# SUFFICIENT CONDITIONS FOR A REAL POLYNOMIAL TO BE A SUM OF SQUARES

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ABSTRACT. We provide explicit sufficient conditions for a polynomial $f$ to be a sum of squares (s.o.s.), linear in the coefficients of $f$. All conditions are simple and provide an explicit description of a convex polyhedral subcone of the cone of s.o.s. polynomials of degree at most $2d$. We also provide a simple condition to ensure that $f$ is s.o.s., possibly after adding a constant.

## 1. INTRODUCTION

The cone $\Sigma^2 \subset \mathbb{R}[X]$ of real polynomials that are sum of squares (s.o.s.) and its subcone $\Sigma_d^2$ of s.o.s. of degree at most $2d$, play a fundamental role in many areas, and particularly in optimization; see for instance Lasserre [3, 4], Parrilo [8] and Schweighofer [9]. When considered as a convex cone of a finite dimensional euclidean space, $\Sigma_d^2$ has a *lifted semidefinite representation* (such sets are called SDr sets in [1]). That is, $\Sigma_d^2$ is the projection of a convex cone of an euclidean space of higher dimension, defined in terms of the coefficients of the polynomial and additional variables (the "lifting"). However, so far there is no simple description of $\Sigma_d^2$ given *directly* in terms of the coefficients of the polynomial. For more details on SDr sets, the interested reader is referred to e.g. Ben Tal and Nemirovski [1], Helton and Vinnikov [2], Lewis et al. [7].

Of course, one could use Tarski's quantifier elimination to provide a description of $\Sigma_d^2$, solely in terms of the coefficients, but such a description is likely hopeless to be *simple*; in particular, it could be sensitive to the degree $d$. Therefore, a more reasonable goal is to search for simple descriptions of *subsets* (or *subcones*) of $\Sigma_d^2$ only. This is the purpose of this note in which we provide simple sufficient conditions for a polynomial $f \in \mathbb{R}[X]$ of degree at most $2d$, to be s.o.s. All conditions are expressed directly in terms of the coefficients $(f_\alpha)$, with no additional variable (i.e. with no lifting) and define a convex polyhedral subcone of $\Sigma_d^2$. Finally, we also provide a sufficient condition on the coefficients of highest degree to ensure that $f$ is s.o.s., possibly after adding a constant. All conditions stress the importance of the *essential* monomials $(X_i^{2k})$ which also play an important role for approximating nonnegative polynomials by s.o.s., as demonstrated in e.g. [4, 6].

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## 2. Conditions for being s.o.s.

For $\alpha \in \mathbb{N}^n$ let $|\alpha| := \sum_{i=1}^n |\alpha_i|$. Let $\mathbb{R}[X]$ be the ring of real polynomials in the variables $X = (X_1, \ldots, X_n)$, and let $\mathbb{R}[X]_{2d}$ the vector space of real polynomials of degree at most $2d$, with canonical basis of monomials $(X^\alpha) = \{X^\alpha \,:\, \alpha \in \mathbb{N}^n;\, |\alpha| \le 2d\}$. Given a sequence $y = (y_\alpha) \subset \mathbb{R}$ indexed in the canonical basis $(X^\alpha)$, let $L_y : \mathbb{R}[X]_{2d} \to \mathbb{R}$ be the linear mapping

$$f\,(= \sum_\alpha f_\alpha\, X^\alpha) \quad \mapsto \quad L_y(f) \,=\, \sum_\alpha f_\alpha\, y_\alpha, \quad f \in \mathbb{R}[X]_{2d},$$

and let $M_d(y)$ be the *moment* matrix with rows and columns indexed in $(X^\alpha)$, and defined by

$$M_d(y)(\alpha,\beta) \,:=\, L_y(X^{\alpha+\beta}) \,=\, y_{\alpha+\beta}, \qquad \alpha,\beta \in \mathbb{N}^n : \, |\alpha|, |\beta| \,\le\, d. \tag{2.1}$$

Let the notation $M_d(y) \succeq 0$ stand for $M_d(y)$ is positive semidefinite. It is clear that

$$M_d(y) \,\succeq\, 0 \quad \Longleftrightarrow \quad L_y(f^2) \ge 0 \quad \forall\, f \in \mathbb{R}[X]_d.$$

The set $\Sigma_d^2 \subset \mathbb{R}[X]_{2d}$ of s.o.s. polynomials of degree at most $2d$ is a finite-dimensional convex cone, and

$$f \in \Sigma_d^2 \quad \Longleftrightarrow \quad L_y(f) \,\ge\, 0 \quad \forall\, y \;\text{ s.t. }\; M_d(y) \,\succeq\, 0. \tag{2.2}$$

**Remark 1.** To prove that $L_y(f) \ge 0$ for all $y$ such that $M_d(y) \succeq 0$ it suffices to prove that $L_y(f) \ge 0$ for all $y$ such that $M_d(y) \succeq 0$ and $L_y(1) > 0$ (and equivalently, by homogeneity, for all $y$ such that $M_d(y) \succeq 0$ and $L_y(1) = 1$).

Indeed, suppose that $L_y(f) \ge 0$ for all $y$ such that $M_d(y) \succeq 0$ and $L_y(1) > 0$. Next, let $y$ be such that $M_d(y) \succeq 0$ and $L_y(1) = 0$. Fix $\epsilon > 0$ arbitrary and let $y(\epsilon) := y + (\epsilon, 0, \ldots, 0)$ so that $L_{y(\epsilon)}(X^\alpha) = y_\alpha$ if $\alpha \neq 0$ and $L_{y(\epsilon)}(1) = \epsilon > 0$. Therefore $M_d(y(\epsilon)) \succeq 0$ (because $M_d(y) \succeq 0$) and so $0 \le L_{y(\epsilon)}(f) = \epsilon f_0 + L_y(f)$. As $\epsilon > 0$ was arbitrary, letting $\epsilon \downarrow 0$ yields the desired result $L_y(f) \ge 0$.

We first recall a preliminary result whose proof can be found in Lasserre and Netzer [6].

**Lemma 1** ([6])**.** *With $d \ge 1$, let $y = (y_\alpha) \subset \mathbb{R}$ be such that the moment matrix $M_d(y)$ defined in (2.1) is positive semidefinite, and let $\tau_d := \max_{i=1,\ldots,n} L_y(X_i^{2d})$. Then:*

$$|L_y(X^\alpha)| \,\le\, \max[\, L_y(1)\,,\, \tau_d\,], \qquad \forall\, \alpha \in \mathbb{N}^n : \quad |\alpha| \le 2d. \tag{2.3}$$

We next complement Lemma 1.

**Lemma 2.** *Let $y = (y_\alpha) \subset \mathbb{R}$ be normalized with $y_0 = L_y(1) = 1$, and such that $M_d(y) \succeq 0$. Let $\tau_d := \max_{i=1,\ldots,n} L_y(X_i^{2d})$. Then:*

$$|L_y(X^\alpha)|^{1/|\alpha|} \,\le\, \tau_d^{1/2d}, \qquad \forall\, \alpha \in \mathbb{N}^n : \, 1 \le |\alpha| \,\le\, 2d. \tag{2.4}$$

For a proof see §3.1.

2.1. **Conditions for a polynomial to be s.o.s.** With $d \in \mathbb{N}$, let $\Gamma \subset \mathbb{N}^n$ be the set defined by:

$$\Gamma := \{ \alpha \in \mathbb{N}^n \,:\, |\alpha| \leq 2d; \quad \alpha = 2\beta \quad \text{for some } \beta \in \mathbb{N}^n \}. \tag{2.5}$$

We now provide our first condition.

**Theorem 3.** *Let $f \in \mathbb{R}[X]_{2d}$ and write $f$ in the form*

$$f \,=\, f_0 + \sum_{i=1}^n f_{i2d}\, X_i^{2d} \,+ h, \tag{2.6}$$

*where $h \in \mathbb{R}[X]_{2d}$ contains no essential monomial $X_i^{2d}$. If*

$$f_0 \;\geq\; \sum_{\alpha \notin \Gamma} |f_\alpha| \quad - \sum_{\alpha \in \Gamma} \min[0, f_\alpha] \tag{2.7}$$

$$\min_{i=1,\ldots,n} f_{i2d} \;\geq\; \sum_{\alpha \notin \Gamma} |f_\alpha| \, \frac{|\alpha|}{2d} \quad - \sum_{\alpha \in \Gamma} \min[0, f_\alpha] \, \frac{|\alpha|}{2d} \tag{2.8}$$

*then $f \in \Sigma^2_d$.*

For a proof see §3.2. The sufficient conditions (2.7)-(2.8) define a polyhedral convex cone in the euclidean space of coefficients $(f_\alpha)$ of polynomials $f \in \mathbb{R}[X]_{2d}$. This is because the functions,

$$f \mapsto \min_{i=1,\ldots,n} f_{i2d}\,, \quad f \mapsto \min[0, f_\alpha]\,, \quad f \mapsto -|f_\alpha|,$$

are all piecewise linear and concave. The description (2.7)-(2.8) of this convex polyhedral cone is *explicit* and given only in terms of the coefficients $(f_\alpha)$, i.e., with no lifting.

Notice that (2.7)-(2.8) together with $f_{i2d} = 0$ for some $i$, implies $f_\alpha = 0$ for all $\alpha \notin \Gamma$, and $f_\alpha \geq 0$ for all $\alpha \in \Gamma$, in which case $f$ is obviously s.o.s.

Theorem 3 is interesting when $f$ has a few non zero coefficients. When $f$ has a lot of non zero coefficients and contains the essential monomials $X_i^{2k}$ for all $k = 1, \ldots, d$, all with positive coefficients, one provides the following alternative sufficient condition. With $k \leq d$, let

$$\Gamma_k^1 \;:=\; \{ \alpha \in \mathbb{N}^n \;: \quad 2k-1 \,\leq\, |\alpha| \,\leq\, 2k \,\} \tag{2.9}$$

$$\Gamma_k^2 \;:=\; \{ \alpha \in \Gamma_k^1 \;: \quad \alpha = 2\beta \quad \text{for some } \beta \in \mathbb{N}^n \}. \tag{2.10}$$

**Corollary 4.** *Let $f \in \mathbb{R}[X]_{2d}$ and write $f$ in the form*

$$f \,=\, f_0 + h + \sum_{k=1}^d \sum_{i=1}^n f_{i2k}\, X_i^{2k}, \tag{2.11}$$

*where $h \in \mathbb{R}[X]_{2d}$ contains no essential monomial $X_i^{2k}$. If*

$$\frac{f_0}{d} \;\geq\; \sum_{\alpha\in\Gamma_k^1\setminus\Gamma_k^2} |f_\alpha| \quad - \sum_{\alpha\in\Gamma_k^2} \min[0, f_\alpha] \tag{2.12}$$

$$\min_{i=1,\ldots,n} f_{i2k} \;\geq\; \sum_{\alpha\in\Gamma_k^1\setminus\Gamma_k^2} |f_\alpha|\,\frac{|\alpha|}{2k} \quad - \sum_{\alpha\in\Gamma_k^2} \min[0, f_\alpha]\,\frac{|\alpha|}{2k} \tag{2.13}$$

*for all $k = 1, \ldots, d$, then $f \in \Sigma_d^2$.*

For a proof see §3.3. Notice that (2.12)-(2.13) together with $f_{i2k} = 0$ for some $i$ and some $k \in \{1, \ldots d\}$, implies $f_\alpha = 0$ for all $\alpha \in \Gamma_k^1 \setminus \Gamma_k^2$, and $f_\alpha \geq 0$ for all $\alpha \in \Gamma_2^k$.

Several variants of Corollary 4 can be derived; for instance, any other way to distribute the constant term $f_0$ as $\sum_{k=1}^d f_{0k}$ with $f_{0k} \neq f_0/d$, is valid and also provides another set of sufficient conditions. Consider also the case when $f$ can be written as

$$f \;=\; f_0 + h + \sum_{k\in\mathbf{K}} \sum_{i=1}^n f_{i2k}\, X_i^{2k},$$

where $\mathbf{K} := \{k \in \{1, \ldots, d\} : \min_{i=1,\ldots,n} f_{i2k} > 0\}$, $d \in \mathbf{K}$, and $h \in \mathbb{R}[X]_{2d}$ contains no essential monomial $X_i^{2k}$, $k \in \mathbf{K}$. Then one may easily derive a set of sufficient conditions in the spirit of Corollary 4; see e.g. [5].

Finally, one provides a simple condition for a polynomial to be s.o.s., possibly after adding a constant.

**Corollary 5.** *Let $f \in \mathbb{R}[X]_{2d}$ and write $f$ in the form*

$$f \;=\; f_0 + h + \sum_{i=1}^n f_{i2d}\, X_i^{2d}, \tag{2.14}$$

*where $h \in \mathbb{R}[X]$ contains no essential monomial $X_i^{2d}$. If*

$$\min_{i=1,\ldots,n} f_{i2d} > \sum_{\alpha\notin\Gamma;\, |\alpha|=2d} |f_\alpha| - \sum_{\alpha\in\Gamma;\, |\alpha|=2d} \min[0, f_\alpha] \tag{2.15}$$

*with $\Gamma$ as in (2.5), then $f + M \in \Sigma_d^2$ for some $M \geq 0$.*

*Proof.* Let $-M := \min\left[0, \inf_y \{L_y(f) \,:\, M_d(y) \succeq 0;\, L_y(1) = 1\}\right]$. We prove that $M < +\infty$. Assume that $M = +\infty$, and let $y^j$ be a minimizing sequence. One must have $\tau_{jd} := \max_{i=1,\ldots,n} L_{y^j}(X_i^{2d}) \to \infty$, as $j \to \infty$, otherwise if $\tau_{jd}$ is bounded by, say $\rho$, by Lemma 1 one would have $|L_{y^j}(X^\alpha)| \leq \max[1, \rho]$ for all $|\alpha| \leq 2d$, and so $L_{y^j}(f)$ would be bounded, in contradiction with $L_{y^j}(f) \to -\infty$. But then from Lemma 2, for sufficiently large $j$, one obtains the contradiction

$$0 > \frac{L_{y^j}(f)}{\tau_{jd}} \geq \min_{i=1,\ldots,n} f_{i2d} - \sum_{\alpha\notin\Gamma;\,|\alpha|=2d} |f_\alpha| + \sum_{\alpha\in\Gamma;\,|\alpha|=2d} \min[0, f_\alpha]$$
$$- \sum_{0\leq|\alpha|<2d} |f_\alpha|\, \tau_{jd}^{(|\alpha|-2d)/2d} \geq 0,$$

where the last inequality follows from (2.15) and $\tau_{jd}^{(|\alpha|-2d)/2d} \to 0$ as $j \to \infty$.

Hence $M < +\infty$ and so, $L_y(f+M) \geq 0$ for every $y$ such that $M_d(y) \succeq 0$ and $L_y(1) = 1$ . But then, in view of Remark 1, $L_y(f+M) \geq 0$ for all $y$ such that $M_d(y) \succeq 0$, which in turn implies that $f+M$ is s.o.s. □

In Theorem 3, Corollary 4 and 5, it is worth noticing the crucial role played by the constant term and the essential monomials $(X_i^\alpha)$, as was already the case in [4, 6] for approximating nonnegative polynomials by s.o.s.

## 3. Proofs

The proof of Lemma 2 first requires the following auxiliary result.

**Lemma 6.** *Let $d \geq 1$, and $y = (y_\alpha) \subset \mathbb{R}$ be such that the moment matrix $M_d(y)$ defined in (2.1) is positive semidefinite, and let $\tau_d := \max_{i=1,\ldots,n} L_y(X_i^{2d})$. Then: $L_y(X^{2\alpha}) \leq \tau_d$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = d$.*

*Proof.* The proof is by induction on the number $n$ of variables. The case $n = 1$ is trivial and the case $n = 2$ is proved in Lasserre and Netzer [6, Lemma 4.2].

Let the claim be true for $k = 1, \ldots, n-1$ and consider the case $n > 2$. By the induction hypothesis, the claim is true for all $L_y\left(X^{2\alpha}\right)$, where $|\alpha| = d$ and $\alpha_i = 0$ for some $i$. Indeed, $L_y$ restricts to a linear form on the ring of polynomials with $n-1$ indeterminates and satisfies all the assumptions needed. So the induction hypothesis gives the boundedness of all those values $L_y\left(X^{2\alpha}\right)$.

Now take $L_y\left(X^{2\alpha}\right)$, where $|\alpha| = d$ and all $\alpha_i \geq 1$. With no loss of generality, assume $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. Consider the two elements

$$\gamma := (2\alpha_1, 0, \alpha_3 + \alpha_2 - \alpha_1, \alpha_4, \ldots, \alpha_n) \in \mathbb{N}^n \text{ and}$$

$$\gamma' := (0, 2\alpha_2, \alpha_3 + \alpha_1 - \alpha_2, \alpha_4, \ldots, \alpha_n) \in \mathbb{N}^n.$$

We have $|\gamma| = |\gamma'| = d$ and $\gamma_2 = \gamma'_1 = 0$, and from what precedes,

$$L_y(X^{2\gamma}) \leq \tau_d \text{ and } L_y(X^{2\gamma'}) \leq \tau_d.$$

As $M_d(y) \succeq 0$, one also has

$$L_y(X^{2\alpha})^2 = L_y(X^{\gamma+\gamma'})^2 \leq L_y(X^{2\gamma}) \cdot L_y(X^{2\gamma'}) \leq \tau_d^2,$$

which yields the desired result $|L_y(X^{2\alpha})| \leq \tau_d$. □

3.1. **Proof of Lemma 2.** The proof is by induction on $d$. Assume it is true for $k = 1, \ldots, d$, and write $M_{d+1}(y)$ in the following block form below with appropriate matrices $V, U_i, V_i, S_i$:

| | | | |
|---|---|---|---|
| $M_{d-2}(y)$ | $U_1$ | $U_2$ | $V$ |
| $U_1^T$ | $S_{2d-2}$ | $V_{2d-1}$ | $V_{2d}$ |
| $U_2^T$ | $V_{2d-1}^T$ | $S_{2d}$ | $V_{2d+1}$ |
| $V^T$ | $V_{2d}^T$ | $V_{2d+1}^T$ | $S_{2d+2}$ |

When $d = 1$, the blocks $M_{d-2}(y)$, and $U_1, U_2, U_1^T, U_2^T, V$ disappear.

• The case $|\alpha| = 2d + 2$. By Lemma 6, all diagonal elements of $S_{2d+2}$ satisfy $y_\alpha \leq \tau_{d+1}$, and so do all other elements of $S_{2d+2}$ because $S_{2d+2} \succeq 0$.

• Consider an arbitrary $y_\alpha$ with $|\alpha| = 2d$. From the definition of the moment matrix, one may choose a pair $(i, j)$ such that the position $(i, j)$ in the matrix $M_{d+1}(y)$ lies in the submatrix $V_{2d}$, and the corresponding entry is $y_\alpha$. From $M_{d+1}(y) \succeq 0$,

$$M_{d+1}(y)(i,i)\, M_{d+1}(y)(j,j) \;\geq\; y_\alpha^2,$$

As $M_{d+1}(y)(i,i)$ is an element $y_\beta$ of $S_{2d-2}$ with $|\beta| = 2d - 2$, invoking the induction hypothesis yields $M_{d+1}(y)(i,i) \leq \tau_d^{(2d-2)/2d}$. On the other hand, $M_{d+1}(y)(j,j)$ is a diagonal element $y_{2\beta}$ of $S_{2d+2}$ with $|\beta| = d + 1$. From Lemma 6, every diagonal element of $S_{2d+2}$ is dominated by $\tau_{d+1}$, and so $M_{d+1}(y)(j,j) \leq \tau_{d+1}$. Combining the two yields

$$y_\alpha^2 \;\leq\; \tau_d^{(d-1)/d} \tau_{d+1}, \qquad \forall\, \alpha : \quad |\alpha| \;=\; 2d.$$

Next, picking up the element $\alpha$ such that $y_\alpha = \tau_d$ one obtains

$$\tau_d^2 \;\leq\; \tau_d^{1-1/d} \tau_{d+1} \quad \Rightarrow \quad \tau_d^{1/d} \leq \tau_{d+1}^{1/(d+1)}, \tag{3.1}$$

and so,

$$y_\alpha^2 \;\leq\; \tau_d^{(d-1)/d} \tau_{d+1}\,; \qquad |y_\alpha|^{1/|\alpha|} \;\leq\; \tau_{d+1}^{1/(2d+2)}, \quad \forall\, \alpha \;:\; |\alpha| = 2d.$$

• Next, consider an arbitrary $y_\alpha$ with $|\alpha| = 2d + 1$. Again, one may choose a pair $(i, j)$ such that the position $(i, j)$ in the matrix $M_{d+1}(y)$

lies in the submatrix $V_{2d+1}$, and the corresponding entry is $y_\alpha$. The entry $M_{d+1}(y)(i,i)$ corresponds to an element $y_{2\beta}$ of $S_{2d}$ with $|\beta| = d$, and so, by Lemma 6, $M_{d+1}(y)(i,i) \leq \tau_d$; similarly the entry $M_{d+1}(y)(j,j)$ corresponds to an element $y_{2\beta}$ of $S_{2d+2}$ with $|\beta| = d+1$, and so, by Lemma 6 again, $M_{d+1}(y)(j,j) \leq \tau_{d+1}$. From $M_{d+1}(y) \succeq 0$, we obtain

$$\tau_{d+1}\,\tau_d \;\geq\; M_{d+1}(y)(i,i)\,M_{d+1}(y)(j,j) \;\geq\; y_\alpha^2,$$

which, using (3.1), yields $|y_\alpha|^{1/|\alpha|} = |y_\alpha|^{1/(2d+1)} \leq \tau_{d+1}^{1/(2d+2)}$ for all $\alpha$ with $|\alpha| = 2d+1$.

• Finally, for an arbitrary $y_\alpha$ with $1 \leq |\alpha| < 2d$, use the induction hypothesis $|y_\alpha|^{1/|\alpha|} \leq \tau_d^{1/2d}$ and (3.1) to obtain $|y_\alpha|^{1/|\alpha|} \leq \tau_{d+1}^{1/2(d+1)}$. This argument is also valid for the case $|\alpha| = 2d$, but this latter case was treated separately to obtain (3.1).

It remains to prove that the induction hypothesis is true for $d = 1$. This easily follows from the definition of the moment matrix $M_1(y)$. Indeed, with $|\alpha| = 1$ one has $y_\alpha^2 \leq y_{2\alpha} \leq \tau_1$ (as $L_y(1) = 1$), so that $|y_\alpha| \leq \tau_1^{1/2}$ for all $\alpha$ with $|\alpha| = 1$. With $|\alpha| = 2$, say with $\alpha_i + \alpha_j = 2$, one has

$$\tau_1^2 \;\geq\; L_y(X_i^2)\,L_y(X_j^2) \;\geq\; L_y(X_iX_j)^2 \;=\; y_\alpha^2,$$

and so $|y_\alpha| \leq \tau_1$ for all $\alpha$ with $|\alpha| = 2$. □

3.2. **Proof of Theorem 3.** From (2.2), it suffices to show that $L_y(f) \geq 0$ for *any* $y$ such that $M_d(y) \succeq 0$, and by Remark 1, we may and will assume that $L_y(1) = 1$.

So let $y$ be such that $M_d(y) \succeq 0$ with $L_y(1) = 1$. Let $\tau_d$ be as in Lemma 1 and consider the two cases $\tau_d \leq 1$ and $\tau_d > 1$.

• The case $\tau_d \leq 1$. By Lemma 1, $|L_y(X^\alpha)| \leq 1$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2d$. Therefore,

$$L_y(f) \;\geq\; f_0 \;-\sum_{\alpha\notin\Gamma}|f_\alpha| \;+\sum_{\alpha\in\Gamma}\min[0,f_\alpha] \;\geq 0,$$

where the last inequality follows from (2.7).

• The case $\tau_d > 1$. Recall that $L_y(1) = 1$, and from Lemma 2, one has $|L_y(X^\alpha)|^{1/|\alpha|} \leq \tau_d^{1/2d}$ for all $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq 2d$. Therefore,

$$\begin{aligned} L_y(f) \;&\geq\; f_0 + (\min_{i=1,\dots,n} f_{i2d})\,\tau_d \\ &\quad - \sum_{\alpha\notin\Gamma}|f_\alpha|\,\tau_d^{|\alpha|/2d} + \sum_{\alpha\in\Gamma}\min[0,f_\alpha]\,\tau_d^{|\alpha|/2d} \end{aligned}$$

Consider the univariate polynomial $t \mapsto p(t)$, with

$$p(t) = f_0 + (\min_{i=1,\dots,n} f_{i2d})\,t^{2d} - \sum_{\alpha\notin\Gamma}|f_\alpha|\,t^{|\alpha|} + \sum_{\alpha\in\Gamma}\min[0,f_\alpha]\,t^{|\alpha|},$$

and denote $p^{(k)} \in \mathbb{R}[X]$, its $k$-th derivative.

By (2.8), $\min_{i=1,\ldots,n} f_{i2d} \geq 0$ and so by (2.7), $p(1) \geq 0$. By (2.8) again, $p'(1) \geq 0$. In addition, with $1 \leq k \leq 2d$, (2.8) also implies

$$\begin{aligned}\min_{i=1,\ldots,n} f_{i2d} &\geq \sum_{\alpha\notin\Gamma;\,|\alpha|\geq k} |f_\alpha| \frac{|\alpha|}{2d}\,\frac{(|\alpha|-1)}{2d-1}\cdots\frac{(|\alpha|-(k-1))}{2d-(k-1)} \\ &\quad - \sum_{\alpha\in\Gamma;\,|\alpha|\geq k} \min[0,f_\alpha] \frac{|\alpha|}{2d}\,\frac{(|\alpha|-1)}{2d-1}\cdots\frac{(|\alpha|-(k-1))}{2d-(k-1)}\end{aligned}$$

because $|\alpha| - j \leq 2d - j$, for all $j = 1, \ldots, k-1$, and so

$$\begin{aligned}\left(\prod_{j=0}^{k-1}(2d-j)\right)\min_{i=1,\ldots,n} f_{i2d} &\geq \sum_{\alpha\notin\Gamma;\,|\alpha|\geq k} |f_\alpha| \left(\prod_{j=0}^{k-1}(|\alpha|-j)\right) \\ &\quad - \sum_{\alpha\in\Gamma;\,|\alpha|\geq k} \min[0,f_\alpha] \left(\prod_{j=0}^{k-1}(|\alpha|-j)\right),\end{aligned}$$

which implies $p^{(k)}(1) \geq 0$. Therefore, $p^{(k)}(1) \geq 0$ for all $k = 0, 1, \ldots, 2d$, and so, $p$ has no root in $(1, +\infty)$; indeed, the (non trivial) polynomial $p(t-1)$ has all its coefficients nonnegative and so has no root in $(0, +\infty)$. Hence, $p \geq 0$ on $(1, +\infty)$ and as $\tau_d > 1$, $L_y(f) \geq p(\tau_d^{1/2d}) \geq 0$. $\square$

3.3. **Proof of Corollary 4.** Let $y$ be such that $M_d(y) \succeq 0$. Again in view of Remark 1, we may and will assume that $L_y(1) = 1$.

Then $L_y(f) \geq \sum_{k=1}^{d} A_k$, with

$$\begin{aligned}A_k &:= \frac{f_0}{d} + \sum_{i=1}^{n} f_{i2k}\,L_y(X_i^{2k}) + \sum_{\alpha\in\Gamma_k^2} \min[0,f_\alpha]\,L_y(X^\alpha) \\ &\quad - \sum_{\alpha\in\Gamma_k^1\setminus\Gamma_k^2} |f_\alpha|\,|L_y(X^\alpha)|, \qquad k = 1,\ldots,d.\end{aligned} \tag{3.2}$$

Fix $k$ arbitrary in $\{1, \ldots, d\}$ and consider the moment matrix $M_k(y) \succeq 0$, which is a submatrix of $M_d(y)$.

- Case $\tau_k \leq 1$. By Lemma 1 applied to $M_k(y)$, $|L_y(X^\alpha)| \leq 1$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2k$. Therefore, with $A_k$ as in (3.2),

$$A_k \geq \frac{f_0}{d} - \sum_{\alpha\in\Gamma_k^1\setminus\Gamma_k^2} |f_\alpha| + \sum_{\alpha\in\Gamma_k^2} \min[0,f_\alpha] \geq 0,$$

where the last inequality follows from (2.12).

• Case $\tau_k > 1$. From Lemma 2 applied to $M_k(y)$, $|L_y(X^\alpha)|^{1/|\alpha|} \leq \tau_k^{1/2k}$ for all $\alpha$ with $|\alpha| \leq 2k$. Therefore, $A_k \geq p_k(\tau_k^{1/2k})$, where $p_k \in \mathbb{R}[t]$, and

$$p_k(t) = \frac{f_0}{d} + t^{2k}\left(\min_{i=1,\ldots,n} f_{i2k} + \sum_{\alpha\in\Gamma_k^2} \min[0, f_\alpha]\right) - \sum_{\alpha\in\Gamma_k^1\setminus\Gamma_k^2} |f_\alpha|\, t^{|\alpha|}.$$

As in the proof of Theorem 3, but now using (2.12)-(2.13), one has $p_k^{(j)}(1) \geq 0$ for all $j = 0, 1, \ldots, 2k$, and so $p_k$ has no root in $(1, +\infty)$. Therefore, $p_k \geq 0$ on $(1, +\infty)$ which in turn implies $A_k \geq p_k(\tau_k^{1/2k}) \geq 0$ because $\tau_k > 1$. Finally, $L_y(f) \geq \sum_{k=1}^d A_k \geq 0$, as $A_k \geq 0$ in both cases $\tau_k \leq 1$ and $\tau_k > 1$. □

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