

Covering models with time-dependent demand ^{*}

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Abstract

In this paper a covering model for locating facilities with time-dependent demand is introduced. Not only the facility locations, but also the instants at which such facilities become operative, are considered as decision variables in order to determine the maximal-profit decision.

Expressed as a mixed nonlinear integer program, structural properties are derived for particular demand patterns, and a metaheuristic procedure is proposed as optimization tool.

Key Words: covering models, competitive location, dynamic demand, maximal profit, variable neighbourhood search.

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1 Introduction

Maximum covering problems have a long tradition in the literature of Operations Research and Management Science, see e.g. [1, 3, 6, 32, 33, 34] and the references therein. In the basic model, p facilities are to be located to serve a set V of users; each user $v \in V$ has a demand ω_v , representing its population, buying power, etc.; the demand of user v is considered to be *covered* if there exists a facility open at a distance not greater than a threshold value \underline{d}_v ; the goal is to determine the location of the p facilities maximizing the demand covered.

Maximum covering models have been used both for public-sector problems, mainly for locating emergency services, e.g. [24], and also for locating facilities in a competitive environment, [4, 9, 12, 13, 15, 18, 29], with the following motivation: a firm is planning to locate p facilities to compete against a set of already operating facilities in order to maximize its profit, usually equivalent to maximizing the market share; market capture is modeled by the *binary* choice rule: consumers demand is fully captured by the closest facility, or, more generally, by the most attractive facility, where attractiveness is measured by a function decreasing in distance and price, and increasing in certain facility attributes such as size, e.g. [7, 8, 20, 21, 27, 28, 30, 35].

Several attempts have been made to model maximum capture location problems in different metric spaces (a discrete set, a transportation network or the plane), or, in the case of location with competition, different choice rules (e.g. the above-mentioned binary rule, as well as rules relying upon the assumption that consumer demand is split into the different facilities, each capturing a fraction of the demand, this amount of demand being decreasing in distance, etc.). However, most models assume that demand remains constant along the full planning horizon, which may be a rather unrealistic assumption for new goods or high seasonality products. See e.g. [2, 11, 14, 16, 17, 19, 25, 31, 36, 37, 38] for facility location models which do accommodate time-dependent demand, and [5, 14, 19, 22, 38, 39] for models in which other aspects such as transportation costs, travel distances and times, facilities capacities, prices, number of facilities, etc., are

considered to be time-dependent.

In this paper we introduce a covering model for locating facilities in which demand is time-dependent. This implies that, as e.g. in [11], not only the facility sites, but also the instants at which facilities become operative, are decision variables, to be chosen to optimize a certain performance measure.

The remaining of the paper is structured as follows: in Section 2 the model is formally introduced and then expressed as a nonlinear mixed integer program. General properties concerning the optimal sites and opening times are derived in Section 3. These results are strengthened in Section 4, in which different particular instances of demand patterns are considered, namely, the cases in which the demand of a given consumer is constant, varies linearly in time, is increasing or is decreasing. In Section 5, a heuristic procedure is proposed to determine the optimal policy, and a numerical example is provided. The paper ends with a discussion of possible extensions and future lines of research in Section 6.

2 The model and its formulation

We consider a set of users $V = \{v_i\}_{i=1}^n$, to be served by a set of at most q facilities within the set $F = \{f_j\}_{j=1}^m$ of candidate sites. Location is assumed to be sequential, in the sense that there exists a time interval $[0, T]$, $0 < T < +\infty$, within which the facilities will start to be operating; both the sites for the facilities and the times at which they will be located must be determined. We assume that facilities can be open at any instant time t within $[0, T]$, but, once a facility is open at time t , it will remain operating in the whole interval $[t, T]$, which may be a realistic assumption when set-up costs are important.

Demand is assumed to be time-dependent: In any given infinitesimal time interval $[t, t + \Delta t]$, demand at $v \in V$ is of the form $\omega_v(t)\Delta t$, where $\omega_v(t)$ is called hereafter demand rate function.

Demand capture is modeled via a binary rule: given $v \in V$, if, at time $t \in [0, T]$, some $f \in F$ is such that the travel distance $d(v, f)$ from v to f is smaller than the threshold value \underline{d}_v and a facility at f is open,

then the demand of v at instant t will be fully captured; else, such demand will be lost. In other words, if, for each $v \in V$ we denote by A_v the set of candidate sites which *cover* v , i.e., which will capture the demand from v , we have that

$$A_v = \{f \in F : d(v, f) < \underline{d}_v\}.$$

The interpretation of the parameter \underline{d}_v depends on the nature of the problem. For instance, for locating emergency units, \underline{d}_v is the highest travel distance considered to be acceptable for user v . For locating competitive facilities, we may assume that, previous to our entrance in the market, a set E of competing stores are already located; demand is assumed to be inelastic, and consumer preferences follow a binary rule oriented to the old facilities: customers use all their buying power in the closest facility, (prices are not influenced by producers or consumers, firms serve the same type of product), ties in the distance between customers and firms are broken in favor of the existing facilities. In other words, for each user v , \underline{d}_v is given by

$$\underline{d}_v = \min_{f \in E} d(v, f). \quad (1)$$

For simplicity we assume that the threshold values \underline{d}_v remain constant in $[0, T]$, although the results in this paper extend to the case in which the \underline{d}_v are also time-dependent, but independent of the previous locational decisions. In the competing-facilities case, this implies that competing firms do not react opening new facilities in $[0, T]$. Although it is a strong assumption, it may apply when opening facilities is regulated by rules; moreover, our model can anyway be considered as the first step to addressing the more realistic model in which strategies of the competing firms (in sites and times) are affected by the strategies of their competitors.

The net profit margin at time t per demand unit is $\rho(t)$, and thus the revenue generated by v within the infinitesimal time interval $[t, t + \Delta t]$, is given by $\rho(t)\omega_v(t)\Delta t$, if some $f \in A_v$ is operating at t , and zero otherwise.

Fixed operating costs of a facility at $f \in F$ within the infinitesimal time interval $[t, t + \Delta t]$ have the

form $c_f(t)\Delta t$.

Hereafter we assume that functions ω_v, ρ and c_f are continuous on the interval $[0, T]$.

We seek the sites and opening times for a set of at most q facilities maximizing in $[0, T]$ the *total profit*, i.e. revenue generated by demand points minus operating costs.

To express this as a mathematical program, we will first consider the much easier case in which the opening times are fixed, thus yielding a dynamic location problem, similar to those addressed e.g. in [10, 38, 39], and later these will also be considered to be decision variables.

Suppose then that facilities are scheduled to start operating at *fixed* instants τ_1, \dots, τ_r , with

$$0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_r \leq \tau_{r+1} = T,$$

and $0 \leq r \leq q$.

For $f \in F$, $v \in V$ and k with $1 \leq k \leq r$, define the binary variables

$$y_f^k = \begin{cases} 1, & \text{if the facility at } f \text{ is operating in the interval } [\tau_k, T] \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

$$x_v^k = \begin{cases} 1, & \text{if } v \text{ is covered by a facility in the interval } [\tau_k, T] \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Obviously, once the values of the variables y_f^k are fixed, those for the variables x_v^k become also fixed:

$$x_v^k = 1 \quad \text{iff} \quad y_f^k = 1 \quad \text{for some } f \in A_v. \quad (4)$$

However, in order to come up with a linear program, these are also considered to be decision variables.

With this notation, for opening instants τ_1, \dots, τ_r fixed, the covering location problem to be solved is the following linear integer program

$$\Pi_r(\tau_1, \dots, \tau_r) = \max_{\mathbf{x}, \mathbf{y}} \sum_{k=1}^r \left\{ \sum_{v \in V} x_v^k \int_{\tau_k}^{\tau_{k+1}} \rho(t) w_v(t) dt - \sum_{f \in F} y_f^k \int_{\tau_k}^{\tau_{k+1}} c_f(t) dt \right\}$$

$$\sum_{f \in F} y_f^r \leq q \quad (5)$$

$$x_v^k \leq \sum_{f \in A_v} y_f^k, \quad \forall v \in V, 1 \leq k \leq r \quad (6)$$

$$y_f^{k-1} \leq y_f^k, \quad \forall f \in F, 2 \leq k \leq r \quad (7)$$

$$y_f^k, x_v^k \in \{0, 1\}, \quad \forall f \in F, \forall v \in V, 1 \leq k \leq r. \quad (8)$$

We briefly discuss the correctness of the formulation. For the objective, within the time interval $[\tau_0, \tau_1)$, no benefit or cost is incurred, since no plants are operating. The interval $[\tau_1, \tau_{r+1}]$ is split into the subintervals $[\tau_k, \tau_{k+1})$, $k = 1, \dots, r$. Within an interval $[\tau_k, \tau_{k+1}]$, the total revenue obtained from consumer at v is $x_v^k \int_{\tau_k}^{\tau_{k+1}} \rho(t) w_v(t) dt$, whereas the total operating cost incurred by facility at f is given by $y_f^k \int_{\tau_k}^{\tau_{k+1}} c_f(t) dt$.

Constraint (5) imposes that the number of open facilities cannot exceed q . Constraints (6) impose that, if $x_v^k = 1$, i.e., if v is counted as captured in time interval $[\tau_k, T]$, then there must exist at least one facility $f \in A_v$ operating within such interval.

With constraints (7) we express that, if plant f is operating within $[\tau_{k-1}, T]$, then it must also be so in $[\tau_k, T]$ (recall that closing of facilities is not allowed).

Finally, constraints (8) express the binary character of the variables x_v^k and y_f^k for $v \in V$, $f \in F$ and $1 \leq k \leq r$.

Since the variables x_v^k and y_f^k are binary and the functions $\rho(t)w_v(t)$ and $c_f(t)$ are, by assumption, continuous, the optimization problem above is well defined, and its optimal value $\Pi_r(\tau_1, \dots, \tau_r)$ is attained.

The continuity of $\rho(t)w_v(t)$ and $c_f(t)$ enables us also to define their primitives,

$$\begin{aligned} g_v(t) &= \int_0^t \rho(s) w_v(s) ds \\ h_f(t) &= \int_0^t c_f(s) ds, \end{aligned}$$

which are differentiable functions. Moreover, for each k , $1 \leq k \leq r$, one has

$$\int_{\tau_k}^{\tau_{k+1}} \rho(t) w_v(t) dt = g_v(\tau_{k+1}) - g_v(\tau_k)$$

$$\int_{\tau_k}^{\tau_{k+1}} c_f(t) dt = h_f(\tau_{k+1}) - h_f(\tau_k).$$

With this notation, we can express the optimal profit $\Pi_r(\tau_1, \dots, \tau_r)$ for opening times τ_1, \dots, τ_r as

$$\Pi_r(\tau_1, \dots, \tau_r) = \max_{(\mathbf{x}, \mathbf{y}) \in S_r} \sum_{k=1}^r \left\{ \sum_{v \in V} x_v^k [g_v(\tau_{k+1}) - g_v(\tau_k)] - \sum_{f \in F} y_f^k [h_f(\tau_{k+1}) - h_f(\tau_k)] \right\}, \quad (9)$$

where S_r denotes the set of pairs (\mathbf{x}, \mathbf{y}) satisfying constraints (5)-(8).

Considering times τ_1, \dots, τ_r as decision variables to be optimized, yields the optimal planning for locating at most q facilities in *at most* r different opening instants. Indeed, for $\tau_1 < \dots < \tau_r$ the facilities will become operative in exactly r different instants. Allowing different τ_i to coincide collapses the number of different instants considered. In other words, the problem of determining sites for at most q facilities, to become operative in at most r different instants, can be written as the bilevel problem

$$\begin{aligned} \max \quad & \Pi_r(\tau_1, \dots, \tau_r) \\ & 0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_r \leq T, \end{aligned} \quad (10)$$

or equivalently as the bilevel problem

$$\begin{aligned} \max_{(\mathbf{x}, \mathbf{y}) \in S_r} \quad & \max \sum_{k=1}^r \left\{ \sum_{v \in V} x_v^k [g_v(\tau_{k+1}) - g_v(\tau_k)] - \sum_{f \in F} y_f^k [h_f(\tau_{k+1}) - h_f(\tau_k)] \right\} \\ & 0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_r \leq T. \end{aligned} \quad (11)$$

Since (10) and (11) are simply different writings of the same problem, we will use one or other at our convenience.

Our problem, namely, the determination of optimal locations and opening times for at most q facilities corresponds to (10) or (11) with $r = q$.

3 General properties

By (9), Π_r is the maximum of differentiable functions, and thus Π_r is continuous. Since the feasible region of (10) is compact, we have that its optimal value is attained, i.e., optimal times exist, and thus optimal

values for the variables $(\mathbf{x}, \mathbf{y}) \in S_r$ also exist for each r .

Finding an optimal policy via directly solving (10) is hard not only due to the fact that we are dealing with a nonlinear mixed integer problem, but also due to the fact that (many) optimal equivalent solutions may exist. Indeed, if an optimal solution exists with some facilities becoming operating at instant T , (thus $\tau_k = T$ for some $k \leq r$), then just by considering as closed such facilities, we obtain another feasible solution with same sales and same cost, and thus also optimal. Similarly, if some $k \leq r$ exists with $\tau_k = \tau_{k+1}$, then, for any f for which $y_f^k = 0$, $y_f^{k+1} = 1$, setting $y_f^k = 1$ we obtain an equivalent optimal solution.

In order to simplify the analysis and gain insight into the problem, these degenerate cases are ruled out, as described in the following proposition, whose proof is straightforward and thus omitted.

Proposition 1 *Let $\tau = (\tau_1, \tau_2, \dots, \tau_r)$, $(\mathbf{x}, \mathbf{y}) \in S_r$ be an optimal solution to (10). Let r^* be the number of different opening times strictly smaller than T in this solution, i.e., r^* is the cardinality of the set $\{\tau_1, \dots, \tau_r\} \setminus \{T\}$. Let $s_0 = 0 < s_1 < \dots < s_{r^*} = r$ be the indices satisfying*

$$\begin{aligned} \tau_1 &= \tau_2 = \dots = \tau_{s_1} < \\ < \tau_{s_1+1} &= \tau_{s_1+2} = \dots = \tau_{s_2} < \\ &\vdots &\vdots \\ < \tau_{s_{r^*-1}+1} &= \tau_{s_{r^*-1}+2} = \dots = \tau_{s_{r^*}} < \\ < \tau_{s_{r^*}+1} &= T. \end{aligned}$$

Define $\bar{\tau} = (\bar{\tau}_1, \dots, \bar{\tau}_{r^*})$, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in S_{r^*}$ as

$$\begin{aligned} \bar{\tau}_k &= \tau_{s_k}, \quad k = 1, 2, \dots, r^*, \\ \bar{y}_f^k &= y_f^{s_k}, \quad k = 1, 2, \dots, r^*, \quad f \in F \\ \bar{x}_v^k &= x_v^{s_k}, \quad k = 1, 2, \dots, r^*, \quad v \in V. \end{aligned}$$

Then, $\bar{\tau}, (\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is optimal solution to the problem (10) of locating at most q facilities in at most r^*

time instants. Moreover,

$$\Pi_r(\tau_1, \dots, \tau_r) = \Pi_{r^*}(\bar{\tau}_1, \dots, \bar{\tau}_{r^*}).$$

■

Now, given an optimal solution to (10), let $\bar{\tau}, (\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in S_{r^*}$ be the optimal solution constructed in Proposition 1. Then, $\bar{\tau}$ must also be an optimal solution to the optimization problem

$$\begin{aligned} \max \quad & \sum_{k=1}^{r^*} \left\{ \sum_{v \in V} \bar{x}_v^k [g_v(\tau_{k+1}) - g_v(\tau_k)] - \sum_{f \in F} \bar{y}_f^k [h_f(\tau_{k+1}) - h_f(\tau_k)] \right\} = \sum_{k=1}^{r^*+1} \varphi_k(\tau_k) \\ & 0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_{r^*} \leq T, \end{aligned} \quad (12)$$

in which the variables \mathbf{x}, \mathbf{y} are fixed at their (optimal) values $\bar{\mathbf{x}}, \bar{\mathbf{y}}$, and each φ_k is defined as

$$\varphi_k(t) = \begin{cases} \sum_{f \in F} h_f(t) \bar{y}_f^1 - \sum_{v \in V} g_v(t) \bar{x}_v^1 & \text{if } k = 1 \\ \sum_{f \in F} h_f(t) (\bar{y}_f^k - \bar{y}_f^{k-1}) - \sum_{v \in V} g_v(t) (\bar{x}_v^k - \bar{x}_v^{k-1}) & \text{if } 2 \leq k \leq r^* \\ - \sum_{f \in F} h_f(t) \bar{y}_f^{r^*} + \sum_{v \in V} g_v(t) \bar{x}_v^{r^*} & \text{if } k = r^* + 1 \end{cases}$$

being $\bar{\tau}_{r^*+1} = T$.

By construction, each φ_k is differentiable on $[0, T]$, its derivative φ'_k being given by

$$\varphi'_k(t) = \begin{cases} \sum_{f \in F} c_f(t) \bar{y}_f^1 - \rho(t) \sum_{v \in V} w_v(t) \bar{x}_v^1 & \text{if } k = 1 \\ \sum_{f \in F} c_f(t) (\bar{y}_f^k - \bar{y}_f^{k-1}) - \rho(t) \sum_{v \in V} w_v(t) (\bar{x}_v^k - \bar{x}_v^{k-1}) & \text{if } 2 \leq k \leq r^* \\ - \sum_{f \in F} c_f(t) \bar{y}_f^{r^*} + \rho(t) \sum_{v \in V} w_v(t) \bar{x}_v^{r^*} & \text{if } k = r^* + 1. \end{cases}$$

Using Proposition 1 and expression (12), let us derive the Karush-Kuhn-Tucker optimality conditions for such a solution.

Proposition 2 *If $\bar{\tau} = (\bar{\tau}_1, \dots, \bar{\tau}_{r^*})$, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in S_{r^*}$, with $0 \leq \bar{\tau}_1 < \bar{\tau}_2 < \dots < \bar{\tau}_{r^*-1} < \bar{\tau}_{r^*} < T$, is an optimal solution to problem (10), then*

1.

$$\begin{cases} \varphi'_k(\bar{\tau}_k) = 0 \text{ for } 1 \leq k \leq r^* & \text{if } \bar{\tau}_1 > 0 \\ \varphi'_1(0) \leq 0 \text{ and } \varphi'_k(\bar{\tau}_k) = 0, 2 \leq k \leq r^* & \text{if } \bar{\tau}_1 = 0. \end{cases} \quad (13)$$

2. φ_k has a maximum at $\bar{\tau}_k$, for $1 \leq k \leq r^*$.**Proof.**

Given $\bar{\mathbf{y}} = (\bar{y}_f^k)_{f \in F, 1 \leq k \leq r^*}$ and $\bar{\mathbf{x}} = (\bar{x}_v^k)_{v \in V, 1 \leq k \leq r^*}$, the necessary optimality conditions for problem (12) are

$$\varphi'_k(\tau_k) - \lambda_k + \lambda_{k-1} = 0, \quad 1 \leq k \leq r^*, \quad (14)$$

$$\lambda_k(\tau_{k+1} - \tau_k) = 0, \quad 0 \leq k \leq r^*, \quad (15)$$

$$0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_{r^*} \leq \tau_{r^*+1} = T, \quad (16)$$

$$\lambda_k \geq 0, \quad 0 \leq k \leq r^*. \quad (17)$$

As $\bar{\tau}_k < \bar{\tau}_{k+1}$ for $1 \leq k \leq r^*$, by equations (15), $\lambda_k = 0$ for $1 \leq k \leq r^*$. Therefore, from equation (14), it follows that $\varphi'_1(\bar{\tau}_1) + \lambda_0 = 0$ and $\varphi'_k(\bar{\tau}_k) = 0$ for $2 \leq k \leq r^*$. Then, if $\bar{\tau}_1 = 0$, $\lambda_0 \geq 0$ and, consequently, $\varphi'_1(\bar{\tau}_1) \leq 0$. If $\bar{\tau}_1 > 0$, $\lambda_0 = 0$ and $\varphi'_1(\bar{\tau}_1) = 0$. This completes the proof of result 1.

As $\bar{\tau}_{k-1} < \bar{\tau}_k < \bar{\tau}_{k+1}$, $2 \leq k \leq r^*$, if there is an index k such that $\varphi_k(\bar{\tau}_k)$ is not a maximum, then a value $\epsilon > 0$ exists such that $\varphi_k(\bar{\tau}_k - \delta) > \varphi_k(\bar{\tau}_k)$ or $\varphi_k(\bar{\tau}_k + \delta) > \varphi_k(\bar{\tau}_k)$, for $\delta < \epsilon$. Suppose $\varphi_k(\bar{\tau}_k - \delta) > \varphi_k(\bar{\tau}_k)$ for $\delta < \epsilon$. Then $\Pi(\bar{\tau}) = \sum_{1 \leq j \leq r^*} \varphi_j(\bar{\tau}_j) + \varphi_{r^*+1}(T)$ is less than $\Pi(\bar{\tau}_1, \dots, \bar{\tau}_{k-1}, \bar{\tau}_k - \delta, \bar{\tau}_{k+1}, \dots, \bar{\tau}_{r^*}) = \sum_{1 \leq j \leq r^*, j \neq k} \varphi_j(\bar{\tau}_j) + \varphi_k(\bar{\tau}_k - \delta) + \varphi_{r^*+1}(T)$ for $\delta < \epsilon$, which contradicts the optimality of $\bar{\tau}$. A similar reasoning leads to the same contradiction if $\varphi_k(\bar{\tau}_k + \delta) > \varphi_k(\bar{\tau}_k)$, for $\delta < \epsilon$. This proves result 2. ■

Observe that, given $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, the necessary optimality condition (13) is equivalent to

$$\left\{ \begin{array}{ll} \sum_{f \in F} c_f(\bar{\tau}_1) \bar{y}_f^1 \leq \sum_{v \in V} \rho(\bar{\tau}_1) w_v(\bar{\tau}_1) \bar{x}_v^1 & \text{if } \bar{\tau}_1 = 0 \\ \sum_{f \in F} c_f(\bar{\tau}_1) \bar{y}_f^1 = \sum_{v \in V} \rho(\bar{\tau}_1) w_v(\bar{\tau}_1) \bar{x}_v^1 & \text{if } \bar{\tau}_1 > 0 \\ \sum_{v \in V} \rho(\bar{\tau}_k) w_v(\bar{\tau}_k) (\bar{x}_v^k - \bar{x}_v^{k-1}) = \sum_{f \in F} c_f(\bar{\tau}_k) (\bar{y}_f^k - \bar{y}_f^{k-1}) & \text{for } 2 \leq k \leq r^*, \end{array} \right.$$

which means that, for $k > 1$ ($k \geq 1$ if $\bar{\tau}_1 > 0$), the increase in the revenue produced by the facilities whose opening occurs at time $\bar{\tau}_k$ is equal to the increase in operating costs. This result corresponds to the economic profit maximizing condition "marginal revenue is equal to marginal cost".

4 Different demand patterns

Under stronger assumptions on demand, profit margin and cost functions, a much deeper insight in the problem and the optimal locations strategy can be obtained. Henceforth, we assume constant marginal profit and cost functions, $\rho(t) = \rho$ and $c_f(t) = c_f$, $\forall t \in [0, T]$, $f \in F$, and we analyse different demand patterns. More general demand patterns will be addressed in Section 6.

4.1 Constant demand

Assume now that the demand remains constant in time, i.e., $w_v(t) = \alpha_v$ for $v \in V$. It follows that the profit provided by a facility at f in $[\tau, T]$ from the captured nodes is given by

$$\left(\rho \sum_v \alpha_v - c_f \right) (T - \tau).$$

This is a linear function in τ . Hence, only two situations can occur: either a facility is located at f at the beginning of the period (at time $\tau = 0$) or no facility is opened at f during the planning horizon.

4.2 Demand varying linearly in time

In this section, we analyse the competitive location dynamic problem for increasing linear demand functions.

Let

$$w_v(t) = \alpha_v + \beta_v t, \quad \alpha_v, \beta_v \in \mathbb{R}, \quad \alpha_v \geq 0, \beta_v > 0, \quad \text{for } v \in V, t \in [0, T].$$

Assuming r^* different opening times in the market and $\bar{\tau}_1 > 0$, the necessary optimality conditions are

$$\varphi'_1(\bar{\tau}_1) = \sum_{f \in F} c_f \bar{y}_f^1 - \sum_{v \in V} \rho(\alpha_v + \beta_v \bar{\tau}_1) \bar{x}_v^1 = 0$$

$$\varphi'_k(\bar{\tau}_k) = \sum_{f \in F} c_f (\bar{y}_f^k - \bar{y}_f^{k-1}) - \sum_{v \in V} \rho(\alpha_v + \beta_v \bar{\tau}_k) (\bar{x}_v^k - \bar{x}_v^{k-1}) = 0, \quad 2 \leq k \leq r^*.$$

From these expressions one obtains the following instants:

$$\bar{\tau}_1 = \frac{\sum_{f \in F} c_f \bar{y}_f^1 - \rho \sum_{v \in V} \alpha_v \bar{x}_v^1}{\rho \sum_{v \in V} \beta_v \bar{x}_v^1} \quad (18)$$

$$\bar{\tau}_k = \frac{\sum_{f \in F} c_f (\bar{y}_f^k - \bar{y}_f^{k-1}) - \rho \sum_{v \in V} \alpha_v (\bar{x}_v^k - \bar{x}_v^{k-1})}{\rho \sum_{v \in V} \beta_v (\bar{x}_v^k - \bar{x}_v^{k-1})}, \quad 2 \leq k \leq r^*, \quad (19)$$

which must satisfy the order condition $\bar{\tau}_1 < \dots < \bar{\tau}_{r^*}$.

Now, replacing $\bar{\tau} = (\bar{\tau}_1, \dots, \bar{\tau}_{r^*})$ by its value in the profit function, we obtain

$$\Pi = \sum_{k=1}^{r^*+1} \varphi_k(\bar{\tau}_k)$$

$$\begin{aligned}
&= \frac{1}{2\rho} \frac{[\sum_{f \in F} c_f \bar{y}_f^1 - \rho \sum_{v \in V} \alpha_v \bar{x}_v^1]^2}{\sum_{v \in V} \beta_v \bar{x}_v^1} - \sum_{f \in F} \bar{y}_f^{r^*} c_f T + \sum_{v \in V} \bar{x}_v^{r^*} \rho (\alpha_v T + \frac{\beta_v}{2} T^2) + \\
&\quad + \sum_{k=2}^{r^*} \frac{1}{2\rho} \frac{[\sum_{f \in F} c_f (\bar{y}_f^k - \bar{y}_f^{k-1}) - \rho \sum_{v \in V} \alpha_v (\bar{x}_v^k - \bar{x}_v^{k-1})]^2}{\sum_{v \in V} \beta_v (\bar{x}_v^k - \bar{x}_v^{k-1})}.
\end{aligned} \tag{20}$$

Example 1 Consider the network shown in Figure 1 representing a location problem with competing firms.

The length of any edge is 1, the set of facilities from the competitor is $E = \{v_3\}$, threshold values are thus given by (1), and the set of feasible locations is $F = \{v_i\}_{1 \leq i \leq 8, i \neq 3}$. Let $\rho(t) = 1$, $w_v(t) = \frac{1}{4}t + t$ and $c_f(t) = 4$, $\forall t \in [0, T]$, $v \in V$, $f \in F$, with $T = 7$.

1. For $r = q = 1$, assuming that a facility will be located at vertex v_5 , we have $y_{v_5}^1 = 1$ and $y_f^1 = 0$ if $f \neq v_5$, and $x_v^1 = 1$ if $v = v_i$ for $i = 4, 5, 6, 7, 8$, and $x_v^1 = 0$ otherwise. Then $g_v(t) = \frac{1}{4}t + \frac{1}{2}t^2$, $\forall v \in V$, $h_f(t) = 4t$, $\forall f \in F$, and $\varphi_1(t) = 4t - 5(\frac{1}{4}t + \frac{1}{2}t^2)$. The optimal time is $\bar{\tau}_1 = \frac{11}{20}$ and the maximum total profit is $\Pi^* = \frac{16641}{160} \approx 104.006$, which is the area shown in Figure 2.
2. Let $r \leq q = 2$ and assume that the facilities will be located at vertices v_5 and v_1 . If the facilities are to be located at different times, $\tau_1 < \tau_2$, the first one at node v_5 and the second one at v_1 , we have $y_{v_5}^1 = 1$ and $y_f^1 = 0$ if $f \neq v_5$, $y_{v_1}^2 = y_{v_5}^2 = 1$ and $y_f^2 = 0$ for $f \neq v_1, v_5$, $x_v^1 = 1$ if $v = v_i$ for $i = 4, 5, 6, 7, 8$, and $x_v^1 = 0$ otherwise, and $x_v^2 = 1$ if $v = v_i$ for $i = 1, 4, 5, 6, 7, 8$, and $x_v^2 = 0$ otherwise. Then $\varphi_1(t) = 4t - 5(\frac{1}{4}t + \frac{1}{2}t^2)$ and $\varphi_2(t) = 4t - (\frac{1}{4}t + \frac{1}{2}t^2)$. The optimal times are $\bar{\tau}_1 = \frac{11}{20}$ and $\bar{\tau}_2 = \frac{15}{4}$, and the maximum total profit is $\Pi^* = \frac{8743}{80} \approx 109.287$, which is the area shown in Figure 3.
3. If facilities at nodes v_5 and v_1 are opened at the same time, the optimal instant is $\tau_1 = \frac{13}{12}$ and the total profit is $\Pi = \frac{5041}{48} \approx 105.021$. This means that different entry times provide a greater profit than the opening of both facilities simultaneously.

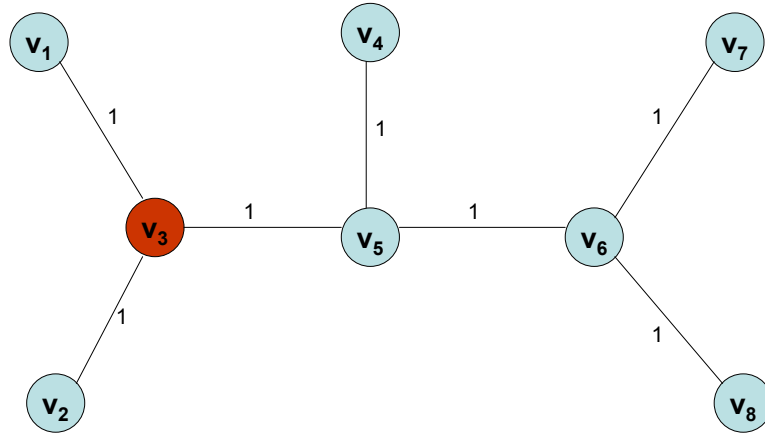


Figure 1: Network used in Example 1

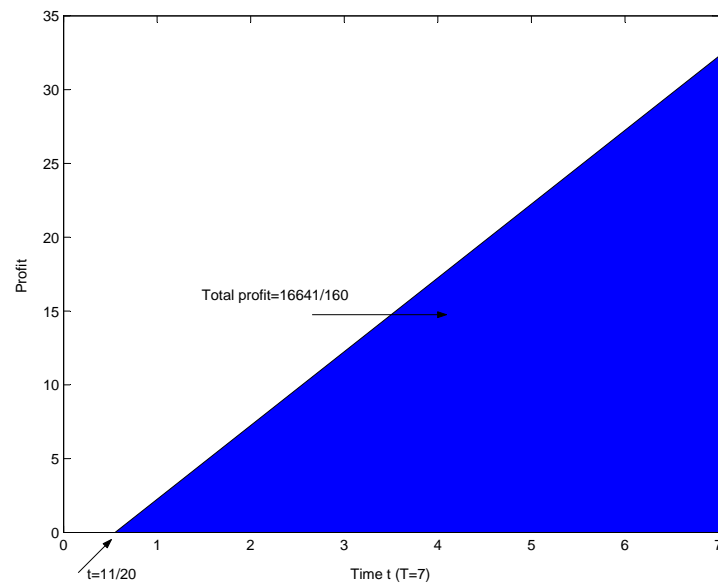


Figure 2: Maximum profit in Example 1.1

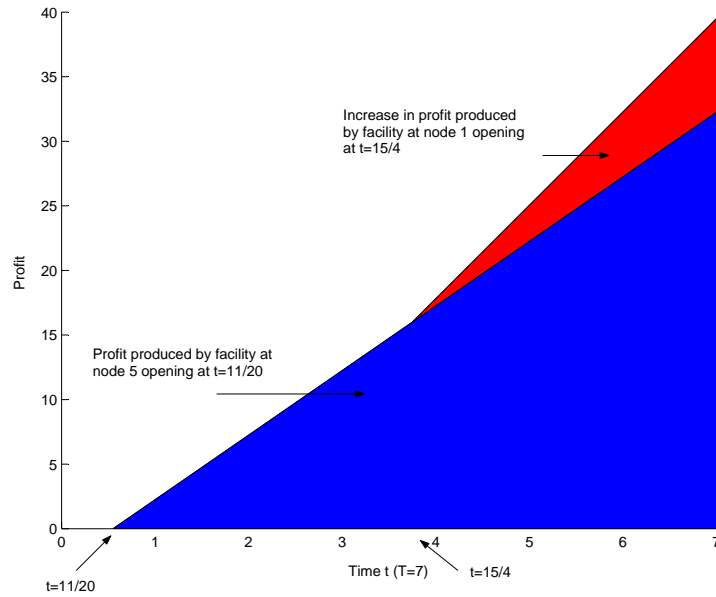


Figure 3: Maximum profit in Example 1.2

4.3 Demand increasing in time

Since profit margins and costs have been assumed to be constant, if demands, $w_v(t)$, $v \in V$, are increasing, functions φ_k , $1 \leq k \leq r^* + 1$, are concave and, consequently, the profit function is also concave. Hence, assuming the locations to be fixed, the necessary optimality conditions are also sufficient, and any local maximum is also a global maximum. Therefore, for \mathbf{y} fixed, any local search algorithm used to optimize the total profit will yield the globally optimal times. A particular case is the linear demands scenario analysed in Section 4.2.

4.4 Demand decreasing in time

For constant profit margins and costs, if demands functions, $w_v(t)$, $v \in V$, are decreasing, functions φ_k are convex and, consequently, the profit function is also convex. Then, we must maximize a convex function on the set $\mathcal{R} = \{\tau = (\tau_1, \dots, \tau_r) : 0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_r \leq T\}$. It then follows that, for any candidate location

f , either a facility is operating at f from the beginning of the planning horizon or no facility is open at f , leading to a static problem.

5 A VNS algorithm to find optimal policies

Although the aim of the paper is methodological (we have proposed a new model, and different theoretical properties are analyzed), we now illustrate how the resulting problems can be handled using a rather standard metaheuristic strategy. Extensive numerical experience or the use of more sophisticated (meta)heuristics are important topics beyond the scope of the present paper.

In our experiments, the following stronger assumptions have been added:

$$\rho(t) = \rho, \quad t \in [0, T] \quad (21)$$

$$c_f(t) = c_f, \quad \forall f \in F, t \in [0, T] \quad (22)$$

$$w_v(t) = \alpha_v + \beta_v t, \quad \alpha_v \geq 0, \beta_v > 0, \forall v \in V, t \in [0, T], \quad (23)$$

that is, we will consider the problem where the profit margin is constant (constraint (21)), the operating costs are constant for each candidate site f (constraint (22)) and the demand at each node v has a linear behaviour (constraint (23)).

Since Problem (10) is a nonlinear mixed integer program, heuristic methods seem to be the only possible choice. We propose as resolution technique the so-called Variable Neighborhood Search algorithm (see [26]), to take advantage of the combinatorial structure of the problem. This method is a recent metaheuristic based on systematic change of neighborhood within a local search, for solving combinatorial and global optimization problems.

The general scheme of a VNS algorithm is the following [26]:

Algorithm 1 (Mladenovic and Hansen)

- **Initialization step.**

- Select the set of neighbourhood structures \mathcal{N}_k , $k = 1, \dots, k_{max}$ to be used in the search.
- Find an initial solution x .
- Choose a stopping condition.

- **Main step.**

1. Set $k := 1$.
2. Until $k = k_{max}$, repeat the following steps:
 - Generate a feasible solution x' at random from the k^{th} neighbourhood of x (that is, $x' \in \mathcal{N}_k(x)$).
 - Apply some local search method with x' as initial solution (the new local optimum will be denoted by x'').
 - If the solution obtained x'' is better than x , move there ($x := x''$) and continue the search with \mathcal{N}_1 ; otherwise, set $k := k + 1$.

We describe below the search space, the strategies to choose an initial solution, the neighborhood structure and the local search used for applying the algorithm to our setting.

5.1 Search space

The different solutions in the search space are given in terms of the variables y_f^k , defined in (2), with $f \in F$, $k = 1, \dots, r^*$, with r^* the number of different opening instants for the facilities. We denote by $Y := (y_f^k)_{k,f}$ an $(r^* + 1) \times m$ boolean matrix, m being the number of candidate sites, which represents any feasible selection of facilities to be opened in r^* different opening times during a planning horizon.

To simplify the implementation of the program, a final $(r^* + 1)^{th}$ row has been added to the matrix Y , with all the components taking the value 1,

$$y_f^{r^*+1} = 1 \quad \forall f, \quad (24)$$

that is, all the facilities are assumed to be opened at the end of the interval $[0, T]$.

Likewise, we denote by $X := (x_v^k)_{k,v}$ an $(r^* + 1) \times n$ boolean matrix, n being the number of demand nodes, which represents the set of demand points $v \in V$ captured by the new facilities in each period of time during the whole planning horizon. This matrix is built directly from the value of the corresponding matrix Y by using the rule (4). Their role is not important for constructing the set of feasible solutions, although they are necessary to compute the value of the objective function and, therefore, to study the improvement of this function for a new solution.

5.2 Initial solution

Two different strategies have been approached to obtain an initial solution for the algorithm.

In the first approach, a random order is imposed on the elements of the set F of facilities. We select the q first facilities to be opened and we compute the opening times by using formulae (18)-(19).

In the second approach, a rank is assigned to the elements of F according to the total profit that a facility would produce if it was the only one to be opened and it was operating during the whole interval $[0, T]$. That profit is computed by using expression (20) for $\tau_1 = 0$. Afterwards, we compute the opening times of the q first facilities with (18)-(19).

The ordered list of candidate sites (obtained via any of the two approaches described before) will be kept for the rest of the algorithm. Moreover, for the two approaches (and also for the rest of the algorithm), whenever the opening time for one facility is bigger than T , that candidate site is rejected and sent to the end of the list and then, the following candidate location in the ordered list becomes operating.

5.3 Neighbourhood structure

The neighbourhoods \mathcal{N}_k are defined by considering the possible choices of facilities to be opened during the planning horizon.

Given a feasible solution Y , the k^{th} neighbourhood of Y , $\mathcal{N}_k(Y)$, with $k \leq q$, being q the maximum number of facilities to be opened, is defined as the set of solutions obtained by closing k of the open facilities in solution Y and opening k of the closed facilities. By (24), this is equivalent to build an element of $\mathcal{N}_k(Y)$ by changing k facilities opened before time T by k facilities opened just at time T .

For example, given the following matrix Y , a feasible solution to our problem,

$$Y = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

the following matrices Y^1 and Y^2 belong, respectively, to the neighbourhoods $\mathcal{N}_1(Y)$ and $\mathcal{N}_2(Y)$,

$$Y^1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad Y^2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

That is, one can generate a k^{th} neighbourhood of Y via a swap of $2k$ columns of Y : k columns with less than r^* zeros swapped with k columns with exactly r^* zeros.

Concerning the opening times, those of the facilities which remain open are kept and the opening time of each replaced facility is assigned to the new one.

5.4 Main step of the algorithm

Given a solution Y , we chose at random another feasible solution Y' from the first neighbourhood of Y , $Y' \in \mathcal{N}_1(Y)$. Let τ_j be the opening time of the operating facility in the initial solution which has been replaced by a new facility. Now, keeping the opening times of the facilities open in both solutions, we try to improve the opening time of the new facility.

By using formulae (18)-(19), we calculate a new opening time t for the new facility. If $\bar{\tau}_{j-1} \leq t \leq \bar{\tau}_{j+1}$, we set $\tau_j := t$. Otherwise, we disregard t and set $\tau_j := \bar{\tau}_{j-1}$ (if $t < \bar{\tau}_{j-1}$) or $\tau_j := \bar{\tau}_{j+1}$ (if $t > \bar{\tau}_{j+1}$). The solution with the new value of τ_j is denoted by Y'' .

Afterwards, we evaluate the objective function for the new solution Y'' . If the objective value has improved, we move to Y'' , that is, we set $Y := Y''$ and we continue the search in $\mathcal{N}_1(Y)$. Otherwise, we set $k := k + 1$ and we continue the search in $N_k(Y)$, until $k = k_{max}$, with k_{max} fixed to q in our problem.

Finally, the stopping rule is given by a maximum number of iterations.

5.5 Examples

The algorithm has been implemented by using Matlab 6.5 on a computer with Pentium IV CPU 3.06 GHz and several numerical experiments of location problems in a competitive environment have been performed.

The initial scenario of an artificial database, built in Matlab, is depicted in Figure 4. In this example, the market is displayed as a square with 100 demand points located inside. Every node has a demand varying linearly in time (condition (23)) and is represented in Figure 4 via a circle whose radius is proportional to the corresponding slope. Both the locations and the radii of the nodes have been generated at random. Some firms are already operating in the market with five existing facilities, represented with asterisks in Figure 4.

Our firm enters into the market with at most $q = 3$ facilities which will compete with the existing ones. The set of candidate locations for these facilities coincides with the set of demand nodes, that is, $F = V$.

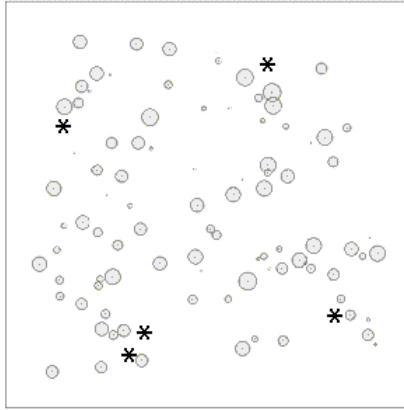


Figure 4: Initial scenario

Profit margins and operating costs are assumed to be constant (conditions (21)-(22)), and furthermore, the operating costs are the same for every facility, that is, $c_f = c, \forall f \in F$.

We use the VNS algorithm to solve this problem, and the two different approaches to select an initial solution are tested. First, we solve the problem when a random order is imposed on the set of candidate sites to choose the initial solution. This initial solution is depicted in Figure 5 (left), where the black circles symbolize the new facilities to enter the market, and the dark-grey circles represent the demand points captured by these new facilities. Figure 5 (right) shows the solution obtained after 1000 iterations of the algorithm.

Secondly, Figure 6 (left) shows the initial solution when the order is assigned according to the total profit provided by a facility operating individually during the whole interval $[0, T]$. In this case, the three facilities to be opened (black circles) are very close each other and the solution is worse than that obtained with a random initial order. However, although the algorithm will escape fast from this type of solutions, it can be helpful to fix at least one facility to open (the one providing the biggest profit). The final solution obtained by the algorithm is shown in Figure 6 (right), with the dark-grey circles representing the captured demand nodes.

Figure 7 displays the improvement in the objective function for the algorithm run twice (for using

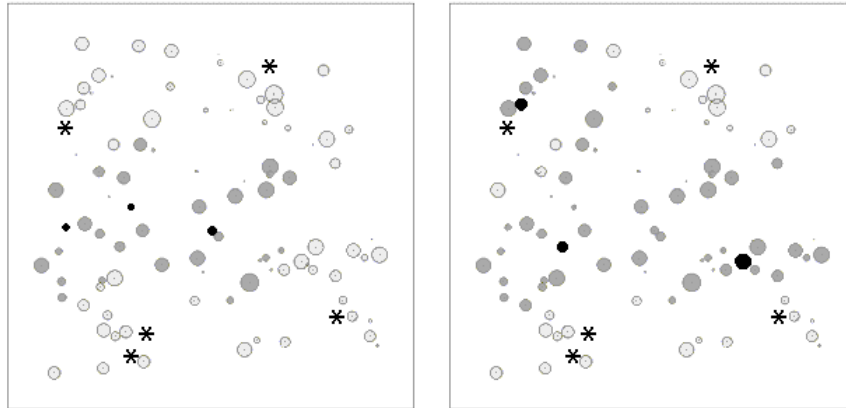


Figure 5: Initial and final solutions (random initial solution)

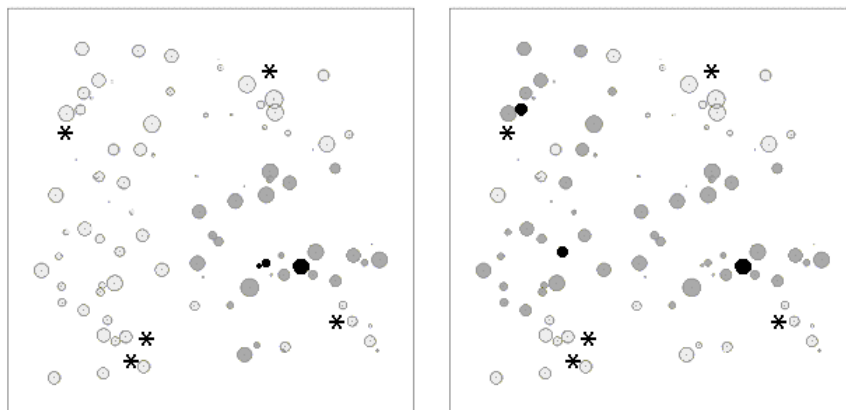


Figure 6: Initial and final solutions (initial solution according to profit)

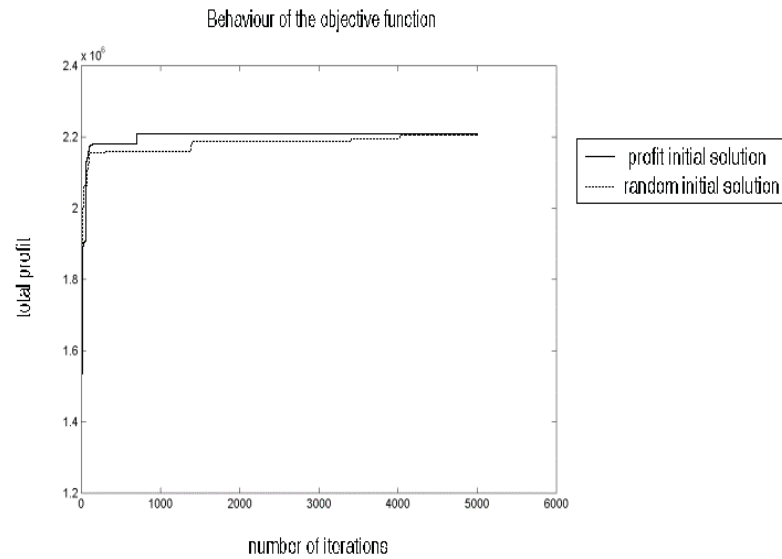


Figure 7: Objective function (100 demand points and 5000 iterations)

the two different strategies to choose an initial solution) with 5000 iterations in each experiment. Dotted and continuous lines represent the objective value when using the random and the biggest profit initial solutions, respectively. In both cases, there is a remarkable improvement for the first iterations, becoming more stable during the rest of the process. As one can observe in Figures 5 and 6, the two different initial solutions give the same final locations. However, the value of the objective function for the solution obtained with the biggest profit initial solution is slightly better. This is due to the different opening times for the opened facilities (black circles), despite their locations coinciding in both solutions.

Finally, Figure 8 shows the objective function in a similar experiment with 1000 demand points and 1000 iterations of the process. In this case, there is also a remarkable improvement at the beginning of the process (especially for the algorithm with the profit initial solution) and it becomes more stable later.

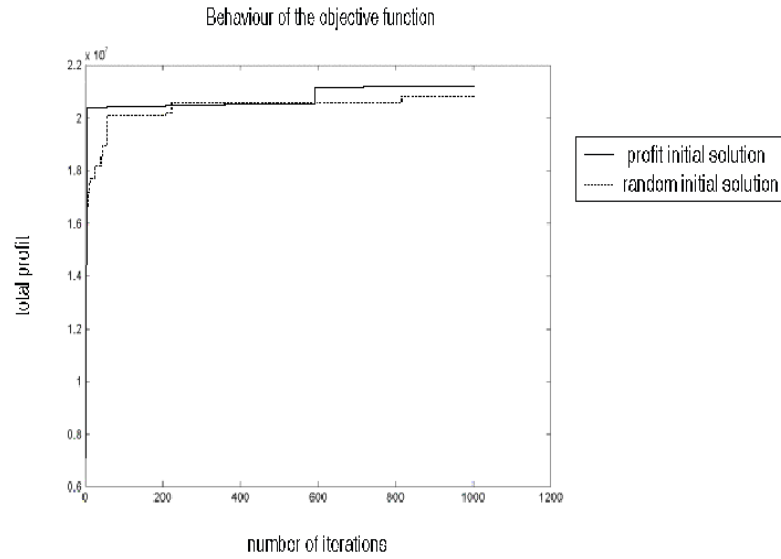


Figure 8: Objective function (1000 demand points and 1000 iterations)

6 Extensions

6.1 General demands

When it is not the case that all demand functions are increasing (or all are decreasing), for fixed values $\mathbf{y} = (y_f^k)_{f \in F, k=1,2,\dots,r}$, the profit function may be concave in some intervals and convex in the remaining parts of $[0, T]$. Let $0 = t_0^k < t_1^k < \dots < t_{u_k-1}^k < t_{u_k}^k = T$ be the local optima of the demand function in $[0, T]$. These points define a partition of the interval $[0, T]$ into pieces in which the function φ_k is either concave or convex. Moreover, this partition defines a partition of the feasible region, $R = \{\tau = (\tau_1, \tau_2, \dots, \tau_r) : 0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_r \leq T\}$, whose elements have the form $R_{i_1, i_2, \dots, i_r} = \prod_{k=1}^r [t_{i_k}^k, t_{i_k+1}^k] \cap R$, $0 \leq i_k \leq u_k - 1$. Since one cannot guarantee convexity or concavity of the profit function in each of the $\prod_{k=1}^r u_k$ sets of the partition of R , and taking into account that the number of such pieces can be huge, the optimization process asks for the use of heuristic procedures.

6.2 Non-constant profit margin or cost

Assuming that benefit margins and fixed operating costs also vary with time, we would be in one of the following situations:

1. All functions φ_k , $1 \leq k \leq r$ are concave. Then, the profit function is also concave, and the optimality conditions are not only necessary, but they are also sufficient. Hence, solving the problem for r^* different opening times amounts to solving system (14)-(17); therefore, for $\bar{\mathbf{y}}$ given, the optimal sequence of times is the solution of (13). This happens, for instance, when the functions $\rho(t)w_v(t)$, $v \in V$, are increasing and the functions $c_f(t)$, $f \in F$, are decreasing.
2. All functions φ_k , $1 \leq k \leq r$ are convex. In this case, the profit function is also convex, and hence an optimal solution can be found in the set of extreme points of the feasible region. Under these assumptions, the facilities which are opened enter the market in the first possible instant ($\tau_0 = 0$), and a static location problem results. This will be the case, for instance, when the functions $\rho(t)w_v(t)$, $v \in V$, are decreasing and the cost functions $c_f(t)$, $f \in F$, are increasing.
3. Not all functions φ_k , $1 \leq k \leq r$ are convex or concave. In this case, we would be in a situation similar to that in which demands are not all increasing or decreasing, as addressed in Section 6.1.

6.3 Long-run analysis

In order to analyse the competitive dynamic location model when the planning horizon is very long, we consider the problem of finding the optimal opening times in $[0, +\infty)$. We use the term *short-run* to refer the problem of finding the optimal times in $[0, T]$, $0 < T < \infty$, when the marginal profit, demand and cost functions are $\rho(t)$, $w_v(t)$, $v \in V$, and $c_f(t)$, $f \in F$, defined on $[0, +\infty)$. Now, using a discount factor, we introduce the *long-run* problem, which is the dynamic location problem for $\rho(t)$, and demand and cost functions defined as $\hat{w}_v(t) = e^{-\gamma t}w_v(t)$ and $\hat{c}_f(t) = e^{-\gamma t}c_f(t)$, with $\gamma > 0$, such that

$\int_0^{+\infty} \rho(t)w_v(t)e^{-\gamma t} < +\infty$ and $\int_0^{+\infty} c_f(t)e^{-\gamma t} < +\infty$. The following proposition shows that, under certain conditions, the optimal solution to the short-run problem coincides with the set of optimal times to the long-run problem in the interval $[0, T]$. A similar result can be found in [23].

Proposition 3 *Let $\rho(t)$, $w_v(t)$ and $c_f(t)$, $v \in V$, $f \in F$, be continuous functions on $[0, \infty)$. Then, $\bar{\tau} = (\bar{\tau}_1, \dots, \bar{\tau}_r^*)$ is an optimal solution to the short-run problem if and only if $\{\bar{\tau}_i\}_{i=1}^{r^*}$ are the optimal times to the long-run problem in the interval $[0, T]$.*

Proof.

Let φ_k and $\hat{\varphi}_k$, $1 \leq k \leq r^*$, be the functions defined in Section 3 for the short-run and long-run problems, respectively. The result follows from the equations $\hat{\varphi}'_k = e^{-\gamma\tau} \varphi'_k$, $1 \leq k \leq r^*$, and the optimality conditions. ■

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