SIMPLE EXPLICIT FORMULA FOR COUNTING LATTICE POINTS OF POLYHEDRA

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Abstract. Given $z \in \mathbb{C}^n$ and $A \in \mathbb{Z}^{m\times n}$, we consider the problem of evaluating the counting function $h(y;z) := \sum\{\, z^x \mid x{\in}\mathbb{Z}^n;\, Ax{=}y, x{\geq}0\}$. We provide an explicit expression for $h(y;z)$ as well as an algorithm with possibly numerous but very simple calculations. In addition, we exhibit *finitely many* fixed convex cones of $\mathbb{R}^n$ explicitly and exclusively defined by A such that for *any* $y \in \mathbb{Z}^m$, the sum $h(y;z)$ can be obtained by a simple formula involving the evaluation of $\sum z^x$ over the integral points of those cones only. At last, we also provide an alternative (and different) formula from a decomposition of the generating function into simpler rational fractions, easy to invert.
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1. introduction

Consider the (not necessarily compact) polyhedron

$$\mathbf{\Omega}(y) \;:=\; \{x \in \mathbb{R}^n \mid Ax = y;\; x \geq 0\}, \tag{1.1}$$

with $y \in \mathbb{Z}^m$ and $A \in \mathbb{Z}^{m\times n}$ of maximal rank for $n \geq m$; besides, given $z \in \mathbb{C}^n$, let $h : \mathbb{Z}^m \to \mathbb{C}$ be the *counting* function

$$y \;\mapsto\; h(y;z) \;:=\; \sum_{x\in\mathbf{\Omega}(y)\cap\mathbb{Z}^n} z^x \tag{1.2}$$

(where z^x stands for $\prod_k {z_k}^{x_k}$). The complex vector $z \in \mathbb{C}^n$ may be chosen close enough to zero in order to ensure that $h(y;z)$ is well defined even when $\mathbf{\Omega}(y)$ is not compact. If $\mathbf{\Omega}(y)$ is compact, then $y \mapsto h(y;z)$ provides us with the exact number of points in the set $\mathbf{\Omega}(y)\cap\mathbb{Z}^n$ by either evaluating $h(y,1)$, or even rounding $h(y;z)$ up to the nearest integer when all the entries of z are close enough to one.

Computation of h has attracted a lot of attention in recent years, from both theoretical and practical computation viewpoints. Barvinok and Pommersheim [4], Brion and Vergne [8], have provided nice exact (theoretical) formulas for $h(y;z)$; see also Szenes and Vergne [15]. For instance, Barvinok considers $z \mapsto h(y;z)$ as the generating function (evaluated at $z := \mathrm{e}^c \in \mathbb{C}^n$) of the indicator function $x \mapsto I_{\mathbf{\Omega}(y)\cap\mathbb{Z}^n}(x)$ of the set $\mathbf{\Omega}(y)\cap\mathbb{Z}^n$ and provides a decomposition into a sum of simpler generating functions associated with supporting cones (themselves having a signed decomposition into unimodular cones). We call this a *primal* approach because y is *fixed*, and one works in the primal space $\mathbb{R}^n$ in which $\mathbf{\Omega}(y)$ is defined. Remarkably, Barvinok's counting algorithm which is implemented in the software `LattE` (see De Loera et al. [10]) runs in time polynomial in the problem size when the dimension n is fixed. The software developed by Verdoolaege [17] extends the `LattE` software to handle *parametric polytopes*. On the other hand, Brion and Vergne [8] consider the generating function $H : \mathbb{C}^m \to \mathbb{C}$ of $y \mapsto h(y; z)$, that is,

$$w \;\mapsto\; H(w) \;:=\; \sum_{y\in\mathbb{Z}^m} h(y;z)w^y \;=\; \prod_{k=1}^{n} \frac{1}{1-z_k w^{A_k}}. \tag{1.3}$$

They provide a generalized residue formula, and so obtain $h(y; z)$ in closed form by *inversion*. We call this latter approach *dual* because z is fixed, and one works in the space $\mathbb{C}^m$ of variables w associated with the m constraints $Ax = y$.

As a result of both primal and dual approaches, $h(y; z)$ is finally expressed as a weighted sum over the vertices of $\mathbf{\Omega}(y)$. Similarly, Beck [5], and Beck, Diaz and Robins [6] provided a complete analysis based on residue techniques for the case of a tetrahedron ($m = 1$). Despite its theoretical interest, Brion and Vergne's formula is not directly *tractable* because it contains many products with complex coefficients (roots of unity) which makes the formula difficult to evaluate numerically. However, in some cases, this formula can be exploited to yield an efficient algorithm as e.g. in [2] for flow polytopes, in [7] for transportation polytopes, and more generally when the matrix A is totally unimodular as in [9]. Finally, in [12, 13], we have provided two algorithms based on Cauchy residue techniques to invert H in (1.3), and an alternative algebraic technique based on partial fraction expansion of H. A nice feature of the latter technique of [13] is to avoid computing residues.

Contribution: Our contribution is twofold as it is concerned with both primal and dual approaches. On the primal side, we provide an explicit expression of $h(y; z)$ and an algorithm which involves only elementary operations. It uses Brion's identity along with an explicit description of the supporting cones at the vertices of $\mathbf{\Omega}(y)$. It also has a simple equivalent formulation as a (finite) *group problem*. Finally, we exhibit *finitely many* fixed convex *cones* of $\mathbb{R}^n$, explicitly and exclusively defined from A, such that for *any* $y \in \mathbb{Z}^m$, the sum $h(y; z)$ is obtained by a simple formula which evaluates $\sum z^x$ over the integral points of those cones only.

On the dual side, we analyze the *counting* function h, via its generating function H in (1.3). Inverting H is difficult in general, except if an appropriate expansion of H into simple fractions is available, as in e.g. [13]. In their landmark paper [8], Brion and Vergne provided a *generalized residue* formula which yields the generic expansion

$$H(w) \;=\; \sum_{\sigma\in\mathbb{J}_A} \sum_{g\in G_\sigma} \widehat{Q}_{g,\sigma} \prod_{k\in\sigma} \frac{[z_k w^{A_k}]^{\delta_{k,\sigma}}}{1-\rho_q^{g_k}[z_k w^{A_k}]^{1/q}}. \tag{1.4}$$

Here, $\sigma \in \mathbb{J}_A$ whenever A_σ is invertible, q is the smallest common multiple of all $|\det A_\sigma| \neq 0$, $\rho_q = e^{2\pi i/q}$ is the q-root of unity, $\delta_{k,\sigma} \in \{0, 1/q\}$, and $\widehat{Q}_{g,\sigma} \in \mathbb{C}$. The finite group G_σ has q^m elements. The coefficients $\widehat{Q}_{g,\sigma}$ are difficult to evaluate. Our contribution is to expand H in (1.3) in the form

$$(1.5)\quad H(w) = \sum_{\sigma\in\mathbb{J}_A} \left[\prod_{j\in\sigma} \frac{1}{1 - z_j w^{A_j}}\right] \times \frac{1}{R_2(\sigma; z)} \sum_{u_{\not\sigma}\in\mathbb{Z}^{n-m}_{\mu_\sigma}} z^{\eta[\sigma,u_{\not\sigma}]}\, w^{A\eta[\sigma,u_{\not\sigma}]},$$

where: $\mathbb{Z}_{\mu_\sigma} = \{0, 1, \ldots, \mu_\sigma - 1\}$, $\mu_\sigma = |\det A_\sigma|$, each $\eta[\sigma, u_{\not\sigma}] \in \mathbb{Z}^n$ and:

$$(1.6)\qquad z \mapsto R_2(\sigma; z) := \prod_{k\notin\sigma} \left[1 - \left(z_k z_\sigma^{-A_\sigma^{-1}A_k}\right)^{\mu_\sigma}\right].$$

Identity (1.5) is a nontrivial simplification of the residue formula (1.4) because the $\eta[\sigma, u_{\not\sigma}]$'s are given *explicitly*. And so the coefficients of the rational fraction (1.5) in w are very simple to evaluate with no root of unity involved (it can also be done symbolically); however this task can be tedious as for each $\sigma \in \mathbb{J}_A$ one has $|\det A_\sigma|^{n-m}$ terms $\eta[\sigma, u_{\not\sigma}]$ to determine. But once determined, (1.5) is easy to invert and provides $h(y; z)$ for *any* $y \in \mathbb{Z}^m$.

2. Brion's decomposition

2.1. **Notation and definitions.** The notation $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ stand for the usual sets of complex, real and integer numbers, respectively. Moreover, the set of natural numbers $\{0, 1, 2, \ldots\}$ is denoted by $\mathbb{N}$, and for every natural number $\mu \in \mathbb{N}$, the finite set $\{0, 1, \ldots, \mu - 1\}$ of cardinality μ is denoted by $\mathbb{Z}_\mu$. The notation B' stands for the transpose of a matrix (or vector) $B \in \mathbb{R}^{s\times t}$; and the kth column of the matrix B is denoted by $B_k := (B_{1,k}, \ldots, B_{s,k})'$. When $y = 0$, the cone $\mathbf{\Omega}(0)$ in (1.1) is convex, and its *dual* cone is given by,

$$(2.1)\qquad \mathbf{\Omega}(0)^* := \{b \in \mathbb{R}^n \mid b'x \geq 0 \text{ for every } x \in \mathbf{\Omega}(0)\}.$$

Notice that $\mathbf{\Omega}(0)^* \equiv \mathbb{R}^n$ if $\mathbf{\Omega}(0) = \{0\}$, which is the case if $\mathbf{\Omega}(y)$ is compact.

Definition 1. Let $A \in \mathbb{Z}^{m\times n}$ be of maximal rank. An ordered set $\sigma = \{\sigma_1, \ldots, \sigma_m\}$ of natural numbers is said to be a *basis* if it has cardinality $|\sigma| = m$, the sequence of inequalities $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_m \leq n$ holds, and the square $[m \times m]$ submatrix :

$$(2.2)\qquad A_\sigma := [A_{\sigma_1}|A_{\sigma_2}|\cdots|A_{\sigma_m}] \quad \text{is invertible.}$$

We denote the set of all bases σ by $\mathbb{J}_A$.

Definition 2. Given a maximal rank matrix $A \in \mathbb{Z}^{m\times n}$, and any basis $\sigma \in \mathbb{J}_A$, the complementary matrices $A_\sigma \in \mathbb{Z}^{m\times n}$ and $A_{\not\sigma} \in \mathbb{Z}^{m\times(n-m)}$ stand for $[A_k]_{k\in\sigma}$ and $[A_k]_{k\notin\sigma}$, respectively. Similarly, given $z \in \mathbb{C}^n$, the complementary vectors $z_\sigma \in \mathbb{C}^m$ and $z_{\not\sigma} \in \mathbb{C}^{n-m}$ stand for $(z_k)_{k\in\sigma}$ and $(z_k)_{k\notin\sigma}$, respectively.

For each basis $\sigma \in \mathbb{J}_A$ with associated matrix $A_\sigma \in \mathbb{Z}^{m\times m}$, introduce the *indicator* function $\delta_\sigma : \mathbb{Z}^m \to \mathbb{N}$ defined by :

$$y \mapsto \delta_\sigma(y) := \left\{ \begin{array}{ll} 1 & \text{if } A_\sigma^{-1} y \in \mathbb{Z}^m, \\ 0 & \text{otherwise.} \end{array} \right. \tag{2.3}$$

Notice that δ_σ is a multi-periodic function with periods A_σ and $\mu_\sigma := |\det A_\sigma|$, meaning that $\delta_\sigma(y + A_\sigma q) = \delta_\sigma(y + \mu_\sigma q) = \delta_\sigma(y)$ for all $y, q \in \mathbb{Z}^m$. Finally, given a triplet $(z, x, u) \in \mathbb{C}^n \times \mathbb{Z}^n \times \mathbb{R}^n$, introduce the notation :

$$\begin{array}{rcl} z^x & := & z_1^{x_1} z_2^{x_2} \cdots z_s{}^{x_n}, \\ \|z\| & := & \max\{|z_1|, |z_2|, \ldots, |z_n|\}, \\ \ln\langle z\rangle & := & (\ln(z_1), \ln(z_2), \ldots, \ln(z_n)). \end{array} \tag{2.4}$$

Notice that $z^x = z_\sigma^{x_\sigma} z_{\sigma'}^{x_{\sigma'}}$, for all bases $\sigma \in \mathbb{J}_A$ and all $z \in \mathbb{C}^n$, $x \in \mathbb{Z}^n$.

2.2. **Brion's decomposition.** Let $\mathbf{\Omega}(y)$ be the convex polyhedron in (1.1) with $y \in \mathbb{Z}^m, A \in \mathbb{Z}^{m\times n}$ being of maximal rank, and let $h : \mathbb{Z}^m \to \mathbb{C}$ be the counting function in (1.2), with $\|z\| < 1$.

Obviously $h(y; z) = 0$ whenever the equation $Ax = y$ has no solution $x \in \mathbb{N}^n$. The main idea is to decompose the function h following Brion's ideas. Given any convex rational polyhedron $P \subset \mathbb{R}^n$, let $[P] : \mathbb{R}^n \to \{0, 1\}$ be its characteristic function, and $f[P] : \mathbb{C} \to \mathbb{C}$ its associated rational function, such that

$$z \mapsto f[P, z] := \sum_{x \in P\cap\mathbb{Z}^n} z^x, \tag{2.5}$$

holds whenever the sum converges absolutely. For every vertex V of P, define $\mathrm{Co}(P, V) \subset \mathbb{R}^n$ to be the supporting cone of P at V. Then, Brion's formula yields the decomposition :

$$[P] = \sum_{\text{vertices } V} [\mathrm{Co}(P, V)], \tag{2.6}$$

modulo the group generated by the characteristic functions of convex polyhedra which contain affine lines. And so,

$$f[P, z] = \sum_{\text{vertices } V} f[\mathrm{Co}(P, V), z]. \tag{2.7}$$

The above summation is *formal* because in general there is no $z \in \mathbb{C}^n$ for which the series

$$\sum\{z^x \mid x \in P \cap \mathbb{Z}^n\} \quad \text{and} \quad \sum\{z^x \mid x \in \mathrm{Co}(P, V) \cap \mathbb{Z}^n\}$$

converge absolutely for all vertices V. The notation $\sum E$ stands for the sum of all elements of a countable set $E \subset \mathbb{C}$. It is a complex number whenever the resulting series converges absolutely; otherwise it stands for a formal series.

Example: Let $P := [0,1] \subset \mathbb{R}$ so that $\mathrm{Co}(P,\{0\}) = [0,+\infty)$ and $\mathrm{Co}(P,\{1\}) = (-\infty,1]$. Simple enumeration yields $f[P,z] = z^0 + z = 1 + z$, but one also has:

$$f[P,z] \;=\; f[\mathrm{Co}(P,\{0\}),z] + f[(\mathrm{P},\{1\}),\mathrm{z}] \;=\; 1/(1-\mathrm{z}) + \mathrm{z}^2/(\mathrm{z}-1) \;=\; 1+\mathrm{z}.$$

3. Computing $h(y;z)$: A primal approach

Let $C(\mathbb{J}_A) := \{Ax \,|\, x \in \mathbb{N}^n\} \subset \mathbb{R}^m$ be the cone generated by the columns of A, and for any basis $\sigma \in \mathbb{J}_A$, let $C(\sigma) \subset \mathbb{R}^m$ be the cone generated by the columns A_k with $k \in \sigma$. As A has maximal rank, $C(\mathbb{J}_A)$ is the union of all $C(\sigma)$, $\sigma \in \mathbb{J}_A$. With any $y \in C(\mathbb{J}_A)$ associate the intersection of all cones $C(\sigma)$ that contain y. This defines a subdivision of $C(\mathbb{J}_A)$ into polyhedral cones. The interiors of the maximal subdivisions are called *chambers*. In each chamber γ, the polyhedron $\mathbf{\Omega}(y)$ is *simple*, i.e. $A_\sigma^{-1}y > 0$ for all $\sigma \in \mathbb{J}_A$ such that $A_\sigma^{-1}y \geq 0$.

For any chamber γ, define,

$$\mathcal{B}(\mathbb{J}_A,\gamma) \;:=\; \{\sigma \in \mathbb{J}_A \,|\, \gamma \subset C(\sigma)\}. \tag{3.1}$$

The intersection of all $C(\sigma)$ with $\sigma \in \mathcal{B}(\mathbb{J}_A,\gamma)$ is the closure $\overline{\gamma}$ of γ.

Back to our original problem, and setting $P := \mathbf{\Omega}(y)$, the rational function $f[P,z]$ is equal to $h(y;z)$ in (1.2) whenever $\|z\| < 1$. We next provide an explicit description of the rational function $f[\mathrm{Co}(P,V),z]$ for every vertex V of P.

Let δ_σ be the function defined in (2.3), and let $\mathbb{Z}_{\mu_\sigma} := \{0,1,\ldots,\mu_\sigma - 1\}$ with $\mu_\sigma := |\det A_\sigma|$. A vector $V \in \mathbb{R}^n$ is a vertex of $P = \mathbf{\Omega}(y)$ if and only if there exists a basis $\sigma \in \mathbb{J}_A$ such that :

$$V_\sigma \;=\; A_\sigma^{-1}y \;\geq\; 0 \quad \text{and} \quad V_{\not\sigma} = 0, \tag{3.2}$$

where V_σ and $V_{\not\sigma}$ are given in Definition 2. Moreover, the supporting cone of P at the vertex V is described by :

$$\mathrm{Co}(\mathbf{\Omega}(y),V) \;:=\; \{x \in \mathbb{R}^n \mid Ax = y;\; x_k \geq 0 \text{ if } V_k = 0\}\,. \tag{3.3}$$

Let us now define the larger set

$$C(\mathbf{\Omega}(y),\sigma) \;:=\; \{x \in \mathbb{R}^n \,|\, A_\sigma x_\sigma + A_{\not\sigma} x_{\not\sigma} = y;\; x_{\not\sigma} \geq 0\}, \tag{3.4}$$

so that $\mathrm{Co}(\mathbf{\Omega}(y),V)$ is a subcone of $C(\mathbf{\Omega}(y),\sigma)$ for all bases $\sigma \in \mathbb{J}_A$ and vertex V of $\mathbf{\Omega}(y)$ which satisfy $V_{\not\sigma} = 0$ (recall (3.2)). Besides, when $V_{\not\sigma} = 0$ and $y \in \gamma$ for some chamber γ, then $C(\mathbf{\Omega}(y),\sigma)$ and $\mathrm{Co}(\mathbf{\Omega}(y),V)$ are identical because $\mathbf{\Omega}(y)$ is a simple polytope, and so $A_\sigma^{-1}y > 0$ for all $\sigma \in \mathbb{J}_A$.

Recall that $A_\sigma \in \mathbb{Z}^{m\times n}$ and $A_{\not\sigma} \in \mathbb{Z}^{m\times(n-m)}$ stand for $[A_k]_{k\in\sigma}$ and $[A_k]_{k\notin\sigma}$, respectively. Similarly, given a vector $x \in \mathbb{Z}^n$, the vectors x_σ and $x_{\not\sigma}$ stand for $(x_k)_{k\in\sigma}$ and $(x_k)_{k\notin\sigma}$ respectively. The following result is from [8, p. 818].

Proposition 1. *Let $y \in \mathbb{R}^m$ and let $\mathbf{\Omega}(y)$ be as in (1.1), and let $y \in \overline{\gamma}$ for some chamber γ. Then,*

$$[\mathbf{\Omega}(y)] \;=\; \sum_{\sigma\in\mathcal{B}(\mathbb{J}_A,\gamma)} [C(\mathbf{\Omega}(y),\sigma)], \tag{3.5}$$

modulo the group generated by the characteristic functions of convex polyhedra which contain affine lines.

Proof. Using notation of [8, p. 817], define the linear mapping $p:\mathbb{R}^n\to\mathbb{R}^m$ with $p(x)=Ax$, so that the polyhedra $P_\Delta(y)$ and $\mathbf{\Omega}(y)$ are identical. Moreover, for every basis $\sigma\in\mathcal{B}(\mathbb{J}_A,\gamma)$, $v_\sigma:\mathbb{R}^m\to\mathbb{R}^n$ is the linear mapping:

$$y\mapsto\quad [v_\sigma(y)]_\sigma \;=\; A_\sigma^{-1}y \quad\text{and}\quad [v_\sigma(y)]_{\not\sigma} \;=\; 0,\qquad y\in\mathbb{R}^m.$$

Finally, for every $\hat{x}\in\mathbb{R}^n$ with $\hat{x}\geq 0$, $\rho_\sigma(\hat{x}):=\hat{x}-v_\sigma(A\hat{x})$ satisfies,

$$[\rho_\sigma(\hat{x})]_\sigma = -A_\sigma^{-1}A_{\not\sigma}\hat{x}_{\not\sigma} \quad\text{and}\quad [\rho_\sigma(\hat{x})]_{\not\sigma} \;=\; \hat{x}_{\not\sigma}.$$

Therefore, the cone $[v_\sigma(y)+\rho_\sigma(C)]$ in [8] is the set of points $x\in\mathbb{R}^m$ such that $x_{\not\sigma}\geq 0$ and $x_\sigma = A_\sigma^{-1}(y-A_{\not\sigma}x_{\not\sigma})$; and so this cone is just $[C(\mathbf{\Omega}(y),\sigma)]$ in (3.4). Therefore a direct application of (3.2.1) in [8, p. 818] yields (3.5). □

Theorem 2. *Let $y\in\mathbb{Z}^m$, $z\in\mathbb{C}^n$ with $\|z\|<1$, and let $y\in\overline{\gamma}$ for some chamber γ. Recall the set of bases $\mathcal{B}(\mathbb{J}_A,\gamma)$ defined in (3.1). With $P:=\mathbf{\Omega}(y)$, the rational function h defined in (1.2) can be written:*

$$h(y;z) \;=\; \sum_{\sigma\in\mathcal{B}(\mathbb{J}_A,\gamma)} f[C(\mathbf{\Omega}(y),\sigma),z] \;=\; \sum_{\sigma\in\mathcal{B}(\mathbb{J}_A,\gamma)} \frac{R_1(y,\sigma;z)}{R_2(\sigma;z)}, \tag{3.6}$$

$$\text{with}\qquad z\;\mapsto\; R_1(y,\sigma;z) \;:=\; z_\sigma^{A_\sigma^{-1}y} \sum_{u\in\mathbb{Z}^{n-m}_{\mu_\sigma}} \frac{\delta_\sigma(y-A_{\not\sigma}u)\, z_{\not\sigma}^u}{z_\sigma^{A_\sigma^{-1}A_{\not\sigma}u}}, \tag{3.7}$$

$$\text{and}\qquad z\;\mapsto\; R_2(\sigma;z) \;:=\; \prod_{k\notin\sigma}\left[1-\left(z_k z_\sigma^{-A_\sigma^{-1}A_k}\right)^{\mu_\sigma}\right]. \tag{3.8}$$

The pair $\{R_1,R_2\}$ is well defined whenever $z\in\mathbb{C}^n$ satisfies $z_k\neq 0$ and $z_k\neq {z_\sigma}^{A_\sigma^{-1}A_k}$ for every basis $\sigma\in\mathbb{J}_A$ which does not contain the index $k\notin\sigma$.

Proof. By a direct application of Brion's theorem to the sum (3.5), the associated rational functions $f[\mathbf{\Omega}(y),z]$ and $f[C(\mathbf{\Omega}(y),\sigma),z]$ satisfy:

$$h(y,z) \;=\; f[\Omega(y),z] \;=\; \sum_{\sigma\in\mathcal{B}(\mathbb{J}_A,\gamma)} f[C(\mathbf{\Omega}(y),\sigma),z]. \tag{3.9}$$

Therefore, in order to show (3.6), one only needs to prove that the rational function $\frac{R_1(y,\sigma;z)}{R_2(\sigma;z)}$ is equal to $f[C(\mathbf{\Omega}(y),\sigma),z]$, i.e.,

$$\frac{R_1(y,\sigma;z)}{R_2(\sigma;z)} \;=\; \sum\{z^x \,|\, x\in C(\Omega(y),\sigma)\cap\mathbb{Z}^n\}, \tag{3.10}$$

on the domain $D_\sigma = \{z \in \mathbb{C}^n \,|\, 1 > |z_k z_\sigma^{-A_\sigma^{-1}A_k}| \text{ for each } k \notin \sigma\}$. Notice that

$$\begin{aligned}\frac{1}{R_2(\sigma;z)} &= \prod_{k\notin\sigma} \frac{1}{1-\left(z_k z_\sigma^{-A_\sigma^{-1}A_k}\right)^{\mu_\sigma}} = \\ &= \prod_{k\notin\sigma}\sum_{v_k\in\mathbb{N}}\left[\frac{z_k}{z_\sigma^{A_\sigma^{-1}A_k}}\right]^{\mu_\sigma v_k} = \sum_{v\in\mathbb{N}^{n-m}} \frac{z_{\not\sigma}^{\mu_\sigma v}}{z_\sigma^{\mu_\sigma A_\sigma^{-1}A_{\not\sigma} v}},\end{aligned}$$

on D_σ. On the other hand, according to (3.4), the integer vector $x \in \mathbb{Z}^n$ lies inside the cone $C(P, V_\sigma)$ if and only if :

$$\begin{aligned} x_\sigma &= A_\sigma^{-1}(y - A_{\not\sigma} x_{\not\sigma}), \quad \delta_\sigma(y - A_{\not\sigma}x_{\not\sigma}) = 1 \quad \text{and} \\ x_{\not\sigma} &= u + \mu_\sigma v, \quad \text{with} \quad u \in \mathbb{Z}^{n-m}_{\mu_\sigma} \quad \text{and} \quad v \in \mathbb{N}^{n-m}.\end{aligned}$$

But from the definition (3.7) of $R_1(y,\sigma;z)$ and $z^x = z_{\not\sigma}^{x_{\not\sigma}} z_\sigma^{x_\sigma} = z_\sigma^{A_\sigma^{-1}(y-A_{\not\sigma}x_{\not\sigma})}$,

$$\begin{aligned}\frac{R_1(y,\sigma;z)}{R_2(\sigma;z)} &= z_\sigma^{A_\sigma^{-1}y} \sum_{u\in\mathbb{Z}^{n-m}_{\mu_\sigma}} \sum_{v\in\mathbb{N}^{n-m}} \frac{\delta_\sigma(y-A_{\not\sigma}u)\, z_{\not\sigma}^{x_{\not\sigma}}}{z_\sigma^{A_\sigma^{-1}A_{\not\sigma}x_{\not\sigma}}}, \\ &= \sum \{z^x \,|\, x \in C(\mathbf{\Omega}(y),\sigma) \cap \mathbb{Z}^n\} \;=\; f[C(\mathbf{\Omega}(y)\sigma), z],\end{aligned} \tag{3.11}$$

which is exactly (3.10). Notice that $x_{\not\sigma} = u + \mu_\sigma v$, and so $\delta_\sigma(y - A_{\not\sigma}u) = \delta_\sigma(y - A_{\not\sigma}x_{\not\sigma})$ because of the definition (2.3) of δ_σ. Finally, using (3.11) in (3.9) yields that (3.6) holds whenever $\|z\| < 1$ and $R_1(y,\sigma;z)$ and $R_2(\sigma;z)$ are all well defined. □

Notice that R_2 is constant with respect to y, and from the definition (2.3) of δ_σ, R_1 is *quasiperiodic* with periods A_σ and μ_σ, meaning that

$$\begin{aligned} R_1(y + A_\sigma q, \sigma; z) &= R_1(y,\sigma;z)\, z_\sigma^q \quad \text{and} \\ R_1(y + \mu_\sigma q, \sigma; z) &= R_1(y,\sigma;z)\left(z_\sigma^{A_\sigma^{-1}q}\right)^{\mu_\sigma}\end{aligned} \tag{3.12}$$

hold for all $y, q \in \mathbb{Z}^m$. Obviously, the more expensive part in calculating $R_2(\cdot)$ in (3.8) is to compute the determinant $\mu_\sigma = |\det A_\sigma|$. On the other hand, computing $R_1(\cdot)$ in (3.7) may become quite expensive when μ_σ is large, as one must evaluate μ_σ^{n-m} terms, the cardinality of $\mathbb{Z}^{n-m}_{\mu_\sigma}$. However, as detailed below, a more careful analysis of (3.7) yields some simplifications.

3.1. **Simplifications via group theory.** From the proof of Theorem 2, the closed forms (3.7)–(3.8) for $R_1(\cdot)$ and $R_2(\cdot)$ are deduced from (3.11), i.e.,

$$\frac{R_1(y,\sigma;z)}{R_2(\sigma;z)} = z_\sigma^{A_\sigma^{-1}y} \sum_{x_{\not\sigma}\in\mathbb{Z}^{n-m}} \frac{\delta_\sigma(y-A_{\not\sigma}x_{\not\sigma})\, z_{\not\sigma}^{x_{\not\sigma}}}{z_\sigma^{A_\sigma^{-1}A_{\not\sigma}x_{\not\sigma}}},$$

after setting $x_{\not\sigma} = u + \mu_\sigma v$ and recalling that $\delta_\sigma(y)$ is a periodic function, i.e., $\delta_\sigma(y + \mu_\sigma q) = \delta_\sigma(y)$ for all $y, q \in \mathbb{Z}^m$. However, we have not used yet that $\delta_\sigma(y + A_\sigma q) = \delta_\sigma(y)$ as well. For every $\sigma \in \mathbb{J}_A$, consider the lattice :

$$\Lambda_\sigma := \bigoplus_{j\in\sigma} A_j\mathbb{Z} \subset \mathbb{Z}^m, \tag{3.13}$$

generated by the columns A_j, $j \in \sigma$. The following quotient group

$$\begin{aligned} \mathcal{G}_\sigma &:= \mathbb{Z}^m/\Lambda_\sigma = \mathbb{Z}^m \Big/ \bigoplus_{j\in\sigma} A_j\mathbb{Z} \\ &= \{\mathrm{Ec}[0,\sigma], \mathrm{Ec}[2,\sigma], \ldots, \mathrm{Ec}[\mu_\sigma - 1, \sigma]\} \end{aligned} \tag{3.14}$$

is commutative, with $\mu_\sigma = |\det A_\sigma|$ elements (or, equivalence classes) $\mathrm{Ec}[j,\sigma]$, and so, $\mathcal{G}_\sigma$ is isomorphic to a finite Cartesian product of cyclic groups $\mathbb{Z}_{\eta_k}$, i.e.,

$$\mathcal{G}_\sigma \cong \mathbb{Z}_{\eta_1} \times \mathbb{Z}_{\eta_2} \times \cdots \times \mathbb{Z}_{\eta_s}.$$

Obviously, $\mu_\sigma = \eta_1\eta_2\cdots\eta_s$, and so, $\mathcal{G}_\sigma$ is isomorphic to the cyclic group $\mathbb{Z}_{\mu_\sigma}$ whenever μ_σ is a prime number. Actually, $\mathcal{G}_\sigma = \{0\}$ whenever $\mu_\sigma = 1$. Notice that the Cartesian product $\mathbb{Z}_{\eta_1} \times \cdots \times \mathbb{Z}_{\eta_s}$ can be seen as the integer space $\mathbb{Z}^s$ modulo the vector $\eta := (\eta_1, \eta_2, \cdots, \eta_s)' \in \mathbb{N}^s$.

Hence, for every finite commutative group $\mathcal{G}_\sigma$, there exist a positive integer $s_\sigma \geq 1$, a vector $\eta_\sigma \in \mathbb{N}^{s_\sigma}$ with positive entries, and a group isomorphism,

$$g_\sigma : \mathcal{G}_\sigma \to \mathbb{Z}^{s_\sigma} \bmod \eta_\sigma, \tag{3.15}$$

where $g_\sigma(\xi) \bmod \eta_\sigma$ means evaluating $[g_\sigma(\xi)]_k \bmod [\eta_\sigma]_k$, for all indices $1 \leq k \leq s_\sigma$. For every $y \in \mathbb{Z}^m$, there exists a unique equivalence class $\mathrm{Ec}[j_y, \sigma]$ which contains y, and so we can define the following group epimorphism,

$$\begin{aligned} \hat{h}_\sigma &: \mathbb{Z}^m \to \mathbb{Z}^{s_\sigma} \quad \bmod \eta_\sigma, \\ y &\mapsto \hat{h}_\sigma(y) := g_\sigma(\mathrm{Ec}[j_y, \sigma]). \end{aligned} \tag{3.16}$$

On the other hand, the unit element of $\mathcal{G}_\sigma$ is the equivalence class $\mathrm{Ec}[0,\sigma]$ which contains the origin, that is, $\mathrm{Ec}[0,\sigma] = \{A_\sigma q \mid q \in \mathbb{Z}^m\}$.

Hence, $\hat{h}_\sigma(y) = 0$ if and only if there exists $q \in \mathbb{Z}^m$ such that $y = A_\sigma q$. We can then redefine the function δ_σ as follows,

$$y \mapsto \delta_\sigma(y) := \begin{cases} 1 & \text{if } \hat{h}_\sigma(y) = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.17}$$

One also needs the following additional notation; given any matrix $B \in \mathbb{Z}^{m\times t}$,

$$\hat{h}_\sigma(B) := [\hat{h}_\sigma(B_1)|\hat{h}_\sigma(B_2)|\cdots|\hat{h}_\sigma(B_t)] \in \mathbb{Z}^{s_\sigma\times t}. \tag{3.18}$$

And so, from (3.7), $\hat{h}_\sigma(y - A_{\not\sigma}u) \equiv \hat{h}_\sigma(y) - \hat{h}_\sigma(A_{\not\sigma})u \bmod \eta_\sigma$. Finally, using (3.17) in (3.7), one obtains a simplified version of $R_1(\cdot)$ in the form:

$$R_1(y,\sigma;z) = \sum \left\{ \frac{z_\sigma^{A_\sigma^{-1}y} z_{\not\sigma}^u}{z_\sigma^{A_\sigma^{-1}A_{\not\sigma}u}} \;\middle|\; \begin{array}{l} u \in \mathbb{Z}_{\mu_\sigma}^{n-m}; \\ \hat{h}_\sigma(y) \equiv \hat{h}_\sigma(A_{\not\sigma})u \bmod \eta_\sigma \end{array} \right\}. \tag{3.19}$$

Next, with $q \in \mathbb{Z}^m$ fixed, $\nu_q A_\sigma^{-1} q \in \mathbb{Z}^m$ for some integer ν_q, if and only if $\nu_q \hat{h}_\sigma(q) = 0 \bmod \eta_\sigma$. If we set $\nu_q = \mu_\sigma$, then $\mu_\sigma A_\sigma^{-1} q \in \mathbb{Z}^m$, and $\mu_\sigma \hat{h}_\sigma(q) = 0 \bmod \eta_\sigma$, because $\mathcal{G}_\sigma$ has $\mu_\sigma = |\det A_\sigma|$ elements. Nevertheless, μ_σ may not be the smallest positive integer with that property. So, given $\sigma \in \mathbb{J}_A$ and $k \notin \sigma$, define $\nu_{k,\sigma} \geq 1$ to be *order* of $\hat{h}_\sigma(A_k)$. That is, $\nu_{k,\sigma}$ is the smallest positive integer such that $\nu_{k,\sigma} \hat{h}_\sigma(A_k) = 0 \bmod \eta_\sigma$, or equivalently :

$$\nu_{k,\sigma} A_\sigma^{-1} A_k \in \mathbb{Z}^m. \tag{3.20}$$

Obviously $\nu_{k,\sigma} \leq \mu_\sigma$. Moreover, μ_σ is a multiple of $\nu_{k,\sigma}$ for it is the order of an element in $\mathcal{G}_\sigma$. For example, the group $\mathbb{Z}^2$ modulo $\eta = \binom{2}{7}$ has 14 elements; and the elements $b_1 = \binom{1}{0}$, $b_2 = \binom{0}{1}$ and $b_3 = \binom{1}{1}$ have respective orders : 2, 7 and 14. Notice that, $2b_1 \equiv 7b_2 \equiv 14b_3 \equiv 0 \bmod \eta$. But, $2b_3 \equiv 2b_2 \not\equiv 0$ and $7b_3 \equiv b_1 \not\equiv 0 \bmod \eta$.

The important observation is that $\delta_\sigma(y - \nu_{k,\sigma} A_k q) = \delta_\sigma(y)$ for all $q \in \mathbb{Z}^m$ and $k \notin \sigma$, which follows from (3.20) and (2.3). Thus, following step by step the proof of Theorem 2, we obtain:

Corollary 3. *Let $y \in \mathbb{Z}^m$, $z \in \mathbb{C}^n$ with $\|z\| < 1$, and let $y \in \overline{\gamma}$ for some chamber γ. Recall the set of bases $\mathcal{B}(\mathbb{J}_A, \gamma)$ defined in (3.1). With $\sigma \in \mathcal{B}(\mathbb{J}_A, \gamma)$, let R_1 and R_2 be as in Theorem 2. Then*

$$\frac{R_1(y, \sigma; z)}{R_2(y; z)} = \frac{R_1^*(y, \sigma; z)}{R_2^*(y; z)}, \tag{3.21}$$

$$\text{where :} \qquad R_2^*(\sigma; z) := \prod_{k \notin \sigma} \left[1 - \left(z_k z_\sigma^{-A_\sigma^{-1} A_k} \right)^{\nu_{k,\sigma}} \right], \tag{3.22}$$

$$R_1^*(y, \sigma; z) := z_\sigma^{A_\sigma^{-1} y} \sum_{u_{\not\sigma} \in U_{\not\sigma}} \frac{\delta_\sigma(y - A_{\not\sigma} u_{\not\sigma})\, z_{\not\sigma}^{u_{\not\sigma}}}{z_\sigma^{A_\sigma^{-1} A_{\not\sigma} u_{\not\sigma}}} = \tag{3.23}$$

$$= \sum \left\{ \frac{z_\sigma^{A_\sigma^{-1} y} z_{\not\sigma}^{u_{\not\sigma}}}{z_\sigma^{A_\sigma^{-1} A_{\not\sigma} u_{\not\sigma}}} \;\middle|\; \begin{array}{l} u_{\not\sigma} \in U_{\not\sigma}; \\ \hat{h}_\sigma(y) \equiv \hat{h}_\sigma(A_{\not\sigma}) u_{\not\sigma} \mod \eta_\sigma \end{array} \right\},$$

with $U_{\not\sigma} := \{ u_{\not\sigma} \in \mathbb{N}^{n-m} \mid 0 \leq u_k \leq \nu_{k,\sigma} - 1;\ k \notin \sigma \}$.

One can also obtain (3.21) by noticing that:

$$\frac{R_1(y, \sigma; z)}{R_1^*(y, \sigma; z)} = \frac{R_2(\sigma; z)}{R_2^*(\sigma; z)} = \prod_{k \notin \sigma} \left(1 + \beta^{\nu_{k,\sigma}} + \cdots + \beta^{\mu_\sigma - \nu_{k,\sigma}} \right),$$

where $\beta_{k,\sigma} = z_k z_\sigma^{-A_\sigma^{-1} A_k}$, and μ_σ is a multiple of $\nu_{k,\sigma}$.

3.2. **Simplifications via finite number of generators.** Decompose $\mathbb{Z}^m$ into $\mu_\sigma := |\det A_\sigma|$ disjoint equivalent classes, where $y, \xi \in \mathbb{Z}^m$ are equivalent if and only if $\delta_\sigma(y - \xi) = 1$. For every basis $\sigma \in \mathbb{J}_A$, let $\mathcal{G}_\sigma$ be the quotient group defined in (3.14), that is,

$$\mathcal{G}_\sigma \;=\; \mathbb{Z}^m \Big/ \bigoplus_{j\in\sigma} A_j\mathbb{Z} \;=\; \{\mathrm{Ec}[0,\sigma],\ldots,\mathrm{Ec}[\mu_\sigma-1,\sigma]\}.$$

Notice that $y,\xi\in\mathbb{Z}^n$ belong to $\mathrm{Ec}[j,\sigma]$ if and only if $A_\sigma^{-1}(y-\xi)\in\mathbb{Z}^n$, and that $\mathbb{Z}^m$ is equal to the disjoint union of all classes $\mathrm{Ec}[j,\sigma]$.

Next, pick up a *minimal* representative element of every class, i.e., fix

$$\xi[j,\sigma]\in\mathrm{Ec}[j,\sigma] \quad \text{such that} \quad A_\sigma^{-1}y \ge A_\sigma^{-1}\xi[j,\sigma]\ge 0, \tag{3.24}$$

for every $y\in\mathrm{Ec}[j,\sigma]$ with $A_\sigma^{-1}y\ge 0$. The minimal representative elements $\xi[j,\sigma]$ in (3.24) can be computed as follows: Let $d\in\mathrm{Ec}[j,\sigma]$, arbitrary, and let $d^*\in\mathbb{Z}^m$ be such that his k-entry d_k^* is the smallest integer greater than or equal to the k-entry of $-A_\sigma^{-1}d$. The vector $\xi[j,\sigma]$ defined by $d+A_\sigma d^*$ satisfies (3.24).

Notice that $d^*+A_\sigma^{-1}d\ge 0$. Besides, let $d,y\in\mathrm{Ec}[j,\sigma]$ with $A_\sigma^{-1}y\ge 0$. There exists $q\in\mathbb{Z}^m$ such that $y=d+A_\sigma q$. Hence $q\ge -A_\sigma^{-1}d$; in addition, $q\ge d^*$ follows from the above definition of d^*, and so $A_\sigma^{-1}y\;\ge\; d^*+A_\sigma^{-1}d\;\ge$ 0.

Therefore, the vector $\xi[j,\sigma]:=d+A_\sigma d^*$ satisfies (3.24). In particular, if $\mathrm{Ec}[0,\sigma]$ is the class which contains the origin of $\mathbb{Z}^m$, then $\xi[0,\sigma]=0$. Notice that for every integer vector $y\in\mathbb{Z}^m$, there exists a unique $\xi[j,\sigma]$ such that :

$$y \;=\; \xi[j,\sigma] \,+\, A_\sigma\, q, \quad \text{for} \quad q\in\mathbb{Z}^m.$$

Moreover, the extra condition $A_\sigma^{-1}y\ge 0$ holds if and only if:

$$y \;=\; \xi[j,\sigma] \,+\, A_\sigma\, q \quad \text{with} \quad q\in\mathbb{N}^m. \tag{3.25}$$

We obtain a compact form of $h(y;z)$ when $y\in\mathbb{Z}^m\cap\overline{\gamma}$, for some chamber γ.

Theorem 4. *Let h and $\xi[j,\sigma]$ be as in (1.2) and (3.24), respectively. Let $y\in\mathbb{Z}^m\cap\overline{\gamma}$, for some chamber γ. Recall the set of bases $\mathcal{B}(\mathbb{J}_A,\gamma)$ defined in (3.1). For every basis $\sigma\in\mathcal{B}(\Delta,\gamma)$ there is a unique index $0\le \jmath[y,\sigma]<\mu_\sigma$ such that y is contained in the equivalence class $\mathrm{Ec}[\jmath[y,\sigma],\sigma]$ defined in (3.14), and so:*

$$h(y;z) \;=\; \sum_{\sigma\in\mathcal{B}(\Delta,\gamma)} \frac{R_1(\xi[\jmath[y,\sigma],\sigma],\sigma;z)}{R_2(\sigma;z)}\; z_\sigma^{\lfloor A_\sigma^{-1}y\rfloor}, \tag{3.26}$$

where $\lfloor {A_\sigma}^{-1}y\rfloor\in\mathbb{Z}^m$ is such that his k-entry is the largest integer less than or equal to the k-entry of $A_\sigma^{-1}y$.

Proof. Recall that if $y\in\mathbb{Z}^m\cap\overline{\gamma}$

$$h(y;z) \;=\; \sum_{\sigma\in\mathcal{B}(\Delta,\gamma)} \frac{R_1(y,\sigma;z)}{R_2(y;z)}$$

Next, recalling the definition (3.1) of $\mathcal{B}(\mathbb{J}_A,\gamma)$, $A_\sigma^{-1}y\ge 0$ for every basis $\sigma\in\mathcal{B}(\mathbb{J}_A,\gamma)$ with $y\in\overline{\gamma}$. Recall that there is a unique index $\jmath[y,\sigma]<\mu_\sigma$

such that $y = \xi[\jmath[y,\sigma],\sigma] + A_\sigma q$ with $q \in \mathbb{N}^m$; see (3.25) and the comment just before.

To obtain the vector $q \in \mathbb{N}^m$, recall that the minimal representative element $\xi[\jmath[y,\sigma],\sigma]$ in (3.24) is the sum $y + A_\sigma y^*$ where $y^* \in \mathbb{Z}^m$ is such that his k-entry y^*_k is the smallest integer greater than or equal to $-A_\sigma^{-1}y$, for we only need to fix $d = y$ in the paragraph that follows (3.24). In particular, $\lfloor A_\sigma{}^{-1}y\rfloor = -y^*$, and $\xi[\jmath[y,\sigma],\sigma] = y - A_\sigma\lfloor A_\sigma{}^{-1}y\rfloor$, which when used in (3.7) and (3.12), yields,

$$R_1(y,\sigma;z) \;=\; R_1(\xi[\jmath[y,\sigma],\sigma],\sigma;z)\, z_\sigma^{\lfloor A_\sigma^{-1}y\rfloor}.$$

And so (3.6) implies (3.26). □

Theorem 4 explicitly shows that it suffices to compute $R_1(v,\sigma;z)$ for finitely many values $v = \xi[j,\sigma]$, with $\sigma \in \mathcal{B}(\Delta,\gamma)$ and $0 \le j < \mu_\sigma$, in order to calculate $h(y;z)$ for arbitrary values $y \in \mathbb{Z}^m \cap \overline{\gamma}$, via (3.26).

In other words, in the closure $\overline{\gamma}$ of a chamber γ, one only needs to consider *finitely many* fixed convex cones $C(\mathbf{\Omega}(\xi[j,\sigma]),\sigma) \subset \mathbb{R}^n$, where $\sigma \in \mathcal{B}(\Delta,\gamma)$ and $0 \le j < \mu_\sigma$, and compute their associated rational function (3.26). The counting function $h(y;z)$ is then obtained as follows.

Input: $y \in \mathbb{Z}^m \cap \overline{\gamma}$, $z \in \mathbb{C}^n$.
Output $\rho = h(y;z)$.
Set $\rho := 0$. For every $\sigma \in \mathcal{B}(\Delta,\gamma)$:

- Compute $\xi[\jmath[y,\sigma],\sigma] := y - A_\sigma\lfloor A_\sigma^{-1}y\rfloor \in \mathbb{Z}^m$.
- Read the value $R_1(\xi[\jmath[y,\sigma],\sigma],\sigma;z)/R_2(\sigma;z)$, and update ρ by:

$$\rho := \rho + \frac{R_1(\xi[\jmath[y,\sigma],\sigma],\sigma;z)}{R_2(\sigma;z)}\, z_\sigma^{\lfloor A_\sigma^{-1}y\rfloor}.$$

For the whole space $\mathbb{Z}^m$ it suffices to consider *all* chambers γ and all cones $C(\mathbf{\Omega}(\xi[j,\sigma]),\sigma) \subset \mathbb{R}^n$, where $\sigma \in \mathcal{B}(\Delta,\gamma)$ and $0 \le j < \mu_\sigma$.

Finally, in view of (3.7)-(3.8), the above algorithm can be symbolic, i.e., $z \in \mathbb{C}^m$ can be treated symbolically, and ρ becomes a rational fraction of z.

4. Generating function

An appropriate tool for computing the exact value of $h(y;z)$ in (1.2) is the formal generating function $H : \mathbb{C}^m \to \mathbb{C}$,

$$s \mapsto H(s) \;:=\; \sum_{y\in\mathbb{Z}^m} h(y;z)\, s^y \;=\; \prod_{k=1}^{n} \frac{1}{1 - z_k s^{A_k}}, \tag{4.1}$$

where s^y is defined in (2.4) and the sum is understood as a formal power series, so that we need not consider conditions for convergence. This generating function was already considered in Brion and Vergne [8] with $\lambda = \ln\langle s\rangle$.

Following notation of [8, p. 805], let $0 \le \hat{x} \in \mathbb{R}^n$ be a *regular* vector, i.e., no entry $[A_\sigma^{-1}A\hat{x}]_j$ vanishes for any basis $\sigma \in \mathbb{J}_A$ or index $1 \le j \le m$.

Define :

$$\varepsilon_{j,\sigma} := \begin{cases} 1 & \text{if} \quad [A_\sigma^{-1}A\hat{x}]_j > 0, \\ -1 & \text{if} \quad [A_\sigma^{-1}A\hat{x}]_j < 0. \end{cases} \tag{4.2}$$

Next, for every basis $\sigma \in \mathbb{J}_A$, index $j \in \sigma$ and vector $u_{\not\sigma} \in \mathbb{Z}^{n-m}$, fix :

$$\begin{array}{ll} \theta[j,\sigma,u_{\not\sigma}] \in \mathbb{Z}: & \text{the smallest integer greater} \\ \text{than or equal to} & -\,\varepsilon_{j,\sigma}[A_\sigma^{-1}A_{\not\sigma}u_{\not\sigma}]_j. \end{array} \tag{4.3}$$

Define also the vector $\eta[\sigma,u_{\not\sigma}] \in \mathbb{Z}^n$ by :

$$\eta[\sigma,u_{\not\sigma}]_j = \begin{cases} u_j & \text{if } j \notin \sigma; \\ \theta[j,\sigma,u_{\not\sigma}] & \text{if } j \in \sigma,\ \varepsilon_{j,\sigma} = 1; \\ 1-\theta[j,\sigma,u_{\not\sigma}] & \text{if } j \in \sigma,\ \varepsilon_{j,\sigma} = -1. \end{cases} \tag{4.4}$$

The following expansion can be deduced from [8].

Theorem 5. *Let $0 \le \hat{x} \in \mathbb{R}^n$ be regular and consider the vectors $\eta[\sigma,u_{\not\sigma}] \in \mathbb{Z}^n$ defined in (4.4) for $\sigma \in \mathbb{J}_A$ and $u_{\not\sigma} \in \mathbb{Z}^{n-m}$. The following expansion holds:*

$$\prod_{k=1}^n \frac{1}{1-z_k s^{A_k}} = \sum_{\sigma\in\mathbb{J}_A} \left[\prod_{j\in\sigma} \frac{1}{1-z_j s^{A_j}}\right] \times \times \frac{1}{R_2(\sigma;z)} \sum_{u_{\not\sigma}\in\mathbb{Z}_{\mu_\sigma}^{n-m}} z^{\eta[\sigma,u_{\not\sigma}]}\, s^{A\eta[\sigma,u_{\not\sigma}]}, \tag{4.5}$$

where $\mathbb{Z}_{\mu_\sigma} = \{0,1,\ldots,\mu_\sigma-1\}$, $\mu_\sigma = |\det A_\sigma|$ and:

$$z \mapsto R_2(\sigma;z) := \prod_{k\notin\sigma} \left[1-\left(z_k z_\sigma^{-A_\sigma^{-1}A_k}\right)^{\mu_\sigma}\right]. \tag{4.6}$$

Proof. From Brion and Vergne's identity [8, p. 813],

$$\prod_{j=1}^n \frac{1}{1-\mathrm{e}^{w_k}} = \sum_{\sigma\in\mathbb{J}_A} \left[\prod_{j\in\sigma}\varepsilon_{j,\sigma}\right] F(C_{\hat{x}}^\sigma + \rho_\sigma(C), L), \tag{4.7}$$

where $F(C_{\hat{x}}^\sigma + \rho_\sigma(C), L)$ is the formal power series $\sum_l \mathrm{e}^l$ added over all elements l in the intersection of the cone $C_{\hat{x}}^\sigma + \rho_\sigma(C)$ with the integer lattice $L = \mathbb{Z}[w_1,\ldots,w_n]$. Moreover, the coefficients $\varepsilon_{j,\sigma}$ are defined in (4.2) and the cone $C_{\hat{x}}^\sigma$ is defined by the following formula [8, p. 805],

$$C_{\hat{x}}^\sigma = \left\{ \sum_{j\in\sigma} \varepsilon_{j,\sigma}\, x_j\, w_j \;\middle|\; x_\sigma \in \mathbb{R}^m,\ x_\sigma \ge 0 \right\}. \tag{4.8}$$

Finally, given the real vector space $W = \mathbb{R}[w_1,\ldots,w_n]$, every $\rho_\sigma : W \to W$ is a linear mapping defined by its action on each basis element w_k of W,

$$\rho_\sigma(w_k) := w_k - \sum_{j\in\sigma} [A_\sigma^{-1}A_k]_j w_j. \tag{4.9}$$

Hence, $\rho_\sigma(w_j) = 0$ for every $j \in \sigma$, and the cone $\rho_\sigma(C)$ is given by

$$\rho_\sigma(C) \;=\; \left\{ \sum_{k\notin\sigma} x_k w_k \;-\; \sum_{j\in\sigma} [A_\sigma^{-1} A_{\not\sigma} x_{\not\sigma}]_j w_j \;\middle|\; \begin{array}{l} x_{\not\sigma} \in \mathbb{R}^{n-m}, \\ x_{\not\sigma} \geq 0 \end{array} \right\}; \tag{4.10}$$

see [8, p.805]. Thus, every element in the intersection of the cone $C^\sigma_{\hat{x}} + \rho_\sigma(C)$ with the lattice $\mathbb{Z}[w_1, \ldots, w_n]$ must be of the form :

$$\sum_{k\notin\sigma} x_k w_k \;+\; \sum_{j\in\sigma} \varepsilon_{j,\sigma}\, \xi_j\, w_j, \quad \text{with} \quad x_{\not\sigma} \in \mathbb{N}^{n-m}, \tag{4.11}$$
$$\xi_\sigma \in \mathbb{Z}^m \quad \text{and} \quad \xi_j \geq -\varepsilon_{j,\sigma} [A_\sigma^{-1} A_{\not\sigma} x_{\not\sigma}]_j.$$

On the other hand, for every basis σ, define $\mu_\sigma = |\det A_\sigma|$ and :

$$x_{\not\sigma} \;=\; u_{\not\sigma} + \mu_\sigma v_{\not\sigma}, \quad \text{with} \quad u_{\not\sigma} \in \mathbb{Z}^{n-m}_{\mu_\sigma} \quad \text{and} \quad v_{\not\sigma} \in \mathbb{N}^{n-m}. \tag{4.12}$$

Moreover, as in (4.3), fix $\theta[j, \sigma, u_{\not\sigma}] \in \mathbb{Z}$ to be the smallest integer greater than or equal to $-\varepsilon_{j,\sigma}[A_\sigma^{-1} A_{\not\sigma} u_{\not\sigma}]_j$. Thus, we can rewrite (4.11) so that the intersection of the cone $C^\sigma_{\hat{x}} + \rho_\sigma(C)$ with the lattice $\mathbb{Z}[w_1, \ldots, w_n]$ must be of the form :

$$\sum_{k\notin\sigma} \big[u_k w_k + v_k \mu_\sigma \rho(w_k)\big] \;+\; \sum_{j\in\sigma} \varepsilon_{j,\sigma} w_j \big[\theta[j, \sigma, u_{\not\sigma}]_j + q_j\big], \tag{4.13}$$
$$\text{with} \quad u_{\not\sigma} \in \mathbb{Z}^{n-m}_{\mu_\sigma}, \quad v_{\not\sigma} \in \mathbb{N}^{n-m} \quad \text{and} \quad q_\sigma \in \mathbb{N}^m.$$

We can deduce (4.13) from (4.11) by recalling the definition (4.9) of $\rho_\sigma(w_k)$ and letting :

$$\xi_j \;:=\; \theta[j, \sigma, u_{\not\sigma}] \;+\; q_j \;-\; \varepsilon_{j,\sigma} \mu_\sigma [A_\sigma^{-1} A_{\not\sigma} v_{\not\sigma}]_j.$$

Since $F(C^\sigma_{\hat{x}} + \rho_\sigma(C), L)$ is the formal power series $\sum_l \mathrm{e}^l$ with summation over all elements l in (4.13), one obtains

$$F(C^\sigma_{\hat{x}} + \rho_\sigma(C), L) \;= \tag{4.14}$$
$$\sum_{u_{\not\sigma} \in \mathbb{Z}^{n-m}_{\mu_\sigma}} \left[\prod_{j\in\sigma} \frac{\mathrm{e}^{\varepsilon_{j,\sigma}\theta[j,\sigma,u_{\not\sigma}] w_j}}{1 - \mathrm{e}^{\varepsilon_{j,\sigma} w_j}} \right] \left[\prod_{k\notin\sigma} \frac{\mathrm{e}^{u_k w_k}}{1 - \mathrm{e}^{\mu_\sigma \rho_\sigma(w_k)}} \right].$$

With $\eta[\sigma, u_{\not\sigma}] \in \mathbb{Z}^n$ being as in (4.4), using (4.14) into (4.7) yields the expansion

$$\prod_{j=1}^{n} \frac{1}{1 - \mathrm{e}^{w_k}} \;=\; \sum_{\sigma\in\mathbb{J}_A} \sum_{u_{\not\sigma} \in \mathbb{Z}^{n-m}_{\mu_\sigma}} \left[\prod_{j\in\sigma} \frac{1}{1 - \mathrm{e}^{w_j}} \right] \times \tag{4.15}$$
$$\times \left[\prod_{j=1}^{n} \mathrm{e}^{\eta[\sigma, u_{\not\sigma}]_j w_j} \right] \left[\prod_{k\notin\sigma} \frac{1}{1 - \mathrm{e}^{\mu_\sigma \rho_\sigma(w_k)}} \right].$$

Finally, we defined $w_k := \ln(z_k) + \ln\langle s\rangle A_k$ for every index $1 \leq k \leq n$, where the vectors $s, z \in \mathbb{C}^n$ have all their entries different from zero and $\ln\langle s\rangle$ is the $[1 \times n]$ matrix defined in (2.4). So $\mathrm{e}^{w_k} = z_k s^{A_k}$. Moreover, recalling the definition (4.9) of $\rho_\sigma(w_k)$, the following identities hold for all $1 \leq k \leq n$,

$$\rho_\sigma(w_k) \;=\; \ln(z_k) \;-\; \sum_{j\in\sigma} \ln(z_j)[A_\sigma^{-1}A_k]_j. \tag{4.16}$$

Notice $\sum_{j\in\sigma} A_j[A_\sigma^{-1}A_k]_j \;=\; A_k$. A direct application of (4.16) and the identities $\mathrm{e}^{w_k} = z_k s^{A_k}$ yields (4.5), i.e.:

$$\prod_{k=1}^{n} \frac{1}{1 - z_k s^{A_k}} = \sum_{\sigma\in\mathbb{J}_A} \sum_{u_{\not\sigma}\in\mathbb{Z}_{\mu_\sigma}^{n-m}} \frac{z^{\eta[\sigma,u_{\not\sigma}]} s^{A\eta[\sigma,u_{\not\sigma}]}}{R_2(\sigma;z)} \prod_{j\in\sigma} \frac{1}{1 - z_j s^{A_j}},$$

$$\textit{with} \quad R_2(\sigma;z) \;=\; \prod_{k\notin\sigma} \left[1 - \left(z_k z_\sigma^{-A_\sigma^{-1}A_k}\right)^{\mu_\sigma}\right].$$

□

A direct expansion of (4.5) yields the following:

Theorem 6. *Let $0 \leq \hat{x} \in \mathbb{R}^n$ be regular, and let h and η be as in (1.2) and (4.4), respectively. Let $\mathbb{J}_A$ be the set of bases associated with A. Then for every pair of $(y,z) \in \mathbb{Z}^m \times \mathbb{C}^n$ with $\|z\| < 1$:*

$$h(y;z) \;=\; \sum_{\sigma\in\mathbb{J}_A,\, A_\sigma^{-1}y\geq 0} \frac{z_\sigma^{A_\sigma^{-1}y}}{R_2(\sigma;z)} \sum_{u\in\mathbb{Z}_{\mu_\sigma}^{n-m}} \frac{z_{\not\sigma}^{u}}{z_\sigma^{A_\sigma^{-1}A_{\not\sigma}u}} \times \times \begin{cases} 1 & \textit{if } A_\sigma^{-1}\big(y - A\eta[\sigma,u]\big) \in \mathbb{N}^m, \\ 0 & \textit{otherwise,} \end{cases} \tag{4.17}$$

where: $\mathbb{Z}_{\mu_\sigma} = \{0,1,\ldots,\mu_\sigma - 1\}$, $\mu_\sigma = |\det A_\sigma|$,

$$0 \;\leq\; \big[A_\sigma^{-1}A\eta[\sigma,u]\big]_j \;\leq\; 1 \quad \textit{for each} \quad j\in\sigma \tag{4.18}$$

$$\textit{and} \quad R_2(\sigma;z) \;:=\; \prod_{k\notin\sigma} \left[1 - \left(z_k z_\sigma^{-A_\sigma^{-1}A_k}\right)^{\mu_\sigma}\right]. \tag{4.19}$$

Proof. Recall the expansion of $H(s)$ as a formal power series :

$$H(s) \;=\; \sum_{y\in\mathbb{Z}^m} h(y;z)\, s^y \;=\; \prod_{k=1}^{n} \frac{1}{1 - z_k s^{A_k}}. \tag{4.20}$$

We also have a similar formal power series for the product :

$$\prod_{j\in\sigma} \frac{1}{1 - z_j s^{A_j}} = \prod_{j\in\sigma} \left[\sum_{q_j=1}^{\infty} z_j^{q_j} s^{A_j q_j}\right] = \sum_{q_\sigma\in\mathbb{N}^m} z_\sigma^{q_\sigma} s^{A_\sigma q_\sigma}.$$

Combining the latter with (4.5) yields:

$$H(s) \;=\; \sum_{\sigma\in\mathbb{J}_A} \sum_{q_\sigma\in\mathbb{N}^m} \sum_{u_{\not\sigma}\in\mathbb{Z}_{\mu_\sigma}^{n-m}} \frac{z^{\eta[\sigma,u_{\not\sigma}]} z_\sigma^{q_\sigma} s^{A\eta[\sigma,u_{\not\sigma}]} s^{A_\sigma q_\sigma}}{R_2(\sigma;z)}. \tag{4.21}$$

Notice that (4.20) and (4.21) are identical. Hence, if we want to obtain the exact value of $h(y;z)$ from (4.21), we only have to sum up all terms with exponent $A\eta[\sigma,u_{\not\sigma}]+A_\sigma q_\sigma$ equal to y. That is, recalling that each A_σ is invertible,

$$(4.22)\qquad h(y;z) \;=\; \sum_{\sigma\in\mathbb{J}_A}\;\sum_{u_{\not\sigma}\in\mathbb{Z}^{n-m}_{\mu_\sigma}} \frac{z^{\eta[\sigma,u_{\not\sigma}]}z_\sigma^{q_\sigma}}{R_2(\sigma;z)}\times \\ \times\begin{cases} 1 & \text{if } A_\sigma^{-1}\big(y-A\eta[\sigma,u_{\not\sigma}]\big)\in\mathbb{N}^m,\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, setting $q_\sigma := A_\sigma^{-1}(y-A\eta[\sigma,u_{\not\sigma}])$ and recalling the definition (4.4),

$$(4.23)\qquad z^{\eta[\sigma,u_{\not\sigma}]}z_\sigma^{q_\sigma} \;=\; \frac{z_\sigma^{\eta[\sigma,u_{\not\sigma}]_\sigma}\,z_{\not\sigma}^{u_{\not\sigma}}\,z_\sigma^{A_\sigma^{-1}y}}{z_\sigma^{A_\sigma^{-1}A\eta[\sigma,u_{\not\sigma}]}} \;=\; \frac{z_\sigma^{A_\sigma^{-1}y}\,z_{\not\sigma}^{u_{\not\sigma}}}{z_\sigma^{A_\sigma^{-1}A_{\not\sigma}u_{\not\sigma}}}.$$

We finally prove that the vector $A_\sigma^{-1}A\eta[\sigma,u_{\not\sigma}]$ is bounded, so that (4.18) holds,

$$(4.24)\qquad 0 \;\le\; \big[A_\sigma^{-1}A\eta[\sigma,u_{\not\sigma}]\big]_j \;\le\; 1 \quad\text{for each}\quad j\in\sigma.$$

From (4.3) and (4.4), the following identity

$$(4.25)\qquad \big[A_\sigma^{-1}A\eta[\sigma,u_{\not\sigma}]\big]_j = \big[A_\sigma^{-1}A_{\not\sigma}u_{\not\sigma}\big]_j + \begin{cases} \theta[j,\sigma,u_{\not\sigma}] & \text{if } \varepsilon_{j,\sigma}=1;\\ 1-\theta[j,\sigma,u_{\not\sigma}] & \text{if } \varepsilon_{j,\sigma}=-1\end{cases}$$

holds for every index $j\in\sigma$.

Next, suppose that $[A_\sigma^{-1}A_{\not\sigma}u_{\not\sigma}]_j = R+r$ where $R\in\mathbb{Z}$ and $0\le r<1$ are the respective integer and fractional parts. We can obtain $\theta[j,\sigma,u_{\not\sigma}]$ in (4.3) as follows: If $\varepsilon_{j,\sigma}=1$ then $\theta[j,\sigma,u_{\not\sigma}]=-R$ is the smallest integer greater than or equal to $-R-r$, and so (4.25) yields

$$\big[A_\sigma^{-1}A\eta[\sigma,u_{\not\sigma}]\big]_j \;=\; r.$$

If $\varepsilon_{j,\sigma}=-1$ and $0<r<1$, then $\theta[j,\sigma,u_{\not\sigma}]=R+1$ and :

$$\big[A_\sigma^{-1}A\eta[\sigma,u_{\not\sigma}]\big]_j \;=\; r.$$

At last, if $\varepsilon_{j,\sigma}=-1$ and $r=0$, then $\theta[j,\sigma,u_{\not\sigma}]=R$ and so,

$$\big[A_\sigma^{-1}A\eta[\sigma,u_{\not\sigma}]\big]_j \;=\; 1.$$

Therefore (4.24) holds because $0\le r<1$. Finally, a direct application of (4.23) into (4.22) yields the following version of (4.17):

$$h(y;z) \;=\; \sum_{\sigma\in\mathbb{J}_A,\,A_\sigma^{-1}y\ge 0}\frac{1}{R_2(\sigma;z)}\sum_{u_{\not\sigma}\in\mathbb{Z}^{n-m}_{\mu_\sigma}}\frac{z_\sigma^{A_\sigma^{-1}y}\,z_{\not\sigma}^{u_{\not\sigma}}}{z_\sigma^{A_\sigma^{-1}A_{\not\sigma}u_{\not\sigma}}}\times \\ \times\begin{cases} 1 & \text{if } A_\sigma^{-1}\big(y-A\eta[\sigma,u_{\not\sigma}]\big)\in\mathbb{N}^m,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that the first sum is calculated only on those bases $\sigma \in \mathbb{J}_A$ which satisfy $A_\sigma^{-1}y \geq 0$. This follows from (4.22) combined with (4.24). □

Observe that (4.17) is different from (3.6) or (3.21) because in (3.6) and (3.21), the summation is over bases σ in the subset $\mathcal{B}(\Delta, \gamma) \subset \{\mathbb{J}_A;\, A_\sigma^{-1}y \geq 0\}$.

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