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Abstract. This paper is devoted to optimality conditions for nonsmooth quasiconcave programming. Arrow and Enthoven [2] formulate several economic problems into quasiconcave programming, and give a sufficient condition for smooth quasiconcave programming in their epoch-making and comprehensive paper. In this paper, generalized necessary and sufficient conditions for nonsmooth quasiconcave programming have been derived in terms of the Clarke subdifferential [10]. A Slater-type constraint qualification is introduced to derive the generalized Kuhn-Tucker necessary condition effectively. An application to the Rawsian social welfare function is discussed.

Keywords: Arrow-Enthoven's sufficient optimality theorem, quasiconcave programming, locally Lipschitz, the Rawlsian social welfare function

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1 Introduction

Arrow and Enthoven [2] paid attention to the importance of quasiconcave programming, and derived its theory and applications early in 1961. Indeed, the quasiconcavity of the consumer's utility function is equal to the convexity of the indifference curves of the ordinary utility function to the origin, and the quasiconcavity of the production functions allows increasing returns to scale. These observations should suffice to motivate a study of quasiconcave programming in the framework of economic theory (Intriligator [16]).

Quasiconcave functions are originally defined by employing their level sets by De Finetti and intrinsically characterized by Fenchel [12] in the initial stage. Several researchers (see, e.g., Arrow and Enthoven [2], Kannai [17], Diewert, Avriel and Zang [11], Avriel, Diewert, Schaible and Zang [6], Tanaka [29], Mangasarian [20], Penot and Sach [24]) have introduced quasiconcavity or generalized concavity to extend the notion of concavity which plays a central role in both theoretical and practical aspects in economics and other disciplines, since then. Recently, notions of quasi-differentials (Martínez-Legaz and Sach [21]) have been proposed in quasiconcave analysis, aimed to exploit quasiconcave properties for unconstrained optimization.

In the meantime, nonsmooth optimality conditions have been considered by many researchers (see, e.g., Clarke [10], Hiriart-Urruty [13], Hiriart-Urruty [14], Lemaréchal [19], Bonnans and Shapiro [8]) to extend the scope of Kuhn-Tucker conditions, though these results seem to be somewhat too general or restrictive.

Locally Lipschitz functions are a natural generalization of convex or smooth functions (Clarke [9], Clarke [10]), and they have some favorable properties in both theoretical (Clarke [10], Wang [31]) and practical (Xu [32]) aspects. Nonsmooth analysis for locally Lipschitz functions has been established by Clarke ([9], [10]) based on subgradient, of which results are employed in economic analysis by Krishna and Maenner [18], and so on.

In this paper, we are concerned with optimality conditions for nonsmooth quasiconcave programming, that was firstly introduced by Arrow et al. [2] for smooth case, which can be found in textbooks (see, e.g., Takayama [28], Beavis and Dobbs [7], Mas-Colell, Whinston and Green [22]) for graduate students or professionals. Though several studies have been made on nonsmooth quasiconcave programming (see, e.g., Plastria [25],Thach [30], Anónio et al. [1], and Martínez-Legaz and Sach [21]), only a few attempts to deal with it in the framework of Clarke [10]. We aim at deriving optimality conditions for quasiconcave programming when the class of functions is extended to locally Lipschitz continuous functions in terms of the

Clarke subdifferential [10].

The following notation is used throughout this paper. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Let the generalized directional derivative be defined by

$$f^\circ(x; d) = \limsup_{y \to x, t \downarrow 0} [f(y + td) - f(y)]/t, \tag{1}$$

and if furthermore f is Gâteaux differentiable for $\forall x$ and $d \in \mathbb{R}^n$, the directional derivative be defined by

$$f'(x; d) = \lim_{t \downarrow 0} [f(x + td) - f(x)]/t.$$

We remark that

$$f^\circ(x; d) = \max\{\langle \xi, d\rangle \mid \xi \in \partial f(x)\}, \quad \text{for } \forall x \text{ and } d, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ is an inner product, $\partial f(\cdot)$ denotes the Clarke subdifferential. It is noted that the Clarke-Rockafellar subdifferential (Clarke [10], Aussel, Corvellec, and Lassonde [5]) coincides the Clarke subdifferential if f is locally Lipschitz continuous (Clarke [10]), and that there exists the usual one-sided directional derivative $f'(x; d)$ and $f^\circ(x; d) = f'(x; d)$ when f is regular at x (Clarke [10]), though it is not generally postulated in this paper.

2 Optimality conditions for quasiconcave programming problems

We first give definitions of quasiconcavity and pseudoconcavity.

Definition 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *quasiconcave* on X, if

$$f(z) \geq \min[f(x),\ f(y)], \qquad \forall x, y \in X \text{ and } \forall z \in [x,\ y].$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *pseudoconcave* on X if $\langle \xi, x - x^0 \rangle \leq 0$, for any $x, x^0 \in X$, $\xi \in \partial f(x^0)$, then $f(x) \leq f(x^0)$ holds.

Remark. The pseudoconcave function defined above is more general than semiconcave (semiconvex) function in Mifflin [23] since the latter postulates the regularity (quasidifferentiability).

We are concerned with the quasiconcave programming problem:

$$\text{(P)} \qquad \begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && g_i(x) \geq 0, \quad i \in I \equiv \{1, ..., m\}, \\ &&& x \geq 0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$, are locally Lipschitz continuous functions which are quasiconcave.

Before proceeding further, we would like to mention the following results.

Lemma 1 *Suppose that $f : X \to \mathbb{R}$ is locally Lipschitz continuous. Then, the following statements are equivalent:*

(i) f is quasiconcave on a Banach space X;

(ii) $\exists a \in \partial f(x)$, $\langle a, y - x \rangle < 0 \Longrightarrow f(z) \geq f(y)$, $\forall z \in [x, y]$.

Proof. Immediate from Theorem 2.1 in Aussel [4] since $(-f)$ is quasiconvex. ■

Let $L_f^{\geq}(\alpha) \equiv \{x \in \mathbb{R}^n \mid f(x) \geq \alpha\}$ be the upper level set of level $\alpha \in \mathbb{R}$, and $L_f^{>}(\alpha) \equiv \{x \in \mathbb{R}^n \mid f(x) > \alpha\}$ be the strict upper level set of level $\alpha \in \mathbb{R}$. It is noteworthy that the solution set of (P) is connected if the feasibility of (P) is assumed since f is quasiconcave if and only if $L_f^{\geq}(\alpha)$ is convex for every α and the feasible region is convex. It also possesses global properties under additional assumptions on f, which we will deal with later.

In order to discuss topological properties in the manner of Hausdorff, we introduce the distance function $d_S(\cdot) : X \to \mathbb{R}$ defined by

$$d_S(x) \equiv \inf\{\| x - c \| | \, c \in S\},$$

where S is a nonempty subset of X (see Clarke [10]).

Let $\mathcal{T}(S; x)$ be a closed convex cone at x defined by

$$\mathcal{T}(S; x) \equiv \{d \in \mathbb{R}^n \mid d_S^\circ(x; d) = 0\},$$

and let $N(S; x)$ be a polar cone of $\mathcal{T}(S; x)$ at x defined by

$$N(S; x) \equiv \{\zeta \in \mathbb{R}^n \mid \langle \zeta, d \rangle \leq 0, \quad \forall d \in \mathcal{T}(S; x)\}, \tag{3}$$

where S is a convex set.

For $S_i = L^{\geq}_{g_i}(g_i(x))$, let $T_i(x)$, $T(x)$ also be closed convex cones at x defined by

$$T_i(x) \equiv \{d \in \mathbb{R}^n \mid \exists t_k \downarrow 0,\ d_k \to d,\ \text{with}\ \ x + t_k d_k \in S_i \quad \forall k\}, \tag{4}$$

$$T(x) \equiv \cap_{j \in I(x)} T_j(x),$$

where $I(x) \equiv \{i \in I \mid g_i(x) = 0\}$, and let $N_i(x)$, $N(x)$ be polar cones defined by

$$N_i(x) \equiv \{\zeta \in \mathbb{R}^n \mid \langle \zeta, d\rangle \leq 0, \quad \forall d \in T_i(x)\}, \tag{5}$$

$$N(x) \equiv \cup_{j \in I(x)} N_j(x).$$

We will show the following lemma that holds for locally Lipschitz continuous quasiconcave function at any x^0 which is not necessarily a solution.

Lemma 2 *Let $g_i : \mathbb{R}^n_+ \to \mathbb{R}$, $i = 1, ..., m$, be locally Lipschitz continuous quasiconcave functions. Suppose that $0 \notin \partial g_i(x^0)$. Then, one has*

$$N_i(x^0) \subset -\mathbb{R}_+ \partial g_i(x^0). \tag{6}$$

Furthermore, if g_i is regular, then it holds that

$$N_i(x^0) = -\mathbb{R}_+ \partial g_i(x^0). \tag{7}$$

Proof. It follows from its definition, the local Lipschitzness of g_i, and the convexity of $S_i = L^{\geq}_{g_i}(g_i(x^0))$ that $T_i(x^0)$ can be expressed as

$$T_i(x^0) = \mathcal{T}(S_i; x^0) = \text{cl}\ \{\mathbb{R}_+(\text{cl}\ S_i - x^0)\} \tag{8}$$

as in Remark 2 to Theorem 2 in Hiriart-Urruty [14].
Then, by regarding $-g_i$ as g_i in Remark 1 to Proposition 4 in [14], one has

$$N_i(x^0) \subset -\mathbb{R}_+ \partial g_i(x^0).$$

where $\partial g_i(x^0)$ is nonempty from the local Lipschitzness of g_i.

Conversely, if g_i is regular, it follows from Theorem 2.4.7 in Clarke [10] that

$$\begin{aligned}\{d \in \mathbb{R}^n \mid (-g_i)^\circ(x^0; d) \le 0\} &= \{d \in \mathbb{R}^n \mid g_i^\circ(x^0; d) \le 0\} \\ &= \{d \in \mathbb{R}^n \mid \max_{\xi_i \in \partial g_i(x^0)} \langle \xi_i,\ -d\rangle \le 0\} \\ &= \{d \in \mathbb{R}^n \mid \max_{\xi_i \in \partial g_i(x^0)} \langle -\xi_i,\ d\rangle \le 0\} \\ &= T_{S_i}(x^0) =\ T_i(x^0)\end{aligned}$$

for $S_i = \{x \in \mathbb{R}^n \mid (-g_i)(x) \le (-g_i)(x^0)\} = \{x \in \mathbb{R}^n \mid g_i(x^0) \le g_i(x)\} = L^{\ge}_{g_i}(g_i(x^0))$. Then it holds that

$$\langle -\xi_i,\ d\rangle \le 0, \quad \forall \xi_i \in \partial g_i(x^0), \quad \forall d \in T_i(x^0),$$

which implies

$$-\mathbb{R}_+ \partial g_i(x^0) \subset N_i(x^0).$$

And hence, if g_i is regular, then it holds that

$$N_i(x^0) = -\mathbb{R}_+ \partial g_i(x^0).$$

■

The generalized Fritz John conditions for (P) which are associated to any local solution can be expressed as follows (Clarke [10]):

$$\begin{aligned}0 \ \in\ & -\partial(\bar{u}_0 f + \sum_{i\in I} \bar{u}_i g_i)(\bar{x}) + N(\mathbb{R}^n_+; \bar{x}), \\ & \bar{u}_i g_i(\bar{x}) = 0, \quad i \in I, \\ & \bar{u}_0, \bar{u}_i \ge 0, \quad i \in I, \\ & g_i(\bar{x}) \ge 0,\ i \in I, \quad \bar{x} \ge 0.\end{aligned}$$

We will define the following abstract constraint qualification for (P):

Abstract Constraint Qualification (ACQ): There does not exist $(\lambda, \mu) \in \mathbb{R}^{m+1}_+$ at the local maximizer $\bar{x}$ of (P) with the exception of $(\lambda, \mu) = (0, 0) \in \mathbb{R}^{m+1}_+$ such that

$$0 \in \partial(\sum_{i \in I(\bar{x})} \lambda_i g_i)(\bar{x}) + \sum_{\{j \mid x^*_j = 0\}} \mu_j e_j. \tag{9}$$

If the abstract constraint qualification (ACQ) holds, the generalized Kuhn-Tucker conditions can be rewritten as follows:

$$
\begin{aligned}
(GKT) \qquad\qquad 0 \;\in\; & -\partial(f+\sum_{i\in I}\bar{\lambda}_i g_i)(\bar{x}) + N(\mathbb{R}^n_+;\bar{x}),\\
& \bar{\lambda}_i g_i(\bar{x}) = 0, \quad i\in I,\\
& \bar{\lambda}_i \geq 0, \quad i\in I,\\
& g_i(\bar{x}) \geq 0,\; i\in I, \quad \bar{x}\geq 0.
\end{aligned}
$$

Remark. The abstract constraint qualification (ACQ) is weaker than the Hiriart-Urruty constraint qualification [13] in section 3 later. In fact, if (9) holds for $\bar{x} \in \text{int}\,\mathbb{R}^n_+$ and $\exists\lambda \neq 0 \in \mathbb{R}^m_+$, namely, $0 \in \partial(\sum_{i\in I(\bar{x})}\lambda_i g_i)(\bar{x})$ which equals to $0 \in -\partial(\sum_{i\in I(\bar{x})}\lambda_i g_i)(\bar{x}) \subset \sum_{i\in I(\bar{x})}\lambda_i\partial(-g_i)(\bar{x})$, then there must holds that $0 = \langle 0, d_0\rangle = \sum_{i\in I(\bar{x})}\langle\lambda_i\xi_i, d_0\rangle \leq \sum_{i\in I(\bar{x})}\lambda_i(-g)^\circ(\bar{x}, d_0)$ for $\exists\xi_i \in \partial g_i(\bar{x})$, which contradicts the Hiriart-Urruty constraint qualification. Hiriart-Urruty also proposes the Hiriart-Urruty constant rank condition [15] based on Clarke's generalized Jacobians, which is stronger than (ACQ) and becomes the linearly independance condition in a smooth case.

Lemma 3 *Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a locally Lipschitz continuous function, and let $g_i : \mathbb{R}^n_+ \to \mathbb{R}$, $i = 1, ..., m$, be locally Lipschitz continuous quasiconcave functions. Suppose that $0 \notin \partial g_j(\bar{x})$, $j \in I(\bar{x})$. Let the abstract constraint qualification (ACQ) holds, and $\bar{x}$ and $\bar{\lambda}$ satisfy (GKT). Then,*

$$(\exists a \in \partial f(\bar{x}))(\forall y \in \mathbb{R}^n_+, y - \bar{x} \in T(\bar{x})) : \quad \langle a, y - \bar{x}\rangle \leq 0.$$

Proof. Under the (ACQ), we obtain from (GKT) and Corollary 2 to Proposition 2.3.3 in Clarke [10] that

$$\nu \in \partial(f + \sum_{i\in I}\bar{\lambda}_i g_i)(\bar{x}) \subset \partial f(\bar{x}) + \sum_{i\in I}\bar{\lambda}_i\partial g_i(\bar{x}).$$

for $\exists \nu \in N(\mathbb{R}^n_+;\bar{x})$.
Then it follows that

$$(\exists a \in \partial f(\bar{x}))(\exists \xi \in \sum_{i\in I} \bar{\lambda}_i \partial g_i(\bar{x}))(\forall y \in \mathbb{R}^n_+, y-\bar{x} \in T(\bar{x})) :$$
$$0 \geq \langle \nu,\ y-\bar{x}\rangle = \langle a+\xi, y-\bar{x}\rangle.$$

And hence,

$$(\exists a \in \partial f(\bar{x}))(\forall y \in \mathbb{R}^n_+, y-\bar{x} \in T(\bar{x})) : \quad \langle a, y-\bar{x}\rangle \leq 0$$

holds, since

$$(\exists \xi \in \sum_{i\in I} \bar{\lambda}_i \partial g_i(\bar{x}))(\forall y \in \mathbb{R}^n_+, y-\bar{x} \in T(\bar{x})) : \quad \langle \xi, y-\bar{x}\rangle \geq 0,$$

because there exists a $p_i \in \partial g_i(\bar{x})$, $i \in I(\bar{x})$ such that $p_i \in -N_i(\bar{x}) \subset \mathbb{R}_+\partial g_i(\bar{x})$, $p_i \perp T(\bar{x})$, $i \in I(\bar{x})$ by Lemma 2, so we can take $\xi = \sum_{i\in I(\bar{x})} \bar{\lambda}_i p_i \in -N(\bar{x})$. ■

We can now establish the main theorem which ensures global optimality for quasiconcave programming under the abstract constraint qualification (ACQ).

Theorem 1 *Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a locally Lipschitz continuous quasiconcave function, and let $g : \mathbb{R}^n_+ \to \mathbb{R}^m$ be a locally Lipschitz continuous quasiconcave function, and $X = \{x \mid g(x) \geq 0, x \geq 0\}$. Suppose that the abstract constraint qualification (ACQ), and $0 \notin \partial g_j(\bar{x})$, $j \in I(\bar{x})$ hold. Then $\bar{x}$ maximizes $f(x)$ subject to the constraints $g(x) \geq 0$, $x \geq 0$, if and only if $\bar{x}$ and $\bar{\lambda}$ satisfy (GKT) and one of the following conditions be satisfied:*

(a) $0 \notin \partial f(\bar{x})$,

(b) $f(x)$ is pseudoconcave on X.

Proof. To prove the necessity, we need to show if $\bar{x}$ is a global maximum on X and $0 \in \partial f(\bar{x})$ then f is pseudoconcave on X, which has been proven in Theorem 4.1 in Aussel [4].

To prove sufficiency, we need only to consider the following two cases, which are (a) $0 \notin \partial f(\bar{x})$ and (b′) $0 \in \partial f(\bar{x})$ and f is pseudoconcave.

Case (a): If $0 \notin \partial f(\bar{x})$, then f $\bar{x}$ and $\bar{\lambda}$ satisfy (GKT) under the (ACQ), it follows from Lemma 3 that a ($a \in \partial f(\bar{x})$) belongs to $N(\bar{x})$ and that ($\forall y \in \mathbb{R}^n_+$, $y-\bar{x} \in T(\bar{x})$): $\langle a, y-\bar{x}\rangle \leq 0$.

If $\langle a, y - \bar{x} \rangle < 0$, we obtain $f(\bar{x}) \geq f(z) \geq f(y)$ by applying Lemma 1.
If $\langle a, y - \bar{x} \rangle = 0$, then it follows $f(\bar{x}) \geq f(y)$, since the set $L_f^{\geq}(f(\bar{x}))$ is convex (see Mangasarian [20]).

Case (b′): If $0 \in \partial f(\bar{x})$ and f is pseudoconcave, then f has its global maximum at $\bar{x}$ from Theorem 4.1 in Aussel [4].
Therefore, $\bar{x}$ is a global maximum of (P) in both cases. ■

Remark. Theorem 1 generalizes and simplifies the sufficiency result of Theorem 1 in Arrow et al. [2]. Indeed, Theorem 1 above does not need regularity nor even directional differentiability of functions. For differential functions, the result is reduced to Theorem 21.22 in Simon and Blume [27]. Case (b) has been well known in a smooth case since the early stage (see Mangasarian [20]).

3 Constraint qualification

We have utilized the abstract constraint qualification (ACQ) in the previous section. Clarke [10] considers the perturbed problem for the general locally Lipschitz problem, and proposes calmness as a constraint qualification. In this section, however, we will derive the normality condition ($u^0 = 1$) employing pseudoconcavity, which allows us to use the generalized Kuhn-Tucker conditions.

We remind from Theorem 6 in Hiriart-Urruty [13] and (8) that if

$$\exists d_0 \in \mathcal{T}(\mathbb{R}^n_+, \bar{x}), \quad (-g_j)^\circ(\bar{x}; d) < 0, \ j \in I(\bar{x}) \equiv \{i \in I \mid g_i(\bar{x}) = 0\},$$

then it is said that *the Hiriart-Urruty constraint qualification* is satisfied.

Theorem 2 *Let $g : \mathbb{R}^n_+ \to \mathbb{R}^m$ be a locally Lipschitz continuous quasiconcave function. Let $g(x') > 0$ and $g(\bar{x}) \geq 0$ for some $x', \bar{x} \geq 0$, and for each $j \in I(\bar{x})$, let*

$$0 \notin \partial g_j(\bar{x}), \quad j \in I(\bar{x}), \tag{10}$$

hold. Then $g(x)$ satisfies the Hiriart-Urruty constraint qualification.

Proof. If we take $(0 \neq)\ x' - \bar{x} \in \text{int}\ T(\bar{x})$, we can obtain $g_j^\circ(\bar{x}; x' - \bar{x}) > 0$, $j \in I(\bar{x})$, since $g_j^\circ(\bar{x}; x' - \bar{x}) = \max\{\langle \xi_j, x' - \bar{x}\rangle \mid \xi_j \in \partial g_j(\bar{x})\} \geq \langle \xi_j, x' - \bar{x}\rangle$, $j \in I(\bar{x})$, $x' - \bar{x} \in \text{int}\ T(\bar{x})$, and $\langle \xi_j^0, x' - \bar{x}\rangle > 0$ as $\langle -\xi_j^0, x' - \bar{x}\rangle < 0$ for $0 \neq \exists \xi_j^0 \in -\partial g_j(\bar{x}) \cap (-N(\bar{x}))$ from Lemma 2. In fact, the Hiriart-Urruty constraint qualification holds since $(-g_j)^\circ(\bar{x}; x' - \bar{x}) = g_j^\circ(\bar{x}; \bar{x} - x') < 0$ from Proposition 2.1.1(c) in Clarke [10], because $g_j^\circ(\bar{x}; \bar{x} - x') = \max\{\langle \xi_j, \bar{x} - x'\rangle \mid \xi_j \in \partial g_j(\bar{x})\} = \max\{\langle -\xi_j, x' - \bar{x}\rangle \mid \xi_j \in \partial g_j(\bar{x})\} < 0$, $j \in I(\bar{x})$, $\bar{x} - x' \in \text{int}\ T(\bar{x})$, and $\langle -\xi_j^0, x' - \bar{x}\rangle < 0$ from Lemma 2 as above. ■

Remark. The assumption of Theorem 2 generalizes that of Theorem 2 in Arrow et al. [2], and the Slater assumption. In other words, the proposed constraint qualification is the weakest among these three assumptions. If g_j is differentiable, the result of Theorem 2 is close to Arrow-Hurwicz-Uzawa's constraint qualification (Arrow, Hurwicz, and Uzawa [3]).

Corollary 1 *Let $g : \mathbb{R}_+^n \to \mathbb{R}^m$ be a locally Lipschitz continuous quasiconcave function. Let $g(x') > 0$ and $g(\bar{x}) \geq 0$ for some $x', \bar{x} \geq 0$, and for each $j \in I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$, let*

$$g_j \quad \text{is pseudoconcave}, \quad j \in I(\bar{x}) \tag{11}$$

hold. Then $g(x)$ satisfies the Hiriart-Urruty constraint qualification.

Proof. If g_j, $j \in I(\bar{x})$ is pseudoconcave, it follows that

$$(\exists \eta \in \partial g_j(\bar{x}), j \in I(\bar{x}))(\forall x', \bar{x} \in \mathbb{R}^n): \quad \langle \eta, x' - \bar{x}\rangle \leq 0 \ \Rightarrow\ g_j(x') \leq g_j(\bar{x})$$

from its definition. And hence $0 \notin \partial g_j(\bar{x})$, $j \in I(\bar{x})$ since $g_j(x') > g_j(\bar{x})$, which satisfies the hypothesis of Theorem 2. ■

We remark that the result of Corollary 1 generalizes the ordinary Slater assumption.

The following theorem is an immediate consequence of the result of the previous section and the above corollary.

Theorem 3 *Let $f : \mathbb{R}_+^n \to \mathbb{R}$ be a locally Lipschitz continuous pseudoconcave function and $g : \mathbb{R}_+^n \to \mathbb{R}^m$ be a locally Lipschitz continuous pseudoconcave function, and $X = \{x \mid g(x) \geq 0, x \geq 0\}$. Suppose that there exists an x' such that $g(x') > 0$. Then $\bar{x}$ maximizes $f(x)$ subject to the constraints $g(x) \geq 0$, $x \geq 0$, if and only if $\bar{x}$ and $\bar{\lambda}$ satisfy (GKT).*

Proof. The result is obtained by combining Theorem 1 with Corollary 1. ■

4 Economics application

Consider the following consumer's utility maximization problem in this section:

$$\text{(UM)} \qquad V(p) \equiv \begin{array}{ll} \text{maximize} & u(x) \\ \text{subject to} & \langle p, x \rangle \leq I, \\ & x \geq 0. \end{array}$$

where utility function $u : \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to any nonnegative consumer vector $x \in \mathbb{R}^n_+$, $p = (p_1, ..., p_n) > 0$ is a positive vector of prices, and $I > 0$ is a value of income. We recall that $V(p)$ is called an indirect utility function.

It is usual that additional monotonicity assumptions are imposed on utility functions so that it may exclude the cases in which contour surfaces arise or in which the satiation is attained.

Definition 2. A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be *monotone* if

$$u(x'_1, x'_2, ..., x'_n) > u(x_1, x_2, ..., x_n), \quad \forall x'_i > x_i, \; i = 1, ..., n.$$

Under the monotonicity assumption, we obtain $\langle p, x \rangle = I$ since the satiation does not occur.

Then we can derive the optimality condition which holds if and only if $\bar{x}$ is a local maximizer of (UM) as follows:

$$\begin{cases} \bar{\lambda} p_i \in (\partial u(\bar{x}))_i, & \text{for } \bar{x}_i > 0, \\ \bar{\lambda} p_i \in (\partial u(\bar{x}))_i + \bar{\mu}, \; \bar{\mu} \geq 0, & \text{for } \bar{x}_i = 0, \\ \bar{\lambda} \geq 0, \quad \bar{x} \geq 0. \end{cases} \tag{12}$$

It is noteworthy that the monotonicity condition (or *nonsatiation*) satisfies the condition of Theorem 1, which ensures the existence of an optimal solution.

It is one of our motivations to handle the Rawlsian social welfare function (Rawls [26]) defined as a minimum of utility functions. Then, the optimal solutions of (UM) maximize the welfare of society's poor people, which gives priority to egalitarianism.

Example. Consider the utility maximization problem

$$\begin{aligned} \text{maximize} \quad & u(x_1,x_2) \equiv \min\{u_1(x_1,x_2),\ u_2(x_1,x_2)\} \\ \text{subject to} \quad & \langle p,x\rangle \le I, \\ & x \ge 0, \end{aligned}$$

where $u_i(x_1,x_2) = x_1^{\alpha_i} x_2^{\beta_i}$, $\alpha_i + \beta_i = 1$, $\alpha_i > 0$, $\beta_i > 0$, $i = 1,2$, $\alpha_1 \ge \alpha_2$ are Cobb-Douglas utility functions, so the objective function u is locally Lipschitz and quasiconcave.

The solution $\bar{x}$ and $\partial u(\bar{x})$ can be obtained by solving a simple condition (12), which are

if $p_2/p_1 < \beta_1/\alpha_1$, then $(\bar{x}_1,\bar{x}_2)^T = ((\alpha_1/p_1)I, (\beta_1/p_2)I)^T$ and $\partial u(\bar{x}_1,\bar{x}_2) = \nabla u(\bar{x}_1,\bar{x}_2) = (\alpha_1^{\alpha_1}\beta_1^{\beta_1}(p_1/p_2)^{\beta_1}, \alpha_1^{\alpha_1}\beta_1^{\beta_1}(p_2/p_1)^{\alpha_1})^T$,

if $\beta_1/\alpha_1 \le p_2/p_1 \le \beta_2/\alpha_2$, then $(\bar{x}_1,\bar{x}_2)^T = (I/(p_1+p_2), I/(p_1+p_2))^T$ and $\partial u(\bar{x}_1,\bar{x}_2) = \operatorname{conv}\{(\alpha_1,\beta_1)^T, (\alpha_2,\beta_2)^T\}$,

if $\beta_2/\alpha_2 < p_2/p_1$, then $(\bar{x}_1,\bar{x}_2)^T = ((\alpha_2/p_1)I, (\beta_2/p_2)I)^T$ and $\partial u(\bar{x}_1,\bar{x}_2) = \nabla u(\bar{x}_1,\bar{x}_2) = (\alpha_2^{\alpha_2}\beta_2^{\beta_2}(p_1/p_2)^{\beta_2}, \alpha_2^{\alpha_2}\beta_2^{\beta_2}(p_2/p_1)^{\alpha_2})^T$.

We remark that the optimal point $\bar{x}$ is insensitive to prices p in the case $\beta_1/\alpha_1 \le p_2/p_1 \le \beta_2/\alpha_2$. In other words, the problem is stable if the difference between α_1 and α_2 is large.

5 Conclusion

We have established the necessary and sufficient conditions for nonsmooth quasiconcave programming, which generalizes Arrow and Enthoven's sufficient ones for smooth quasiconcave programming. We also proposed the constraint qualification which significantly generalizes Slater assumption and is meant for nonsmooth quasiconcave programming. Nonsmoothness of a function in quasiconcave programming, similar to strong quasiconcavity, brings certain stability into the problem which might be a favorable property in economic circumstances. Nonsmooth quasiconcave programming may play an important role in oligopoly theory related to sensitivity analysis. Further sensitivity analysis seems to be acutely required in this field and will be our future task.

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