MIR Closures of Polyhedral Sets

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February 28, 2007

Abstract

We study the mixed-integer rounding (MIR) closures of polyhedral sets. The MIR closure of a polyhedral set is equal to its split closure and the associated separation problem is NP-hard. We describe a mixed-integer programming (MIP) model with linear constraints and a non-linear objective for separating an arbitrary point from the MIR closure of a given mixed-integer set. We linearize the objective using additional variables to produce a linear MIP model that solves the separation problem approximately, with an accuracy that depends on the number of additional variables used. Our analysis yields a short proof of the result of Cook, Kannan and Schrijver (1990) that the split closure of a polyhedral set is again a polyhedron. We also present some computational results with our approximate separation model.

1 Introduction

In this paper we study the mixed-integer rounding (MIR) closure of a given mixed-integer set

$$P = \{v \in R^{|J|},\ x \in Z^{|I|}\ :\ Cv + Ax \geq b,\ v, x \geq 0\}$$

where all numerical data is rational. In other words, we are interested in the set of points that satisfy all MIR inequalities

$$(\lambda C)^+ v + (-\lambda)^+(Cv + Ax - b) + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r \lfloor \lambda A \rfloor x \geq r \lceil \lambda b \rceil$$

that can be generated by some λ of appropriate dimension. Here $r = \lambda b - \lfloor \lambda b \rfloor$, $(\cdot)^+$ denotes $\max\{0, \cdot\}$, $\mathbf{1}$ is an all-ones vector, and all operators are applied to vectors component-wise. In Section 2, we discuss in detail how these inequalities are derived and why they are called MIR inequalities.

The term *mixed-integer rounding* was first used by Nemhauser and Wolsey [20, pp.244] to denote valid inequalities that can be produced by what they call the MIR procedure.

These authors in [19] strengthen and redefine the MIR procedure and the resulting inequality. The same term was later used to denote seemingly simpler inequalities in Marchand and Wolsey [18], and Wolsey [22]. The definition of the MIR inequality we use in this paper is equivalent to the one in [19], though our presentation is based on [22].

Split cuts were defined by Cook, Kannan and Schrijver in [10], and are a special case of the disjunctive cuts introduced by Balas [2]. In [19], Nemhauser and Wolsey show that MIR cuts are equivalent to split cuts in the sense that, for a given polyhedral set, every MIR cut is a split cut and vice-versa. In [10], Cook, Kannan and Schrijver show that the split closure (the set of points satisfying all split cuts) of a polyhedral set is again a polyhedron. In this paper, we present a short proof of the same fact by analyzing MIR closures of polyhedral sets. This is not a new result but our proof is significantly easier to follow and present.

The problem of separating an arbitrary point from the MIR closure of a polyhedral set is NP-hard; this was shown (using split cuts) by Caprara and Letchford [8]. A similar property was shown by Eisenbrand [15] for the Chvátal closure of a polyhedral set. In [17], Fischetti and Lodi show that, in practice, it is possible to separate points from the Chvátal closure in a reasonable amount of time. Their approach involves formulating the separation problem as an MIP, and solving it with a black-box MIP solver. By repeatedly applying their separation algorithm, they are able to approximately optimize over the Chvátal closures of MIPLIB instances and obtain very tight bounds on the value of optimal solutions. Motivated by their work, and the fact that the MIR closure is contained in the Chvátal closure (usually strictly), we describe an MIP model for separating from the MIR closure of a polyhedral set. We also present computational results on approximately optimizing over the MIR closure for problems in the MIPLIB 3.0 test set.

Our work is also closely related with two recent papers written independently. The first one is a paper [4] by Balas and Saxena who experiment with a parametric MIP model to find violated split cuts. The second one is the paper by Vielma [21] which presents a proof of the fact that the split closure of a polyhedral set is again a polyhedron.

The paper is organized as follows. In Section 2, we define MIR inequalities and discuss how our definition is related to earlier definitions. In Section 3, we present a mixed-integer programming model that approximately separates an arbitrary point from the MIR closure of a given polyhedral set. In Section 4, we present a simple proof that the MIR (or, split) closure of a polyhedral set is again a polyhedron. In Sections 5 and 6 we discuss computational issues and present a summary of our computational experiments with the approximate separation model.

2 Mixed-integer rounding inequalities

In [22], Wolsey develops the MIR inequality as the only non-trivial facet of the following simple mixed-integer set:

$$Q^0 = \Big\{ v \in R,\ x \in Z \ :\ v + x \ \geq\ b,\ \ v \geq 0 \Big\}$$

where $b \notin Z$. It is easy to see that

$$v \geq \hat{b}(\lceil b \rceil - x) \tag{1}$$

where $\hat{b} = b - \lfloor b \rfloor$ is valid and facet defining for Q^0. In [22] this inequality is called the *basic mixed-integer* inequality.

To apply this idea to more general sets defined by a single inequality, one needs to group variables in a way that resembles Q^0. More precisely, given a set

$$Q^1 = \Big\{ v \in R^{|J|},\ x \in Z^{|I|}\ :\ \sum_{j \in J} c_j v_j + \sum_{i \in I} a_i x_i \ \geq\ b,\ \ v,\ x \geq 0 \Big\}$$

the defining inequality is relaxed to obtain

$$\big(\sum_{j \in J} \max\{0, c_j\} v_j + \sum_{i \in I'} \hat{a}_i x_i\big) + \big(\sum_{i \in I \setminus I'} x_i + \sum_{i \in I} \lfloor a_i \rfloor x_i\big) \geq\ b$$

where $\hat{a}_i = a_i - \lfloor a_i \rfloor$ and $I' \subseteq I$. As the first part of the left hand side of this inequality is non-negative, and the second part is integral, the MIR inequality

$$\sum_{j \in J} \max\{0, c_j\} v_j + \sum_{i \in I'} \hat{a}_i x_i \geq\ \hat{b}\big(\lceil b \rceil - \sum_{i \in I \setminus I'} x_i - \sum_{i \in I} \lfloor a_i \rfloor x_i\big)$$

is valid for Q^1. Notice that $I' = \{i \in I\ :\ \hat{a}_i < \hat{b}\}$ gives the strongest inequality of this form and therefore the MIR inequality can also be written as

$$\sum_{j \in J} (c_j)^+ v_j + \sum_{i \in I} \min\{\hat{a}_i, \hat{b}\} x_i + \hat{b} \sum_{i \in I} \lfloor a_i \rfloor x_i \geq\ \hat{b} \lceil b \rceil \tag{2}$$

where $(\cdot)^+$ denotes $\max\{0, \cdot\}$ as defined earlier.

To apply this idea to sets defined by $m > 1$ inequalities, the first step is to combine them to obtain a single *base* inequality and then apply inequality (2). Let

$$P = \Big\{ v \in R^l,\ x \in Z^n\ :\ Cv + Ax \geq b,\ v, x \geq 0 \Big\}$$

be a mixed-integer set where C, A and b are vectors of appropriate dimension. To obtain the base inequality, one possibility is to use a vector $\lambda \in R^m$, $\lambda \geq 0$ to combine the inequalities defining P. This approach leads to the base inequality

$$\lambda C v + \lambda A x \ \geq\ \lambda b$$

and the corresponding MIR inequality

$$(\lambda C)^{+}v + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r\lceil \lambda b \rceil, \tag{3}$$

where operators $(\cdot)^{+}$, $\lfloor \cdot \rfloor$ and $\min\{\cdot,\cdot\}$ are applied to vectors component-wise, and $r = \lambda b - \lfloor \lambda b \rfloor$.

Alternatively, it is also possible to first introduce slack variables to the set of inequalities defining P and combine them using a vector λ which is not necessarily non-negative. This leads to the base inequality

$$\lambda C v + \lambda A x - \lambda s \;=\; \lambda b$$

and the corresponding MIR inequality

$$(\lambda C)^{+}v + (-\lambda)^{+}s + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r\lceil \lambda b \rceil, \tag{4}$$

where s denotes the (non-negative) slack variables. Finally, substituting out the slack variables gives the following MIR inequality in the original space of P:

$$(\lambda C)^{+}v + (-\lambda)^{+}(Cv + Ax - b) + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r\lceil \lambda b \rceil. \tag{5}$$

These inequalities are what we call MIR inequalities in this paper.

Notice that when $\lambda \geq 0$, inequality (5) reduces to inequality (3). When $\lambda \ngeq 0$, however, there are inequalities (5) which cannot be written in the form (3). We present an example to emphasize this point.

Example 1 *Consider the following simple mixed-integer set*

$$T = \{v \in R,\; x \in Z \;:\; -v - 4x \geq -4, \;\; -v + 4x \geq 0, \;\; v, x \geq 0\}$$

and the base inequality generated by $\lambda = [-1/8, 1/8]$

$$x + s_1/8 - s_2/8 \geq 1/2$$

where s_1 *and* s_2 *denote the slack variables for the first and second constraint, respectively. The corresponding MIR inequality is* $1/2x + s_1/8 \geq 1/2$, *which after substituting out* s_1, *becomes* $-v/8 \geq 0$ *or simply* $v \leq 0$. *This inequality defines the only non-trivial facet of* T.

Notice that it is not possible to generate this inequality using non-negative multipliers. Any base inequality generated by $\lambda_1, \lambda_2 \geq 0$ *has the form*

$$(-\lambda_1 - \lambda_2)v + (-4\lambda_1 + 4\lambda_2)x \geq -4\lambda_1$$

where variable v *has a negative coefficient. Therefore, the MIR inequality generated by this base inequality would have a coefficient of zero for the* v *variable, establishing that* $v \leq 0$ *cannot be generated as an MIR inequality (3)*

We note that a similar example is also independently developed in [6].

2.1 Basic properties of MIR inequalities

Let P^{LP} denote the continuous relaxation of P. A linear inequality $hv + gx \geq d$ is called a *split cut* for P if it is valid for both $P^{LP} \cap \{\bar{\alpha}x \leq \bar{\beta}\}$ and $P^{LP} \cap \{\bar{\alpha}x \geq \bar{\beta} + 1\}$, where $\bar{\alpha}$ and $\bar{\beta}$ are integral. Inequality $hv + gx \geq d$ is said to be derived from the *disjunction* $\bar{\alpha}x \leq \bar{\beta}$ and $\bar{\alpha}x \geq \bar{\beta} + 1$. Obviously all points in P satisfy any split cut for P. Note that multiple split cuts can be derived from the same disjunction.

The basic MIR inequality (1) is a split cut for Q^0 derived from the disjunction $x \leq \lfloor b \rfloor$ and $x \geq \lfloor b \rfloor + 1$. Therefore, the MIR inequality (5) is also a split cut for P derived from the disjunction $\bar{\alpha}x \leq \bar{\beta}$ and $\bar{\alpha}x \geq \bar{\beta} + 1$ where $\bar{\beta} = \lfloor \lambda b \rfloor$ and

$$\bar{\alpha}_i = \begin{cases} \lceil (\lambda A)_i \rceil & \text{if } (\lambda A)_i - \lfloor (\lambda A)_i \rfloor \geq \lambda b - \lfloor \lambda b \rfloor \\ \lfloor (\lambda A)_i \rfloor & \text{otherwise.} \end{cases}$$

We note that this observation also implies that if a point $(v^*, x^*) \in P^{LP}$ violates the MIR inequality (5) then $\bar{\beta} + 1 > \bar{\alpha}x^* > \bar{\beta}$.

Furthermore, Nemhauser and Wolsey [19] showed that every split cut for P can be derived as an MIR cut for P. As we show later, what we call MIR inequalities in this paper are equivalent to the MIR inequalities defined in [19]. We next formally define the MIR closure of a polyhedral set.

Definition 2 *The MIR closure of P is the set of points satisfying all MIR inequality (5) that can be generated by some multiplier vector $\lambda \in R^m$.*

Thus, the split closure of a polyhedral set is the same as its MIR closure.

2.2 Original MIR procedure of Nemhauser and Wolsey

In their book [20, Section II.1.6], Nemhauser and Wolsey develop the MIR inequalities for mixed-integer sets. Their original discussion uses the "$\leq$" form, both for the inequalities that define set and the MIR inequalities derived for it. To compare their inequality with what we call the MIR inequality in this paper, we present their results in the "$\geq$" form.

The MIR procedure of Nemhauser and Wolsey starts with two vectors $\lambda^1, \lambda^2 \geq 0$ of appropriate dimension to generate two implied inequalities

$$\lambda^1 Cv + \lambda^1 Ax \geq \lambda^1 b \quad \text{and} \quad \lambda^2 Cv + \lambda^2 Ax \geq \lambda^2 b.$$

Using these two *base* inequalities, the procedure then generates the following valid MIR inequality:

$$\lambda^1 Ax + r\lceil \lambda^2 A - \lambda^1 A \rceil x + \max\{\lambda^1 C, \lambda^2 C\} v \geq r\lceil \lambda^2 b - \lambda^1 b \rceil + \lambda^1 b$$

where $r = \lambda^2 b - \lambda^1 b - \lfloor \lambda^2 b - \lambda^1 b \rfloor$. This inequality can also be written as follows:

$$\big((\lambda^2 - \lambda^1)C\big)^+ v + \lambda^1(Cv + Ax - b) + r\lceil(\lambda^2 - \lambda^1)A\rceil x \ge r\lceil(\lambda^2 - \lambda^1)b\rceil. \tag{6}$$

Notice that, given a vector λ and the associated MIR inequality (5), it is possible to construct two non-negative vectors $\lambda^2 = (\lambda)^+$ and $\lambda^1 = (-\lambda)^+$ and produce the corresponding inequality (6). The two inequalities would look identical, except some of the coefficients of the integer variables would be stronger in inequality (5) due to the term $\min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x$. Similarly, given two vectors $\lambda^1, \lambda^2 \ge 0$, it is possible to show that MIR inequality (5) generated by $\lambda = \lambda^2 - \lambda^1$ dominates inequality (6).

2.3 Revised MIR procedure of Nemhauser and Wolsey

Later, in a paper [19], Nemhauser and Wolsey show that for $\mu^1, \mu^2 \ge 0$, inequality (6) remains valid for the relaxed set $P' = \{v \in R^{|J|},\ x \in Z^{|I|} : C'v + A'x \ge b'\}$ provided that $\mu^1 C' = \mu^2 C'$ and $(\mu^2 - \mu^1)A'$ is integral. In this case, inequality (6) becomes

$$\mu^1(C'v + A'x - b') + r'(\mu^2 - \mu^1)A'x \ge r'\lceil(\mu^2 - \mu^1)b'\rceil \tag{7}$$

where $r' = (\mu^2 - \mu^1)b' - \lfloor(\mu^2 - \mu^1)b'\rfloor$. It is easy to see that inequality (7) can be strengthened if for some index i both μ_i^1 and μ_i^2 are strictly positive. It is therefore possible to let $\mu = \mu^2 - \mu^1$ and write inequality (7) as

$$(-\mu)^+(C'v + A'x - b') + r'\mu A'x \ge r'\lceil \mu b' \rceil \tag{8}$$

where the vector μ is not restricted in sign and it satisfies (i) $\mu C' = 0$ and (ii) $\mu A'$ is integral.

We next show that inequality (8) and the MIR inequality (5) are equivalent when applied to the set P in the sense that for any λ it is possible to construct an appropriate μ that would give the same inequality and vice-versa. Notice that the non-negativity requirements are not explicitly present in the definition of P'. It is possible to represent the set P in this form by defining

$$C' = \begin{bmatrix} C \\ I \\ 0 \end{bmatrix}, \quad A' = \begin{bmatrix} A \\ \mathbf{0} \\ I \end{bmatrix}, \quad b' = \begin{bmatrix} b \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

where I and $\mathbf{0}$ respectively denote the identity and zero matrix of appropriate dimension.

Let λ be given and consider $\mu = [\lambda, -\lambda C, \gamma]$ where

$$\gamma_i = \begin{cases} -\hat{a}_i & \text{if } \hat{a}_i < r \\ 1 - \hat{a}_i & \text{otherwise} \end{cases},$$

and $\hat{a} = \lambda A - \lfloor \lambda A \rfloor$. Note that $\mu C' = 0$ and $\mu A'$ is integral. Also notice that $\mu b' = \lambda b$ and therefore $r' = r$. Inequality (8) for this choice of μ is

$$(-\lambda)^+(Cv + Ax - b) + (\lambda C)^+ v + (-\gamma)^+ x + r(\lambda A + \gamma)x \geq r\lceil \lambda b \rceil$$

where the coefficient of x can also be written as

$$(-\gamma)^+ + r\lfloor \lambda A \rfloor + r(\hat{a} + \gamma) \;=\; r\lfloor \lambda A \rfloor + \min\{\hat{a}, r\mathbf{1}\}.$$

Therefore, inequality (8) generated by μ is identical to inequality (5) generated by λ.

Conversely, given $\mu = [\mu_0, \mu_v, \mu_x] \geq 0$ consider the corresponding inequality (8)

$$(-\mu_0)^+(Cv + Ax - b) + (-\mu_v)^+ v + (-\mu_x)^+ x + r'(\mu_0 Ax + \mu_x x) \geq r'\lceil \mu_0 b \rceil$$

and notice that $\mu C' = 0$ implies that $\mu_0 C = -\mu_v$ and therefore $(-\mu_v)^+ = (\mu_0 C)^+$. In addition, $r' = \mu_0 b - \lfloor \mu_0 b \rfloor$. As $\mu A'$ is integral, $(\mu_0 A + \mu_x)$ is integral and therefore $\mu_x = -\hat{a} + t$ where t is an integral vector. Clearly inequality (8) can be strengthened unless $t_i = 0$ if $\hat{a}_i < r$ and $t_i = 1$, otherwise. It is therefore clear that the MIR inequality (5) generated by μ_0 is identical to inequality (8) generated by μ.

We next give a basic property of MIR inequalities (8) for the set P'. This property is known to hold for the Chvátal closure [15] and can easily be extended for MIR cuts.

Proposition 3 *The MIR closure of P' is invariant under the operation $y = Ux + l$ where l is an integer vector and U is a unimodular matrix.*

Proof Let $clo(\cdot)$ denote the MIR closure of a set. We will show that a given point $(\bar{v}, \bar{x}) \in clo(P')$ if and only if $(\bar{v}, \bar{y}) \in clo(T)$ where $\bar{y} = U\bar{x} + l$ and $T = \{v \in R^{|J|},\ y \in Z^{|I|} \,:\, C'v + A'U^{-1}y \geq b' + A'U^{-1}l\}$.

Assume that $(\bar{v}, \bar{x}) \in clo(P')$ and $(\bar{v}, \bar{y}) \notin clo(T)$. Then there exists a μ such that

$$(-\mu)^+(C'\bar{v} + A'U^{-1}\bar{y} - b' - A'U^{-1}l) + r(\mu A'U^{-1})\bar{y} < r\left\lceil \mu(b' + A'U^{-1}l) \right\rceil$$

where r denotes the fractional part of $\mu(b' + A'U^{-1}l)$, and $\mu C' = 0$ and $\mu A'U^{-1}$ is integral. This implies that $\mu A'U^{-1}l$ is integral and therefore r is also equal to the fractional part of $\mu b'$. As $\bar{y} = U\bar{x} + l$, the above inequality can also be written as

$$(-\mu)^+(C'\bar{v} + A'\bar{x} - b') + r(\mu A')\bar{x} + r\mu A'U^{-1}l < r\left\lceil \mu b' + \mu A'U^{-1}l \right\rceil.$$

Furthermore, as $\mu A'U^{-1}l$ is integral, $(-\mu)^+(C'\bar{v} + A'\bar{x} - b') + r(\mu A')\bar{x} < r\lceil \mu b' \rceil$. This, however, cannot be true as $\bar{x}$ must satisfy the MIR inequality generated by the same μ.

Similarly, it is possible to show that $\bar{x} \notin clo(P')$ and $\bar{y} \in clo(T)$ leads to a contradiction. ∎

3 The Separation Problem

In this section, we study the problem of separating an arbitrary point from the MIR closure of the polyhedral set $P = \{v \in R^l,\ x \in Z^n\ :\ Cv + Ax \geq b,\ v, x \geq 0\}$. In other words, for a given point, we are interested in either finding violated inequalities or concluding that none exists. For convenience of notation, we first argue that without loss of generality we can assume P is given in equality form.

Consider the MIR inequality (4) for P,

$$(\lambda C)^+ v + (-\lambda)^+ s + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r \lceil \lambda b \rceil,$$

where s denotes the slack expression $(Cv + Ax - b)$. If we explicitly define the slack variables, by letting $\tilde{C} = (C,\ -I)$ and $\tilde{v} = (v,\ s)$, then the constraints defining P become $\tilde{C}\tilde{v} + Ax = b,\ \tilde{v} \geq 0,\ x \geq 0$, and the MIR inequality can be written as

$$(\lambda \tilde{C})^+ \tilde{v} + \min\{\lambda A - \lfloor \lambda A \rfloor, r\mathbf{1}\}x + r\lfloor \lambda A \rfloor x \geq r \lceil \lambda b \rceil. \tag{9}$$

In other words, all continuous variables, whether slack or structural, can be treated uniformly. In the remainder of this paper we assume that P is given in the equality form

$$P = \{v \in R^l, x \in Z^n : Cv + Ax = b, v, x \geq 0\}.$$

We denote the continuous relaxation of P by P^{LP}.

3.1 Relaxed MIR inequalities

Let

$$\begin{aligned} \Pi = \Big\{(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in R^m \times R^l \times R^n \times Z^n \times R \times Z \quad &: \\ c^+ &\geq \lambda C \\ \hat{\alpha} + \bar{\alpha} &\geq \lambda A \\ \hat{\beta} + \bar{\beta} &\leq \lambda b \\ c^+ &\geq 0 \\ 1 \geq \hat{\alpha} &\geq 0 \\ 1 \geq \hat{\beta} &\geq 0 \end{aligned}$$

Note that for any $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$,

$$c^+ v + (\hat{\alpha} + \bar{\alpha})x \geq \hat{\beta} + \bar{\beta} \tag{10}$$

is valid for P^{LP} as it is a relaxation of $(\lambda C)v + (\lambda A)x = \lambda b$. Furthermore, using the basic mixed-integer inequality (1), we infer that

$$c^+v + \hat{\alpha}x + \hat{\beta}\bar{\alpha}x \geq \hat{\beta}(\bar{\beta} + 1) \tag{11}$$

is a valid inequality for P. We call inequality (11) where $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ a *relaxed MIR inequality* derived using the *base inequality* (10). We next show some basic properties of relaxed MIR inequalities.

Lemma 4 *A relaxed MIR inequality (11) violated by $(v^*, x^*) \in P^{LP}$ satisfies*

(i) $1 > \hat{\beta} > 0$,

(ii) $1 > \Delta > 0$,

(iii) the violation of the inequality is at most $\hat{\beta}(1 - \hat{\beta}) \leq 1/4$,

where $\Delta = \bar{\beta} + 1 - \bar{\alpha}x^*$ and violation is defined to be the right hand side of inequality (11) minus its left hand side.

Proof If $\hat{\beta} = 0$, then the relaxed MIR cut is trivially satisfied by all points in P^{LP}. Furthermore, if $\hat{\beta} = 1$, then inequality (11) is identical to its base inequality (10) which again is satisfied by all points in P^{LP}. Therefore, a non-trivial relaxed MIR cut satisfies $1 > \hat{\beta} > 0$.

For part (ii) of the Lemma, note that if $\bar{\alpha}x^* \geq \bar{\beta} + 1$ then inequality (11) is satisfied, as $c^+, \hat{\alpha}, \hat{\beta} \geq 0$ and $(v^*, x^*) \geq 0$. Furthermore, if (v^*, x^*) satisfies inequality (10) and $\bar{\alpha}x^* \leq \bar{\beta}$, then so is inequality (11) as $\hat{\beta} \leq 1$. Therefore, as the cut is violated, $1 > \Delta > 0$. It is also possible to show this by observing that inequality (11) is a split cut for P derived from the disjunction $\Delta \geq 1$ and $\Delta \leq 0$.

For the last part, let $w = c^+v^* + \hat{\alpha}x^*$ so that the base inequality (10) becomes $w \geq \hat{\beta} + \Delta - 1$ and the relaxed MIR inequality (11) becomes $w \geq \hat{\beta}\Delta$. Clearly

$$\hat{\beta}\Delta - w \leq \hat{\beta}(w + 1 - \hat{\beta}) - w = \hat{\beta}(1 - \hat{\beta}) - (1 - \hat{\beta})w \leq \hat{\beta}(1 - \hat{\beta}).$$

The last inequality follows from the fact that $w \geq 0$ and $\hat{\beta} \leq 1$. ■

Next, we relate MIR inequalities to relaxed MIR inequalities.

Lemma 5 *For any $\lambda \in R^m$, the MIR inequality (9) is a relaxed MIR inequality.*

Proof For a given multiplier vector λ, define α to denote λA. Further, set $c^+ = (\lambda C)^+$, $\bar{\beta} = \lceil \lambda b \rceil$ and $\hat{\beta} = \lambda b - \lfloor \lambda b \rfloor$. Also, define $\hat{\alpha}$ and $\bar{\alpha}$ as follows:

$$\hat{\alpha}_i = \begin{cases} \alpha_i - \lfloor \alpha_i \rfloor & \text{if } \alpha_i - \lfloor \alpha_i \rfloor < \hat{\beta} \\ 0 & \text{otherwise} \end{cases}, \qquad \bar{\alpha}_i = \begin{cases} \lfloor \alpha_i \rfloor & \text{if } \alpha_i - \lfloor \alpha_i \rfloor < \hat{\beta} \\ \lceil \alpha_i \rceil & \text{otherwise} \end{cases},$$

Clearly, $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ and the corresponding relaxed MIR inequality (11) is the same as the MIR inequality (9). ■

Lemma 6 *MIR inequalities dominate relaxed MIR inequalities.*

Proof Let $(v^*, x^*) \in P^{LP}$ violate a relaxed MIR inequality $\mathcal{I}$ generated with $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$. We will show that (v^*, x^*) also violates the MIR inequality (9).

Due to Lemma 4, we have $\bar{\beta} + 1 - \bar{\alpha}x^* > 0$ and therefore increasing $\hat{\beta}$ only increases the violation of the relaxed MIR inequality. Assuming $\mathcal{I}$ is the most violated relaxed MIR inequality, $\hat{\beta} = \min\{\lambda b - \bar{\beta}, 1\}$. By Lemma 4, we know that $\hat{\beta} < 1$, and therefore $\hat{\beta} = \lambda b - \bar{\beta}$ and $\bar{\beta} = \lfloor \lambda b \rfloor$.

In addition, due to the definition of Π we have $c^+ \geq (\lambda C)^+$ and $\hat{\alpha} + \hat{\beta}\bar{\alpha} \geq \min\{\lambda A - \lfloor \lambda A \rfloor, \hat{\beta}\mathbf{1}\} + \hat{\beta} \lfloor \lambda A \rfloor$. As $(v^*, x^*) \geq 0$, the violation of the MIR inequality is at least as much as the violation of $\mathcal{I}$. ■

Combining Lemmas 5 and 6, we observe that a point in P^{LP} satisfies all MIR inequalities, if and only if it satisfies all relaxed MIR inequalities. Therefore, we can define the MIR closure of a polyhedral set using relaxed MIR inequalities and thus without using operators that take minimums, maximums or extract fractional parts of numbers. Let $\bar{\Pi}$ be the projection of Π in the space of $c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}$ and $\bar{\beta}$ variables. In other words, $\bar{\Pi}$ is obtained by projecting out the λ variables. We now describe the MIR closure of P as follows:

$$P^{MIR} = \Big\{(v, x) \in P^{LP} \ : \ c^+v + \hat{\alpha}x + \hat{\beta}\bar{\alpha}x \geq \hat{\beta}(\bar{\beta} + 1) \text{ for all } (c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \bar{\Pi}\Big\}.$$

We would like to emphasize that $\bar{\Pi}$ is not the polar of P^{MIR} and therefore even though $\bar{\Pi}$ is a polyhedral set (with a finite number of extreme points and extreme directions), we have not yet shown that the polar of P^{MIR} is polyhedral. The polar of a polyhedral set is defined to be the set of points that yield valid inequalities for the original set. If the original set is defined in R^n, its polar is defined in R^{n+1} and the first n coordinates of any point in the polar give the coefficients of a valid inequality for the original set, and the last coordinate gives the right hand side of the valid inequality. Therefore, the polar of P^{MIR} is the collection of points $(c^+, \hat{\alpha} + \hat{\beta}\bar{\alpha}, \hat{\beta}(\bar{\beta} + 1)) \in R^{l+n+1}$ where $(c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \bar{\Pi}$. A set is polyhedral if and only if its polar is polyhedral.

For a given point $(v^*, x^*) \in P^{LP}$, testing if $(v^*, x^*) \in P^{MIR}$ can be achieved by solving the following non-linear integer program (MIR-SEP):

$$\begin{aligned} \max \quad & \hat{\beta}(\bar{\beta}+1) - (c^+ v^* + \hat{\alpha} x^* + \hat{\beta}\bar{\alpha} x^*) \\ \text{s.t.} \quad & \\ & (\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi. \end{aligned}$$

If the optimal value of this program is non-positive, then $(v^*, x^*) \in P^{MIR}$. On the other hand, if the optimal value is positive, the optimal solution gives a most violated MIR inequality.

3.2 An Approximate separation model

We next (approximately) linearize the nonlinear terms that appear in the objective function of MIR-SEP. To this end, we first define a new variable Δ that stands for the term $(\bar{\beta} + 1 - \bar{\alpha}x)$. We then approximate $\hat{\beta}$ by a number $\tilde{\beta} \le \hat{\beta}$ representable over some $\mathcal{E} = \{\epsilon_k : k \in K\}$. We say that a number δ is *representable* over $\mathcal{E}$ if $\delta = \sum_{k \in \bar{K}} \epsilon_k$ for some $\bar{K} \subseteq K$. We can therefore write $\tilde{\beta}$ as $\sum_{k\in K} \epsilon_k \pi_k$ using binary variables π_k and approximate $\hat{\beta}\Delta$ by $\tilde{\beta}\Delta$ which can now be written as $\sum_{k\in K} \epsilon_k \pi_k \Delta$. Finally, we linearize terms $\pi_k \Delta$ using standard techniques as π_k is binary and $\Delta \in (0, 1)$ for any violated inequality.

An approximate MIP model APPX-MIR-SEP for the separation of the most violated MIR inequality reads as follows:

$$\max \quad \sum_{k\in K} \epsilon_k \Delta_k \; - \; (c^+ v^* + \hat{\alpha} x^*) \tag{12}$$

$$\text{s.t.} \quad (\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi \tag{13}$$

$$\hat{\beta} \ge \sum_{k\in K} \epsilon_k \pi_k \tag{14}$$

$$\Delta = (\bar{\beta} + 1) - \bar{\alpha} x^* \tag{15}$$

$$\Delta_k \le \Delta \qquad \forall k \in K \tag{16}$$

$$\Delta_k \le \pi_k \qquad \forall k \in K \tag{17}$$

$$\pi \in \{0,1\}^{|K|} \tag{18}$$

Let z^{sep} and $z^{apx-sep}$ denote the optimal value of MIR-SEP and APPX-MIR-SEP, respectively. For any integral solution of APPX-MIR-SEP, we have $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$

and

$$\sum_{k \in K} \epsilon_k \Delta_k \;\leq\; \sum_{k \in K} \epsilon_k \Delta \pi_k \;\leq\; \hat{\beta}\Delta \;=\; \hat{\beta}(\bar{\beta} + 1 - \bar{\alpha}x^*)$$

establishing that $z^{sep} \;\geq\; z^{apx-sep}$. In other words, APPX-MIR-SEP is a restriction of MIR-SEP and if the approximate separation problem finds a solution with objective function value $z^{apx-sep} > 0$, the corresponding MIR cut is violated by at least as much.

In our computational experiments, we use $\mathcal{E} = \{2^{-k} : k = 1, \ldots, \bar{k}\}$ for some small number $\bar{k}$. We next show that with this choice of $\mathcal{E}$, APPX-MIR-SEP yields a violated MIR cut provided that there is an MIR cut with a "large enough" violation. Notice that for any $\hat{\beta}$ there exists a $\tilde{\beta}$ representable over $\mathcal{E}$ such that $2^{-\bar{k}} \geq \hat{\beta} - \tilde{\beta} \geq 0$.

Theorem 7 *Let $\mathcal{E} = \{2^{-k} : k = 1, \ldots, \bar{k}\}$ for some positive integer $\bar{k}$, then*

$$z^{sep} \;\geq\; z^{apx-sep} \;>\; z^{sep} - 2^{-\bar{k}} \tag{19}$$

where z^{sep} and $z^{apx-sep}$ denote the optimal values of MIR-SEP and APPX-MIR-SEP, respectively.

Proof The first inequality holds as APPX-MIR-SEP is a restriction of MIR-SEP. For the second inequality, note that $z^{apx-sep} \geq 0$ as we can get a feasible solution of APPX-MIR-SEP with objective 0 by setting Δ to 1, and the remaining variables to 0. Therefore the second inequality in (19) holds if $z^{sep} \leq 0$. Assume that $z^{sep} > 0$. Let $(\lambda, c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ be an optimal solution of MIR-SEP. For the variables in APPX-MIR-SEP common with MIR-SEP, set their values to the above optimal solution of MIR-SEP. Let $\tilde{\beta}$ be the largest number representable over $\mathcal{E}$ less than or equal to $\hat{\beta}$. Clearly, $2^{-\bar{k}} \geq \hat{\beta} - \tilde{\beta} \geq 0$. Choose $\pi \in \{0,1\}^{\bar{k}}$ such that $\tilde{\beta} = \sum_{k \in K} \epsilon_k \pi_k$. Set $\Delta = \bar{\beta} + 1 - \bar{\alpha}x^*$. Set $\Delta_k = 0$ if $\pi_k = 0$, and $\Delta_k = \Delta$ if $\pi_k = 1$. Then $\Delta_k = \pi_k \Delta$ for all $k \in K$, and $\tilde{\beta}\Delta = \sum_{k \in K} \epsilon_k \Delta_k$. Therefore,

$$2^{-\bar{k}} \;>\; 2^{-\bar{k}}\Delta \;\geq\; \hat{\beta}\Delta - \tilde{\beta}\Delta = \hat{\beta}\Delta - \sum_{k \in K} \epsilon_k \Delta_k.$$

The second inequality in (19) follows. ∎

In the next section (Theorem 13) we show that APPX-MIR-SEP becomes an exact model for finding violated MIR cuts when $\mathcal{E}$ is chosen as $\{\epsilon_k = 2^k/\Phi, \forall k = \{1, \ldots, \lceil log\Phi \rceil\}\}$ where Φ is the least common multiple of all subdeterminants of $A|C|b$.

4 A simple proof that the MIR closure is a polyhedron

In this section we give a short proof that the MIR closure of a polyhedral set is a polyhedron. As MIR cuts are equivalent to split cuts, this result obviously follows from the work of Cook, Kannan and Schrijver (1990) on split cuts. Andersen, Cornuéjols and Li (2005), and Vielma (2006) give alternative proofs that the split closure of a polyhedral set is a polyhedron. We believe our proof is simpler than the previous proofs; further it is framed in the language of MIR cuts and not split cuts.

The main tool in the proof is a finite bound on the multipliers λ needed for non-redundant MIR cuts given in Lemma 10. The bounds on λ can be tightened if the MIP is a pure integer program, and we give these tighter bounds first, in the next lemma. In this section we assume that the coefficients in $Cv + Ax = b$ are integers. Denote the ith equation of $Ax + Cv = b$ by $c_i v + a_i x = b_i$. An equation $c_i v + a_i x = b_i$ is a *pure integer* equation if $c_i = 0$.

Lemma 8 *If some MIR inequality is violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) derived using $\lambda_i \in [0, 1)$ for every pure integer equation.*

Proof: (sketch) Let $(\lambda, (\lambda C)^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ define an MIR inequality where $\lambda_i \notin [0, 1)$ for a pure integer equation $c_i v + a_i x = b_i$ where $c_i = 0$. It is possible to show that the MIR inequality defined by

$$(\lambda - \lfloor \lambda_i \rfloor e_i, (\lambda C)^+, \hat{\alpha}, \bar{\alpha} - \lfloor \lambda_i \rfloor a_i, \hat{\beta}, \bar{\beta} - \lfloor \lambda_i \rfloor b_i) \in \Pi$$

has precisely the same violation. ■

Definition 9 *We define Ψ to be the largest absolute value of subdeterminants of C, and $1/m$ if $C = 0$, where m is the number of rows in $Ax + Cv = b$.*

Lemma 10 *If there is an MIR inequality violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) with $\lambda_i \in (-m\Psi, m\Psi)$, where m is the number of rows in $Ax + Cv = b$.*

Proof: Let the MIR cut

$$(\lambda C)^+ v + \hat{\alpha} x + \hat{\beta} \bar{\alpha} x \geq \hat{\beta}(\bar{\beta} + 1) \tag{20}$$

be violated by (v^*, x^*). Then $(\lambda, (\lambda C)^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \Pi$ with $0 < \hat{\beta} < 1$. Let C_j stand for the jth column of C. Let $S_1 = \{j : \lambda C_j > 0\}$ and $S_2 = \{j : \lambda C_j \leq 0\}$.

Consider the following cone:

$$\mathcal{C} = \{v \in R^m : vC_i \leq 0 \ \ \forall i \in S_1, \ \ vC_i \geq 0 \ \ \forall i \in S_2\}.$$

Obviously λ belongs to $\mathcal{C}$. We will find a vector λ' in $\mathcal{C}$, such that $\bar{\lambda} = \lambda - \lambda'$ is integral and belongs to $\mathcal{C}$. $\mathcal{C}$ is a polyhedral cone, and is generated by a finite set of vectors $\mu_1, \ldots, \mu_t$, for some $t > 0$. (Observe that if $C = 0$, then $\mathcal{C} = R^m$, and $\mu_1, \ldots, \mu_t$ can be chosen to be the unit vectors times ± 1.) We can assume these vectors are integral (by scaling); we can also assume the coefficients of $\mu_1, \ldots, \mu_t$ have absolute value at most Ψ. Further, we can assume that $\mu_1, \ldots, \mu_k$ (here $k \leq m$) are linearly independent vectors such that

$$\lambda = \sum_{j=1}^{k} v_j \mu_j, \text{ with } v_j \in R,\ v_j > 0.$$

If $v_j < 1$ for $j = 1, \ldots, k$, then each coefficient of λ has absolute value less than $m\Psi$, and there is nothing to prove. If $v_j \geq 1$ for any $j \in \{1, \ldots, k\}$, then let

$$\lambda' = \sum_{j=1}^{k} \hat{v}_j \mu_j \Rightarrow \lambda - \lambda' = \sum_{j=1}^{k} \lfloor v_j \rfloor \mu_j,$$

where $\hat{v}_j = v_j - \lfloor v_j \rfloor$. Clearly λ' belongs to $\mathcal{C}$, and has coefficients with absolute value at most $m\Psi$. Also, $\lambda' \neq 0$ as $\lambda' = 0 \Rightarrow \lambda$ is integral $\Rightarrow \hat{\beta} = 0$. Let $\bar{\lambda} = \lambda - \lambda'$; obviously $\bar{\lambda}$ belongs to $\mathcal{C}$ and is integral. Further,

$$(\lambda C)^+ - (\lambda' C)^+ = (\bar{\lambda} C)^+.$$

Therefore $(\lambda', (\lambda' C)^+, \hat{\alpha}, \bar{\alpha} - \bar{\lambda} A, \hat{\beta}, \bar{\beta} - \bar{\lambda} b) \in \Pi$. It follows that the multipliers λ' lead to the MIR

$$(\lambda' C)^+ v + \hat{\alpha} x + \hat{\beta}(\bar{\alpha} - \bar{\lambda} A) x \geq \hat{\beta}(\bar{\beta} - \bar{\lambda} b + 1). \tag{21}$$

The rhs of the old MIR minus the rhs of the new MIR equals

$$\begin{aligned} \hat{\beta}\bar{\lambda} b = \hat{\beta}\bar{\lambda}(Ax^* + Cv^*) &= \hat{\beta}\bar{\lambda} A x^* + \hat{\beta}\bar{\lambda} C v^* \\ &\leq \hat{\beta}\bar{\lambda} A x^* + \hat{\beta}(\bar{\lambda} C)^+ v^*. \end{aligned} \tag{22}$$

The lhs of the old MIR (with v^*, x^* substituted) minus the lhs of the new MIR equals the last term in (22). Therefore the new MIR is violated by at least as much as the old MIR and the lemma follows. ∎

As the multipliers λ are bounded, there are only a finite number of choices for $\bar{\alpha}$ and $\bar{\beta}$ for non-redundant MIR cuts, see (23).

Theorem 11 *If there is an MIR inequality violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) for which $\hat{\beta}$ and the components of $\lambda, \hat{\alpha}$ are rational numbers with denominator equal to a subdeterminant of $A|C|b$, and each component of λ is contained in the interval $[-m\Psi, m\Psi]$.*

Proof Let (v^*, x^*) be a point in P^{LP} which violates an MIR cut. Let this MIR cut be defined by $(\lambda_o, c_o^+, \hat{\alpha}_o, \bar{\alpha}_o, \hat{\beta}_o, \bar{\beta}_o) \in \Pi$. By Lemma 10 , we can assume each component of λ_o lies in the range $(-m\Psi, m\Psi)$. Define $\Delta_o = \bar{\beta}_o + 1 - \bar{\alpha}_o^T x^*$. Then

$$\hat{\beta}_o \Delta_o - c_o^+ v^* - \hat{\alpha}_o x^* > 0.$$

Consider the following LP:

$$\begin{aligned} \max \hat{\beta}\Delta_o - c^+ v^* \quad &- \quad \hat{\alpha} x^* \\ (\lambda, c^+, \hat{\alpha}, \bar{\alpha}_o, \hat{\beta}, \bar{\beta}_o) \quad &\in \quad \Pi \\ -m\Psi \le \lambda_i \quad &\le \quad m\Psi \end{aligned}$$

Note that the objective is a linear function as Δ_o is fixed. Further, we have fixed the variables $\bar{\alpha}$ and $\bar{\beta}$ in the constraints defining Π. The bounds on λ come from Lemma 10, except that we weaken them to non-strict inequalities. This LP has at least one solution for $(\lambda, c^+, \hat{\alpha}, \hat{\beta})$ with positive objective value, namely $(\lambda_o, c_o^+, \hat{\alpha}_o, \hat{\beta}_o)$. Therefore a basic optimal solution of this LP has positive objective value. Consider the MIR cut defined by an optimal solution along with $\bar{\alpha}_o$ and $\bar{\beta}_o$. It is obviously an MIR cut with violation at least the violation of the original MIR cut. Therefore, $0 < \hat{\beta} < 1$. Further, it is easy to see that the LP constraints (other than the bounds on the variables) can be written as

$$\begin{bmatrix} A^T & -I & & \\ C^T & & -I & \\ b^T & & & -1 \end{bmatrix} \begin{pmatrix} \lambda \\ \hat{\alpha} \\ c^+ \\ \hat{\beta} \end{pmatrix} \begin{matrix} \le \\ \le \\ \ge \end{matrix} \begin{pmatrix} \bar{\alpha}_o \\ 0 \\ \bar{\beta}_o \end{pmatrix}.$$

The theorem follows. ∎

By Theorem 11, each non-redundant MIR inequality is defined by multipliers $\lambda = (\lambda_i)$ where λ_i is a rational number between $-m\Psi$ and $m\Psi$ with a denominator equal to a subdeterminant of $A|C|b$. Therefore the number of non-redundant MIR inequalities is finite.

Corollary 12 *The MIR closure of a polyhedral set P is a polyhedron.*

As the MIR closure equals the split closure, it follows that the split closure of a polyhedral set is again a polyhedron. Let the split closure of P be denoted by $P_S = \bigcap_{c \in Z^n, d \in Z} P_{(c,d)}$, where for $c \in Z^n$ and $d \in Z$,

$$P_{(c,d)} = conv\{(P \cap \{cx \le d\}) \cup (P \cap \{cx \ge d+1\})\}.$$

Lemma 10 gives a characterization of the useful disjunctions in the definition of the split closure. Define the vector $\mu \in R^m$ by

$$\mu_i = \begin{cases} m\Psi & \text{if } c_i \neq 0 \\ 1 & \text{if } c_i = 0 \end{cases}$$

Define

$$D = \{(c,d) \in Z^n \times Z : -\mu|A| \leq c \leq \mu|A|, \lfloor -\mu|b| \rfloor \leq d \leq \lfloor \mu|b| \rfloor\}. \quad (23)$$

D is clearly a finite set, and

$$P_S = \bigcap_{c \in Z^n, d \in Z} P_{(c,d)} = \bigcap_{(c,d) \in D} P_{(c,d)}.$$

To see this, let x^* be a point in P but not in P_S. Then some split cut, which is also an MIR cut, is violated by x^*. By Lemma 10 , there is an MIR cut with $-\mu < \lambda < \mu$ which is violated by x^*. This MIR cut has the form $(\lambda C)^+ v + \hat{\alpha} x + \hat{\beta} \bar{\alpha} x \geq \hat{\beta}(\bar{\beta} + 1)$, where $(\bar{\alpha}, \bar{\beta}) \in D$. Thus x^* does not belong to $P_{(\bar{\alpha},\bar{\beta})}$. This implies that

$$\bigcap_{(c,d) \in D} P_{(c,d)} \subseteq \bigcap_{c \in Z^n, d \in Z} P_{(c,d)},$$

and the two sets in the expression above are equal as the reverse inclusion is true by definition.

Theorem 13 *Let Φ be the least common multiple of all subdeterminants of $A|C|b$, $K = \{1, \ldots, log\Phi\}$, and $\mathcal{E} = \{\epsilon_k = 2^k/\Phi, \forall k \in K\}$. Then APPX-MIR-SEP is an exact model for finding violated MIR cuts.*

Proof By Theorem 11, $\hat{\beta}$ in a violated MIR cut can be assumed to be a rational number with a denominator equal to a subdeterminant of $A|C|b$ and therefore of Φ. But such a $\hat{\beta}$ is representable over $\mathcal{E}$. ■

5 Computational Issues

We next discus some practical issues that we encountered during our computational experiments.

5.1 Numerical Issues

Assume that the point (v^*, x^*) to be separated from the MIR closure of P is obtained using a practical LP solver, by optimizing a linear function over P^{LP}. Then (v^*, x^*) will only approximately satisfy the linear inequalities defining P^{LP}, i.e., some of these inequalities will be violated by small amounts (usually at most 10^{-6}). This can result in MIR-SEP returning cuts which are not useful, or not really violated. For example, if $v_i^* < 0$ for some index i, then the objective function of MIR-SEP, $\hat{\beta}\Delta - c^+ v^* - \hat{\alpha} x^*$, can be made positive by setting $\lambda = 0$, and letting c_i^+ be a large positive number. Clearly, this choice of λ does not yield a violated MIR cut.

It is also possible that some equation in $Cv + Ax = b$ is not satisfied exactly. Assume $c_i v + a_i x = b_i$ is the ith equation in $Cv + Ax = b$, and assume that $c_i v^* + a_i x^* < b_i$. In such a case, MIR-SEP would choose a large positive value of λ_i, and the resulting base inequality $c^+ v + (\hat{\alpha} + \bar{\alpha})x \geq \hat{\beta} + \bar{\beta}$ would actually be violated by (v^*, x^*). The resulting MIR cut would also be violated, but would not necessarily be violated if we moved to another approximately feasible solution (v', x') of $Cv + Ax = b$ with $c_i v' + a_i x' \geq b_i$. Thus, approximate solutions of $Cv + Ax = b, v, x \geq 0$ create numerical problems.

We deal with these numerical issues by modifying (v^*, x^*) and b to get a truly feasible solution of a different set of constraints. We let $v' = \max\{v^*, \mathbf{0}\}$, and $x' = \max\{x^*, \mathbf{0}\}$, for non-negative variables and then define b' as $Cv' + Ax'$. We then use APPX-MIR-SEP to separate (v', x') from the MIR closure of $Cv + Ax = b', v, x \geq 0, x \in Z$. We use the multipliers λ in the solution of APPX-MIR-SEP to compute an MIR cut for P.

This approach helps in dealing with the numerical issues mentioned above, however, in some cases the cut derived from (v', x') is not violated by (v^*, x^*). This does not happen frequently as the point (v', x') is usually very close to (v^*, x^*).

5.2 Reducing the size of the separation problem

The number of integer variables in APPX-MIR-SEP equals the number of integer variables in P plus the number of variables π_i used in linearizing the objective, and thus solving APPX-MIR-SEP could be as hard as solving the original MIP. However, one can often find violated MIR cuts by solving an MIP with fewer integer variables than in P.

In APPX-MIR-SEP, the variables $c_i^+, \hat{a}_j, \bar{a}_j$ corresponding to $v_i^* = 0$ and $x_j^* = 0$ do not contribute to the objective. One can remove them and the corresponding constraints from APPX-MIR-SEP, solve the reduced APPX-MIR-SEP, and then simply compute their values from the multipliers λ in an optimal solution to the reduced model. The resulting cut would have the same violation as the cut in the reduced set of variables. Further, if $x_j^* = 0$ for an index j, and P has an upper bound constraint for x_j, say $x_j \leq 1$, then the component of λ corresponding to $x_j \leq 1$ can be assumed to be 0. Finally, if $x_j^* = 1$ and $x_j \leq 1$ for points in P, we can replace x_j by $1 - x_j'$ where $0 \leq x_j' \leq 1$, derive an MIR cut for the modified system of constraints (here (v^*, x^*) maps to a point with $x_j' = 0$) and get an MIR cut for P by replacing x'_j by $1 - x_j$.

Consider the situation where $Cv + Ax = b$ has many more variables than constraints. An optimal basic solution (v^*, x^*) of the linear relaxation of P would have at most m non-zero variables, where m is the number of constraints in $Cv + Ax = b$. We can solve an optimization problem with at most $2m + 2\bar{k}$ variables ($\bar{k}$ stands for the size of the set $\mathcal{E}$) to obtain a violated MIR cut if any. For some of the problems in MIPLIB 3.0, this approach is quite crucial in allowing us to use APPX-MIR-SEP at all. For example, the MIP **nw04** has 36 constraints, and over 87000 0-1 variables. If we let $\bar{k} = 5$, the first separation MIP has at most 36+5 integer variables. Even after adding a few hundred MIR cuts, the reduced separation model is still vastly smaller (and solvable) compared to the original separation model.

5.3 Finding good MIR cuts

Given a point $(v^*, x^*) \in P^{LP}$, the separation model MIR-SEP is guaranteed to produce the most violated MIR inequality, if there is one. Similarly, based on Theorem 7, the approximate model is guaranteed to produce an MIR inequality with violation slightly less than the most violated inequality. In both cases violation of a cut defined by $\kappa = (c^+, \hat{\alpha}, \bar{\alpha}, \hat{\beta}, \bar{\beta}) \in \bar{\Pi}$ is defined to be

$$\eta(\kappa) \;=\; \hat{\beta}\Delta - c^+ v^* - \hat{\alpha} x^*$$

where $\Delta = \bar{\beta} + 1 - \bar{\alpha} x^*$. Unfortunately, there is no guarantee that MIR cuts with maximum values of $\eta(\kappa)$ would actually be the most effective MIR cuts in practice.

Example 14 *Consider separating* $(x^*, y^*) = (0.001, 0.5)$ *from the MIR closure of*

$$P = \{x, y \in Z \;:\; 100x - y \geq -0.4,\; 100x + y \geq 0.6\}.$$

First we convert the inequalities defining P *to equations by adding slacks:*

$$\begin{aligned} 100x - y - s_1 &= -0.4, \quad (A) \\ 100x + y - s_2 &= 0.6, \quad (B) \end{aligned}$$

and construct the related point $(x^*, y^*, s_1^*, s_2^*) = (0.001, 0.5, 0, 0)$ *to be separated.*

The base inequality $s_1/2 + y \geq 1/2$ *can obtained by taking* $\lambda_A = -1/2$ *and* $\lambda_B = 1/2$. *The corresponding cut* $s_1/2 + y/2 \geq 1/2$ *has violation* 0.25. *This inequality can also be written as* $x \geq 0.006$ *after substituting out* s_1.

Another base inequality $x \geq 0.001$ *can be obtained by taking* $\lambda_A = \lambda_B = 1/200$. *The resulting MIR cut* $0.001x \geq 0.001$ *(or* $x \geq 1$*) has violation less than* 0.001. ∎

Another possible measure of violation for MIR inequalities is

$$\eta'(\kappa) \ = \ \Delta - \frac{c^+v^* + \hat{\alpha}x^*}{\hat{\beta}};$$

see [22]. In the previous example, η' of $x \geq 1$ is 0.999, whereas η' of $x \geq 0.006$ is only 0.006. This suggests that η' may be a more effective measure than η. However, consider the base inequality $s_1+x+2y \geq 1.001$ obtained by taking $\lambda_A = (-1+1/200)$ and $\lambda_B = (1+1/200)$. The resulting MIR cut $s_1 + 0.001(x + 2y) \geq 0.002$ has a violation of 0.999. However, this inequality is even weaker than $x \geq 0.006$ after substituting out s_1.

Another problem with both these measures is that adding integral multiples of tight constraints without continuous variables to the original base inequality does not change the violation of the resulting MIR cut (see the proof of Lemma 8). For example, if $x^* = 0.5$, the base inequalities $x \geq .5$ and $11x \geq 5.5$ lead to MIR cuts with identical violation for each measure. The first inequality leads to $x \geq 1$ and the second one to $11x \geq 6$ which is clearly weaker than $x \geq 1$.

One could normalize the cut coefficients in some way (e.g., by dividing the cut by the norm of the cut coefficients) and choose the most violated cut after normalization. This approach, though frequently used, is harder to incorporate into a linear separation model.

6 Computational experiments

In this section we discuss our computational experience with the approximate separation model APPX-MIR-SEP. The goal is to approximately optimize over the MIR closure of a given MIP instance by repeatedly solving APPX-MIR-SEP to get violated MIR cuts. The general idea is to start off with the continuous relaxation of the given MIP. Then the following separation step is repeated. APPX-MIR-SEP is solved to find one or more MIR inequalities violated by the optimal solution of the current relaxation of the MIP, and the current relaxation is strengthened by adding these cuts. Even though this procedure is guaranteed to terminate after a finite number of iterations (for any fixed precision), in practice, there is no guarantee that we can actually optimize over the (approximate) MIR closure in a reasonable amount of time. Our approach, therefore, should be considered as a heuristic that tries to find good bounds.

We next discuss some practical issues and heuristic ideas to obtain good bounds faster. Finally, we present numerical results.

6.1 Modeling Issues

Practical MIPs, such as those in MIPLIB 3.0, do not necessarily have the same form as P. Many of the variables have upper bounds in addition to the lower bounds of 0. We simply treat the upper bound constraints as general linear constraints. Further, some of the variables have negative lower bounds. For an integer variable x_i bounded below by l_i, where l_i is a negative integer, we "shift" it by performing the substitution $x'_i = x_i - l_i$. Finally, if an integer variable x_i is free, we replace the constraint $\hat{\alpha}_i + \bar{\alpha}_i \geq (\lambda A)_i$ in (10) by $\bar{\alpha}_i = (\lambda A)_i$. If a continuous variable v_j is free, we replace the constraints $c_j^+ \geq (\lambda C)_i$, $c_j^+ \geq 0$ in (10) by $0 = (\lambda C)_i$. See Section 2.3 for an explanation of why the above modifications are either necessary (in the case of free variables) or do not change the MIR closure (in the case of shifted variables).

6.2 Separation Heuristics

After a preliminary round of testing, we realized that a naive cutting plane algorithm that only solves APPX-MIR-SEP to find violated inequalities reduces the integrality gap very slowly. To speed up this process we implemented several ideas which can essentially be seen as finding heuristic solutions to APPX-MIR-SEP. These solutions might be sub-optimal with respect to the objective function of APPX-MIR-SEP, but they help increase the performance of the algorithm significantly. As discussed in Section 5.3, the objective function used in APPX-MIR-SEP does not necessarily help produce the most effective cuts.

We next describe the components of the final cutting plane algorithm that we have implemented. The final algorithm is the following:

* Add MIR cuts of the form $x_i \leq \bar{\beta}$ or $x_i \geq \bar{\beta}$ for some integer $\bar{\beta}$ (preprocessing)
* Add Gomory mixed-integer cuts cuts to the initial LP relaxation
* Repeat
    - Use heuristics to find MIR cuts from the formulation rows
    - Solve a restriction of APPX-MIR-SEP to find cuts based on pure integer base inequalities
    - Solve APPX-MIR-SEP with limits on the enumeration process

  Until no violated cuts are found or time is up.

6.2.1 Preprocessing

We take a subset S of integer variables, and for every x_i with $i \in S$, we solve LPs to maximize and minimize x_i for $x \in P^{LP}$. If $\beta_1 \le x_i \le \beta_2$, then $x_i \ge \lceil \beta_1 \rceil$ and $x_i \le \lfloor \beta_2 \rfloor$ are Chvátal-Gomory cuts and therefore MIR cuts. This simple bound-strengthening procedure seems to be useful in a few MIPLIB 3.0 instances, especially **p0282**.

6.2.2 Gomory mixed-integer cuts

Gomory mixed-integer cuts for the initial LP-relaxation of the MIP are known to be MIR inequalities [18] where the multipliers used to aggregate the rows of the formulation are obtained from the inverse of the optimal basis. The base inequalities for these cuts are readily available after solving the initial relaxation and the resulting cuts are known to be effective in reducing the integrality gap significantly [3]. We use these cuts only in the first iteration of the cutting plane algorithm as the basis in the following iterations might include cuts from earlier iterations and therefore the resulting Gomory mixed-integer cuts would not necessarily be rank 1 MIR cuts, i.e., MIR cuts derived only from the constraints defining P.

6.2.3 Cuts based on the rows of the formulation

Another heuristic considers rows of the formulation, one at a time, and obtains base inequalities by scaling them. Variables that have upper bounds are sometimes complemented using the bound constraints. More precisely, for a given row of the formulation and a given fractional solution, this procedure generates base inequalities by dividing the row by the coefficient of an integer variable which is currently fractional. Variables with upper bounds are complemented if their current value is closer to their upper bound then the lower bound. After writing the MIR cut, complemented variables are un-complemented to obtain a cut in the original space. This procedure was used in [14] and the authors observed that it produces effective MIR cuts.

We also note that in [18] Marchand and Wolsey describe a more sophisticated procedure that produces violated MIR inequalities by combining several rows as well as complementing variables. They observe that base inequalities obtained by combining only a few rows of the formulation can lead to effective MIR cuts. The procedure we use is motivated by their work and can be considered as a simplification of their algorithm. We noticed that even using a single row of the formulation leads to MIR inequalities that reduce the integrality gap significantly for some instances.

6.2.4 Cuts based on pure integer base inequalities

One way to generate effective MIR cuts is to concentrate on base inequalities that only contain integer variables. To obtain such base inequalities, the multiplier vector λ, used to aggregate the rows of the formulation, is required to satisfy $\lambda C \leq 0$ so that $(\lambda C)^+ = 0$. This can be achieved by fixing c^+ to zero in APPX-MIR-SEP. Note that if the original formulation has inequality constraints, the slack variables associated with these constraints are also treated as continuous variables. Therefore, multipliers associated with these rows are restricted to be non-negative for "$\geq$" constraints and non-positive for "$\leq$" constraints.

This heuristic in a way mimics the procedure to generate the so-called projected Chvátal-Gomory (pro-CG) cuts [7] for mixed integer programs. These cuts are shown to be effective in closing the integrality gap for some general mixed integer programs. Given a multiplier vector λ such that $(\lambda C)^+ = 0$, if we denote the resulting base inequality by $\alpha x = \beta$, where $\alpha = \lambda A$ and $\beta = \lambda b$, the associated pro-CG cut is

$$\sum_{i \in I} \lceil \alpha_i \rceil x_i \;\geq\; \lceil \beta \rceil$$

and the associated MIR cut is

$$\sum_{i \in I} (\min\{\hat{\beta}, \hat{\alpha}_i\} + \hat{\beta} \lfloor \alpha_i \rfloor) x_i \;\geq\; \hat{\beta} \lceil \beta \rceil, \tag{24}$$

where $\hat{\alpha}_i$ and $\hat{\beta}$ denote the fractional part of α_i and β respectively. In other words, MIR cuts that only contain integer variables can be seen as a strengthening of pro-CG cuts.

In our implementation, we also set $\hat{\alpha}$ to zero in the separation model, and divide the objective by $\hat{\beta}$. In such a case, the objective is to maximize Δ alone. We do not then need the variables π_i or Δ_i, as we do not need to model $\hat{\beta}\Delta$. After solving this simplified model (we call this INT-SEP), we use the multipliers λ to write the cut (24). In other words, we find the most violated Chvátal-Gomory (CG) cut (in the case of pure integer programs) or pro-CG cuts (in the case of mixed-integer programs), and then write the corresponding MIR cut, instead of directly finding the most violated MIR cut. The motivation for this simplification is that the resulting model was shown to be efficiently solvable for pure integer programs in [17] and for mixed-integer programs in [7].

6.2.5 Cuts generated by APPX-MIR-SEP

The only parameter which must be specified for the definition and solution of APPX-MIR-SEP is the value of $\bar{k}$, i.e., the parameter responsible for the degree of approximation we use for $\hat{\beta}$. In our computational experiments, we use $\bar{k} = 6$ which is a good compromise between computational efficiency and precision. In such a way, as proved in Theorem 7, our approximate model is guaranteed to find a cut violated by at least $1/64 = .015625$ which can be considered a reasonable threshold value to distinguish violated cuts.

6.3 Piloting the black-box MIP solver

A few tricks can be used to force the black-box MIP solver, in our experiments `ILOG-Cplex` 10.0.1, to return good heuristic solutions of both INT-SEP and APPX-MIR-SEP. Indeed, it has to be stressed that we do not need to solve any of the separation problems to optimality in our cutting plane algorithm but, eventually, a final APPX-MIR-SEP so as to prove that no MIR inequality violated more than the threshold exists, i.e., the MIR closure has been approximately computed. In all other cases, a feasible solution (or better a set of feasible solutions) of the separation problem at hand corresponds to a cutting plane separating the current fractional solution from P, thus the procedure can be iterated.

To find a number of MIR cuts quickly we activate the *RINS* heuristic [13] of `ILOG-Cplex` after every 100 nodes. This approach is similar to [17] and [7]. In addition, to control the runtime in each iteration, we impose the following node limits for the enumeration tree.

- For INT-SEP, the initial node limit is set to 10,000 if no MIR cuts have been found by other heuristics, else, it is set to 1,000. After each integral solution, this limit is reset to 1,000 if the violation is less than 0.2 and 100 nodes otherwise.
- For APPX-MIR-SEP, there is no initial node limit if no MIR cuts have been found by other heuristics, else, it is set to 1,000. After each integral solution, this limit is reset to 1,000 if the violation is less than 0.1 and 100 nodes otherwise.

6.4 Computational results

In the following tables, we give our bounds for problem instances in the MIPLIB 3.0[5] library obtained by running our algorithm with a time limit of one hour. In the first table, we compare our bounds with those obtained by 20 minutes of projected CG cuts separation in [7], and with the bounds obtained by Balas and Saxena, using their MIP model for split cut separation [4]. In the second table we compare our bounds after one hour with bounds obtained for the Chvátal closure after three hours in [17], and the split cut bounds from [4]. In both tables, the percent gap closed refers to the fraction of the integrality gap closed after adding MIR cuts.

For a number of problems, we terminate prematurely because of numerical issues. For example, for **harp2**, ater several iterations APPX-MIR-SEP returned a cut which was not

instance	$\lvert I \rvert$	# iter	# cuts	% gap closed	time MIR	% CG gap closed	time CG	% gap split	time split
air03	10,757	1	36	100.00	1	100.0	1	100.00	3
air04	8,904	5	294	9.18	3,600	30.4	43,200	91.23	864,360
air05	7,195	8	238	12.08	3,600	35.3	43,200	61.98	24,156
cap6000	6,000	120	334	50.55	3,600	22.5	43,200	65.17	1,260
fast0507	63,009	14	330	1.66	3,600	5.3	43,200	19.08	304,331
gt2	188	83	254	98.21	664	91.0	10,800	98.37	599
harp2	2,993	122	796	59.99	260	49.5	43,200	46.98	7,671
l152lav	1,989	57	214	12.66	3,600	59.6	10,800	95.20	496,652
lseu	89	103	306	92.28	3,600	93.3	175	93.75	32,281
mitre	10,724	12	158	100.00	380	16.2	10,800	100.00	5,330
mod008	319	41	173	100.00	11	100.0	12	99.98	85
mod010	2,655	1	39	100.00	0	100.0	1	100.00	264
nw04	87,482	100	301	95.16	3,600	100.0	509	100.00	996
p0033	33	27	115	87.42	2,179	85.3	16	87.42	429
p0201	201	394	1357	74.43	3,600	60.6	10,800	74.93	31,595
p0282	282	223	1474	99.60	3,600	99.9	10,800	99.99	58,052
p0548	548	255	1309	96.35	3,600	62.4	10,800	99.42	9,968
p2756	2,756	83	717	35.32	3,600	42.6	43,200	99.90	12,673
seymour	1,372	1	559	8.35	3,600	33.0	43,200	61.52	775,116
stein27	27	70	325	0.00	3,600	0.0	521	0.00	8,163
stein45	45	420	1930	0.00	3,600	0.0	10,800	0.00	27,624

Table 1: IPs of the MIPLIB 3.0.

violated by the point to be separated while the other separation heuristics did not return any cuts. For **p0033**, we terminate because APPX-MIR-SEP has no solution, and thus there does not exist an MIR cut which is violated by more than 1/64.

Our computed bounds are clearly sensitive to the MIP solver used in solving APPX-MIR-SEP, except where our procedure terminates. Our procedure is also very sensitive to the exact parameters used while solving APPX-MIR-SEP. In Table 3, we give the bounds obtained after one hour of computing time for a few instances if we just turn off the RINS heuristic while solving APPX-MIR-SEP. For **qnet1**, we get a worse bound, and for the remaining four problems, we get better bounds. In the case of **p2756**, we get a substantially better bound, 79.78% versus only 35.34% if we turn on RINS. Notice that because of the time spent on RINS, we only perform 83 iterations and generate 717 cuts for **p2756** in Table 1; the corresponding numbers in Table 3 are 313 and 1640.

instance	$\|I\|$	$\|J\|$	# iter	# cuts	% gap closed	time MIR	% CG gap closed	time CG	% gap split	time split
10teams	1,800	225	338	3341	100.00	3,600	57.14	1,200	100.00	90
arki001	538	850	14	124	33.93	3,600	28.04	1,200	83.05	193,536
bell3a	71	62	21	166	98.69	3,600	48.10	65	65.35	102
bell5	58	46	105	608	93.13	3,600	91.73	4	91.03	2,233
blend2	264	89	723	3991	32.18	3,600	36.40	1,200	46.52	552
dano3mip	552	13,321	1	124	0.10	3,600	0.00	1,200	0.22	73,835
danoint	56	465	501	2480	1.74	3,600	0.01	1,200	8.20	147,427
dcmulti	75	473	480	4527	98.53	3,600	47.25	1,200	100.00	2,154
egout	55	86	37	324	100.00	31	81.77	7	100.00	18,179
fiber	1,254	44	98	408	96.00	3,600	4.83	1,200	99.68	163,802
fixnet6	378	500	761	4927	94.47	3,600	67.51	43	99.75	19,577
flugpl	11	7	11	26	93.68	3,600	19.19	1,200	100.00	26
gen	150	720	11	127	100.00	16	86.60	1,200	100.00	46
gesa2	408	816	433	1594	99.81	3,600	94.84	1,200	99.02	22,808
gesa2_o	720	504	131	916	97.74	3,600	94.93	1,200	99.97	8,861
gesa3	384	768	464	1680	81.84	3,600	58.96	1,200	95.81	30,591
gesa3_o	672	480	344	1278	69.74	3,600	64.53	1,200	95.20	6,530
khb05250	24	1,326	65	521	100.00	113	4.70	3	100.00	33
markshare1	50	12	4781	90628	0.00	3,600	0.00	1,200	0.00	1,330
markshare2	60	14	4612	87613	0.00	3,600	0.00	1,200	0.00	3,277
mas74	150	1	1	12	6.68	0	0.00	0	14.02	1,661
mas76	150	1	1	11	6.45	0	0.00	0	26.52	4,172
misc03	159	1	143	727	33.65	3,600	34.92	1,200	51.70	18,359
misc06	112	1,696	112	1125	99.84	376	0.00	0	100.00	229
misc07	259	1	432	2135	11.03	3,600	3.86	1,200	20.11	41,453
mod011	96	10,862	253	1781	17.30	3,600	0.00	0	72.44	86,385
modglob	98	324	357	2645	60.77	254	0.00	0	92.18	1,594
mkc	5,323	2	112	2745	12.18	3,600	1.27	1,200	36.16	51,519
pk1	55	31	4229	22088	0.00	3,600	0.00	0	0.00	55
pp08a	64	176	246	1400	95.97	3,600	4.32	1,200	97.03	12,482
pp08aCUTS	64	176	143	687	62.99	3,600	0.68	1,200	95.81	5,666
qiu	48	792	847	2243	28.41	3,600	10.71	1,200	77.51	200,354
qnet1	1,417	124	182	805	64.60	3,600	7.32	1,200	100.00	21,498
qnet1_o	1,417	124	90	409	83.78	3,600	8.61	1,200	100.00	5,312
rentacar	55	9,502	92	281	23.41	3,600	0.00	5	0.00	—
rgn	100	80	114	666	99.81	1,200	0.00	0	100.00	222
rout	315	241	2225	17230	16.07	3,600	0.03	1,200	70.70	464,634
set1ch	240	472	156	694	63.39	3,600	51.41	34	89.74	10,768
swath	6,724	81	167	1421	33.96	3,600	7.68	1,200	28.51	2,420
vpm1	168	210	53	241	99.93	158	100.00	15	100.00	5,010
vpm2	168	210	74	314	71.48	224	62.86	1,022	81.05	6,012

Table 2: MILPs of the MIPLIB 3.0.

instance	# iter	# cuts	% gap closed	time MIR
misc03	316	1139	38.95	582
p2756	313	1640	79.78	3,600
qnet1	77	367	57.47	3,600
rentacar	2	26	33.37	16
set1ch	306	1011	82.15	3,600

Table 3: Instances where turning off RINS helps

Our results confirm what other authors have already noticed, i.e., that the MIR closure indeed provides a very tight approximation of the optimal solution of the problems in MIPLIB 3.0. Most of the times we are able to compute bounds comparable with the ones already reported in [4, 7, 17] in a much shorter computing time although sometimes a very large computational effort seems necessary to obtain tight approximations. In a few cases, namely **bell3a, bell5, harp2, swath** and **gesa2**, we have been able to improve over the best bound known so far. Of course, 1 hour of CPU time to strengthen the initial formulation can be too much, but as shown in [4, 17], in a few cases such a preprocessing step allows the solution of hard unsolved problems. We believe that speeding up the MIR separation procedure would be a potentially valuable step.

Acknowledgments

Part of this research was carried out when the third author was Herman Goldstine Fellow of the IBM T.J. Watson Research Center, whose support is strongly acknowledged.

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