduce conservatism. The family of affine time-varying state feedback law appears promising as it includes the fixed-gain parametrization as a special case. However, direct parametrization with affine time-varying state feedback is unappealing as the resulting FH problem is not computationally tractable(Löfberg, 2003). Instead, Löfberg (2003) and van Hessem & Bosgra (2002) propose a parametrization based on disturbance feedback as a means for implementing affine time-varying state feedback law. Specifically, for a horizon of length N, the control is $u_i = \sum_{j=0}^{i-1} M_{i,j}w_j + v_i$ for $i = 0, \cdots, N-1$ where $M_{i,j}$ and $v_i, i = 0, \cdots, N-1$ are the optimization variables. This parametrization has the advantage that the corresponding FH problem is convex and admits efficient computational procedures. Recently, Goulart et al. (2006) shows the equivalence of the feasible domains using $u_i = \sum_{j=0}^{i-1} M_{i,j}w_j + v_i$ and the affine time-varying state feedback law. They also show that, under mild assumptions, the origin of the closed-loop system is input-to-state stable (ISS) under the MPC control law derived using the disturbance feedback parametrization.

This paper proposes a control parametrization based on the popular and intuitive form of $u_i = Kx_i + c_i, i = 0, \cdots, N-1$ where each c_i is an affine function of disturbances. As shown in the sequel, this parametrization admits a feasible domain that is the same as those proposed by Löfberg (2003) and Goulart et al. (2006). Together with a specific choice of the cost function, the proposed approach admits a characterization of the asymptotic behavior of the closed-loop system under mild assumptions. Consequently, the approach provides no less conservatism compared to the other two approaches with a stability result similar to those of fixed-gain parametrization.

The rest of this paper is organized as follows. This section ends with notations used, assumptions needed and a brief review of standard results. Section 2 discusses the proposed control parametrization and establishes the equivalence of its feasible domain to those using disturbance feedback. The properties and choices of objective functions are discussed in section 3. Computational issues of FH optimization problem is given in section 4. Section 5 takes on the feasibility and stability results of the closed-loop system. Numerical examples and conclusions are contents of the last two sections.

The following notations are used. $\mathbb{Z}_k$ denotes the integer set $\{0, 1, \cdots, k\}$; given a vector $x \in \mathbb{R}^n$, matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$: $A \otimes B$ is the Kronecker product of A and B; $vec(A) = \begin{bmatrix} A_1^T & \cdots & A_m^T \end{bmatrix}^T \in \mathbb{R}^{nm}$ is the stacked vector of columns of A. $A \succ (\succeq) 0$ means that square matrix A is positive definite (semi-definite). For any $A \succ 0$, $\|x\|_A^2 = x^T A x$. 1_r is a r-vector with all elements being 1, I_ℓ is the $\ell \times \ell$ identity matrix. Boldface characters are used for collections of vectors or matrices over the horizon.

The system (1)-(2) is assumed to satisfy the following assumptions: (A1) the system (A, B) is controllable, D is full column rank; (A2) the set

$$Y = \{(x, u)|\ Y_x x + Y_u u \leq 1_q\} \subset \mathbb{R}^{n+m} \qquad (3)$$

is compact; (A3) the random variable w_t is uncorrelated from instant to instant, has zero mean, covariance matrix Σ_w and

$$w_t \in W := \{w|\ Hw \leq h\} \subset \mathbb{R}^p \qquad (4)$$

where W is a bounded set, contains the origin and $h \in \mathbb{R}^r$; (A4) A constant feedback gain $K_f \in \mathbb{R}^{m\times n}$ is given such that $\Phi := A + BK_f$ has a spectral radius $\rho(\Phi) < 1$. Assumption (A1) is standard. The characterizations of Y and W in (A2) and (A3) are made out of the need for a concrete computational representation. Assumption (A3) is mild and can be satisfied by many disturbance models. Assumption (A4) is easily satisfied under (A1) and is made for convenience. Under (A1)-(A4), Kolmanovsky & Gilbert (1995), (1998) show that, for sufficiently small W satisfying (A3), a constraint-admissible maximal disturbance invariant set,

$$X_f \triangleq \{x|\ Gx \leq 1_g\}, \qquad (5)$$

exists for some integer g and is bounded. This means that $\Phi x + Dw \in X_f, (x, K_f x) \in Y$ for all $x \in X_f$ and for all $w \in W$. It is also known (Kolmanovsky & Gilbert, 1998) that the state of the system $x_{t+1} = \Phi x_t + Dw_t$ converges to the minimal disturbance invariant set, F_∞, given by $F_\infty = DW + \Phi DW + \Phi^2 DW + \cdots$. and that F_∞ is compact.

2 Control parametrization

Suppose the FH problem has a horizon length N. Let $x_{i|t}, u_{i|t}$ be the i^{th} predicted state and i^{th} predicted control respectively within the horizon at time t. The choice of $u_{i|t}$ takes the form

$$\begin{cases} u_{i|t} = K_f x_{i|t} + c_{i|t} \\ c_{i|t} = d_{i|t} + \sum_{j=1}^{N-1} C_{i|t}^{-j} w_{i-j|t} \end{cases} \qquad i \in \mathbb{Z}_{N-1} \qquad (6)$$

where $d_{i|t} \in \mathbb{R}^m,\ C_{i|t}^{-j} \in \mathbb{R}^{m\times p}$. Clearly, $c_{i|t}$ is an affine function of $w_{i-j|t}, j \in \mathbb{Z}_{N-1}$, a set of $N-1$ disturbances preceding time $t+i$. Also, $w_{i-j|t}$ is a realized disturbance if $i-j < 0$ or a future (unknown) disturbance if $i-j \geq 0$. If (6) is applied onto (1), then it can be shown that

$$x_{i|t} = \Phi^i x_0 + \sum_{j=0}^{i-1} \Phi^{i-1-j} Bc_{j|t} + \sum_{i=0}^{i-1} \Phi^{i-1-j} Dw_{j|t} \quad (7)$$

and $x_{i|t}$ depends linearly on past disturbances $w_{k|t}, k \in \mathbb{Z}_{i-1}$ following the structure of $c_{i|t}$. To simplify notations, let

$$\begin{aligned}
\mathbf{x}_t &= [x_{0|t}^T\ x_{1|t}^T \cdots\ x_{N|t}^T]^T \in \mathbb{R}^{(N+1)n},\\
\mathbf{u}_t &= [u_{0|t}^T\ u_{1|t}^T \cdots\ u_{N-1|t}^T]^T \in \mathbb{R}^{Nm},\\
\mathbf{w}_t^- &= [w_{-(N-1)|t}^T\ \cdots\ w_{-1|t}^T]^T \in \mathbf{W}^- := W \times \cdots \times W,\\
\mathbf{w}_t^+ &= [w_{0|t}^T\ w_{1|t}^T\ \cdots\ w_{(N-1)|t}^T]^T \in \mathbf{W}^+ := W \times \cdots \times W,\\
\mathbf{w}_t &= [(\mathbf{w}_t^-)^T\ (\mathbf{w}_t^+)^T]^T \in \mathbf{W}^- \times \mathbf{W}^+
\end{aligned}$$

where $\mathbf{w}_t^-$ is the collection of realized disturbances at time t and $\mathbf{w}_t^+$ is the collection of future disturbances at time t. The variable K_f is assumed to be specified and the rest of the variables in (6) can be collected in $\mathbf{C}_t \in \mathbb{R}^{Nm\times(2N-1)p}$ and $\mathbf{d}_t \in \mathbb{R}^{Nm}$ as

$$\mathbf{C}_t^- = \begin{bmatrix} C_{0|t}^{-(N-1)} & C_{0|t}^{-(N-2)} & \cdots & C_{0|t}^{-2} & C_{0|t}^{-1} \\ 0 & C_{1|t}^{-(N-1)} & \cdots & C_{1|t}^{-3} & C_{1|t}^{-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{N-2|t}^{-(N-1)} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_t^+ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ C_{1|t}^{-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{N-2|t}^{-(N-2)} & C_{N-2|t}^{-(N-3)} & \cdots & 0 & 0 \\ C_{N-1|t}^{-(N-1)} & C_{N-1|t}^{-(N-2)} & \cdots & C_{N-1|t}^{-1} & 0 \end{bmatrix},$$

$\mathbf{C}_t = [\mathbf{C}_t^- \ \mathbf{C}_t^+]$ and $\mathbf{d}_t = [d_{0|t}^T \ d_{1|t}^T \cdots d_{N-1|t}^T]^T$. Using these notations and others, the FH optimization using the control parametrization of (6), referred hereafter as $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-, K_f)$, is

$$\min_{\mathbf{d}_t, \mathbf{C}_t} \ J_N(\mathbf{d}_t, \mathbf{C}_t)$$

$$\text{s.t. } \mathbf{x}_t = \mathcal{A}x_t + \mathcal{B}\mathbf{u}_t + \mathcal{G}\mathbf{w}_t \tag{8}$$

$$\mathbf{u}_t = \mathcal{K}\mathbf{x}_t + \mathbf{d}_t + \mathbf{C}_t\mathbf{w}_t \tag{9}$$

$$(x_{i|t}, u_{i|t}) \in Y \ \ \forall \ \mathbf{w}_t^+ \in \mathbf{W}^+, i \in \mathbb{Z}_{N-1} \tag{10}$$

$$x_{N|t} \in X_f \qquad \forall \ \mathbf{w}_t^+ \in \mathbf{W}^+. \tag{11}$$

In the above, $\mathcal{K} = [I_N \otimes K_f \ \ 0]$,

$$\mathcal{A} = \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \text{ and}$$

$$\mathcal{G} = [\mathcal{G}^- \ \mathcal{G}^+] = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots & D & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & AD & D & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & A^{N-1}D & A^{N-2}D & \cdots & D \end{bmatrix}$$

where $\mathcal{G}^-$ corresponds to the submatrix of all zero in $\mathcal{G}$ and $\mathcal{G}^+$ the rest; X_f is the maximal disturbance invariant set as given in (5); $J_N(\mathbf{d}_t, \mathbf{C}_t)$ is an appropriate cost function whose details are discussed in Section III.

Clearly, the FH problem depends on design variables $(\mathbf{d}_t, \mathbf{C}_t)$, system variables $x_t, \mathbf{w}_t$ and other system parameters like A, B, D and K_f. The system parameters are assumed fixed and references to them will be omitted for notational simplicity, unless warranted by context. Let the FH feasible set be

$$\Pi_N(x_t, \mathbf{w}_t^-) = \{(\mathbf{d}_t, \mathbf{C}_t) | \mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-) \text{ is feasible.}\}$$

and the set of admissible initial states be

$$\mathcal{X}_N = \{x | \Pi_N(x, \mathbf{w}_t^-) \neq \emptyset \text{ for all } \mathbf{w}_t^- \in \mathbf{W}^-\}. \tag{12}$$

The rest of the MPC formulation is standard: $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-)$ is solved at each time t and the very first term of

$$(\mathbf{d}_t^*, \mathbf{C}_t^*) = \arg\min \ \mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-)$$

is applied to system (1). Hence, the MPC control law is

$$u_t = K_f x_t + c_t := K_f x_t + d_{0|t}^* + \sum_{j=1}^{N-1} (C_{0|t}^{-j})^* w_{0-j|t} \tag{13}$$

The parameterizations by Löfberg (2003) and Goulart et al. (2006) differs from (6) and has the form

$$\mathbf{u}_t = \mathbf{v}_t + \mathbf{M}_t\mathbf{w}_t^+ \tag{14}$$

where

$$
\mathbf{v}_t = \begin{bmatrix} v_{0|t} \\ v_{1|t} \\ \vdots \\ v_{N-1|t} \end{bmatrix}, \ \mathbf{M}_t = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ M^0_{1|t} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ M^0_{N-1|t} & \cdots & M^{N-2}_{N-1|t} & 0 \end{bmatrix}
$$

are the variables of $\mathbf{u}_t$. The connection between (9) and (14) are given in the following result.

Theorem 1 *Let $x_t \in \mathbb{R}^n$, $K_f \in \mathbb{R}^{m\times n}$ and a realization $\mathbf{w}^-_t$ be given. For any choice of $(\mathbf{M}_t, \mathbf{v}_t)$ that defines $\mathbf{u}_t$ in (14) and the corresponding $\mathbf{x}_t$ in (8), there exists a $(\mathbf{d}_t, \mathbf{C}_t)$ in (9) which yields the same sequences $\mathbf{u}_t$ and $\mathbf{x}_t$ and vice versa.*

Proof. For notational simplicity, the subscript t is dropped from all variables and x_t is denoted, without loss of generality, as x_0. Equations (8) and (9) can be rearranged as

$$
\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} I & -\mathcal{B} \\ -\mathcal{K} & I \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{A}x_0 + \mathcal{G}\mathbf{w} \\ \mathbf{d} + \mathbf{C}\mathbf{w} \end{bmatrix}
$$

$$
:= \begin{bmatrix} \mathcal{A}_x x_0 + \mathcal{B}_x \mathbf{d} + \mathcal{B}_x \mathbf{C}^- \mathbf{w}^- + (\mathcal{B}_x \mathbf{C}^+ + \mathcal{G}^+_x)\mathbf{w}^+ \\ \mathcal{A}_u x_0 + \mathcal{B}_u \mathbf{d} + \mathcal{B}_u \mathbf{C}^- \mathbf{w}^- + (\mathcal{B}_u \mathbf{C}^+ + \mathcal{G}^+_u)\mathbf{w}^+ \end{bmatrix}. \quad (15)
$$

The inverse matrix in (15) exists since $\varphi := (I - \mathcal{B}\mathcal{K})^{-1}$ exists due to the block lower triangular structure of $\mathcal{B}$ and $\mathcal{K}$ (see Appendix A for expressions of φ, $\mathcal{A}_x$, $\mathcal{A}_u$, $\mathcal{B}_x$, $\mathcal{B}_u$, $\mathcal{G}^+_x$ and $\mathcal{G}^+_u$). Comparing the expressions of $\mathbf{u}$ of (15) with (14), it follows that

$$
\mathcal{B}_u(\mathbf{d} + \mathbf{C}^-\mathbf{w}^-) = \mathbf{v} - \mathcal{A}_u x_0 \quad (16)
$$

$$
\mathcal{B}_u \mathbf{C}^+ = \mathbf{M} - \mathcal{G}^+_u. \quad (17)
$$

In the above, $\mathbf{C}^+$, $\mathcal{B}_u$, $\mathbf{M}$ and $\mathcal{G}^+_u$ are block lower triangular matrices (see Appendix A) and, hence, $(\mathbf{M}, \mathbf{v})$ can be expressed in terms of $(\mathbf{d}, \mathbf{C})$. To show that $(\mathbf{d}, \mathbf{C})$ can be expressed in terms of $(\mathbf{M}, \mathbf{v})$, note that $\mathcal{B}_u$ is invertible since $\mathcal{B}_u^{-1} = I - \mathcal{K}(\varphi^{-1} + \mathcal{B}\mathcal{K})^{-1}\mathcal{B} = I - \mathcal{K}\mathcal{B}$. Then $\mathbf{C}^+ = (\mathcal{B}_u)^{-1}(\mathbf{M} - \mathcal{G}^+_u)$ and $\mathbf{d} = (\mathcal{B}_u)^{-1}(\mathbf{v} - \mathcal{A}_u x_0) - \mathbf{C}^-\mathbf{w}^-$ for any choice of $\mathbf{C}^-$. □

The above result is not entirely unexpected since (9), except for the $\mathcal{K}$ term, subsume (14). A less obvious result regarding properties of $\mathcal{X}_N$ is stated next.

Lemma 2 *For any pair of $(\hat{\mathbf{w}}^-_t, \tilde{\mathbf{w}}^-_t) \in \mathbf{W}^- \times \mathbf{W}^-$, $\Pi_N(x, \tilde{\mathbf{w}}^-_t) \neq \emptyset$ implies $\Pi_N(x, \hat{\mathbf{w}}^-_t) \neq \emptyset$ and vice versa.*

Proof. $(\Rightarrow)\Pi_N(x, \tilde{\mathbf{w}}^-_t) \neq \emptyset$ implies that there exists $(\tilde{\mathbf{d}}, \tilde{\mathbf{C}}^-, \tilde{\mathbf{C}}^+)$ feasible to $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}^-_t)$. This also means that there exists $(\tilde{\mathbf{v}}, \tilde{\mathbf{M}})$ such that (16)-(17) hold for $(\tilde{\mathbf{d}}, \tilde{\mathbf{C}}^-, \tilde{\mathbf{C}}^+)$. Let $\hat{\mathbf{C}}^- = 0$, $\hat{\mathbf{d}} = \mathcal{B}_u^{-1}(\tilde{\mathbf{v}} - \mathcal{A}x)$ and $\hat{\mathbf{C}}^+ = \mathcal{B}_u^{-1}(\tilde{\mathbf{M}} - \mathcal{G}^+_u)$ and $(\hat{\mathbf{d}}, \hat{\mathbf{C}}^-, \hat{\mathbf{C}}^+)$ is feasible to $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}^-_t)$ following Theorem 1. $(\Leftarrow)$ Obvious by the symmetry of $(\hat{\mathbf{w}}^-_t, \tilde{\mathbf{w}}^-_t)$. □

An immediate consequence of Lemma 2 is that $\mathcal{X}_N$ of (12) can be equivalently stated as

$$
\mathcal{X}_N \triangleq \{x | \exists \mathbf{w}^-_t \in \mathbf{W}^- \text{such that } \Pi_N(x, \mathbf{w}^-_t) \neq \varnothing\}. \quad (18)
$$

Remark 3 *Suppose the FH optimization problem has (9) replaced by (14) and $\mathcal{X}^M_N$ is the corresponding set of admissible initial states. The above result means that $\mathcal{X}_N = \mathcal{X}^M_N$.*

Remark 4 *Characterization (18) allows a simple verification of the condition $x_0 \in \mathcal{X}_N$. For any $\mathbf{w}^-_t \in \mathbf{W}^-$, $x_0 \in \mathcal{X}_N$ if and only if $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}^-_t)$ admits a feasible solution.*

Remark 5 *As $\mathcal{X}_N = \mathcal{X}_N^M$, it may appear that the variable $\mathbf{C}^-$ is superfluous. Its inclusion is needed in ensuring stability of the closed-loop system and will become obvious in sections 3 and 5.*

3 Choice of objective function

The cost function of $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-)$ is

$$J_N(\mathbf{d}_t, \mathbf{C}_t) := \sum_{i=0}^{N-1} J_{i|t}(c_{i|t}) := \sum_{i=0}^{N-1} \left[\|d_{i|t}\|_\Psi^2 + \sum_{j=1}^{N-1} \|vec(C_{i|t}^{-j})\|_\Lambda^2 \right] \quad (19)$$

for some $\Psi \succ 0$ and $\Lambda \succ 0$. This choice of cost function is motivated from consideration of the standard linear quadratic (LQ) cost and hence preserve the use of the 2 norm. It is possible to show, with additional notations, that the results of Theorem 7 remain true if the $1, \infty$ norm or any norm function of $c_{i|t}$ is used for $J_{i|t}$ in (19). The choices of Ψ and Λ can be arbitrary so long as they are positive definite. Insights to their selections can be made by considering the following treatment of standard LQ cost. Consider the expected LQ cost of

$$V_N(x_t, \mathbf{u}_t) = \mathrm{E}_{\mathbf{w}_t}[\sum_{i=0}^{N-1}(\|x_{i|t}\|_Q^2 + \|u_{i|t}\|_R^2) + \|x_{N|t}\|_P^2] \quad (20)$$

where $(\|x_{i|t}\|_Q^2 + \|u_{i|t}\|_R^2)$ and $\|x_{N|t}\|_P^2$ are the stage and terminal costs respectively. $Q \succeq 0$ and $R \succ 0$ are standard weight matrices for state and control deviations respectively and $P \succ 0$ is the unique solution of $P = A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA + Q$, the algebraic Riccati equation. For this choice of P, it is well known that the term $\sum_{i=0}^{N-1}(\|x_{i|t}\|_Q^2 + \|u_{i|t}\|_R^2) + \|x_{N|t}\|_P^2 = \sum_{i=0}^{\infty}(\|x_{i|t}\|_Q^2 + \|u_{i|t}\|_R^2)$ and the meaning of V_N is clear: it is the expected cost-to-infinity, measured by Q, R matrices, at the current time. To see the connection to J_N, let $\hat{K}_f = -(R + B^T PB)^{-1} B^T PA$ be the infinite-horizon optimal feedback gain for the choices of Q and R. With these choices, theorem 11.2 of (Åström & Wittenmark, 1997) shows that

$$\begin{aligned}
&\sum_{i=0}^{N-1}(x_{i|t}^T Q x_{i|t} + u_{i|t}^T R u_{i|t}) + x_{N|t}^T P x_{N|t} \\
&= x_{0|t}^T P x_{0|t} + \sum_{i=0}^{N-1} \|u_{i|t} - \hat{K}_f x_{i|t}\|_{(R+B^T PB)}^2 + \sum_{i=0}^{N-1} w_{i|t}^T P w_{i|t} \\
&+ \sum_{i=0}^{N-1} \left[(Ax_{i|t} + Bu_{i|t})^T P w_{i|t} + w_{i|t}^T P (Ax_{i|t} + Bu_{i|t}) \right]. \quad (21)
\end{aligned}$$

Following (6) and (7), the terms $x_{i|t}$ and $u_{i|t}$ on the right hand side of the preceding equation are linear functions of past disturbances $w_{j|t}, j < i$. Taking the expected value, the last term on the right hand side of (21) vanishes following assumption (A3). In addition, the first and third terms of the right hand side are constants. This yields

$$V_N(\mathbf{x}_t, \mathbf{u}_t) = x_{0|t}^T P x_{0|t} + N trace(\Sigma_w P) + \mathrm{E}_{\mathbf{w}_t}(\sum_{i=0}^{N-1} c_{i|t}^T \psi c_{i|t}) \quad (22)$$

where $\psi = R + B^T PB$. The last term of (22) is exactly $J(\mathbf{d}_t, \mathbf{C}_t)$ of (19) if $\Psi = \psi$ and $\Lambda = \Sigma_w \otimes \psi$. This can be seen from

$$\begin{aligned}
&\sum_{i=0}^{N-1} \mathrm{E}_{\mathbf{w}_t} \left[(d_{i|t} + \sum_{j=1}^{N-1} C_{i|t}^{-j} w_{i-j|t})^T \psi (d_{i|t} + \sum_{j=1}^{N-1} C_{i|t}^{-j} w_{i-j|t}) \right] \\
&= \sum_{i=0}^{N-1} \left[(d_{i|t})^T \psi d_{i|t} + \sum_{j=1}^{N-1} trace \left[\Sigma_w ((C_{i|t}^{-j})^T \psi C_{i|t}^{-j}) \right] \right] \\
&= \sum_{i=0}^{N-1} \left[(d_{i|t})^T \psi d_{i|t} + \sum_{j=1}^{N-1} vec(C_{i|t}^{-j})^T (\Sigma_w \otimes \psi) vec(C_{i|t}^{-j}) \right]. (23)
\end{aligned}$$

The last line results from $\mathrm{E}\left[w^T X w\right] = trace(X\Sigma_w) = trace(\Sigma_w X) = vec(X^T)^T vec(\Sigma_w)$ and $vec(AXB) = (B^T \otimes A) vec(X)$. ψ is positive definite since R is positive definite and P is positive definite. $\Sigma_w \otimes \psi$ is positive (semi)definite since Kronecker product of two positive (semi)definite matrices is also positive (semi)definite (Theorem 4.2.12 of Horn & Johnson (1991)).

As the first two terms of (22) are independent of $(\mathbf{d}_t, \mathbf{C}_t)$, the minimization of $J_N(\mathbf{d}_t, \mathbf{C}_t)$ has the same effect on the system as the minimization of $V_N(x_t, \mathbf{u}_t)$ over $(\mathbf{d}_t, \mathbf{C}_t)$, or, more succinctly, the same effect as the minimization of the expected cost-to-infinity.

4 Computation of FH problem

Inequalities (10) and (11) can be restated, using characterization of Y and X_f in (3) and (5) respectively, as

$$Y_x x_{i|t} + Y_u u_{i|t} \leq 1_q, \forall i = 0, \cdots, N-1$$
$$G x_{N|t} \leq 1_g$$

Using expressions of $\mathbf{x}_t$ and $\mathbf{u}_t$ from (15), the above can be written as

$$\bar{\mathcal{A}} x_t + \bar{\mathcal{B}} \mathbf{d}_t + \bar{\mathcal{F}} vec(\mathbf{C}_t^-) + \max_{\mathbf{w}_t^+ \in \mathbf{W}^+} [\bar{\mathcal{B}} \mathbf{C}_t^+ + \bar{\mathcal{G}}^+] \mathbf{w}_t^+ \leq 1_s \quad (24)$$

where $s = N \cdot q + g$ and expressions of $\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{F}}$, $\bar{\mathcal{G}}^+$ are given in Appendix B. As every row of the above matrix inequality must hold for all $\mathbf{w}_t^+ \in \mathbf{W}^+$, the max operator is meant to be taken element-wise. If W is characterized by (4), $\mathbf{W}^+ = \{\mathbf{w} | \bar{H}\mathbf{w} \leq \bar{h}\}$ with $\bar{H} = I_N \otimes H$ and $\bar{h} = 1_N \otimes h$ where $\bar{h} \in \mathbb{R}^\ell, \ell = N \cdot r$. Let μ_i be the i^{th} row of $(\bar{\mathcal{B}}\mathbf{C}_t^+ + \bar{\mathcal{G}}^+)$ and consider the linear program (LP)

$$\max_{\mathbf{w}_t^+} \{\mu_i \mathbf{w}_t^+ | \bar{H}\mathbf{w}_t^+ \leq \bar{h}\},$$

and its dual problem

$$\min_{\mathbf{z}_i} \{\bar{h}^T \mathbf{z}_i | \bar{H}^T \mathbf{z}_i = \mu_i^T, \mathbf{z}_i \geq 0\}$$

with dual variable $\mathbf{z}_i \in \mathbb{R}^\ell$. It is well known that $\mu_i \mathbf{w}_t^+ \geq \bar{h}^T \mathbf{z}_i$ for any feasible $\mathbf{w}_t^+$ and $\mathbf{z}_i$ and

$$\max_{\mathbf{w}_t^+} \{\mu_i \mathbf{w}_t^+ | \bar{H}\mathbf{w}_t^+ \leq \bar{h}\} = \min_{\mathbf{z}_i} \{\bar{h}^T \mathbf{z}_i | \bar{H}^T \mathbf{z}_i = \mu_i^T, \mathbf{z}_i \geq 0\}.$$

Let $\mathbf{Z} = [\mathbf{z}_1 \cdots \mathbf{z}_s] \in \mathbb{R}^{\ell \times s}$ be matrix of the dual variables. Constraint (24) can therefore be stated equivalently as a set of linear inequalities in $\mathbf{d}_t, \mathbf{C}_t$ and $\mathbf{Z}$ as

$$\begin{aligned} &\bar{\mathcal{A}} x_t + \bar{\mathcal{B}} \mathbf{d}_t + \bar{\mathcal{F}} vec(\mathbf{C}_t^-) + \mathbf{Z}^T \bar{h} \leq 1_s, \\ &\quad \mathbf{Z}^T \bar{H} = \bar{\mathcal{B}} \mathbf{C}_t^+ + \bar{\mathcal{G}}^+, \\ &\qquad \mathbf{z}_i \geq 0, \ i = 1, \cdots, s \end{aligned} \quad (25)$$

With these results, the computation of the solution to $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-)$ corresponds to a convex problem with quadratic cost function (19) and linear constraints (25) in $\mathbf{d}_t, \mathbf{C}_t$ and $\mathbf{Z}$.

5 Feasibility and Stability

This section deals with the existence of feasible solution of $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-)$ at various t followed by the stability of the overall closed-loop system under the feedback law (13).

Theorem 6 *If $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-)$ has an optimal solution, so does $\mathcal{P}_N(\mathbf{d}_{t+1}, \mathbf{C}_{t+1}; x_{t+1}, \mathbf{w}_{t+1}^-)$ under the feedback law (13).*

Proof. The proof follows standard argument. Let $(\mathbf{d}_t^*, \mathbf{C}_t^*)$ be the optimal control of $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_t, \mathbf{w}_t^-)$ at time t and denote the optimal $u_{i|t}$ in (6) by its parameters $c_{i|t}^*$, i.e., $\mathbf{u}_t(\mathbf{d}_t^*, \mathbf{C}_t^*)$ is represented by $[c_{0|t}^*, \cdots, c_{N-1|t}^*]$. Let

$$\hat{\mathbf{u}}_{t+1}(\hat{\mathbf{d}}_{t+1}, \hat{\mathbf{C}}_{t+1}) := \left[c_{1|t}^*, \cdots, c_{N-1|t}^*, 0\right] \quad (26)$$

and it is feasible to $\mathcal{P}_N(\mathbf{d}_{t+1}, \mathbf{C}_{t+1}; x_{t+1}, \mathbf{w}_{t+1}^-)$ following the disturbance invariant property of X_f. Since $J_N(\mathbf{d}_t, \mathbf{C}_t)$ is continuous, coercive and convex, it follows from Weirestrass theorem that the optimal $(\mathbf{d}_{t+1}, \mathbf{C}_{t+1})$ exists at time $t+1$. □

Theorem 7 *Suppose $x_0 \in \mathcal{X}_N$ and assumptions (A1-A4) are satisfied. System (1)-(2) under MPC* *control law (13) has the following properties: (i)* $(x_t, u_t) \in Y$ *for all* $t \geq 0$, *(ii)* $\lim_{t\to\infty} c_t = 0$ *element-wise, (iii)* $x_t \to F_\infty$ as $t \to \infty$, *(iv)* $\lim_{t\to\infty} \mathrm{E}\left[x_t x_t^T\right] = \Sigma_\infty$, *where* $\Sigma_\infty = \Sigma_\infty^T$, $\Phi\Sigma_\infty\Phi^T = \Sigma_\infty - D\Sigma_w D^T$ *and* $\Phi = A + BK_f$.

Proof. (i) Suppose $\mathbf{w}_t^- \in \mathbf{W}^-$ has been realized and $x_0 \in \mathcal{X}_N$. It follows from Lemma 2 and Remark 4 that there exists a feasible solution to $\mathcal{P}_N(\mathbf{d}_t, \mathbf{C}_t; x_0, \mathbf{w}_t^-)$, or an optimal solution since $\mathcal{P}_N$ is convex in $(\mathbf{d}_t, \mathbf{C}_t)$. With this, the stated result follows from Theorem 6.

(ii) Let $J_t^* := J_N(\mathbf{d}_t^*, \mathbf{C}_t^*)$ be the optimal J at time t and $\hat{J}_{t+1} := J_N(\hat{\mathbf{u}}_{t+1}(\hat{\mathbf{d}}_{t+1}, \hat{\mathbf{C}}_{t+1}))$ where $\hat{\mathbf{u}}_{t+1}$ is given by (26). Then the following inequality holds true

$$J_t^* - J_{t+1}^* \geq J_t^* - \hat{J}_{t+1} = J_{0|t}^* \geq 0 \;\forall\, t \qquad (27)$$

where $J_{i|t}$ is as defined in (19). Hence, $\{J_t^*\}$ is a monotonic non-increasing sequence and is bounded from below by zero. This means that $J_\infty := \lim_{t\to\infty} J_t^* \geq 0$ exists. Repeating (27) for t from 0 to ∞ and summing them up, it follows that

$$\infty > J_0^* - J_\infty \geq \sum_{t=0}^{\infty} J_{0|t}^* \geq 0$$

which implies that $\lim_{t\to\infty} J_{0|t}^* = 0$. Since Ψ and Λ are positive definite, this implies that $\lim_{t\to\infty} d_{0|t} = 0$ and $\lim_{t\to\infty} C_{0|t}^{-j} = 0 \;\forall\, j = 1, \cdots, N-1$ and the stated result follows.

(iii) The system state under (13) is

$$x_t = \Phi^t x_0 + \sum_{i=0}^{t-1} \Phi^{t-1-i} B c_i + \sum_{i=0}^{t-1} \Phi^{t-1-i} D w_i.$$

The first term on the right approaches zero as $t \to \infty$ since $\rho(\Phi) < 1$ and the second term approaches zero following property (ii). The last term corresponds to a point in the set $F_t := DW + \cdots + \Phi^{t-1}DW$, which approaches F_∞ as $t \to \infty$. Hence the stated result follows.

(iv) Let $x_\infty := \Sigma_{i=0}^{\infty} \Phi^i D w_i$. Then $E(x_\infty) = 0$ and

$$E(x_\infty x_\infty^T) = \Sigma_\infty = D\Sigma_w D^T + \Phi D \Sigma_w D^T \Phi^T + \cdots \qquad (28)$$

following the assumptions that $E(w_i) = 0$ and $E(w_i w_j^T) = E(w_i)E(w_j)^T = 0$, $i \neq j$ in (A3). By pre- and post- multiplications of Φ and Φ^T of (28) respectively, it is easy to see that Σ_∞ satisfy $\Phi\Sigma_\infty\Phi^T = \Sigma_\infty - D\Sigma_w D^T$ and $\Sigma_\infty = \Sigma_\infty^T$. $\square$

6 Example and discussion

The performance of the proposed MPC control law is illustrated on an example system. The system considered is that used in Chisci et al. (2001) with system parameters and constraints given by:

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1.9992 & -0.2629 \\ -0.2629 & 1.0859 \end{bmatrix}$$

$$K_f = \begin{bmatrix} -0.74343 & -1.0922 \end{bmatrix}, \quad R = 0.01,$$

$$Y = \{(x, u) |\; -1 \leq u \leq 1, x \in \mathbb{R}^2\},$$

$$W = \{w |\; \|w\|_\infty \leq 0.12\}, \quad \Sigma_w = diag[0.048\; 0.048].$$

Terminal set X_f is chosen to be the maximal constraint-admissible disturbance invariant set of (1) under $u = K_f x$. Following the development in Section 3, $\Psi = \psi = 2.5593$ and $\Lambda = \Sigma_w \otimes \psi = diag[0.1228\; 0.1228]$. Comparison with the proposed approach is also made with the approaches of Chisci et al. (2001) and Goulart et al. (2006). The performances of all 3 approaches for the case where N is 9 and $x_0 = [-6.9\;\; 2.3]'$ are shown. Figures 1 and 2 show the state and control trajectories respectively. In Fig. 1, X_f and an outer bound of F_∞, $\hat{F}_\infty$ are shown. The outer bound $\hat{F}_\infty$ is used because the exact F_∞ is not computable. The procedure for computing $\hat{F}_\infty$ follows that given in (Ong & Gilbert, 2006). It is clear from these figures that x_t stays within $\hat{F}_\infty$ as $t \to \infty$ and that the constraints are satisfied at all time.

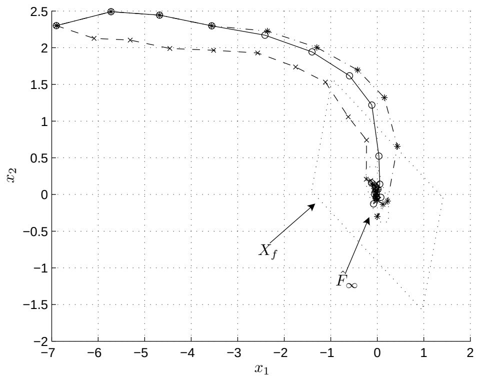

Fig. 1. State trajectories of 3 approaches. ∘ - the proposed approach, × - Chisci et. al. and ⋆ - Goulart et.al.

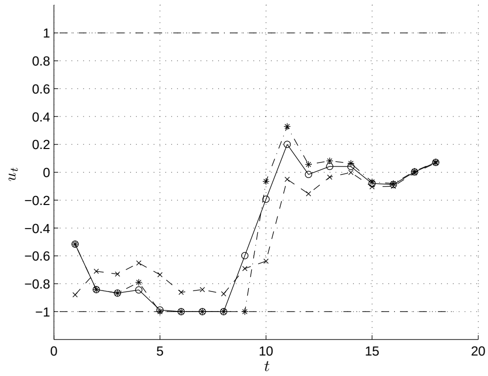

Fig. 2. Control trajectories of 3 approaches. ∘ - the proposed approach, × - Chisci et. al. and ⋆ - Goulart et.al.

Since the three cost functions are attempts to minimize the cost-to-infinity in the LQ sense, it is of interest to compare the total LQ cost for these three approaches. In Chisci et al. (2001), the cost function used (in the terminology of this paper) is $\sum_{i=0}^{N-1} d_{i|t}^T \Psi d_{i|t}$ and Goulart et al. (2006) uses the nominal LQ cost, $\sum_{i=0}^{N-1}(\|\tilde{x}_{i|t}\|_Q^2 + \|\tilde{u}_{i|t}\|_R^2) + \|\tilde{x}_{N|t}\|_P$ where $\tilde{x}_{i|t}$ and $\tilde{u}_{i|t}$ are the nominal state and control respectively without disturbance consideration. For this purpose, Fig. 3 shows the total cost against time. The total cost at the current time is sum of the cost from $i = 0$ till $i = t-1$ and the cost-to-infinity, i.e., $L(x(t); x_0) := \sum_{i=0}^{t-1}(\|x_i\|_Q^2 + \|u_i\|_R^2) + \sum_t^{\infty}(\|x_i\|_Q^2 + \|u_i\|_R^2)$. As the cost-to-infinity depends on the disturbance realization, it is approximated by $L_t(x_0) = \sum_{i=0}^{t-1}(\|x_i\|_Q^2 + \|u_i\|_R^2) + \|x_t\|_P^2$. It is worth noting that Fig. 3 is typical for other starting points and disturbance realizations.

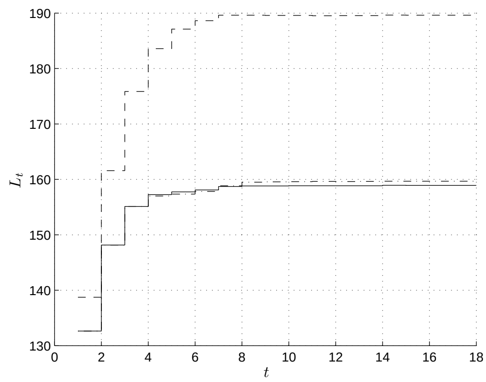

Fig. 3. The total LQ cost for x_0 for the 3 approaches. Solid line - the proposed approach, dash line - Chisci et. al. and dash dot line - Goulart et.al.

The next experiment shows the sizes of $\mathcal{X}_N$ for the proposed approach and that of (Chisci et al., 2001). The intention is to show the differences between the parametrization of (6) and the parametrization of $u_t = K_f x + c_t$ where c_t is a direct variable of the FH problem, exemplified by the work of Chisci et al. (2001) and others. For comparison purpose, let $\mathcal{Q}(X_f) = \{x | \exists u, (x,u) \in Y s.t. Ax + Bu + Dw \in X_f \forall w \in W\}$, the set of states that can be brought into X_f in one step and $\mathcal{Q}_r = \mathcal{Q} \cdots \mathcal{Q}(\mathcal{Q}(X_f))$, the r-times repeated application of $\mathcal{Q}(X_f)$. In general, the computation of $\mathcal{X}_N$ is expensive. An estimate of it can be obtained by checking over a grid of points in the x space according to Remark 4. Fig. 4 shows 3 sets: $\hat{\mathcal{X}}_9$, the approximation of $\mathcal{X}_9$; $\mathcal{X}_9^R$, the feasible domain of (Chisci et al., 2001);

and $\mathcal{Q}_9$, the 9-step $\mathcal{Q}$ set. As shown from the figure, $\mathcal{X}_9$ is almost indistinguishable from $\mathcal{Q}_9$ but is appreciably larger than $\mathcal{X}_9^R$.

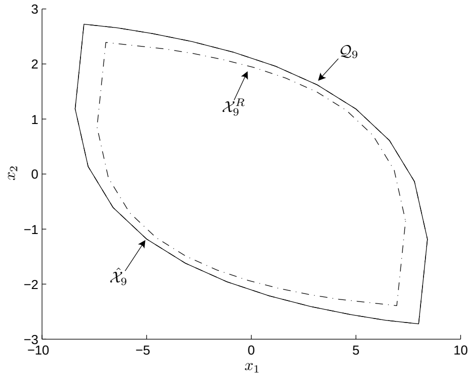

Fig. 4. Comparison of admissible sets

7 Conclusions

A new parametrization is proposed for MPC of constrained linear systems with disturbances. This parametrization has the same feasible domain as that achieved by parametrization using affine time-varying state feedback law but admits a stronger stability result - the closed-loop system state converges to the minimal robust invariant set F_∞ asymptotically. This is achieved using a cost function motivated from the expectation of standard LQ cost.

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A Appendix

$$\varphi = (I - \mathcal{B}\mathcal{K})^{-1} = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ BK_f & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi^{N-2}BK_f & \Phi^{N-3}BK_f & \cdots & I & 0 \\ \Phi^{N-1}BK_f & \Phi^{N-2}BK_f & \cdots & BK_f & I \end{bmatrix}.$$

$$\begin{bmatrix} I & -\mathcal{B} \\ -\mathcal{K} & I \end{bmatrix}^{-1} = \begin{bmatrix} \varphi & \varphi\mathcal{B} \\ \mathcal{K}\varphi & \mathcal{K}\varphi\mathcal{B} + I \end{bmatrix}.$$

$$\mathcal{A}_x = \varphi\mathcal{A}, \quad \mathcal{B}_x = \varphi\mathcal{B}, \quad \mathcal{G}_x^+ = \varphi\mathcal{G}^+,$$

$$\mathcal{A}_u = \mathcal{K}\varphi\mathcal{A}, \quad \mathcal{B}_u = I + \mathcal{K}\varphi\mathcal{B}, \quad \mathcal{G}_u^+ = \mathcal{K}\varphi\mathcal{G}^+.$$

and they are used in the expression of (15).

B Appendix

Let $\bar{Y}_x = I_N \otimes Y_x, \bar{Y}_u = I_N \otimes Y_u, \mathcal{Y} = \begin{bmatrix} \bar{Y}_x & 0 & \bar{Y}_u \\ 0 & G & 0 \end{bmatrix}$.

Then

$$\bar{\mathcal{A}} = \mathcal{Y} \begin{bmatrix} \mathcal{A}_x \\ \mathcal{A}_u \end{bmatrix}, \quad \bar{\mathcal{B}} = \mathcal{Y} \begin{bmatrix} \mathcal{B}_x \\ \mathcal{B}_u \end{bmatrix}$$

$$\bar{\mathcal{G}}^+ = \mathcal{Y} \begin{bmatrix} \mathcal{G}_x^+ \\ \mathcal{G}_u^+ \end{bmatrix}, \quad \bar{\mathcal{F}} = \mathcal{Y}((\mathbf{w}_t^-)^T \otimes \begin{bmatrix} \mathcal{B}_x \\ \mathcal{B}_u \end{bmatrix})$$

where $\mathcal{A}_x$, $\mathcal{A}_u$, $\mathcal{B}_x$, $\mathcal{B}_u$, $\mathcal{G}_x^+$ and $\mathcal{G}_u^+$ are defined in Appendix A.