Separation of convex polyhedral sets with uncertain data

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Abstract

This paper is a contribution to the interval analysis and separability of convex sets. Separation is a familiar principle and is often used not only in optimization theory, but in many economic applications as well. In real problems input data are usually not known exactly. For the purpose of this paper we assume that data can independently vary in given intervals. We study two cases when convex polyhedral sets are described by a system of linear inequalities or by the list of its vertices. For each case we propose a way how to check whether given convex polyhedral sets are separable for some or for all realizations of the interval data. Some of the proposed problems can be checked efficiently, while the others are NP-hard.

Keywords

Separating hyperplane, convex polyhedra, interval analysis, linear interval equation, linear interval inequalities.
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1 Introduction

In this paper we study separability of two convex polyhedral sets ($\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{d} \in \mathbb{R}^l$):

$$\mathcal{M}_1 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\}, \tag{1}$$

$$\mathcal{M}_2 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \le \mathbf{d}\}. \tag{2}$$

There are various kinds of separability of convex sets (cf. [8]). We introduce so called weak and strong separation. Strong separation is dealt with in Section 2 and this kind is especially convenient in order to utilize Theorem 2 and Theorem 3. Weak separation is dealt with in Section 3.

Definition 1. Sets $X, Y \subset \mathbb{R}^n$ are called *weakly separable* if there exists a hyperplane $\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} = s\}$ such that $X \subseteq \overline{\mathcal{R}^-} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \le s\}$, and $Y \subseteq \overline{\mathcal{R}^+} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \ge s\}$ hold. Such a hyperplane $\mathcal{R}$ is called *the separating hyperplane* of the sets X, Y. Sets $X, Y \subset \mathbb{R}^n$ are called *strongly separable* if they are weakly separable and $dim\, X = dim\, Y = n$.

Let us remind the familiar separation theorem (see e.g. [3, 7]):

Theorem 1. *Convex sets X, $Y \subset \mathbb{R}^n$ are strongly separable if and only if $dim\, X = dim\, Y = n$, and $int\, X \cap int\, Y = \emptyset$.*

Let us introduce some notation. Symbol $\mathbf{1}$ will denote a vector all coordinates of which are equal to one, $diag(\mathbf{v})$ is a diagonal matrix with elements $v_1, \ldots, v_n$. Given a matrix $\mathbf{M}$, the expressions $\mathbf{M}_{i,\cdot}$, $\mathbf{M}_{\cdot,j}$ denote the i-th row and the j-th column of the matrix $\mathbf{M}$, respectively. For vectors $\mathbf{a}, \mathbf{b}$ the inequalities $\mathbf{a} \le \mathbf{b}$ or $\mathbf{a} < \mathbf{b}$ are understood componentwise. For any set $\mathcal{X}$ let us denote by $\overline{\mathcal{X}}$, $int\, \mathcal{X}$, $dim\, \mathcal{X}$, and $conv\, \mathcal{X}$ the closure, the interior, the dimension, and the convex hull of $\mathcal{X}$, respectively. A sign of a real number $r \in \mathbb{R}$ is defined as follows: $sgn(r) = 0$ if $r = 0$, $sgn(r) = 1$ if $r > 0$ and $sgn(r) = -1$ if $r < 0$. A sign of a vector is understood componentwise.

Let us introduce the convex polytope

$$\mathcal{Q}^* \equiv \left\{ (\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathbb{R}^{m+l+1} \mid \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T & \mathbf{0} \\ \mathbf{b}^T & \mathbf{d}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix},\ (\mathbf{u}, \mathbf{v}, v_{l+1}) \ge \mathbf{0} \right\}. \tag{3}$$

With the help of the set $\mathcal{Q}^*$ we can describe all separating hyperplanes of $\mathcal{M}_1,\ \mathcal{M}_2$ from (1), (2). The following Theorem 2 and Theorem 3 were proved in [5, 6].

Theorem 2. *Suppose that $dim\,\mathcal{M}_1 = dim\,\mathcal{M}_2 = n$, $int\,\mathcal{M}_1 \cap int\,\mathcal{M}_2 = \emptyset$. Let $(\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*$, $\mathbf{u}^T\mathbf{A} \neq \mathbf{0}^T$, and $\eta \in \langle 0, v_{l+1}\rangle$ is arbitrary. Then*

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = \eta\} \tag{4}$$

represents a separating hyperplane of the convex polyhedral sets $\mathcal{M}_1,\ \mathcal{M}_2$. Conversely, any separating hyperplane $\mathcal{R}$ of $\mathcal{M}_1,\ \mathcal{M}_2$ can be expressed in the form of (4) for a certain $(\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*$, $\mathbf{u}^T\mathbf{A} \neq \mathbf{0}^T$, and $\eta \in \langle 0, v_{l+1}\rangle$.

Theorem 3. *Let $dim\,\mathcal{M}_1 = dim\,\mathcal{M}_2 = n$. Then the convex sets $\mathcal{M}_1$, $\mathcal{M}_2$ are strongly separable if and only if $\mathcal{Q}^* \neq \emptyset$.*

1.1 Some results from interval analysis

Coefficients and right-hand sides of systems of linear equalities and inequalities are rarely known exactly. In interval analysis we suppose that these values vary independently in some real intervals. Let us introduce some notation. Interval matrix is defined as $\mathbf{M}^I = \{\mathbf{M} \in \mathbb{R}^{m\times n} \mid \underline{\mathbf{M}} \leq \mathbf{M} \leq \overline{\mathbf{M}}\}$. Next introduce

$$\mathbf{M}^c \equiv \frac{1}{2}.(\overline{\mathbf{M}} + \underline{\mathbf{M}}), \quad \mathbf{M}^\Delta \equiv \frac{1}{2}.(\overline{\mathbf{M}} - \underline{\mathbf{M}}).$$

Then we can write

$$\mathbf{M}^I = \langle \underline{\mathbf{M}}, \overline{\mathbf{M}}\rangle = \langle \mathbf{M}^c - \mathbf{M}^\Delta, \mathbf{M}^c + \mathbf{M}^\Delta\rangle.$$

From the point of view of interval analysis there are two possibilities how to deal with the problem of finding a solution of interval linear system of equalities and inequalities. The system of interval linear inequalities

$$\mathbf{M}^I\mathbf{x} \leq \mathbf{m}^I \tag{5}$$

is *strongly solvable*, if every system $\mathbf{M}\mathbf{x} \leq \mathbf{m}$ is solvable for all $\mathbf{M} \in \mathbf{M}^I$, $\mathbf{m} \in \mathbf{m}^I$. Vector $\mathbf{x}^0$ is a *strong solution,* if $\mathbf{M}\mathbf{x}^0 \leq \mathbf{m}$ holds for all $\mathbf{M} \in \mathbf{M}^I$, $\mathbf{m} \in \mathbf{m}^I$. The interval system (5) is *weakly solvable,* if $\mathbf{M}\mathbf{x}^1 \leq \mathbf{m}$ holds for certain vector $\mathbf{x}^1$ and $\mathbf{M} \in \mathbf{M}^I$, $\mathbf{m} \in \mathbf{m}^I$ (such a vector $\mathbf{x}^1$ is called *a weak solution*). Similarly, we can define strong and weak solvability for other types of linear interval systems.

Theorem 4. *An interval system $\mathbf{M}^I\mathbf{x} = \mathbf{m}^I,\ \mathbf{x} \geq \mathbf{0}$ is weakly solvable if and only if the system*

$$\underline{\mathbf{M}}\mathbf{x} \leq \overline{\mathbf{m}},\ \overline{\mathbf{M}}\mathbf{x} \geq \underline{\mathbf{m}},\ \mathbf{x} \geq \mathbf{0} \tag{6}$$

is solvable. Moreover, a vector $\mathbf{x}$ is a weak solution of the system $\mathbf{M}^I\mathbf{x} = \mathbf{m}^I,\ \mathbf{x} \geq \mathbf{0}$ if and only if it satisfies (6).

Proof. See [2, Theorem 2.13]. □

Theorem 5. *An interval system $\mathbf{M}^I\mathbf{x} = \mathbf{m}^I, \mathbf{x} \geq \mathbf{0}$ is strongly solvable if and only the system*

$$(\mathbf{M}^c - diag(\mathbf{z})\mathbf{M}^\Delta)\mathbf{x} = \mathbf{m}^c + diag(\mathbf{z})\mathbf{m}^\Delta, \quad \mathbf{x} \geq \mathbf{0}$$

is solvable for each $\mathbf{z} \in \{\pm 1\}^m$.

Proof. See [2, Theorem 2.17]. □

Theorem 6. *An interval system $\mathbf{M}^I\mathbf{x} \leq \mathbf{m}^I$ is weakly solvable if and only if the system*

$$(\mathbf{M}^c - \mathbf{M}^\Delta diag(\mathbf{z}))\mathbf{x} \leq \overline{\mathbf{m}} \tag{7}$$

is solvable for some $\mathbf{z} \in \{\pm 1\}^n$. Moreover, a vector $\mathbf{x}$ is a weak solution of the system $\mathbf{M}^I\mathbf{x} \leq \mathbf{m}^I$ if and only if it satisfies (7).

Proof. See [2, Theorem 2.20]. ☐

Corollary 1. *An interval system* $\mathbf{M}^I\mathbf{x} \le \mathbf{0},\ \mathbf{x} \neq \mathbf{0}$ *is weakly solvable if and only if the system*

$$(\mathbf{M}^c - \mathbf{M}^{\Delta} diag(\mathbf{z}))\mathbf{x} \le \mathbf{0},\ \mathbf{x} \neq \mathbf{0}$$

is solvable for some $\mathbf{z} \in \{\pm 1\}^n$.

Theorem 7. *An interval system* $\mathbf{M}^I\mathbf{x} \le \mathbf{m}^I$ *is strongly solvable if and only if the system*

$$\overline{\mathbf{M}}\mathbf{x}^1 - \underline{\mathbf{M}}\mathbf{x}^2 \le \underline{\mathbf{m}}, \quad \mathbf{x}^1, \mathbf{x}^2 \ge \mathbf{0} \tag{8}$$

is solvable. Moreover, if $\mathbf{x}^1, \mathbf{x}^2$ *is a solution of* (8), *then the vector* $\mathbf{x}^1 - \mathbf{x}^2$ *is a strong solution of the given interval system.*

Proof. See [2, Theorem 1.25]. ☐

2 Separation of interval convex polyhedral sets

In this section we deal with the strong separability of two convex polyhedral sets $\mathcal{M}_1$, $\mathcal{M}_2$ the input data of which can vary in given real intervals. Let us consider two families of convex polyhedral sets

$$\mathcal{M}_1^I \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}^I\mathbf{x} \le \mathbf{b}^I\}, \tag{9}$$

$$\mathcal{M}_2^I \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}^I\mathbf{x} \le \mathbf{d}^I\}, \tag{10}$$

where $\mathbf{A}^I = \{\mathbf{A} \in \mathbb{R}^{m\times n} \mid \underline{\mathbf{A}} \le \mathbf{A} \le \overline{\mathbf{A}}\}$, $\mathbf{C}^I = \{\mathbf{C} \in \mathbb{R}^{l\times n} \mid \underline{\mathbf{C}} \le \mathbf{C} \le \overline{\mathbf{C}}\}$, $\mathbf{b}^I = \{\mathbf{b} \in \mathbb{R}^m \mid \underline{\mathbf{b}} \le \mathbf{b} \le \overline{\mathbf{b}}\}$, $\mathbf{d}^I = \{\mathbf{d} \in \mathbb{R}^l \mid \underline{\mathbf{d}} \le \mathbf{d} \le \overline{\mathbf{d}}\}$. Matrices $\underline{\mathbf{A}}, \overline{\mathbf{A}} \in \mathbb{R}^{m\times n}$, $\underline{\mathbf{C}}, \overline{\mathbf{C}} \in \mathbb{R}^{l\times n}$ and vectors $\underline{\mathbf{b}}, \overline{\mathbf{b}} \in \mathbb{R}^m$, $\underline{\mathbf{d}}, \overline{\mathbf{d}} \in \mathbb{R}^l$ are fixed.

Let us assume that no matrix $\mathbf{A} \in \mathbf{A}^I$ contains a zero row and assume that $dim\,\mathcal{M}_1 = n$ holds for all $\mathcal{M}_1 \in \mathcal{M}_1^I$ (i.e., the interval system $\mathbf{A}^I\mathbf{x} < \mathbf{b}^I$ is strongly solvable). Let us make analogical assumptions about $\mathcal{M}_2^I$.

The former assumption can be verified easily, the latter assumption can be verified in the following way. The dimension of $\mathcal{M}_1$ is equal to n for all $\mathcal{M}_1 \in \mathcal{M}_1^I$ if and only if the interval system $\mathbf{A}^I\mathbf{x} \le \mathbf{b}^I - \boldsymbol{\varepsilon}$ is strongly solvable for any sufficiently small $\boldsymbol{\varepsilon} > \mathbf{0}$. According to Theorem 7 this happens if and only if the system

$$\overline{\mathbf{A}}\mathbf{x}^1 - \underline{\mathbf{A}}\mathbf{x}^2 \le \underline{\mathbf{b}} - \boldsymbol{\varepsilon},\ \mathbf{x}^1, \mathbf{x}^2 \ge \mathbf{0},$$

or, equivalently, the system

$$\overline{\mathbf{A}}\mathbf{x}^1 - \underline{\mathbf{A}}\mathbf{x}^2 < \underline{\mathbf{b}},\ \mathbf{x}^1, \mathbf{x}^2 \ge \mathbf{0} \tag{11}$$

is solvable. Moreover, if vectors $\tilde{\mathbf{x}}^1$, $\tilde{\mathbf{x}}^2$ solve (11), then $\tilde{\mathbf{x}}^1 - \tilde{\mathbf{x}}^2 \in int\,\mathcal{M}_1$ holds for all $\mathcal{M}_1 \in \mathcal{M}_1^I$.

We are interested in two cases. We will study whether the convex polyhedral sets $\mathcal{M}_1$, $\mathcal{M}_2$ are strongly separable either for some, or for all realizations $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$.

2.1 Separability for some realization

The first case can be checked efficiently. According to Theorem 3, there exist two convex polyhedral sets $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ which are strongly separable if and only if the interval system

$$\begin{pmatrix} (\mathbf{A}^I)^T & (\mathbf{C}^I)^T & \mathbf{0} \\ (\mathbf{b}^I)^T & (\mathbf{d}^I)^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix},\ (\mathbf{u}, \mathbf{v}, v_{l+1}) \ge \mathbf{0}. \tag{12}$$

is weakly solvable. From Theorem 4 we have that interval system (12) is weakly solvable if and only if the system

$$\begin{pmatrix} \underline{\mathbf{A}}^T & \underline{\mathbf{C}}^T & \mathbf{0} \\ \underline{\mathbf{b}}^T & \underline{\mathbf{d}}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} \le \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix} \le \begin{pmatrix} \overline{\mathbf{A}}^T & \overline{\mathbf{C}}^T & \mathbf{0} \\ \overline{\mathbf{b}}^T & \overline{\mathbf{d}}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix},\ \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} \ge \mathbf{0}$$

is solvable.

2.2 Separability for all realizations

The problem to verify whether all convex polyhedral sets $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ are strongly separable is equivalent (see Theorem 3) to the problem to verify whether interval system (12) is strongly solvable. Theorem 5 enables us to check strong solvability of the interval system (12) with an exponential complexity. Polynomial algorithm is not likely to exist; we will show that this problem is NP-hard.

Interval system (12) is strongly solvable if and only if the interval system

$$\begin{pmatrix} (\mathbf{A}^I)^T & (\mathbf{C}^I)^T & \mathbf{0} \\ (\mathbf{b}^I)^T & (\mathbf{d}^I)^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \ (\mathbf{u}, \mathbf{v}, v_{l+1}) \gneqq \mathbf{0} \tag{13}$$

is strongly solvable. It follows from Theorem 8 that checking the strong solvability of the interval system (13) is an NP-hard problem.

Lemma 1. *Let $\mathbf{M} \in \mathbb{Q}^{n\times n}$ be a nonnegative positive definite matrix. Checking the solvability of the system*

$$|\mathbf{M}\mathbf{x}| \le \mathbf{1}, \ \mathbf{1}^T|\mathbf{x}| > 1 \tag{14}$$

is an NP-hard problem.

Proof. It is a modification of the proof of Theorem 2.3 from [2] where the NP-hardness of testing the solvabiliy of a system $|\mathbf{M}\mathbf{x}| \le \mathbf{1}, \ \mathbf{1}^T|\mathbf{x}| \ge 1$ was proved. □

Theorem 8. *Let $\mathbf{N}^I \subset \mathbb{R}^{n\times 2n}$. Checking strong solvability of an interval system*

$$\mathbf{N}^I\mathbf{x} = \mathbf{0}, \ \mathbf{x} \gneqq \mathbf{0} \tag{15}$$

is NP-hard problem.

Proof. According to Lemma 1 we know that checking solvability of the system (14) is NP-hard. Thus it is sufficient to prove that the system (14) has a solution if and only if an interval system

$$\langle \mathbf{M}^T - \mathbf{1}\mathbf{1}^T, \mathbf{M}^T + \mathbf{1}\mathbf{1}^T \rangle \mathbf{x}^1 + \langle -\mathbf{M}^T - \mathbf{1}\mathbf{1}^T, -\mathbf{M}^T + \mathbf{1}\mathbf{1}^T \rangle \mathbf{x}^2 = \mathbf{0}, \ (\mathbf{x}^1, \mathbf{x}^2) \gneqq \mathbf{0}$$

is not strongly solvable, or equivalently an interval system

$$\begin{aligned} \langle \mathbf{M}^T - \mathbf{1}\mathbf{1}^T, \mathbf{M}^T + \mathbf{1}\mathbf{1}^T \rangle \mathbf{x}^1 + \langle -\mathbf{M}^T - \mathbf{1}\mathbf{1}^T, -\mathbf{M}^T + \mathbf{1}\mathbf{1}^T \rangle \mathbf{x}^2 &= \mathbf{0}, \\ \mathbf{1}^T\mathbf{x}^1 + \mathbf{1}^T\mathbf{x}^2 = 1, \ (\mathbf{x}^1, \mathbf{x}^2) &\ge \mathbf{0} \end{aligned} \tag{16}$$

is not strongly solvable (it is a special kind of the interval system (15)). The system (16) is not strongly solvable if and only if there exists $\mathbf{y} \in \{\pm 1\}^n$ such that a system

$$\begin{aligned} (\mathbf{M}^T - \mathbf{y}\mathbf{1}^T)\mathbf{x}^1 + (-\mathbf{M}^T - \mathbf{y}\mathbf{1}^T)\mathbf{x}^2 &= \mathbf{0}, \\ \mathbf{1}^T\mathbf{x}^1 + \mathbf{1}^T\mathbf{x}^2 = 1, \ (\mathbf{x}^1, \mathbf{x}^2) &\ge \mathbf{0} \end{aligned}$$

is not solvable (see [2, Theorem 2.17]). From the familiar Farkas Theorem it is sufficient and necessary that there exists a vector $(\mathbf{z}, z') \in \mathbb{R}^{n+1}$ satisfying the system

$$\begin{aligned} (\mathbf{M} - \mathbf{1}\mathbf{y}^T)\mathbf{z} + \mathbf{1}z' &\ge \mathbf{0}, \\ (-\mathbf{M} - \mathbf{1}\mathbf{y}^T)\mathbf{z} + \mathbf{1}z' &\ge \mathbf{0}, \\ z' &< 0, \end{aligned}$$

equivalently

$$\begin{aligned} (\mathbf{M} - \mathbf{1}\mathbf{y}^T)\mathbf{z} &> \mathbf{0}, \\ (-\mathbf{M} - \mathbf{1}\mathbf{y}^T)\mathbf{z} &> \mathbf{0}, \end{aligned}$$

or

$$|\mathbf{Mz}| < -\mathbf{1y}^T\mathbf{z}. \tag{17}$$

We claim that (17) has a solution if and only if the system (14) has a solution. If $\mathbf{x} \in \mathbb{R}^n$ solves (14), then it satisfies $|\mathbf{Mx}| < \mathbf{11}^T|\mathbf{x}|$ and vectors $\mathbf{z} = \mathbf{x}$, $\mathbf{y} = -sgn(\mathbf{x})$ forms a solution of (17). Conversely, if a certain $\mathbf{z} \in \mathbb{R}^n$ a $\mathbf{y} \in \{\pm 1\}^n$ satisfies (17), then

$$|\mathbf{Mz}| < |-\mathbf{1y}^T\mathbf{z}| \leq \mathbf{11}^T|\mathbf{z}|,$$

i.e.

$$|\mathbf{M}\frac{\mathbf{z}}{\mathbf{1}^T|\mathbf{z}|}| < \mathbf{1},$$

and a solution of (14) is a vector $\mathbf{x} = \frac{\mathbf{z}}{\mathbf{1}^T|\mathbf{z}|-\varepsilon}$, where $\varepsilon > 0$ is sufficiently small. □

Another possibility how to verify whether each couple $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ is strongly separable is to use the following sufficient condition. Each two convex polyhedral sets $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ are strongly separable, if convex hulls $conv\left(\cup_{\mathcal{M}_1 \in \mathcal{M}_1^I}\mathcal{M}_1\right)$, $conv\left(\cup_{\mathcal{M}_2 \in \mathcal{M}_2^I}\mathcal{M}_2\right)$ are strongly separable. Moreover, any separating hyperplane of these convex hulls is a separating hyperplane of convex polyhedral sets $\mathcal{M}_1$, $\mathcal{M}_2$ for all $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$. It follows from Theorem 6 that a vector $\mathbf{x} \in \mathbb{R}^n$ solves a system $\mathbf{Ax} \leq \mathbf{b}$ for certain $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{b} \in \mathbf{b}^I$ if and only if the vector $\mathbf{x}$ solves

$$(\mathbf{A}^c - \mathbf{A}^\Delta diag(\mathbf{z}))\mathbf{x} \leq \overline{\mathbf{b}}$$

for some $\mathbf{z} \in \{\pm 1\}^n$. Hence

$$\bigcup_{\mathcal{M}_1 \in \mathcal{M}_1^I} \mathcal{M}_1 = \bigcup_{\mathbf{z} \in \{\pm 1\}^n} \left\{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{A}^c - \mathbf{A}^\Delta diag(\mathbf{z}))\mathbf{x} \leq \overline{\mathbf{b}}\right\}$$

and the problem is reduced to the problem of computing the convex hull of a finite (but exponential) number of convex polyhedral sets (for explicit description of the convex hull of two convex polyhedral sets see [5]).

Note that the reverse implication generally does not hold, i.e. it can occur that all convex polyhedral sets $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ strongly separable and convex hulls $conv\left(\cup_{\mathcal{M}_1 \in \mathcal{M}_1^I}\mathcal{M}_1\right)$, $conv\left(\cup_{\mathcal{M}_2 \in \mathcal{M}_2^I}\mathcal{M}_2\right)$ need not be strongly separable. The reason is that sets $\cup_{\mathcal{M}_1 \in \mathcal{M}_1^I}\mathcal{M}_1$, $\cup_{\mathcal{M}_2 \in \mathcal{M}_2^I}\mathcal{M}_2$ are not convex in general.

3 Convex polytopes

In this section we suppose that convex polytopes (bounded convex polyhedral sets) $\mathcal{M}_1$, $\mathcal{M}_2$ are described by the lists of their vertices as follows

$$\mathcal{M}_1 \text{ has vertices } \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n,\ m \geq 1, \tag{18}$$

$$\mathcal{M}_2 \text{ has vertices } \mathbf{c}_1, \ldots, \mathbf{c}_l \in \mathbb{R}^n,\ l \geq 1. \tag{19}$$

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix, for which $\mathbf{A}_{i,\cdot} = \mathbf{a}_i^T$, $i \in \{1, \ldots, m\}$ holds (i.e. the rows of matrix $\mathbf{A}$ correspond to vectors $\mathbf{a}_i^T$) and by analogy let $\mathbf{C} \in \mathbb{R}^{l \times n}$ be a matrix for which $\mathbf{C}_{j,\cdot} = \mathbf{c}_j^T$, $j \in \{1, \ldots, l\}$. We will also use the more transparent notation $\mathcal{M}_1 \equiv \mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2 \equiv \mathcal{M}_2(\mathbf{C})$.

For this situation it is convenient to study weak separability (Definition 1) of convex polytopes $\mathcal{M}_1$, $\mathcal{M}_2$, since nonemptiness of $\mathcal{M}_1$, $\mathcal{M}_2$ is guaranteed (in comparison with to full dimension).

Checking the existence of separating hyperplane of convex polytopes $\mathcal{M}_1$, $\mathcal{M}_2$ in spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ can be done in expected time $O(\sqrt{m+l})$, which is optimal (see [1]). But for the sake of this paper it is more convenient to use the standard linear programming problem: convex polytopes $\mathcal{M}_1$, $\mathcal{M}_2$ are weakly separable if and only if a convex polyhedral set

$$\mathcal{D} \equiv \left\{(\mathbf{r}, s) \in \mathbb{R}^{n+1} \mid \begin{pmatrix} \mathbf{A} & -\mathbf{1} \\ -\mathbf{C} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ s \end{pmatrix} \leq \mathbf{0},\ \mathbf{r} \neq \mathbf{0}\right\} \tag{20}$$

is nonempty (whereas $\mathbf{r}^T\mathbf{x} = s$ represents a separating hyperplane).

Let two interval matrices $\mathbf{A}^I$, $\mathbf{C}^I$ be given. Like in Section 2 two natural questions arise: Are convex polytopes $\mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2(\mathbf{C})$ weakly separable for some realization $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{C} \in \mathbf{C}^I$? Are $\mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2(\mathbf{C})$ weakly separable for all realizations $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{C} \in \mathbf{C}^I$?

3.1 Separability for some realizations

Theorem 9. *Given an interval matrix $\mathbf{M}^I \subset \mathbb{R}^{m\times n}$. Checking weak solvability of an interval system*

$$\mathbf{M}^I\mathbf{x} \le \mathbf{0},\ \mathbf{x} \ne \mathbf{0} \tag{21}$$

is an NP-hard problem.

Proof. We proceed analogically as in [2, Theorem 2.21]. A vector $\mathbf{x} \in \mathbb{R}^n$ is a weak solution of the interval system (21) if and only if it is a solution of a system

$$\mathbf{M}^c\mathbf{x} \le \mathbf{M}^\Delta|\mathbf{x}|,\ \mathbf{x} \ne \mathbf{0}. \tag{22}$$

It is sufficient to prove that checking solvability of (22) is NP-hard. We known (see [2, Theorem 2.3]) that checking solvability of system

$$|\mathbf{N}\mathbf{x}| \le \mathbf{1},\ 1 \le \mathbf{1}^T|\mathbf{x}| \tag{23}$$

is NP-hard. We claim that system (23) is solvable if and only if the system

$$|\mathbf{N}\mathbf{z}| \le \mathbf{1}z',\ z' \le \mathbf{1}^T|\mathbf{z}|,\ (\mathbf{z}, z') \ne \mathbf{0} \tag{24}$$

is solvable. When $\mathbf{x}$ solves (23), then $(\mathbf{z}, z') = (\mathbf{x}, 1)$ solves (24). Conversely, let $(\mathbf{z}, z')$ be a solution of (24). If $z' \ne 0$, then $z' > 0$ and a vector $\mathbf{x} = \frac{\mathbf{z}}{z'}$ solves system (23). If $z' = 0$, then system (23) is satisfied for a vector $\mathbf{x} = \frac{\mathbf{z}}{\mathbf{1}^T|\mathbf{z}|}$. System (24) can be equivalently rewritten as

$$\mathbf{N}\mathbf{z} - \mathbf{1}z' \le \mathbf{0},\ -\mathbf{N}\mathbf{z} - \mathbf{1}z' \le \mathbf{0},\ z' \le \mathbf{1}^T|\mathbf{z}|,\ (\mathbf{z}, z') \ne \mathbf{0}.$$

By choosing

$$\mathbf{M}^c = \begin{pmatrix} \mathbf{N} & -\mathbf{1} \\ -\mathbf{N} & -\mathbf{1} \\ \mathbf{0}^T & 1 \end{pmatrix},\quad \mathbf{M}^\Delta = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1}^T & 0 \end{pmatrix}$$

we reduce solvability of (24) to solvability of (22). Hence checking weak solvability of the interval system (21) is NP-hard. ☐

Remark 1. Unlike to other types of interval systems, the weak solvability of interval system (21) in case that $m \le n$ can be checked in constant time, since the system

$$\mathbf{M}\mathbf{x} \le \mathbf{0},\ \mathbf{x} \ne \mathbf{0} \tag{25}$$

is solvable for all $\mathbf{M} \in \mathbf{M}^I$. The set of solutions of the system $\mathbf{M}\mathbf{x} + \mathbf{1}x' = \mathbf{0}$ forms a vector space the dimension of which is greater or equal to one. Hence there exists a vector $(\mathbf{x}, x') \ne (\mathbf{0}, 0)$ satisfying this system. If $x' = 0$, then $\mathbf{x}$ solves (25). If $x' \ne 0$, then system (25) has a solution $\frac{\mathbf{x}}{x'}$.

The convex polytopes $\mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2(\mathbf{C})$ are weakly separable for some $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{C} \in \mathbf{C}^I$ if and only if the interval system

$$\begin{pmatrix} \mathbf{A}^I & -\mathbf{1} \\ -\mathbf{C}^I & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{r} \\ s \end{pmatrix} \le \mathbf{0},\ \mathbf{r} \ne \mathbf{0} \tag{26}$$

is weakly solvable. But according to Theorem 9 this is an NP-hard problem, since interval system $\mathbf{M}^I\mathbf{x} \le \mathbf{0}$, $\mathbf{x} \ne \mathbf{0}$ is weakly solvable if and only if the following system is weakly solvable

$$\begin{pmatrix} \mathbf{M}^I & -\mathbf{1} \\ \mathbf{0}^T & -1 \\ \mathbf{0}^T & 1 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ x' \end{pmatrix} \le \mathbf{0},\ \mathbf{x} \ne \mathbf{0}.$$

This is a special kind of interval system (26). For checking (with an exponential complexity) weak solvability of (26) we can use Corollary 1.

3.2 Separability for all realizations

The convex polytopes $\mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2(\mathbf{C})$ are weakly separable for all $\mathbf{A} \in \mathbf{A}^I$ and $\mathbf{C} \in \mathbf{C}^I$ if and only if the interval system (26) is strongly solvable. If the interval system (26) has a strong solution $(\mathbf{r}, s)$, then simply (26) is strongly solvable and we have for the hyperplane $\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} = s\}$

$$\mathcal{M}_1(\mathbf{A}) \subset \overline{\mathcal{R}^-} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \le s\} \quad \forall \mathbf{A} \in \mathbf{A}^I,$$
$$\mathcal{M}_2(\mathbf{C}) \subset \overline{\mathcal{R}^+} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \ge s\} \quad \forall \mathbf{C} \in \mathbf{C}^I.$$

The reverse implication holds only under some additional assumptions – see Theorem 10.

Assertion 1. *The set $\cup_{\mathbf{A}\in\mathbf{A}^I}\mathcal{M}_1(\mathbf{A})$ is convex.*

Proof. Let $\mathbf{A}^1, \mathbf{A}^2 \in \mathbf{A}^I$ and $\mathbf{x}^1 \in \mathcal{M}_1(\mathbf{A}^1)$, $\mathbf{x}^2 \in \mathcal{M}_1(\mathbf{A}^2)$. Denote by $\mathbf{a}_i^1$ and by $\mathbf{a}_i^2$, $i = 1, \dots, m$, vertices of $\mathcal{M}_1(\mathbf{A}^1)$ and $\mathcal{M}_1(\mathbf{A}^2)$, respectively. Then vectors $\mathbf{x}^1,\ \mathbf{x}^2$ can be expressed as convex combinations

$$\mathbf{x}^1 = \sum_{i=1}^m \alpha_i^1 \mathbf{a}_i^1, \quad \mathbf{x}^2 = \sum_{i=1}^m \alpha_i^2 \mathbf{a}_i^2$$

for certain $\alpha_i^1, \alpha_i^2 \ge 0$, $\sum_{i=1}^m \alpha_i^1 = \sum_{i=1}^m \alpha_i^2 = 1$. An arbitrary convex combination of $\mathbf{x}^1,\ \mathbf{x}^2$ in the form $\mathbf{x}^c \equiv c^1\mathbf{x}^1 + c^2\mathbf{x}^2$ (where $c^1, c^2 \ge 0,\ c^1 + c^2 = 1$) is equal to

$$\mathbf{x}^c = c^1\sum_{i=1}^m \alpha_i^1\mathbf{a}_i^1 + c^2\sum_{i=1}^m \alpha_i^2\mathbf{a}_i^2 = \sum_{i=1}^m (c^1\alpha_i^1 + c^2\alpha_i^2)\left(\frac{c^1\alpha_i^1}{c^1\alpha_i^1 + c^2\alpha_i^2}\mathbf{a}_i^1 + \frac{c^2\alpha_i^2}{c^1\alpha_i^1 + c^2\alpha_i^2}\mathbf{a}_i^2\right).$$

Denote

$$\mathbf{a}_i^c \equiv \frac{c^1\alpha_i^1}{c^1\alpha_i^1 + c^2\alpha_i^2}\mathbf{a}_i^1 + \frac{c^2\alpha_i^2}{c^1\alpha_i^1 + c^2\alpha_i^2}\mathbf{a}_i^2$$

(a vector $\mathbf{a}_i^c$ is a convex combination of $\mathbf{a}_i^1$ and $\mathbf{a}_i^2$). Define a matrix $\mathbf{A}^c$ as follows $\mathbf{A}_{i,\cdot}^c \equiv (\mathbf{a}_i^c)^T$. Then $\mathbf{A}^c \in \mathbf{A}^I$. Since $\sum_{i=1}^m (c^1\alpha_i^1 + c^2\alpha_i^2) = c^1\sum_{i=1}^m \alpha_i^1 + c^2\sum_{i=1}^m \alpha_i^2 = 1$, the vector $\mathbf{x}^c = \sum_{i=1}^m (c^1\alpha_i^1 + c^2\alpha_i^2)\mathbf{a}_i^c$ is a convex combination of vectors $\mathbf{a}_i^c$. Thus $\mathbf{x}^c \in \mathcal{M}_1(\mathbf{A}^c)$. ☐

Theorem 10. *Let $dim\,\mathcal{M}_1(\mathbf{A}) = dim\,\mathcal{M}_2(\mathbf{C}) = n$ for all $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{C} \in \mathbf{C}^I$. Then the interval system (26) has a strong solution if and only if (26) is strongly solvable.*

Proof. If (26) has a strong solution, then the interval system (26) is simply strongly solvable. The second implication we prove by contradiction. Suppose that (26) has not any strong solution, it means that the intersection $(\cup_{\mathbf{A}\in\mathbf{A}^I}\mathcal{M}_1(\mathbf{A})) \cap (\cup_{\mathbf{C}\in\mathbf{C}^I}\mathcal{M}_2(\mathbf{C}))$ is of full dimension. Hence there is a vector $\mathbf{x}^1$ belonging to the interior of $(\cup_{\mathbf{A}\in\mathbf{A}^I}\mathcal{M}_1(\mathbf{A})) \cap (\cup_{\mathbf{C}\in\mathbf{C}^I}\mathcal{M}_2(\mathbf{C}))$. This vector $\mathbf{x}^1$ belongs to $\mathcal{M}_1(\mathbf{A}^1)$ for certain $\mathbf{A} \in \mathbf{A}^I$. From assumptions of the theorem we have that $\mathcal{M}_1(\mathbf{A}^1) \cap (\cup_{\mathbf{C}\in\mathbf{C}^I}\mathcal{M}_2(\mathbf{C}))$ is of full dimension. By analogy there is $\mathbf{C} \in \mathbf{C}^I$ such that $int\,\mathcal{M}_1(\mathbf{A}^1) \cap int\,\mathcal{M}_2(\mathbf{C}^1) \ne \emptyset$. Therefore (for choice $\mathbf{A}^1,\ \mathbf{C}^1$) the interval system (26) is not strongly solvable. ☐

The existence of a strong solution of the interval system (26) can be checked by two ways. First, we can compute $\cup_{\mathbf{A}\in\mathbf{A}^I}\mathcal{M}_1(\mathbf{A})$, $\cup_{\mathbf{C}\in\mathbf{C}^I}\mathcal{M}_2(\mathbf{C})$ and check weak separability of these convex polytopes. Denoting by $\mathbf{a}_i^j$, $i \in \{1, \dots, m\},\ j \in J$ ($|J| = 2^n$), vertices of $\mathbf{A}_{i,\cdot}^I$ we have

$$\bigcup_{\mathbf{A}\in\mathbf{A}^I}\mathcal{M}_1(\mathbf{A}) = conv\left(\bigcup_{i\in\{1,\dots,m\}}\bigcup_{j\in J}\{\mathbf{a}_i^j\}\right).$$

We reduced computing the union of infinitely many convex polytopes to computing the convex hull of finitely many points (concretely $m2^n$). By analogy we can compute $\cup_{\mathbf{C}\in\mathbf{C}^I}\mathcal{M}_2(\mathbf{C})$.

The second way is the following one. Interval system (26) has a strong solution if and only if there is $\mathbf{y} \in \{\pm 1\}^n$ such that the interval system

$$\begin{pmatrix} \mathbf{A}^I & -\mathbf{1} \\ -\mathbf{C}^I & \mathbf{1} \\ \mathbf{y}^T & 0 \end{pmatrix}\begin{pmatrix} \mathbf{r} \\ s \end{pmatrix} \le \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ -1 \end{pmatrix} \tag{27}$$

has a strong solution. Interval system (27) has a strong solution (see Theorem 7) if and only if the system

$$\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{1} \\ -\underline{\mathbf{C}} & \mathbf{1} \\ \mathbf{y}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}^1 \\ s^1 \end{pmatrix} - \begin{pmatrix} \underline{\mathbf{A}} & -\mathbf{1} \\ -\overline{\mathbf{C}} & \mathbf{1} \\ \mathbf{y}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}^2 \\ s^2 \end{pmatrix} \leq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ -1 \end{pmatrix}, \quad (\mathbf{r}^1, s^1, \mathbf{r}^2, s^2) \geq \mathbf{0}$$

or, equivalently the system

$$\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{1} \\ -\underline{\mathbf{C}} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r}^1 \\ s^1 \end{pmatrix} - \begin{pmatrix} \underline{\mathbf{A}} & -\mathbf{1} \\ -\overline{\mathbf{C}} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r}^2 \\ s^2 \end{pmatrix} \leq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{y}^T(\mathbf{r}^1 - \mathbf{r}^2) < 0, \quad (\mathbf{r}^1, s^1, \mathbf{r}^2, s^2) \geq \mathbf{0}$$

is solvable for some $\mathbf{y} \in \{\pm 1\}^n$. On the whole we obtain that there exists a strong solution of interval system (26) if and only if the system

$$\begin{pmatrix} \overline{\mathbf{A}} & -\underline{\mathbf{A}} & -\mathbf{1} \\ -\underline{\mathbf{C}} & \overline{\mathbf{C}} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^2 \\ s \end{pmatrix} \leq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{r}^1 \neq \mathbf{r}^2, \quad \mathbf{r}^1, \mathbf{r}^2 \geq \mathbf{0} \tag{28}$$

is solvable. Moreover, if $(\mathbf{r}^1, \mathbf{r}^2, s)$ solves (28), then vector $(\mathbf{r}^1 - \mathbf{r}^2, s)$ is the required strong solution of the interval systems (27) and (26).

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