# On a Generalization of the Master Cyclic Group Polyhedron

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**Abstract**

We study the Master Equality Polyhedron (MEP) which generalizes the Master Cyclic Group Polyhedron and the Master Knapsack Polyhedron.

We present an explicit characterization of the polar of the nontrivial facet-defining inequalities for MEP. This result generalizes similar results for the Master Cyclic Group Polyhedron by Gomory [9] and for the Master Knapsack Polyhedron by Araóz [1]. Furthermore, this characterization also gives a polynomial time algorithm for separating an arbitrary point from MEP.

We describe how facet-defining inequalities for the Master Cyclic Group Polyhedron can be lifted to obtain facet-defining inequalities for the MEP, and also present facet-defining inequalities for MEP that cannot be obtained in such a way. Finally, we study the mixed-integer extension of MEP and present an interpolation theorem that produces valid inequalities for general mixed integer programming problems using facets of MEP.

## 1 Introduction

We study the Master Equality Polyhedron (MEP), which we define as:

$$K(n,r) = conv\left\{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+ : \sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = r\right\} \tag{1}$$

where $n, r \in \mathbb{Z}$ and $n > 0$. Without loss of generality we assume that $r \geq 0$. To the best of our knowledge, $K(n,r)$ was first defined by Uchoa [16] in a slightly different form, although its polyhedral structure was not studied in that paper.

Two well-known families of polyhedra are lower dimensional faces of MEP: the Master Cyclic Group Polyhedron (MCGP), which is defined as

$$P(n,r) = conv\left\{(x,y) \in \mathbb{Z}^{n-1}_+ \times \mathbb{Z}_+ : \sum_{i=1}^{n-1} ix_i - ny_n = r\right\}, \tag{2}$$

where $r, n \in \mathbb{Z}$, and $n > r > 0$; and the Master Knapsack Polyhedron (MKP), which is defined as

$$K(r) = conv\left\{x \in \mathbb{Z}^r_+ : \sum_{i=1}^{r} ix_i = r\right\}, \tag{3}$$

where $r \in \mathbb{Z}$ and $r > 0$.

Facets of $P(n,r)$ are a useful source of cutting planes for general MIPs. The Gomory mixed-integer cut (also known as the mixed-integer rounding (MIR) inequality) can be derived from a facet of $P(n,r)$ [10]. For work on other properties and facets of the Master Cyclic Group Polyhedron; see [2, 4, 5, 6, 7, 8, 11, 12, 13]. In particular, several relationships between facet-defining inequalities of MCGP and facet-defining inequalities of MKP were established in [2]. The Master Cyclic Group Polyhedron is usually presented as

$$P'(n,r) \;=\; conv\left\{x \in \mathbb{Z}_+^{n-1} : \sum_{i=1}^{n-1} ix_i \equiv r \mod n\right\}$$

which is the projection of $P(n,r)$ in the space of $x$ variables. We use (2) as it makes the comparison to $K(n,r)$ simpler.

Gomory [9] and Aráoz [1] give an explicit characterization of the polar of the nontrivial facets of MCGP and MKP. In this paper, we give a similar description of the nontrivial facets of MEP for $n \geq r \geq 0$. Using this result, we obtain a polynomial time algorithm to separate over MEP for all $r \geq 0$ (including $r > n$). We also analyze some structural properties of MEP and relate it to MCGP.

For $n > r > 0$, it is easy to obtain valid (facet-defining) inequalities for MEP using valid (facet-defining) inequalities for MCGP. Notice that

$$\sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = r \iff \sum_{i=1}^{n-1} i(x_i + y_{n-i}) - n(\sum_{i=1}^{n} y_i - x_n) = r. \tag{4}$$

Furthermore, $\sum_{i=1}^{n} y_i - x_n \geq 0$ for all $(x,y) \in K(n,r)$. Therefore, any valid inequality $\sum_{i=1}^{n-1} \pi_i x_i + \rho_n y_n \geq \beta$ for $P(n,r)$ leads to a valid inequality $\sum_{i=1}^{n-1} \pi_i x_i + \sum_{i=1}^{n-1} (\pi_{n-i} + \rho_n) y_i + \rho_n y_n - \rho_n x_n \geq \beta$ for $K(n,r)$. In fact, as MCGP is a lower dimensional face of MEP, any valid inequality for MCGP can lead to multiple valid inequalities (including the inequality presented above) for MEP via lifting. We study lifted inequalities from MCGP and show that not all facets of MEP can be obtained in this way.

In addition, we describe how to obtain valid inequalities for general MIPs using facet-defining inequalities for MEP. Another motivation to study MEP is that it also arises as a natural structure in a reformulation of the Fixed-Charge Network Flow problem, which has recently been used in [17] to derive strong cuts for the Capacitated Minimum Spanning Tree Problem and can also be used in other problems such as the Capacitated Vehicle Routing Problem. Such reformulation gives rise to the following polyhedron, almost identical to MEP:

$$conv\left\{y \in \mathbb{Z}_+^{C}, z \in \mathbb{Z}_+^{C-1} : \sum_{d=1}^{C} dy^d - \sum_{d=1}^{C-1} dz^d = d(S)\right\}.$$

In [17], valid inequalities for the above polyhedron are called Homogeneous Extended Capacity cuts, and just using simple rounded Chvátal-Gomory cuts, they were able to reduce the integrality gap by more than 50% on average. Therefore the benefit of using strong inequalities for MEP can be quite substantial to those problems as well.

In the next section, we present our characterization of the polar of the nontrivial facets of $K(n,r)$, for any $n > 0$ and any $r$ satisfying $0 < r \leq n$. In Section 3 we discuss how to lift facets of $P(n,r)$ to obtain facets of $K(n,r)$ and in Section 4, we show that not all facets of $K(n,r)$ can be obtained by lifting. In Section 5, we study $K(n,r)$ when $r = 0$. In Section 6, we describe how to separate an arbitrary point from $K(n,r)$ for any $r$, including the case $r > n$. Section 7 is dedicated to extending the results of $K(n,r)$ to the mixed-integer case. In Section 8, we follow the approach of Gomory and Johnson [10] to derive valid inequalities for mixed-integer programs from facets of $K(n,r)$ via *interpolation*. We conclude in Section 9 with some remarks on directions for further research on $K(n,r)$.

## 2 Polyhedral analysis of $K(n,r)$ when $0 < r \leq n$

Throughout this section, we assume $0 < r \leq n$. The case $r = 0$ is studied in Section 5. We start with some notation and some basic polyhedral properties of $K(n,r)$.

Let $e_i \in \mathbb{R}^{2n}$ be the unit vector with a one in the component corresponding to $x_i$ and let $f_i \in \mathbb{R}^{2n}$ be the unit vector with a one in the component corresponding to $y_i$, for $i = 1, \ldots, n$.

**Lemma 2.1** *$dim(K(n,r)) = 2n - 1$.*

**Proof.** Clearly $dim(K(n,r)) \leq 2n - 1$ as all points in $K(n,r)$ satisfy $\sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = r$. Let $U$ be the set of $2n$ points $p_1 = re_1$, $p_i = re_1 + e_i + if_1$ for $i = 2, \ldots, n$, and $q_i = (r+i)e_1 + f_i$ for $i = 1, \ldots, n$. The vectors of $U$ are affinely independent, as $\{u - p_1 : u \in U, u \neq p_1\}$ is a set of linearly independent vectors. As $U \subseteq K(n,r)$, $dim(K(n,r)) \geq 2n - 1$. ∎

**Lemma 2.2** *The nonnegativity constraints of $K(n,r)$ are facet-defining if $n \geq 2$.*

**Proof.** Let $U$ be defined as in the proof of Lemma 2.1. For any $i \neq 1$, the vectors in $U \setminus \{p_i\}$ and $U \setminus \{q_i\}$ are affinely independent, and satisfy $x_i = 0$ and $y_i = 0$, respectively. Therefore $x_i \geq 0$ and $y_i \geq 0$ define facets of $K(n,r)$ for $i \geq 2$. To see that $y_1 \geq 0$ is facet-defining, replace $p_i$ $(2 \leq i \leq n)$ in $U$ by $p'_i = re_1 + ne_i + if_n$ to get a set of affinely independent vectors $U' \subseteq K(n,r)$. All points in $U'$ other than $q_1$ satisfy $y_1 = 0$. Finally, let $V$ be the set of points $t_0 = e_n + (n-r)f_1$, $t_i = t_0 + ie_n + nf_i$ for $i = 1, \ldots, n$, and $s_i = t_0 + e_i + if_1$ for $i = 2, \ldots, n-1$. $V$ is contained in $K(n,r)$, and its elements are affinely independent vectors as $\{v - t_0 : v \in V, v \neq t_0\}$ is a set of linearly independent vectors. The points in $V$ also satisfy $x_1 = 0$. ∎

Clearly, $K(n,r)$ is an unbounded polyhedron. We next characterize all the extreme rays (one-dimensional faces of the recession cone) of $K(n,r)$. We represent an extreme ray $\{u + \lambda v : u, v \in \mathbb{R}^{2n}_{+}, \lambda \geq 0\}$ of $K(n,r)$ simply by the vector $v$. Let $r_{ij} = je_i + if_j$ for any $i, j \in \{1, \ldots, n\}$.

**Lemma 2.3** *The set of extreme rays of $K(n,r)$ is given by $R = \{r_{ij} : 1 \leq i, j \leq n\}$.*

**Proof.** Let $(c, d)$ be an extreme ray of $K(n, r)$, that is, $\sum_{i=1}^{n} ic_i - \sum_{j=1}^{n} jd_j = 0$. Since $(c, d)$ is a one-dimensional face of the recession cone, at least $2n - 2$ of the nonnegativity constraints hold as equality. As $(c, d)$ can not have a single nonzero component, it must have exactly two nonzero components. Thus, $(c, d)$ is of the form $\alpha e_i + \beta f_j$ and since $\sum_{i=1}^{n} ic_i - \sum_{j=1}^{n} jd_j = 0$, we have that $i\alpha - j\beta = 0 \Rightarrow \alpha = \frac{j}{i}\beta$. Since $(c, d)$ is a ray, we may scale it so that $\beta = i$, and we have shown that $R$ contains all extreme rays of $K(n, r)$.

To complete the proof, it suffices to notice that a conic combination of 2 or more rays in $R$ gives a ray with at least 3 nonzero entries and therefore all rays in $R$ are extreme. ∎

As $K(n, r)$ is not a full-dimensional polyhedron, any valid inequality $\pi x + \rho y \geq \pi_o$ for $K(n, r)$ has an equivalent representation with $\rho_n = 0$. If a valid inequality does not satisfy this condition, one can add an appropriate multiple of the equation $\sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = r$ to it. We state this formally in Observation 2.4, and subsequently assume all valid inequalities have $\rho_n = 0$. This is one of many possible choices of normalization and it was chosen to make the relation to facets of $P(n, r)$ easier, since Gomory's characterization also satisfies that property.

**Observation 2.4** *If $\pi x + \rho y \geq \pi_o$ defines a valid inequality for $K(n, r)$, we may assume $\rho_n = 0$.*

We classify the facets of $K(n, r)$ as *trivial* and *nontrivial* facets.

**Definition 2.5** *The following facet-defining inequalities of $K(n, r)$ are called trivial:*

$$x_i \geq 0, \quad \forall i = 1, \ldots, n, \tag{5}$$

$$y_i \geq 0, \quad \forall i = 1, \ldots, n-1. \tag{6}$$

*All other facet-defining inequalities of $K(n, r)$ are called nontrivial.*

According to this definition, the inequality $y_n \geq 0$ defines a nontrivial facet. There is nothing special about the $y_n \geq 0$ inequality except that it is the only nonnegativity constraint that does not comply directly with the $\rho_n = 0$ assumption. With this distinction between $y_n \geq 0$ and the other trivial facets, our results are easier to state and prove.

## 2.1 Characterization of the nontrivial facets

Let $N = \{1, \ldots, n\}$. We next state our main result:

**Theorem 2.6** *Consider an inequality $\pi x + \rho y \geq \pi_o$ with $\rho_n = 0$. It defines a nontrivial facet of $K(n, r)$ if and only if the following conditions hold: (i) $\pi_o > 0$, and, (ii) $(\pi, \rho, \pi_o)/\pi_o$ is an extreme* *point of $T \subseteq \mathbb{R}^{2n+1}$ where $T$ is defined by the following linear equations and inequalities:*

$$
\begin{aligned}
\pi_i + \rho_j &\geq \pi_{i-j}, && \forall i, j \in N, && i > j, && \text{(SA1)}\\
\pi_i + \pi_j &\geq \pi_{i+j}, && \forall i, j \in N, && i + j \leq n, && \text{(SA2)}\\
\rho_k + \pi_i + \pi_j &\geq \pi_{i+j-k}, && \forall i, j, k \in N, && 1 \leq i + j - k \leq n, && \text{(SA3)}\\
\pi_i + \pi_{r-i} &= \pi_o, && \forall i \in N, && i < r, && \text{(EP1)}\\
\pi_r &= \pi_o, && && && \text{(EP2)}\\
\pi_i + \rho_{i-r} &= \pi_o, && \forall i \in N && i > r, && \text{(EP3)}\\
\rho_n &= 0, && && && \text{(NC1)}\\
\pi_o &= 1. && && && \text{(NC2)}
\end{aligned}
$$

The proof of this theorem requires several preliminary results which will be presented throughout this section. Note that $\rho_n = 0$ is not a restrictive assumption in the above theorem as any valid inequality has an equivalent representation with $\rho_n = 0$.

We call constraints (SA1)-(SA3) *relaxed subadditivity* conditions as they are implied by the first four of the following pairwise subadditivity conditions:

$$
\begin{aligned}
\pi_i + \rho_i &\geq 0, && \forall i \in N, && && \text{(SA0)}\\
\pi_i + \rho_j &\geq \pi_{i-j}, && \forall i, j \in N, && i > j, && \text{(SA1)}\\
\pi_i + \rho_j &\geq \rho_{j-i}, && \forall i, j \in N, && i < j, && \text{(SA1')}\\
\pi_i + \pi_j &\geq \pi_{i+j}, && \forall i, j \in N, && i + j \leq n, && \text{(SA2)}\\
\rho_i + \rho_j &\geq \rho_{i+j}, && \forall i, j \in N, && i + j < n. && \text{(SA2')}\\
\rho_i + \rho_j &\geq \rho_{i+j}, && \forall i, j \in N, && i + j = n. && \text{(SA2'')}
\end{aligned}
$$

We distinguish the cases in (SA2') and (SA2") as (SA2") is not satisfied by the nonnegativity constraint on $y_n$ and we will need this distinction to establish certain structural properties later on.

**Lemma 2.7** *Let $(\pi, \rho) \in \mathbb{R}^{2n}$ satisfy the pairwise subadditivity conditions* (SA0), (SA1), (SA1'), *and* (SA2), *then $(\pi, \rho)$ satisfies* (SA3) *as well.*

**Proof.** Let $i, j, k \in N$ be such that $1 \leq i + j - k \leq n$ and without loss of generality assume that $i \leq j$. If $i + j \leq n$, then using (SA2) and (SA1) we have $\rho_k + \pi_i + \pi_j \geq \rho_k + \pi_{i+j} \geq \pi_{i+j-k}$.

If, on the other hand, $i + j > n$, we consider three cases:

*Case 1: $k < j$.* Using (SA1) and (SA2), we have: $\rho_k + \pi_i + \pi_j \geq \pi_{j-k} + \pi_i \geq \pi_{i+j-k}$.

*Case 2: $k > i$.* Using (SA1') and (SA1), we have: $\rho_k + \pi_i + \pi_j \geq \rho_{k-i} + \pi_j \geq \pi_{i+j-k}$.

*Case 3: $k = i = j$.* Using (SA0), we have: $\rho_k + \pi_k \geq 0 \Longleftrightarrow \rho_k + \pi_k + \pi_k \geq \pi_k$. ∎

As we show later, any nontrivial facet-defining inequality $\pi x + \rho y \geq \pi_o$ for $K(n, r)$ satisfies (SA1)-(SA3) as well as (SA0), (SA1') and (SA2'). Based on Gomory's characterization of MCGP using pairwise subadditivity condition, it would seem more natural to have a characterization of the nontrivial facets using the pairwise subadditivity conditions above instead of the relaxed subadditivity conditions. We show in Section 2.4 that Theorem 2.6 does not hold if (SA3) is replaced with (SA0), (SA1') and (SA2').

The equations (EP1)-(EP3) essentially state that the following $n - \lfloor \frac{r-1}{2} \rfloor$ affinely independent points, which we call the *elementary points* of $K(n,r)$,

$$\{e_i + e_{r-i} \,:\, 1 \leq i < r\} \cup \; e_r \; \cup \{e_i + f_{i-r}, \,:\, r < i \leq n\}$$

lie on every nontrivial facet of $K(n,r)$. In other words, $K(n,r)$ has a face of dimension at least $n - \lfloor \frac{r-1}{2} \rfloor - 1$ where all nontrivial facets intersect.

The last two constraints (NC1) and (NC2) are normalization constraints that are necessary to have a unique representation of nontrivial facets.

Note that the definition of $T$ in Theorem 2.6 is similar to that of a polar. However, $T$ is not the polar of $K(n,r)$, as it does not contain extreme points of the polar that correspond to the trivial inequalities. In addition, some of the extreme rays of the polar are not present in $T$. It is possible to interpret $T$ as an important subset of the polar that contains all extreme points of the polar besides the ones that lead to the trivial inequalities. In the rest of this section we develop the required analysis to prove Theorem 2.6.

## 2.2 Basic properties of $T$

We start with a basic observation which states that any valid inequality for $K(n,r)$ has to be valid for its extreme rays and elementary points.

**Observation 2.8** *Let $\pi x + \rho y \geq \pi_o$ be a valid inequality for $K(n,r)$, then the following holds:*

$$j\pi_i + i\rho_j \geq 0, \qquad \forall i,j \in N \tag{R1}$$

$$\pi_i + \pi_{r-i} \geq \pi_o, \quad \forall i \in N, i < r, \tag{P1}$$

$$\pi_r \geq \pi_o, \tag{P2}$$

$$\pi_i + \rho_{i-r} \geq \pi_o, \quad \forall i \in N, i > r. \tag{P3}$$

We next show that nontrivial facet-defining inequalities satisfy the relaxed subadditivity conditions and they are tight at the elementary points of $K(n,r)$.

**Lemma 2.9** *Let $\pi x + \rho y \geq \pi_o$ be a nontrivial facet-defining inequality of $K(n,r)$, then it satisfies* (SA1)-(SA3) *as well as* (SA0), (SA1'), (SA2') *and* (EP1)-(EP3).

**Proof.** Due to Observation 2.8, $i\pi_i + i\rho_i \geq 0$ for all $i \in N$ and therefore (SA0) holds. Next, let $(x^*, y^*) \in K(n,r)$ be such that $\pi x^* + \rho y^* = \pi_o$ and $x^*_{i-j} > 0$. Such a point exists because the facet-defining inequality we consider is nontrivial. Then $(x^*, y^*) + (e_i + f_j - e_{i-j})$ is also contained in $K(n,r)$. Therefore, (SA1) holds. Proofs for (SA2), (SA3), (SA1') and (SA2') are analogous.

Finally, let $(x', y')$ and $(x'', y'')$ be integral points in $K(n,r)$ lying on the facet defined by $\pi x + \rho y \geq \pi_o$ such that $x'_i > 0$ and $x''_{r-i} > 0$. Then $(\bar{x}, \bar{y}) = (x', y') + (x'', y'') - e_i - e_{r-i} \in K(n,r)$. Therefore

$$\pi\bar{x} + \rho\bar{y} = \pi x' + \rho y' + \pi x'' + \rho y'' - \pi_i - \pi_{r-i} = 2\pi_o - \pi_i - \pi_{r-i} \geq \pi_o.$$

The last inequality above implies that $\pi_i + \pi_{r-i} \leq \pi_o$ and therefore (P1) $\Rightarrow$ (EP1). The proofs of (EP2) and (EP3) are analogous. ∎

**Observation 2.10** *If $\pi x + \rho y \geq \pi_o$ is a facet-defining inequality of $K(n,r)$ which is not a nonnegativity constraint, then it also satisfies* (SA2").

We next show that the normalization condition (NC2) does not eliminate any nontrivial facets.

**Lemma 2.11** *Let $\pi x + \rho y \geq \pi_o$ be a nontrivial facet-defining inequality of $K(n,r)$, that satisfies $\rho_n = 0$. Then $\pi_o > 0$.*

**Proof.** By (R1), we have, for all $i \in N$, $n\pi_i + i\rho_n \geq 0$ and therefore $\pi_i \geq 0$ since $\rho_n = 0$. Also by (EP2), we have $\pi_o = \pi_r$ which implies that $\pi_o \geq 0$.

Assume $\pi_o = 0$. As $\pi \geq 0$, using (EP1) we have $\pi_i = 0$ for $i = 1, \ldots, r$. But then, (SA2) implies that

$$0 + \pi_{i-1} \geq \pi_i \geq 0, \text{ for } i = 2, \ldots, n.$$

Starting with $i = r+1$, we can inductively show that $\pi_i = 0$ for all $i \in N$. This also implies that $\rho_k = 0$ for $1 \leq k \leq n - r$ by (EP3). In addition $\rho_k \geq 0$ for $n - r + 1 \leq k \leq n$ by (SA3).

Therefore, if $\pi_o = 0$, then $\pi = 0$, $\rho \geq 0$ and therefore $\pi x + \rho y \geq 0$ can be written as a conic combination of the nonnegativity facets, which is a contradiction. Thus $\pi_o > 0$. ∎

Combining Lemmas 2.9 and 2.11 we have therefore established the following.

**Corollary 2.12** *Let $\pi x + \rho y \geq \pi_o$ be a nontrivial facet-defining inequality of $K(n,r)$, that satisfies $\rho_n = 0$. Then $\frac{1}{\pi_o}(\pi, \rho, \pi_o) \in T$.*

In the following result, we show that a subset of the conditions presented in Theorem 2.6 suffices to ensure validity of an inequality for $K(n,r)$.

**Lemma 2.13** *Let $(\pi, \rho, \pi_o)$ satisfy* (SA1), (SA2), (SA3) *and* (EP2). *Then $\pi x + \rho y \geq \pi_o$ defines a valid inequality for $K(n,r)$.*

**Proof.** We will prove this by contradiction. Assume that $\pi x + \rho y \geq \pi_o$ satisfies (EP2), (SA1), (SA2) and (SA3) but $\pi x + \rho y \geq \pi_o$ does not define a valid inequality for $K(n,r)$, $r > 0$. Let $(x^*, y^*)$ be an integer point in $K(n,r)$ that has minimum $L_1$ norm among all points violated by $\pi x + \rho y \geq \pi_o$.

Clearly, $(x^*, y^*) \neq 0$ and therefore $||(x^*, y^*)||_1 > 0$. If $||(x^*, y^*)||_1 = 1$, then $x^* = e_r$ and $y^* = 0$, and as $\pi_r = \pi_o$, $(x^*, y^*)$ does not violate the inequality, a contradiction. Therefore $||(x^*, y^*)||_1 \geq 2$. We next consider three cases.

*Case 1:* Assume that $y^* = 0$. Then $\sum_{i=1}^{n} ix_i^* = r$. By successively applying (SA2), we obtain

$$\pi_o > \sum_{i=1}^{n} \pi_i x_i^* \geq \sum_{i=1}^{n} \pi_{ix_i^*} \geq \pi_{\sum_{i=1}^{n} ix_i^*} = \pi_r$$

which contradicts (EP2).

*Case 2:* Assume that there exists $i > j$ such that $x_i^* > 0$ and $y_j^* > 0$. Let $(x', y') = (x^*, y^*) + (e_{i-j} - e_i - f_j)$. Clearly, $(x', y') \in K(n, r)$, and $||(x', y')||_1 = ||(x^*, y^*)||_1 - 1$. Moreover, as $\pi x + \rho y \geq \pi_o$ satisfies (SA1), $\pi x' + \rho y' = \pi x^* + \rho y^* + \pi_{i-j} - \pi_i - \rho_j \leq \pi x^* + \rho y^* < \pi_o$, which contradicts the choice of $(x^*, y^*)$.

*Case 3:* Assume that for any $i, k \in N$, $x_i^* > 0$ and $y_k^* > 0$ imply that $i \leq k$. Let $i \in N$ be such that $x_i^* > 0$, then either there exists another index $j \neq i$ such that $x_j^* > 0$, or, $x_i^* \geq 2$ (in which case, let $j = i$). If $i + j \leq n$, let $(x', y') = (x^*, y^*) + (e_{i+j} - e_i - e_j)$. If $i + j > n$, as $y^* \neq 0$, there exists $k$ such that $y_k^* > 0$ and $k \geq i$, and therefore $1 \leq i + j - k \leq n$. Then let $(x', y') = (x^*, y^*) + (e_{i+j-k} - e_i - e_j - f_k)$. In either case, $(x', y') \in K(n, r)$ and $||(x', y')||_1 < ||(x^*, y^*)||_1$. Moreover, as $(\pi, \rho, \pi_o)$ satisfy (SA2) and (SA3), in either case $\pi x' + \rho y' \leq \pi x^* + \rho y^* < \pi_o$, which contradicts the choice of $(x^*, y^*)$. ∎

**Corollary 2.14** *Let $(\pi, \rho, \pi_o) \in T$, then $\pi x + \rho y \geq \pi_o$ is a valid inequality for $K(n, r)$.*

Note that by Lemma 2.7 an inequality satisfying pairwise subadditivity conditions and (EP2) is valid for $K(n, r)$. We next determine the extreme rays of $T$.

**Lemma 2.15** *The extreme rays of $T$ are $(f_k, 0) \in \mathbb{R}^{2n+1}$ for $n - r < k < n$ .*

**Proof.** First note that $(f_k, 0)$ is indeed an extreme ray of $T$ for $n - r < k < n$.

Let $(\pi, \rho, \pi_o)$ be an extreme ray of $T$ that is not equivalent to $(f_k, 0)$ for some $n - r < k < n$. Clearly $\pi_o = 0$. In this case, $\pi x + \rho y \geq 0$ is a valid inequality for $K(n, r)$. Using the same arguments presented in the proof of Lemma 2.11, it is straightforward to establish that $\pi_i = 0$ for all $i \in N$, $\rho_k = 0$ for $1 \leq k \leq n - r$ and $\rho_k \geq 0$ for $n - r + 1 \leq k \leq n$. But then, $(\pi, \rho, \pi_o)$ can be written as a conic combination of the rays $(f_k, 0)$ for $n - r < k < n$, a contradiction. ∎

## 2.3 Facet characterization

Let

$$\mathcal{F} = \left\{ (\pi^k, \rho^k, \pi_o^k) \right\}_{k=1}^{M}$$

be the set of coefficients of nontrivial facets of $K(n, r)$ with $\rho_n = 0$ and $\pi_o = 1$. Note that by Lemma 2.11 these two assumptions do not eliminate any nontrivial facets. Also, as $y_n \geq 0$ is a nontrivial facet, $\mathcal{F} \neq \emptyset$. By Lemma 2.9, $\mathcal{F} \subseteq T$.

We now proceed to prove Theorem 2.6 in two steps.

**Lemma 2.16** *If $(\pi, \rho, \pi_o) \in \mathcal{F}$, then $(\pi, \rho, \pi_o)$ is an extreme point of $T$.*

**Proof.** Assume that $(\pi, \rho, \pi_o) \in \mathcal{F}$ but is not an extreme point of $T$. It can be written as a convex combination of two distinct points in $T$. The normalization conditions $\rho_n = 0$ and $\pi_o = 1$ imply that any two distinct points in $T$ represent two distinct valid inequalities for $K(n,r)$ in the sense that neither inequality can be obtained from the other by scaling or by adding multiples of the equation $\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i = r$. Therefore $\pi x + \rho y \geq \pi_o$ can be written as a combination of two distinct valid inequalities for $K(n,r)$, and therefore does not define a facet of $K(n,r)$. ∎

**Lemma 2.17** *If $(\pi, \rho, \pi_o)$ is an extreme point of $T$, then $(\pi, \rho, \pi_o) \in \mathcal{F}$.*

**Proof.** Let $(\hat{\pi}, \hat{\rho}, 1)$ be an extreme point of $T$. By Lemma 2.13, $(\hat{\pi}, \hat{\rho}, 1)$ defines a valid inequality for $K(n,r)$ and therefore it is implied by a conic combination of facet-defining inequalities plus a multiple of the equation $\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i = r$. In other words, there exists multipliers $\lambda \in \mathbb{R}_+^M$ and $\alpha \in \mathbb{R}$ such that

$$\hat{\pi}_i \geq \sum_{k=1}^M \lambda_k \pi_i^k + i\alpha, \qquad \forall\, i \in N \tag{7}$$

$$\hat{\rho}_i \geq \sum_{k=1}^M \lambda_k \rho_i^k - i\alpha, \qquad \forall\, i \in N \setminus \{n\} \tag{8}$$

$$\hat{\rho}_n = \sum_{k=1}^M \lambda_k \rho_n^k - n\alpha, \tag{9}$$

$$1 \leq \sum_{k=1}^M \lambda_k + r\alpha \tag{10}$$

hold. The inequalities in (7) follow from the fact that $\hat{\pi}_i = \sum_{k=1}^M \lambda_k \pi_i^k + i\alpha + \mu_i e_i$, where $\mu_i \geq 0$, for $i = 1, \ldots, n$. The term $\mu_i e_i$ corresponds to adding $\mu_i$ times the trivial facet-defining inequality $x_i \geq 0$. The inequalities in (8) can be derived in a similar manner, from the fact that $y_i \geq 0$ for $i = 1, \ldots, n-1$ are trivial facet-defining inequalities. The equality in (9) is due to the fact that $y_n \geq 0$ is considered to be nontrivial. As $\hat{\rho}_n = 0$ and $\rho_n^k = 0$ for all nontrivial facet-defining inequalities, (9) implies that $\alpha = 0$. Furthermore, $\hat{\rho}_r = 1$ and $\rho_r^k = 1$, for all $k$, and combining (8) and (10) we can conclude that $\sum_{k=1}^M \lambda_k = 1$.

For any $i < r$, inequality (7) for $i$ and $r-i$ combined with the equation (EP1) implies that

$$1 = \hat{\pi}_i + \hat{\pi}_{r-i} \geq \sum_{i=1}^M \lambda_k (\pi_i^k + \pi_{r-i}^k) = 1$$

which can hold only if $\hat{\pi}_i = \sum_{i=1}^M \lambda_k \pi_i^k$ for all $i < r$.

Similarly, for $i > r$, we use the equation (EP3) to observe that

$$1 = \hat{\pi}_i + \hat{\rho}_{i-r} \geq \sum_{i=1}^M \lambda_k (\pi_i^k + \rho_{i-r}^k) = 1$$

and therefore $\hat{\pi}_i = \sum_{i=1}^{M} \lambda_k \pi_i^k$ and $\hat{\rho}_{i-r} = \sum_{i=1}^{M} \lambda_k \rho_{i-r}^k$ for all $i > r$,.

Finally as $\hat{\rho}_i \geq \sum_{i=1}^{M} \lambda_k \rho_i^k$ for $i > n-r$, we can write $(\hat{\pi}, \hat{\rho}, 1)$ as a convex combination of points of $\mathcal{F}$ plus a conic combination of extreme rays of $T$. This can only be possible if $(\hat{\pi}, \hat{\rho}, 1) \in \mathcal{F}$. Thus, $(\hat{\pi}, \hat{\rho}, 1)$ is a nontrivial facet. ∎

As a final remark, it is interesting to note that conditions (R1) do not appear in the description of $T$ even though they are necessary for any valid inequality. This is because conditions (R1) are implied by (SA1), (SA2) and (SA3). The proof is analogous to the proof of Lemma 2.13, so we just state it as an observation.

**Observation 2.18** *Let $(\pi, \rho, \pi_o) \in T$. Then $j\pi_i + i\rho_j \geq 0, \forall i, j \in N$.*

We next show that coefficients of facet defining inequalities are bounded by small numbers.

**Lemma 2.19** *Let $(\pi, \rho, \pi_o)$ be an extreme point of $T$, then*

$$0 \leq \pi_k \leq \lceil k/r \rceil \quad \text{and} \quad -\lceil k/r \rceil \leq \rho_k \leq \lceil n/r \rceil$$

*for all $k \in N$.*

**Proof.** Using Observation 2.18 with $j = n$ and the fact that $\rho_n = 0$, we have $\pi \geq 0$.

For $k < r$, combining inequality (EP1) $\pi_k + \pi_{r-k} \leq 1$ with $\pi \geq 0$ gives $\pi_k \leq \lceil k/r \rceil = 1$. For $k > r$, let $k = \lfloor k/r \rfloor r + q$, where $0 \leq q < r$. If $q = 0$, by (SA2) we have $\pi_k \leq \lfloor k/r \rfloor \pi_r = \lceil k/r \rceil$ since $\frac{k}{r}$ is integer. Similarly, if $q > 0$ we have $\pi_k \leq \lfloor k/r \rfloor \pi_r + \pi_q = \lfloor k/r \rfloor + \pi_q$, where $\pi_q \leq 1$. Therefore, $0 \leq \pi_k \leq \lceil k/r \rceil$.

The inequality (SA3) with $i = 1$ and $j = k$ implies that $\rho_k \geq -\pi_k$ and therefore $\rho_k \geq -\lceil k/r \rceil$ for all $k \in N$ . If $k \leq n-r$, (EP3) implies that $\rho_k = \pi_r - \pi_{k+r} \leq 1 \leq \lceil n/k \rceil$. If $k > n-r$, then as $(\pi, \rho, \pi_o)$ is an extreme point of $T$, at least one of (SA1) and (SA3) must hold with equality, in which case, $\rho_k \leq \pi_i$, for some $i \in N$. Thus $\rho_k \leq \lceil n/r \rceil$ ∎

## 2.4 Pairwise subadditivity conditions

Next we give an example that demonstrates that using pairwise subadditivity conditions instead of the relaxed subadditivity conditions in the description of the coefficient polyhedron $T$ leads to extreme points that do not give facet defining inequalities for $K(n, r)$. To generate this example, we used PORTA developed by Thomas Christof and Andreas Löbel at ZIB.

**Example 2.20** *Consider $K(3, 2)$ and denote an extreme point of $T$ as $p = (\pi_1, \pi_2, \pi_3, \rho_1, \rho_2, \rho_3)$. We do not include $\pi_o$ in the description of $p$ since $\pi_o = 1$ for all points in $T$. It can be checked that $T$ has the following two extreme points: $p_1 = (1/2, 1, 0, 1, 1/2, 0)$ and $p_2 = (1/2, 1, 3/2, -1/2, -1, 0)$. Point $p_2$ corresponds to the nonnegativity constraint for $y_n$.*

*Let $T'$ be a restriction of $T$ obtained by replacing* (SA3) *with the missing pairwise subadditivity conditions* (SA0) *and* (SA1'). *It can be checked that in addition to $p_1$ and $p_2$, the set $T'$ has the following extreme point: $p_3 = (1/2, 1, 3/2, -1/2, 1/2, 0)$.*

*Note that $p_2 \leq p_3$ and $p_2 \neq p_3$. Therefore $p_3^T(x,y) \geq 1$ is strictly implied by $p_2^T(x,y) \geq 1$. Therefore $T'$ indeed has extreme points that do not correspond to facets of $K(3,2)$.*

One could think that by using (SA2') and/or (SA2") it would be possible to obtain a characterization of either nontrivial facets or all facets except nonnegativity constraints. However this is not the case, as using (SA2') and/or (SA2") in addition to (SA0) and (SA1'), we get counterexamples similar to the above one.

# 3 Lifting facets of $P(n,r)$ when $n > r > 0$

Lifting is a general principle for constructing valid (facet-defining) inequalities for higher dimensional sets using valid (facet-defining) inequalities for lower dimensional sets. Starting with the early work of Gomory [9], the lifting approach was generalized by Wolsey [18], Balas and Zemel [3] and Gu et. al [14], among others.

As $P(n,r)$ is an $n-1$ dimensional face of $K(n,r)$ obtained by setting $n$ variables to their lower bounds, any facet-defining inequality for $P(n,r)$

$$\sum_{i=1}^{n-1} \bar{\pi}_i x_i \geq 1 \tag{11}$$

can be lifted to obtain one or more facet-defining inequalities of the form

$$\sum_{i=1}^{n-1} \bar{\pi}_i x_i + \pi'_n x_n + \sum_{i=1}^{n-1} \rho'_i y_i \geq 1 \tag{12}$$

for $K(n,r)$. We call inequality (12) a *lifted* inequality. Throughout this section we assume that $n > r > 0$, as this assumption holds true for $P(n,r)$.

For any valid inequality $\pi x + \rho y \geq \beta$ for $K(n,r)$, if $\pi_i = 0$ for some $i \in N$, then Lemma 2.3 implies that $\rho \geq 0$. This, in turn, implies that $\pi \geq 0$. Therefore, a (trivial) facet of $P(n,r)$ defined by a nonnegativity inequality can only yield a conic combination of nonnegativity inequalities for $K(n,r)$ when lifted. Consequently, we only consider nontrivial facets of $P(n,r)$ for lifting.

## 3.1 The restricted coefficient polyhedron $T^{\bar{\pi}}$

We start with a result of Gomory [9] that gives a complete characterization of the nontrivial facets (i.e., excluding the nonnegativity inequalities) of $P(n,r)$. In this description facet defining inequalities are normalized so that $y_n$ has a coefficient of zero and the righthand side is 1.

**Theorem 3.1 (Gomory [9])** *The inequality $\bar{\pi} x \geq 1$ defines a nontrivial facet of $P(n,r)$, for $n > r > 0$, if and only if $\bar{\pi} \in \mathbb{R}^{n-1}$ is an extreme point of*

$$Q = \begin{cases} \pi_i + \pi_j \geq \pi_{(i+j) \bmod n} & \forall i,j \in \{1,\ldots,n-1\}, \\ \pi_i + \pi_j = \pi_r & \forall i,j \text{ such that } r = (i+j) \bmod n, \\ \pi_j \geq 0 & \forall j \in \{1,\ldots,n-1\}, \\ \pi_r = 1. & \end{cases}$$

Clearly nontrivial facets of $P(n, r)$ would give nontrivial facets of $K(n, r)$ when lifted. Using Gomory's characterization above, the lifting of nontrivial facets of $P(n, r)$ can be seen as a way of extending an extreme point $\bar{\pi}$ of $Q$ to obtain an extreme point $(\bar{\pi}, \pi'_n, \rho', 0)$ of $T$.

Let $p = (\bar{\pi}, \pi'_n, \rho', 0)$ be an extreme point of $T$. Then, $p$ also has to be an extreme point of the lower dimensional polyhedron

$$T^{\bar{\pi}} \; = \; T \cap \Big\{ \pi_i = \bar{\pi}_i, \; \forall i \in \{1, \ldots, n-1\} \Big\}.$$

Let $L = \{n - r + 1, \ldots, n - 1\}$.

**Lemma 3.2** *If* (11) *defines a nontrivial facet of* $P(n, r)$, *then* $T^{\bar{\pi}} \neq \emptyset$ *and has the form*

$$T^{\bar{\pi}} = \begin{cases} \tau \;\geq\; \pi_n \;\geq\; 0 & \\ \rho_k \geq l_k & \forall\; k \in L \\ \rho_k + \pi_n \geq t_k & \forall\; k \in L \\ \rho_k - \pi_n \geq f_k & \forall\; k \in L \\ \pi_n + \rho_{n-r} = 1 & \\ \rho_n = 0 & \\ \rho_k = \bar{\pi}_{n-k} & \forall k \in \{1, \ldots, n-r-1\} \\ \pi_i = \bar{\pi}_i & \forall i \in \{1, \ldots, n-1\} \end{cases}$$

*where numbers* $l_k, t_k, f_k$ *and* $\tau$ *can be computed easily using* $\bar{\pi}$.

**Proof.** First note that $\bar{\pi} \in Q$ and therefore $\bar{\pi}$ satisfies inequality (SA2) as well as equations (EP1) and (EP2). In addition, as $\bar{\pi}_i + \bar{\pi}_j = 1$ for all $i, j$ such that $r = (i + j) \bmod n$, equality (EP3) can be rewritten as $\rho_i = \pi_{n-i}$ for all $1 \leq i \leq n - r$. Further, as $\bar{\pi}$ is subadditive (in the modular sense), inequalities (SA1) and (SA3) are satisfied for all $k \in \{1, \ldots, n - r - 1\}$. Therefore, setting

$$\pi_n = 0 \quad \text{and} \quad \rho_k = \begin{cases} \bar{\pi}_{n-k} & \text{if } k \in \{1, \ldots, n-r-1\}, \\ 1 & \text{otherwise}, \end{cases}$$

produces a feasible point for $T^{\bar{\pi}}$, establishing that the set is not empty.

We next show that $T^{\bar{\pi}}$ has the proposed form, and also compute the values of $l_k, t_k, f_k$ and $\tau$.

**Inequality (SA1):** If $i = n$, (SA1) becomes $\pi_n + \rho_k \geq \bar{\pi}_{n-k}$. If $i \neq n$, it becomes $\rho_k \geq \bar{\pi}_{i-k} - \bar{\pi}_i$, and therefore $\rho_k \geq l_k^1 = \max_{n>i>k}\{\bar{\pi}_{i-k} - \bar{\pi}_i\}$.

**Inequality (SA2):** The only relevant case is $i+j = n$ when (SA2) becomes $\pi_n \leq \bar{\pi}_i + \bar{\pi}_{n-i}$. When combined, these inequalities simply become $\pi_n \leq \tau^1 = \min_{n>i>0}\{\bar{\pi}_i + \bar{\pi}_{n-i}\}$.

**Inequality (SA3):** Without loss of generality assume $i \geq j$. We consider 3 cases.

*Case 1, $k = n$:* In this case the inequality reduces to $\pi_i + \pi_j \geq \pi_{i+j-n}$ which is satisfied by $\bar{\pi}$ when $i, j < n$. For $i = n$, this inequality simply becomes $\pi_n \geq 0$.

*Case 2, $k < n$ and $i + j - k = n$:* In this case the inequality becomes $\rho_k - \pi_n \geq -\pi_i - \pi_j$. If $i, j < n$ these inequalities can be combined to obtain $\rho_k - \pi_n \geq f_k^1 = \max_{1 \leq i,j<n, \; k=i+j-n} \big\{ -\bar{\pi}_i - \bar{\pi}_j \big\}$. If, $i = n$, then $j = k$ and the inequality becomes $\rho_k \geq -\bar{\pi}_k$.

*Case 3, $k < n$ and $i + j - k < n$:* If $i, j < n$ the inequality becomes $\rho_k \geq \pi_{i+j-k} - \pi_i - \pi_j$. These inequalities can be combined to obtain $\rho_k \geq l_k^2 = \max_{1 \leq i,j<n:\ k<i+j<n+k} \left\{ \bar{\pi}_{i+j-k} - \bar{\pi}_i - \bar{\pi}_j \right\}$. If $i = n$ then $j < n$, so the inequality becomes $\pi_n + \rho_k \geq \pi_{n+j-k} - \pi_j$, implying $\pi_n + \rho_k \geq t_k^1 = \max_{k>j} \left\{ \bar{\pi}_{n+j-k} - \bar{\pi}_j \right\}$.

Therefore, combining these observations, it is easy to see that $T^{\bar{\pi}}$ has form given in Lemma 3.2 where $l_k, t_k, f_k$ and $\tau$ are computed as follows:

$$
\begin{aligned}
l_k &= \max \left\{ l_k^1, l_k^2, -\bar{\pi}_k \right\}, \\
t_k &= \max \left\{ t_k^1, \bar{\pi}_{n-k} \right\} \ = \ \bar{\pi}_{n-k}, \\
f_k &= f_k^1, \\
\tau &= \min \left\{ \tau^1,\ 1 - l_{n-r},\ (1 - f_{n-r})/2 \right\}.
\end{aligned}
$$

The second equality in the description of $t_k$ states that $t_k = \bar{\pi}_{n-k}$ comes from the fact that $\bar{\pi}$ is sub-additive and therefore $\bar{\pi}_{n-k} + \bar{\pi}_j \geq \bar{\pi}_{n+j-k}$ for all $j < k$. The $1 - l_{n-r}$ and $(1 - f_{n-r})/2$ terms in the last equation come from using the bounds on $\rho_{n-r}$ together with the equations and inequalities of $T^{\bar{\pi}}$ to obtain implied bounds for $\pi_n$.

■

We next make a simple observation that will help us show that $T^{\bar{\pi}}$ has a polynomial number of extreme points.

**Lemma 3.3** *If $p = (\bar{\pi}, \pi'_n, \rho', 0)$ is an extreme point of $T^{\bar{\pi}}$, then*

$$\rho'_k = \max \left\{ l_k, t_k - \pi'_n, f_k + \pi'_n \right\}$$

*for all $k \in L$.*

**Proof.** Assume that the claim does not hold for some $k \in L$ and let $\theta = \max \left\{ l_k, t_k - \pi'_n, f_k + \pi'_n \right\}$. As $p \in T^{\bar{\pi}}$, $\rho'_k \geq \theta$ and therefore $\epsilon = \rho'_k - \theta > 0$. In this case, two distinct points in $T^{\bar{\pi}}$ can be generated by increasing and decreasing the associated coordinate of $p$ by $\epsilon$, establishing that $p$ is not an extreme point, a contradiction. ■

We next characterize the set possible values $\pi'_n$ can take at an extreme point of $T^{\bar{\pi}}$.

**Lemma 3.4** *Let $p = (\bar{\pi}, \pi'_n, \rho', 0)$ be an extreme point of $T^{\bar{\pi}}$, if $\pi'_n \notin \{0, \tau\}$, then*

$$\pi'_n \ \in \ \Lambda \ := \ \left( \bigcup_{k \in L_1} \left\{ t_k - l_k, l_k - f_k \right\} \right) \bigcup \left( \bigcup_{k \in L_2} \left\{ (t_k - f_k)/2 \right\} \right)$$

*where $L_1 \ = \ \{k \in L \ : \ t_k + f_k < 2l_k\}$ and $L_2 \ = \ L \setminus L_1$.*

**Proof.** Note that the description of $T^{\bar{\pi}}$ consists of $3(r-1)$ inequalities that involve $\rho_k$ variables and upper and lower bound inequalities for $\pi'_n$. Being an extreme point, $p$ has to satisfy $r$ of these inequalities as equality. Therefore, if $\pi'_n \notin \{0, \tau\}$ then, there exists an index $k \in L$ for which at least two of the following inequalities

$$\rho_k \geq l_k \tag{a}$$

$$\rho_k + \pi_n \geq t_k \tag{b}$$

$$\rho_k - \pi_n \geq f_k \tag{c}$$

hold as equality. Clearly, this uniquely determines the value of $\pi'_n$ and therefore

$$\pi'_n \in \Lambda^+ = \bigcup_{k \in L} \Big\{ t_k - l_k, l_k - f_k, (t_k - f_k)/2 \Big\}.$$

Furthermore, for any fixed $k \in L$, adding inequalities (b) and (c) gives $2\rho_k \geq t_k + f_k$. Therefore if $t_k + f_k > 2l_k$ inequality (a) is implied by inequalities (b) and (c) and it cannot hold as equality. Similarly, if $t_k + f_k < 2l_k$, inequalities (b) and (c) cannot hold simultaneously. Finally, if $t_k + f_k = 2l_k$ then it is easy to see that $t_k - l_k = l_k - f_k = (t_k - f_k)/2$. Therefore letting

$$L_1 = \{k \in S : t_k + f_k < 2l_k\}, \quad L_2 = L \setminus L_1$$

proves the claim. ∎

Combining the previous Lemmas, we have the following result:

**Theorem 3.5** *Given a nontrivial facet-defining inequality (11) for $P(n, r)$, there are at most $2r$ lifted inequalities that define facets of $K(n, r)$.*

**Proof.** The set $L$ has $r - 1$ members and therefore together with 0 and $\tau$, there are at most $2r$ possible values for $\pi'_n$ in a facet-defining lifted inequality (12). As the value of $\pi'_n$ uniquely determines the remaining coefficients in the lifted inequality, by Lemma 3.3, the claim follows. ∎

In general, determining all possible lifted inequalities is a hard task. However, the above results show that obtaining all possible facet-defining inequalities lifted from a facet of $P(n, r)$ is straightforward and can be performed in polynomial time. In Figure 1, we display two facets of $K(16, 13)$ obtained by lifting the same facet of $P(16, 13)$. The facet coefficient of each variable is a function of its coefficient in $K(n, r)$, that is, variable $x_i$ will have a facet coefficient $f(i)$ and variable $y_i$ will have a facet coefficient $f(-i)$. We marked all of these coefficients as discs in the figure. Also, note that the displayed functions are obtained by interpolating the facet coefficients; we explain their significance in Section 8.

Note that the second facet has the same coefficient values for $x_i, i = 1, \ldots, n - 1$ as the first, and a larger coefficient for $x_n$, and therefore it has coefficient values for the $y$ variables which are less than (sometimes strictly less than) the corresponding coefficients for the first facet. To make comparison easier, we plot the first facet in dashed lines behind the second facet. Later, in Figure 2, we display facets of $K(16, 13)$ which cannot be obtained by lifting.

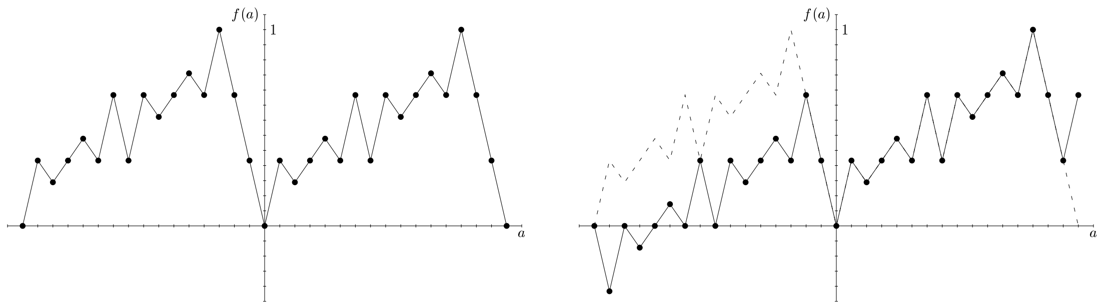


Figure 1: Example of two facets of $K(16, 13)$ obtained by lifting

### 3.2 Sequential lifting

Sequential lifting is a procedure that introduces the missing variables one at a time to obtain the lifted inequality. Depending on the order in which missing variables are lifted, one obtains different inequalities. We note that not all lifted facets can be obtained by sequential lifting, see [15].

In Lemma 3.2 we have established that (regardless of the lifting sequence) $\rho'_k = \pi_{n-k}$ for $k \in K' = \{1, \ldots, n-r-1\}$. Furthermore, in Lemma 3.3 we established that the coefficient of variable $x_n$ determines the coefficients of the remaining variables. Therefore, given a lifting sequence, if $K''$ denotes the set of indices of $y_k$ variables that are lifted after either one of $x_n$ or $y_{n-r}$, then the lifted inequality does not depend on the order in which variables $y_k$ for $k \in K'$ or $k \in K''$ are lifted. We next present a result adapted from Wolsey [18].

**Lemma 3.6 (Wolsey [18])** *Given a facet-defining inequality (11) for $P(n, r)$ and a lifting sequence for the variables $x_n$ and $y_i$ for $i = 1, \ldots, n-1$, the sequential lifting procedure produces a facet-defining inequality for $K(n, r)$.*

*Furthermore, when a variable is lifted, it is assigned the smallest value among all coefficients for that variable in lifted inequalities having the same coefficients for previously lifted variables.*

Therefore, given a nontrivial facet defining inequality $\bar{\pi}x \geq 1$, for $P(n, r)$, the lifting coefficient of the variable currently being lifted can simply be computed by solving a linear program that minimizes the coefficient of that variable subject to the constraint that $T^{\bar{\pi}}$ has a point consistent with the coefficients of the variables that have already been lifted.

We conclude this section by showing that the simple mapping discussed in the Introduction (see inequality (4)) leads to a particular lifted inequality.

**Lemma 3.7** *If variable $x_n$ is lifted before all $y_k$ for $k \in \{n-r, \ldots, n-1\}$, then independent of the rest of the lifting sequence the lifted inequality is*

$$\sum_{i=1}^{n-1} \bar{\pi}_i x_i + \sum_{i=1}^{n-1} \bar{\pi}_{n-i} y_i \geq 1.$$

**Proof.** By Lemma 3.6, we know that variable $x_n$ will be assigned the smallest possible coefficient in the lifted inequality. As $\pi_n \geq 0$ in the description of $T^{\bar{\pi}}$ and as $T^{\bar{\pi}}$ does contain a point with $\pi_n = 0$ (as described in the proof of Lemma 3.2), we conclude that $\pi_n = 0$ in the lifted inequality.

Therefore, by Lemma 3.3, $\rho'_k = \min\left\{l_k, \bar{\pi}_{n-k}, f_k\right\}$ and we need to show that $\bar{\pi}_{n-k} \geq l_k, t_k$ for all $k \in \{1, \ldots, n-1\}$. First, observe that $0 \geq t_k$ and therefore $\bar{\pi}_{n-k} \geq t_k$ for all $k \in \{1, \ldots, n-1\}$. Finally, recall that $\bar{\pi}$ is subadditive (in the modular sense), and therefore $\bar{\pi}_{n-k} + \bar{\pi}_i \geq \bar{\pi}_{i-k}$ for all $n > i > k$ and $\bar{\pi}_{n-k} + \bar{\pi}_i + \bar{\pi}_j \geq \bar{\pi}_{i+j-k}$ for all $n > i, j$ and $n + k > i + j > k$. ∎

The first facet in Figure 1 has the form given in the previous Lemma; it is obtained from a facet of $P(n, r)$ by first lifting $x_n$.

# 4 Mixed integer rounding inequalities for $K(n, r)$ when $r > 0$

In this section we study MIR inequalities in the context of $K(n, r)$. Our analysis also provides an example that shows that lifting facets of $P(n, r)$ cannot give all facets of $K(n, r)$. Throughout, we will use the notation $\hat{x} := x - \lfloor x \rfloor$ and $(x)^+ = \max\{x, 0\}$. Recall that, for a general single row system of the form: $\left\{w \in \mathbb{Z}^p_+ : \sum_{i=1}^p a_i w_i = b\right\}$ where $\hat{b} > 0$, the MIR inequality is:

$$\sum_{i=1}^{p} \left( \lfloor a_i \rfloor + \min\left( \hat{a}_i / \hat{b}, 1 \right) \right) w_i \geq \lceil b \rceil .$$

We define the $\frac{1}{t}$-MIR (for $t \in \mathbb{Z}_+$) to be the MIR inequality obtained from the following equivalent representation of $K(n, r) = \left\{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+ : \sum_{i=1}^n (i/t) x_i - \sum_{i=1}^n (i/t) y_i = r/t\right\}$.

**Lemma 4.1** *Given $t \in \mathbb{Z}$ such that $2 \leq t \leq n$, the $\frac{1}{t}$-MIR inequality*

$$\sum_{i=1}^{n} \left( \left\lfloor \frac{i}{t} \right\rfloor + \min\left( \frac{i \mod t}{r \mod t}, 1 \right) \right) x_i + \sum_{i=1}^{n} \left( - \left\lceil \frac{i}{t} \right\rceil + \min\left( \frac{(t-i) \mod t}{r \mod t}, 1 \right) \right) y_i \geq \left\lceil \frac{r}{t} \right\rceil$$

*is facet-defining for $K(n, r)$ provided that $r/t \notin \mathbb{Z}$.*

**Proof.** Let $\pi x + \rho y \geq \pi_o$ denote the $\frac{1}{t}$-MIR inequality and let $F$ denote the set of points that are on the face defined by this inequality. Also let $q^i$ denote $(i \mod t)$. Using this definition $i = q^i + \lfloor i/t \rfloor t$.

For $i \in N \setminus \{1, t\}$, consider the point

$$w^i = e_i + (\lfloor r/t \rfloor - \lfloor i/t \rfloor)^+ e_t + (\lfloor i/t \rfloor - \lfloor r/t \rfloor)^+ f_t + (q^r - q^i)^+ e_1 + (q^i - q^r)^+ f_1$$

and observe that $w^i \in K(n, r)$. Moreover,

$$(\pi, \rho)^T w^i = (\lfloor i/t \rfloor + \min\{q^i/q^r, 1\}) + (\lfloor r/t \rfloor - \lfloor i/t \rfloor) + \frac{(q^r - q^i)^+}{q^r} = \lfloor r/t \rfloor + 1 = \pi_o$$

and therefore $w^i \in F$. Similarly, let $-i = -\lceil i/t \rceil + m^i$, with $0 \leq m^i < t$ and consider the point

$$z^i = f_i + (\lfloor r/t \rfloor + \lceil i/t \rceil) e_t + (q^r - m^i)^+ e_1 + (m^i - q^r)^+ f_1$$

for $i \in N \setminus \{1,t\}$. Clearly $x^i \in K(n,r)$. Furthermore,

$$(\pi,\rho)^T z^i = (-\lceil i/t \rceil + \min\{m/q^r, 1\}) + (\lfloor r/t \rfloor + \lceil i/t \rceil) + \frac{(q^r - m)^+}{q^r} = \lfloor r/t \rfloor + 1 = \pi_o$$

and therefore $z^i \in F$.

Additionally the following three points are also in $K(n,r) \cap F$: $u^1 = \lfloor r/t \rfloor e_t + q^r e_1$, $u^2 = (\lfloor r/t \rfloor + 1)e_t + (t - q^r) f_1$, $u^3 = (\lfloor r/t \rfloor + 1)e_t + f_t + q^r e_1$. Therefore, $\{u^i\}_{i=1}^3 \cup \{w^i\}_{i \in N \setminus \{1,t\}} \cup \{z^i\}_{i \in N \setminus \{1,t\}}$ is a set of $2n-1$ affinely independent points in $F$. ∎

We next show that $\frac{1}{t}$-MIR inequalities are not facet-defining unless they satisfy the conditions of Theorem 4.1. First, observe that the inequality is not defined if $t$ divides $r$. Next, we show that $1/n$-MIR inequality dominates all $\frac{1}{t}$-MIR inequalities with $t > n$.

**Lemma 4.2** *If $t > n$, then $\frac{1}{t}$-MIR inequality is not facet-defining for $K(n,r)$.*

**Proof.** When $t > n$, $\frac{1}{t}$-MIR inequality becomes

$$\sum_{i \in N} \min\{i/r, 1\} x_i - \sum_{i: i > t-r} \left(1 - \frac{t-i}{r}\right) y_i \geq 1$$

and is dominated by the $1/n$-MIR:

$$\sum_{i \in N} \min\{i/r, 1\} x_i - \sum_{i: i > n-r} \left(1 - \frac{n-i}{r}\right) y_i \geq 1.$$

∎

We conclude this section by showing that $\frac{1}{t}$-MIR inequalities give facets that cannot be obtained by lifting facets of $P(n,r)$.

**Theorem 4.3** *For $n \geq 9$ and $n-2 \geq r > 0$, there are facet-defining inequalities for $K(n,r)$ that cannot be obtained by lifting facet-defining inequalities for $P(n,r)$.*

**Proof.** When $0 < r \leq n-4$, consider the facet induced by the $\frac{1}{n-2}$-MIR inequality $\pi x + \rho y \geq \pi_o$ where

$$\rho_n = -2 + \min\left(\frac{n-4}{r}, 1\right) = -1.$$

We therefore subtract $\frac{1}{n}$ times $\sum_{i=1}^n i x_i - \sum_{i=1}^n i y_i = r$ to the inequality to obtain $\pi' x + \rho' y \geq \pi'_o$ where $\rho'_n = 0$ and therefore it satisfies the normalization condition (NC1). Notice that

$$\begin{aligned} \pi'_{r+1} + \pi'_{n-1} &= \left(1 - \frac{r+1}{n}\right) + \left(1 + \frac{1}{r} - \frac{n-1}{n}\right) \\ &= 1 - \frac{r}{n} + \frac{1}{r} \end{aligned}$$

whereas $\pi'_r = 1 - r/n < \pi'_{r+1} + \pi'_{n-1}$. This proves the claim for $0 < r \leq n-4$ as all facet-defining inequalities for $P(n,r)$ have to satisfy $\pi'_{r+1} + \pi'_{n-1} = \pi'_r$.

For $r \in \{n-3, n-2\}$, the $\frac{1}{r-1}$-MIR provides such an example. ∎

For $r = n-1$, all points in $T$ automatically satisfy all equations in $Q$. Therefore, any given facet-defining inequality of $K(n,r)$ can be obtained by lifting a point in $Q$. However, this point is not necessarily an extreme point of $Q$.

## 5 Polyhedral analysis of $K(n, 0)$

Observe that $LK(n, 0)$, the linear relaxation of $K(n, 0)$, is a pointed cone (as it is contained in the nonnegative orthant) and has a single extreme point $(x, y) = (0, 0)$. Therefore $LK(n, 0)$ equals its integer hull, i.e., $LK(n, 0) = K(n, 0)$. In Lemma 2.3, we characterized the extreme rays of $K(n, r)$ and thereby showed that the recession cone of $K(n, r)$ is generated by the vectors $\{r_{ij}\}$. But the recession cone of $K(n, r)$ for some $r > 0$ is just $K(n, 0)$. Therefore, $LK(n, 0)$ is generated by the vectors $\{r_{ij}\}$, and the next result follows.

**Theorem 5.1** *The inequality $\pi x + \rho y \geq \pi_o$ is facet-defining for $K(n, 0)$ if and only if $(\pi, \rho, \pi_o)$ is a minimal face of*

$$T_o = \begin{cases} j\pi_i + i\rho_j \geq 0 & , \forall i, j \in N, \\ \pi_o = 0. \end{cases}$$

In his work on MCGP, Gomory also studied $P(n, 0)$, the convex hull of non-zero integral solutions that satisfy $\sum_{i=1}^{n-1} ix_i - ny_n = 0$, and showed that Theorem 3.1 also holds for $r = 0$. For the sake of completeness, we now consider a similar modification of $K(n, 0)$ and study the set:

$$\bar{K}(n, 0) = conv\left\{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+ : \sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = 0, (x, y) \neq 0\right\}.$$

For this case it will be more convenient to consider all nonnegativity inequalities (including $y_n \geq 0$) as trivial. We will next prove that all nontrivial facet-defining inequalities for $\bar{K}(n, 0)$ are given by the extreme points of $\bar{T}_o$ defined below.

**Definition 5.2** *We define $\bar{T}_o \subseteq \mathbb{R}^{2n+1}$ as the set of points that satisfy the following linear equalities and inequalities:*

$$\pi_i + \rho_j \geq \pi_{i-j}, \qquad \forall i, j \in N, \quad i > j, \tag{SA1}$$

$$\pi_i + \rho_j \geq \rho_{j-i}, \qquad \forall i, j \in N, \quad i < j, \tag{SA1'}$$

$$\pi_i + \rho_i = \pi_o, \qquad \forall i \in N, \tag{EP1-R0}$$

$$\pi_o = 1, \tag{N1-R0}$$

$$\rho_n = 0. \tag{N2-R0}$$

It is easy to see that the conditions (SA1), (SA1'), and (EP1-R0) are together equivalent to the conditions (SA2), (SA2'), (SA2") and (EP1-R0). For example, replacing $\pi_i$ by $\pi_o - \rho_i$ and $\pi_{i-j}$ by $\pi_o - \rho_{i-j}$ in (SA1), we get (SA2'). Therefore, a point in $\bar{T}_o$ satisfies all the pairwise subadditivity conditions given in the previous section.

Also note that $\bar{T}_o$ is a bounded polyhedron. To see this notice that after swapping indices $i$ and $j$ in inequality (SA1') and using equation (EP1-R0) to substitute out $\rho$ variables in (SA1) and (SA1') we obtain

$$1 \geq \pi_j + \pi_{i-j} - \pi_i \geq 0 \qquad \forall i, j \in N, \quad i > j.$$

Therefore, any ray in the recession cone of $\bar{T}_o$ must have $d_j + d_{i-j} - d_i = 0$ where $d$ denotes the components of the ray corresponding to $\pi$ variables (note that, by equation (EP1-R0), $-d$ gives the components of the ray for the $\rho$ variables). Taking $j = 1$, it is easy to see that $d_i = id_1$ for all $i \in N$. As $d_n = 0$ by equation (N2-R0), each $d_i$ has to be zero.

**Lemma 5.3** *If $(\pi, \rho, \pi_o) \in \bar{T}_o$ then $\pi x + \rho y \geq \pi_o$ is a valid inequality for $\bar{K}(n, 0)$.*

**Proof.** Suppose $\pi x + \rho y \geq \pi_o$ is not valid for $\bar{K}(n, 0)$. Then, let $(x^*, y^*) \in \bar{K}(n, 0)$ be the integer point in $\bar{K}(n, 0)$ with smallest $L_1$ norm such that $\pi x^* + \rho y^* < \pi_o$. Note that any point in $\bar{K}(n, 0)$ has $L_1$ norm 2 or more.

If $||(x^*, y^*)||_1 = 2$, then $(x^*, y^*) = e_i + f_i$ for some $i \in N$, but by (EP1-R0), $\pi x^* + \rho y^* = \pi_o$, which is a contradiction. So we may assume that $||(x^*, y^*)||_1 > 2$.

As $(x^*, y^*) \in \bar{K}(n, 0)$, there exists $i, j \in N$ such that $x_i^* > 0$ and $y_j^* > 0$. Let

$$(x', y') = (x^*, y^*) - e_i - f_j + \begin{cases} f_{j-i} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ e_{i-j} & \text{if } i > j. \end{cases}$$

Clearly, $(x', y')$ is an integer point in $\bar{K}(n, 0)$ and $||(x', y')||_1 \leq ||(x^*, y^*)||_1 - 1$. Furthermore, as $(\pi, \rho, \pi_o)$ satisfies (SA1), (SA1') and (EP1-R0), we also have $\pi x' + \rho y' \leq \pi x^* + \rho x^* < \pi_o$, which contradicts the choice of $(x^*, y^*)$. ∎

**Theorem 5.4** *Consider an inequality $\pi x + \rho y \geq \pi_o$ with $\rho_n = 0$. It defines a nontrivial facet of $\bar{K}(n, 0)$ if and only if it the following conditions hold: (i) $\pi_o > 0$, and, (ii) $(\pi, \rho, \pi_o)/\pi_o$ is an extreme point of $\bar{T}_o$.*

**Proof.**

($\Rightarrow$):

Let $\pi x + \rho y \geq \pi_o$ define a nontrivial facet of $\bar{K}(n, 0)$. We first show that $(\pi, \rho, \pi_o)$ satisfies (SA1), (SA1') and (EP1-R0), and can be assumed to satisfy (N1-R0) and (N2-R0).

**(SA1) - (SA1'):** Let $i, j$ be indices such that $i, j \in N$ and $i > j$. Let $z = (x^*, y^*)$ be an integral point lying on the above facet such that $x_{i-j}^* > 0$. As $z + (e_i + f_j - e_{i-j})$ belongs to $\bar{K}(n, 0)$, (SA1) is true. The proof of (SA1') is similar.

**(EP1-R0):** Let $\gamma = (\pi, \rho)$. Let $z^1 = (x^1, y^1)$ and $z^2 = (x^2, y^2)$ be integral points lying on the facet such that $x_i^1 > 0$ and $y_i^2 > 0$. Then $z = z^1 + z^2 - e_i - f_i \in \bar{K}(n, 0)$, and therefore $\gamma z = \gamma z^1 + \gamma z^2 - \pi_i - \rho_i = 2\pi_o - \pi_i - \rho_i \geq \pi_o \Rightarrow \pi_i + \rho_i \leq \pi_o$. But as $e_i + f_i \in \bar{K}(n, 0)$, $\pi_i + \rho_i \geq \pi_o$ and the result follows.

**(N1-R0):** Assume $\pi_o < 0$, and let $(x^*, y^*)$ be an integral point in $\bar{K}(n, 0)$ satisfying $\pi x^* + \rho y^* = \pi_o$. As $\alpha(x^*, y^*) \in \bar{K}(n, 0)$ for any positive integer $\alpha$, whereas $\pi\alpha x^* + \rho\alpha y^* = \alpha\pi_o < \pi_o$, we obtain a contradiction to the fact that points in $\bar{K}(n, 0)$ satisfy $\pi x + \rho y \geq \pi_o$.

If $\pi_o = 0$, then (EP1-R0) implies that $\rho_i = -\pi_i$ for all $i \in N$. This fact, along with (SA1) and (SA1') implies that $\pi_i = i\pi_1$ and $\rho_i = -i\pi_1$ for all $i \in N$. But then $\pi x + \rho y \geq \pi_o$ is the same as

$\pi_1(\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i) \geq 0$, and therefore cannot define a proper face of $\bar{K}(n,0)$. Therefore, for any nontrivial facet, $\pi_o > 0$ and can be assumed to be 1 by scaling.

We can assume, by subtracting appropriate multiples of $\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i = 0$ from $\pi x + \rho y \geq \pi_o$, that (N2-R0) holds.

Therefore $(\pi, \rho, \pi_o)$ can be assumed to be contained in $\bar{T}_o$. If it is not an extreme point of $\bar{T}_o$, it can be written as a convex combination of two distinct points of $\bar{T}_o$, different from itself, each of which defines a valid inequality for $\bar{K}(n,0)$ (by Lemma 5.3). As the normalization conditions (N1-R0) and (N2-R0) mean that each nontrivial facet-defining inequality corresponds to a unique point in $\bar{T}_o$, this implies that $(\pi, \rho, \pi_o)$ is an extreme point of $\bar{T}_o$.

($\Leftarrow$):
Let $\mathcal{F} = \{(\pi^k, \rho^k, \pi_o^k)\}_{k=1}^M$ be the set of all nontrivial facets of $\bar{K}(n,0)$ such that $\rho_n^k = 0$ and $\pi_o^k = 1$. Let $(\pi, \rho, 1)$ be an extreme point of $\bar{T}_o$. By Lemma 5.3, $(\pi, \rho, 1)$ defines a valid inequality for $\bar{K}(n,0)$, and therefore there exist numbers $\lambda^k$ and $\alpha$ such that

$$
\begin{array}{lll}
\alpha i + \sum_{k=1}^M \lambda_k \pi_i^k & \leq \pi_i, & \forall i \in N \\
-\alpha i + \sum_{k=1}^M \lambda_k \rho_i^k & \leq \rho_i, & \forall i \in N \\
\sum_{k=1}^M \lambda_k & \geq 1 & \\
\lambda \geq 0, & \alpha \text{ free} &
\end{array}
$$

(EP1-R0) implies that for all $i \in N$, $1 = \pi_i + \rho_i \geq \sum_{k=1}^M \lambda_k(\pi_i^k + \rho_i^k)$ and since all nontrivial facets also satisfy (EP1-R0), we can conclude that $\sum_{k=1}^M \lambda_k = 1$. Therefore, $1 = \pi_i + \rho_i \geq \sum_{k=1}^M \lambda_k(\pi_i^k + \rho_i^k) = 1$ and hence $\pi_i = \sum_{k=1}^M \lambda_k \pi_i^k$ and $\rho_i = \sum_{k=1}^M \lambda_k \rho_i^k$. In other words, $(\pi, \rho, 1)$ can be expressed as a convex combination of the elements of $\mathcal{F}$, each of which is contained in $\bar{T}_o$. This is possible only if $(\pi, \rho, 1)$ is itself an element of $\mathcal{F}$, i.e., it defines a nontrivial facet of $\bar{K}(n,0)$.
■

As $P(n,0)$ is a lower dimensional face of $\bar{K}(n,0)$, it is possible to lift facet defining inequalities of $P(n,0)$ to obtain facet defining inequalities for $\bar{K}(n,0)$. From the description of $\bar{T}_o$, any nontrivial facet-defining inequality $\pi x + \rho y \geq \pi_o$ of $\bar{K}(n,0)$ satisfies $\rho_i = 1 - \pi_i$. Thus, if $\sum_{i=1}^{n-1} \bar{\pi}_i x_i \geq 1$ defines a nontrivial facet of $P(n,0)$, there is a unique way to lift this inequality to obtain a facet-defining inequality for $K(n,0)$, namely:

$$x_n + \sum_{i=1}^{n-1} \bar{\pi}_i x_i + \sum_{i=1}^{n-1} (1 - \bar{\pi}_i) y_i \geq 1.$$

Next, we show that when $n = 3$ not all facet defining inequalities for $\bar{K}(n,0)$ can be obtained via lifting. Remember that, after normalization, coefficients of all facet defining inequalities for $P(n,0)$ are between 0 and 1.

**Example 5.5** *Consider $\bar{K}(3,0)$ and note that all $(x,y) \in \bar{K}(3,0)$ satisfy $\sum_{i=1}^3 x_i \geq 1$. Adding $\sum_{i=1}^3 ix_i - \sum_{i=1}^3 iy_i = 0$ to this inequality yields*

$$2x_1 + 3x_2 + 4x_3 - y_1 - 2y_2 - 3y_3 \geq 1. \tag{13}$$

*Dividing* (13) *by* 2 *and writing the MIR inequality gives* $x_1+2x_2+2x_3-y_2-y_3 \geq 1$ *which becomes*

$$\frac{2}{3}x_1+\frac{4}{3}x_2+x_3+\frac{1}{3}y_1-\frac{1}{3}y_2 \geq 1 \tag{14}$$

*after normalization. Denoting feasible points* $p \in K(3,0)$ *as* $p=(x_1,x_2,x_3,y_1,y_2,y_3)$*, note that the following 5 affinely independent points satisfy* (14) *as equality:* $p_1=(1,0,0,1,0,0)$*,* $p_2=(0,1,0,0,1,0)$*,* $p_3=(0,0,1,0,0,1)$*,* $p_4=(2,0,0,0,1,0)$ *and* $p_5=(0,0,1,1,1,0)$ *and therefore* (14) *defines a facet of* $\bar{K}(3,0)$*.*

*Notice that* (14) *cannot be obtained via lifting as the coefficient of* $x_2$ *is greater than 1.*

**Lemma 5.6** *For any* $n \geq 3$*,* $\bar{K}(n,0)$ *has at least one facet that cannot be obtained by lifting a facet of* $P(n,0)$ *.*

**Proof.** For $n=3$ the example above proves the claim so we consider $n \geq 4$. Let

$$\pi_i = \begin{cases} \frac{i+2}{n} & \text{if } i<n \\ 1 & \text{if } i=n \end{cases} \qquad \rho_i = \begin{cases} \frac{n-i-2}{n} & \text{if } i<n \\ 0 & \text{if } i=n \end{cases}$$

and notice that $\pi_{n-1}>1$. We next show that $p=(\pi,\rho,1) \in \bar{T}_o$. Clearly $p$ satisfies equations (EP1-R0), (N1-R0) and (N2-R0). Using $(i+2)/n \geq \pi_i \geq i/n$ and $(n-i)/n \geq \rho_i \geq (n-i-2)/n$ for all $i \in N$, note that

$$\pi_i+\rho_j-\pi_{i-j} \geq i/n+(n-j-2)/n-(i-j+2)/n \geq 1-4/n \geq 0$$

and therefore $p$ satisfies inequality (SA1). Finally, if we have that $i,j \in N$ and $j>i$, then $j-i<n$ which implies that $\rho_{j-i}=(n-j+i-2)/n$ and hence:

$$\pi_i+\rho_j-\rho_{j-i} \geq i/n+(n-j-2)/n-(n-j+i-2)/n \geq 0$$

Therefore $p$ satisfies inequality (SA1') as well and it is indeed contained in $T_o$. As $T_o$ is a bounded polyhedron, it must therefore have an extreme point $(\bar{\pi},\bar{\rho},1)$ that has $\bar{\pi}_{n-1}>1$. Clearly, the inequality corresponding to this extreme point cannot be obtained by lifting a facet of $P(n,0)$.

■

## 6 Separating from $K(n,r)$

Let $LK(n,r)$ be the linear relaxation of $K(n,r)$. We define the separation problem over $K(n,r)$ as follows: given $(x^*,y^*) \in LK(n,r)$, either verify that $(x^*,y^*) \in K(n,r)$ or find a violated valid inequality for $K(n,r)$. Note that the condition that $(x^*,y^*) \in LK(n,r)$ is easy to satisfy.

From Theorems 2.6, 5.1 and 5.4, it follows that any point $(x^*,y^*) \in LK(n,r)$, with $0<r \leq n$, can be separated from $K(n,r)$ by minimizing $\pi x^*+\rho y^*$ over $T$. Therefore, in this case the separation problem can be solved in polynomial-time using an LP with $O(n)$ variables and $O(n^3)$ constraints. Similarly, by Theorem 5.4, one can separate a point from $\bar{K}(n,0)$ using an LP with $O(n)$ variables and $O(n^2)$ constraints.

We next show that the separation problem over $K(n,r)$ can also be solved for any $r>0$ using an LP with $O(n^2)$ constraints.

**Theorem 6.1** *Given $(x^*, y^*) \in LK(n,r)$, where $r > 0$, the separation problem over $K(n,r)$ can be solved in time polynomial in $\max\{n,r\}$ using an LP with $O(\max\{n,r\})$ variables and $O(\max\{n,r\}^2)$ constraints.*

**Proof.** First we consider the case $r \le n$. Let $T'$ be a restriction of $T$ obtained by replacing the relaxed subadditivity conditions (SA1)-(SA3) with the pairwise subadditivity conditions (SA0), (SA1), (SA1'), (SA2), and (SA2'). Due to Lemma 2.7, $T' \subseteq T$ and therefore if $(\pi, \rho) \in T'$, then $\pi x + \rho y \ge 1$ is a valid inequality for $K(n,r)$. In addition, by Lemma 2.9, if $\pi x + \rho y \ge 1$ defines a nontrivial facet of $K(n,r)$, then $(\pi, \rho) \in T'$.

Let $(\pi', \rho')$ be an optimal solution of

$$z^* = \min\{\pi x^* + \rho y^* : (\pi, \rho) \in T' \}.$$

If $z^* < 1$ then $\pi' x + \rho' y \ge 1$ is a violated valid inequality. If, on the other hand, $z^* \ge 1$, then $(x^*, y^*)$ satisfies all facet defining inequalities and therefore belongs to $K(n,r)$.

Next, assume that $r > n$. Define $(x', y') \in \mathbb{R}^r \times \mathbb{R}^r$ such that $x'_i = x^*_i; y'_i = y^*_i$, for $i = 1, \ldots, n$ and $x'_i = y'_i = 0$, for $i = n+1, \ldots, r$. As $(x^*, y^*) \in K(n,r) \iff (x', y') \in K(r,r)$, separation can be done using $K(r,r)$. If a violated inequality is found, then it can simply be turned into a valid inequality for $K(n,r)$ by dropping the extra coefficients. ∎

## 7 Mixed-integer extension

Consider the mixed-integer extension of $K(n,r)$:

$$K'(n,r) = conv\left\{(v_+, v_-, x, y) \in \mathbb{R}^2_+ \times \mathbb{Z}^{2n}_+ : v_+ - v_- + \sum_{i=1}^{n} i x_i - \sum_{i=1}^{n} i y_i = r\right\}$$

where $n, r \in \mathbb{Z}$ and $n \ge r > 0$. As in the case of the mixed-integer extension of MCGP studied by Gomory and Johnson [10], the facets of $K'(n,r)$ can easily be derived from the facets of $K(n,r)$ when $r$ is an integer. To prove such a result, we introduce a few definitions, and also state some easy results without proof.

The dimension of $K'(n,r)$ is $2n+1$, i.e., one less than the number of variables. The inequalities $x_i \ge 0$ and $y_i \ge 0$, for $i = 1, \ldots, n$, and $v_+ \ge 0$ and $v_- \ge 0$ define facets of $K'(n,r)$. We refer to the facets above – other than $y_n \ge 0$ – as trivial facets, and refer to the remaining facets of $K'(n,r)$ as nontrivial. Finally, note that the recession cone of $K'(n,r)$ contains the vectors $je_i + if_j$, for all $i, j$ satisfying $1 \le i, j \le n$. For $K'(n,r)$, let $e_+$ and $e_-$ be the unit vectors in $\mathbb{R}^{2n+2}$ with ones in, respectively, the $v_+$ component, and the $v_-$ component, and zeros elsewhere. For a vector $\chi$ in the $K'(n,r)$ space, define its restriction to the $K(n,r)$ space by removing the $v_+$ and $v_-$ components, and denote it by $\chi_{re}$.

**Proposition 7.1** *All nontrivial facet-defining inequalities for $K'(n,r)$ have the form*

$$\pi_1 v_+ + \rho_1 v_- + \sum_{i=1}^{n} \pi_i x_i + \sum_{i=1}^{n} \rho_i y_i \ge \pi_o. \tag{15}$$

*Furthermore, inequality (15) is facet-defining if and only if $\pi x + \rho y \geq \pi_o$ defines a nontrivial facet of $K(n,r)$.*

**Proof.** Let $\pi x + \rho y \geq \pi_o$ define a nontrivial facet of $K(n,r)$. We first show that the inequality (15) is valid for $K'(n,r)$. Assume (15) is violated by some integral point $\chi \in K'(n,r)$ (the $x$ and $y$ components of $\chi$ are integral). Then the left-hand-side of (15) evaluated at $\chi$ equals a number $z$ less than $\pi_o$. Let $v'_+ = e_+^T \chi$ and $v'_- = e_-^T \chi$. The property (R1) in Observation 2.8 implies that $\pi_1 + \rho_1 \geq 0$. Therefore, if $\min\{v'_+, v'_-\} = \epsilon > 0$, then (15) is also violated by the point $\chi - \epsilon(e_+ + e_-) \in K'(n,r)$. We can thus assume that $\chi$ satisfies $\min\{v'_+, v'_-\} = 0$. But $\min\{v'_+, v'_-\} = 0$ combined with the integrality of $\chi$ implies that $v'_+$ and $v'_-$ are both integers. Therefore $\chi' = \chi_{re} + v'_+ e_1 + v'_- f_1$ is an integral point contained in $K(n,r)$, and $(\pi,\rho)^T \chi' = z < \pi_o$, which contradicts the fact that $\pi x + \rho y \geq \pi_o$ is satisfied by all points in $K(n,r)$.

To see that (15) defines a facet of $K'(n,r)$, let $\chi^1, \ldots \chi^{2n-1}$ be affinely independent integral points in $K(n,r)$ which satisfy $(\pi,\rho)^T \chi^i = \pi_o$. As the facet defined by $\pi x + \rho y \geq \pi_o$ does not equal the facet defined by either $x_1 \geq 0$ or $y_1 \geq 0$, there are indices $j,k$ such that $e_1^T \chi^j = s > 0$ and $f_1^T \chi^k = t > 0$. Define $2n+1$ affinely independent points in $\mathbb{R}^{2n+2}$ as follows:

$$\psi^i = (0,0,\chi^i) \text{ for } i = 1,\ldots,2n-1;$$

$$\psi^+ = \psi^j + se_+ - se_1;\ \psi^- = \psi^k + te_- - tf_1.$$

These points satisfy (15) at equality, and therefore (15) defines a facet of $K'(n,r)$.

We now show that every nontrivial facet of $K'(n,r)$ has the form in (15). Assume $\eta^T(v_+, v_-, x, y) \geq \eta_o$ defines a nontrivial facet $F$ of $K'(n,r)$. Let $\eta = (\alpha_+, \alpha_-, \pi, \rho)$, where $\alpha_+, \alpha_- \in \mathbb{R}$, and $\pi, \rho \in \mathbb{R}^n$. There exists a point $\chi \in K'(n,r)$ lying on the above facet such that $\chi^T e_1 > 0$. As $\chi - e_1 + e_+ \in K'(n,r)$, we conclude that $\alpha_+ \geq \pi_1$. We can similarly conclude that $\alpha_- \geq \rho_1$ and therefore $\alpha_+ + \alpha_- \geq \pi_1 + \rho_1 \geq 0$. The last inequality is implied by the fact that $e_1 + f_1$ is contained in the recession cone of $K'(n,r)$. If $\alpha_+ + \alpha_- = 0$, then clearly $\alpha_+ = \pi_1$ and $\alpha_- = \rho_1$. Assume $\alpha_+ + \alpha_- > 0$. As $F$ is not the same as the facet $v_+ \geq 0$, there exists an integral point $\chi = (v'_+, v'_-, x', y') \in K'(n,r)$ lying on $F$ such that $v'_+ > 0$. If $v'_- > 0$, let $\min\{v'_+, v'_-\} = \epsilon > 0$. Then $\chi^1 = \chi - \epsilon(e_+ + e_-) \in K'(n,r)$, but $\eta^T \chi^1 = \eta_o - \epsilon(\alpha_+ + \alpha_-) < \eta_o$. This contradicts the fact that $(\eta, \eta_o)$ defines a valid inequality for $K'(n,r)$. We can therefore assume that $v'_- = 0$ and $v'_+ = t$, for some positive integer $t$. Define $\chi^2$ as $\chi - te_+ + te_1$. As $\chi^2 \in K'(n,r)$, it follows that $\eta^T \chi^2 \geq \eta_o \Rightarrow \alpha_+ \leq \pi_1$. We can conclude that $\alpha_+ = \pi_1$; a similar argument shows that $\alpha_- = \rho_1$.

Finally, we show that if (15) defines a facet of $K'(n,r)$, then the inequality $\pi x + \rho y \geq \pi_o$ defines a facet of $K(n,r)$. Firstly, this defines a valid inequality for $K(n,r)$ as any point in $K(n,r)$ can be mapped to a point in $K'(n,r)$ by appending zeros in the $v_+$ and $v_-$ components. If it does not define a facet, then $(\pi,\rho) \geq \sum_i \lambda_i(\pi^i, \rho^i)$ and $\pi_o \leq \sum_i \lambda_i \pi_o^i$ for some nontrivial facet-defining inequalities $\pi^i x + \rho^i y \geq \pi_o^i$ of $K(n,r)$, and some numbers $\lambda_i \geq 0$. But that would imply that $(\pi_1, \rho_1, \pi, \rho) \geq \sum_i \lambda_i(\pi_1^i, \rho_1^i, \pi^i, \rho^i)$. By the first part of the proof, the inequalities $\pi_1^i v_+ + \rho_1^i v_- + \pi^i x + \rho^i y \geq \pi_o^i$ define facets of $K'(n,r)$, and this contradicts the assumption that (15) defines a facet of $K'(n,r)$. ∎

## 8 Using $K(n,r)$ to generate valid inequalities for MIP

Gomory and Johnson used facets of $P(n,r)$ to derive valid inequalities for knapsack sets. In particular, they derived *subadditive functions* from facet coefficients via *interpolation*. We show here how to derive valid inequalities for knapsack sets from facets of $K(n,r)$ via interpolation. Clearly, such inequalities can also be used as valid inequalities for general MIPs by using a knapsack set that is a relaxation of the original MIP. Our main result is presented at the end of the session in Theorem 8.5 and the rest of the session is dedicated to developing auxiliary steps for it. For a real number $v$, we define $\hat{v}$ as $v - \lfloor v \rfloor$.

**Definition 8.1** *Given a facet-defining inequality $\pi x + \rho y \geq \pi_o$ for $K(n,r)$, let $f^z : \mathbb{Z} \cap [-n,n] \to \mathbb{R}$ be defined as:*

$$f^z(s) = \begin{cases} \pi_s & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ \rho_{-s} & \text{if } s < 0 \end{cases}$$

*We say that $f : [-n,n] \to \mathbb{R}$ is a* facet-interpolated function *derived from $(\pi, \rho, \pi_o)$ if*

$$f(v) = (1 - \hat{v}) f^z(\lfloor v \rfloor) + \hat{v} f^z(\lceil v \rceil)$$

The function $f$, as defined above, equals $f^z(v)$ when $v$ is an integer, and therefore satisfies:

$$f(v) = (1 - \hat{v}) f(\lfloor v \rfloor) + \hat{v} f(\lceil v \rceil). \tag{16}$$

In the next result, we show that continuous functions arising via interpolation from facets of $K(n,r)$ satisfy continuous analogues of the pairwise subadditivity conditions.

**Proposition 8.2** *Let $f$ be a facet-interpolated function derived from a facet of $K(n,r)$ that is not a nonnegativity constraint. Then:*

$$f(u) + f(v) \geq f(u+v) \text{ if } u, v, u+v \in [-n,n].$$

**Proof.** The proposition is true when $u$ and $v$ are integers; the condition $f(u) + f(v) \geq f(u+v)$ translates to one of (SA0), (SA1), (SA2), (SA1'), (SA2') or (SA2"). Assume $u$ is not an integer. As $u + v \in [-n,n]$, clearly $\lfloor u+v \rfloor$ and $\lceil u+v \rceil$ also belong to $[-n,n]$.
*Case 1:* $\hat{u} + \hat{v} \leq 1$. Then $\lfloor u+v \rfloor = \lfloor u \rfloor + \lfloor v \rfloor$ and $\lceil u+v \rceil = \lceil u \rceil + \lfloor v \rfloor$. We can rewrite the expression for $f(u)$ in (16) as

$$f(u) = (1 - \hat{u} - \hat{v}) f(\lfloor u \rfloor) + \hat{u} f(\lceil u \rceil) + \hat{v} f(\lfloor u \rfloor). \tag{17}$$

Similarly,

$$f(v) = (1 - \hat{u} - \hat{v}) f(\lfloor v \rfloor) + \hat{v} f(\lceil v \rceil) + \hat{u} f(\lfloor v \rfloor). \tag{18}$$

Adding the right-most terms in the above expressions, and using the fact that the proposition is true when $u$ and $v$ are integers, we obtain

$$f(u) + f(v) \geq (1 - \hat{u} - \hat{v}) f(\lfloor u+v \rfloor) + \hat{u} f(\lceil u+v \rceil) + \hat{v} f(\lfloor u \rfloor + \lceil v \rceil). \tag{19}$$

If $v$ is an integer, then $\hat{v} = 0$ and the right-hand side of (19) the above expression equals $f(u+v)$. If $v$ is not an integer, then $\lceil u+v \rceil = \lfloor u \rfloor + \lceil v \rceil$, and again the right-hand side of (19) equals $f(u+v)$. *Case 2:* $\hat{u}+\hat{v} > 1$. Then $\lfloor u+v \rfloor = \lfloor u \rfloor + \lceil v \rceil = \lceil u \rceil + \lfloor v \rfloor$ and $\lceil u+v \rceil = \lceil u \rceil + \lceil v \rceil$. We can expand $\hat{u} f(\lceil u \rceil)$ in (16) as $(\hat{u}+\hat{v}-1) f(\lceil u \rceil) + (1-\hat{v}) f(\lceil u \rceil)$. We can similarly expand $\hat{v} f(\lceil v \rceil)$. When we add the expressions for $f(u)$ and $f(v)$ in (16) after writing the expanded terms, we get

$$f(u) + f(v) \geq (\hat{u}+\hat{v}-1) f(\lceil u+v \rceil) + (2-\hat{u}-\hat{v}) f(\lfloor u+v \rfloor).$$

The right-hand side of the inequality above equals $f(u+v)$. ∎

We say that functions satisfying the property in Proposition 8.2 are *subadditive over the interval* $[-n, n]$. We will see how to generate valid inequalities for knapsack sets from such functions in Proposition 8.4. Also, we can obtain valid inequalities using slightly more restricted functions: we say that $f$ is a *restricted subadditive* function if $f(u) + f(v) \geq f(u+v)$ for $u \in [-n, n]$, and $v, u+v \in [0, n]$. In the next result, we show that facet-interpolated functions satisfy the continuous analogue of the condition (SA3).

**Proposition 8.3** *Let $f$ be a facet-interpolated function derived from a nontrivial facet of $K(n, r)$. Then:*

$$f(u) + f(v) + f(w) \geq f(u+v+w) \text{ if } u \in [-n, n], \text{ and } v, w, u+v+w \in [0, n].$$

**Proof.**(sketch) The proposition is true when $u, v$ and $w$ are integers; the condition $f(u) + f(v) + f(w) \geq f(u+v+w)$ translates to (SA3). As in the proof of (8.2), we assume either that $\hat{u}+\hat{v}+\hat{w}$ is contained in $(0,1]$ or $(1,2]$ or $(2,3)$. In the first case, we expand $(1-\hat{u}) f(\lfloor u \rfloor)$ as $(1-\hat{u}-\hat{v}-\hat{w}) f(\lfloor u \rfloor) + (\hat{v}+\hat{w}) f(\lfloor u \rfloor)$, and proceed similarly for the terms involving $f(\lfloor v \rfloor)$ and $f(\lfloor w \rfloor)$. In the third case, we expand $\hat{u} f(\lceil u \rceil)$ as $(\hat{u}+\hat{v}+\hat{w}-2) f(\lceil u \rceil) + (2-\hat{v}-\hat{w}) f(\lceil u \rceil)$, and proceed similarly for the terms involving $f(\lceil v \rceil)$ and $f(\lceil w \rceil)$. The second case has a number of sub-cases. For example, in expanding the terms in the definition of $f(u)$ in (16), we need to consider the value of $\hat{v}+\hat{w}$ with respect to 1. If $\hat{v}+\hat{w} \leq 1$, then we write $\hat{u} f(\lceil u \rceil)$ as $(\hat{u}+\hat{v}+\hat{w}-1) f(\lceil u \rceil) + (1-\hat{v}-\hat{w}) f(\lceil u \rceil)$. On the other hand, if $\hat{u}+\hat{v} > 1$, we expand $(1-\hat{u}) f(\lfloor u \rfloor)$ as $((2-\hat{u}-\hat{v}-\hat{w}) + (\hat{v}+\hat{w}-1)) f(\lfloor u \rfloor)$. ∎

It is well-known that subadditive functions yield valid inequalities for knapsack sets; the point we emphasize in the next result is that one does not need subadditivity over the entire real line.

**Proposition 8.4** *Consider the set $K = \{w \in \mathbb{Z}^p : \sum_{i=1}^{p} a_i w_i = b\}$, where the coefficients $a_i$ and $b$ are rational numbers. Let $t$ be a number such that $ta_i, tb \in [-n, n]$ and $tb > 0$. If a function $f$ is (i) subadditive over the interval $[-n, n]$ or (ii) satisfies restricted subadditivity and the condition in Proposition 8.3, then*

$$\sum_{i=1}^{p} f(ta_i) w_i \geq f(tb)$$

*is a valid inequality for $K$.*

**Proof.** We can scale the coefficients $a_i$ and $b$ in the constraint defining $K$ by a rational number $\lambda > 0$ so that they become integers contained in the interval $[-m, m]$, with $m = \lambda \max\{|a_i|, |b|\}$. Define the function $g : [-m, m] \to \mathbb{R}$ by $g(w) = f((t/\lambda)w)$. In case (i), $g$ is subadditive over the domain $[-m, m]$. Therefore the vector $\tilde{g} = (g(-m), g(-m+1), \ldots, g(1), \ldots, g(m))$ satisfies (SA0), (SA1), (SA2) and (SA1') with respect to $K(m, b)$, and (by Lemma 2.7 and Lemma 2.13)

$$\sum_{i=1}^{p} g(\lambda a_i) w_i \geq g(\lambda b)$$

is a valid inequality for $K$. In case (ii), $\tilde{g}$ satisfies (SA1), (SA2) and (SA3) with respect to $K(m, b)$ and by Lemma 2.13 the inequality above is valid for $K$. ∎

Note that Proposition 8.4 implies that facet-interpolated functions derived from facets of $K(n, r)$ can be used to generate valid inequalities for $K$ (and consequently for general MIPs). Figure 2 presents examples of such functions obtained from a facets of $K(n, r)$. Note that the first example displays a feature not be observed in facets of $P(n, r)$, namely negative cut coefficients.

We can now give the mixed-integer extension of the previous result.

**Theorem 8.5** *Let $f$ be a facet-interpolated function derived from a nontrivial facet of $K(n, r)$. Consider the set*

$$Q = \left\{ (s, w) \in \mathbb{R}^q_+ \times \mathbb{Z}^p_+ : \sum_{i=1}^{q} c_i s_i + \sum_{i=1}^{p} a_i w_i = b \right\},$$

*where the coefficients of the knapsack constraint defining $Q$ are rational numbers. Let $t$ be such that $ta_i, tb \in [-n, n]$ and $tb > 0$. Then the inequality*

$$f(1) \sum_{i=1}^{q} (tc_i)^+ s_i + f(-1) \sum_{i=1}^{q} (-tc_i)^+ s_i + \sum_{i=1}^{p} f(ta_i) w_i \geq f(tb),$$

*where $(\alpha)^+ = \max(\alpha, 0)$, is valid for $Q$.*

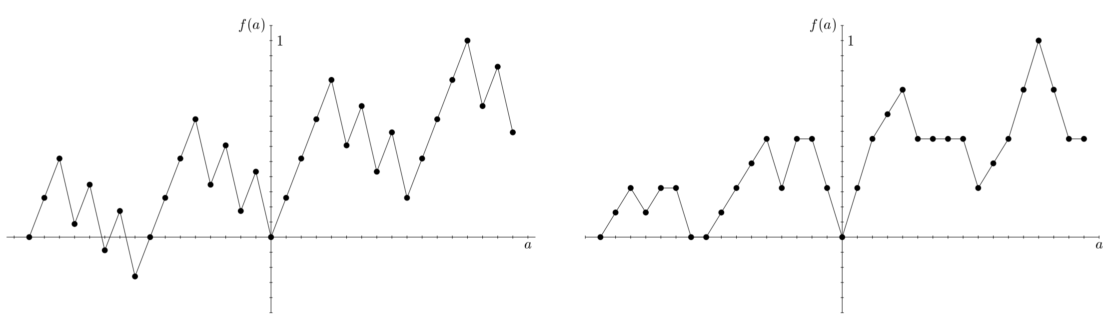


Figure 2: Example of two facet-interpolated functions derived from facets of $K(16, 13)$

## 9 Conclusion

We defined and studied the Master Equality Polyhedron, a generalization of the Master Cyclic Group Polyhedron and presented an explicit characterization of the polar of its nontrivial facet-defining inequalities. We showed that facets of MEP can be used to obtain valid inequalities for a general MIP that cannot be obtained from facets of MCGP. In addition, for mixed-integer knapsack sets with rational data and nonnegative variables without upper bounds, our results yield a pseudo-polynomial time algorithm to separate and therefore optimize over their convex hull. This can be done by scaling their data and aggregating variables to fit into the MEP framework. Our characterization of MEP can also be used to find violated Homogeneous Extended Capacity Cuts efficiently. These cuts were proposed in [17] for solving Capacitated Minimum Spanning Tree problems and Capacitated Vehicle Routing problems.

An interesting topic for further study is the derivation of “interesting” classes of facets for MEP, i.e., facets which cannot be lifted from facets of MCGP and are not defined by rank one mixed-integer rounding inequalities.

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