Pricing A Class of Multiasset Options using Information on Smaller Subsets of Assets*

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Abstract

In this paper, we study the pricing problem for the class of multiasset European options with piecewise linear convex payoff in the asset prices. We derive a simple upper bound on the price of this option by constructing a static super-replicating portfolio using cash and options on smaller subsets of assets. The best upper bound is found by determining the optimal set of strike prices that minimizes the cost of this super-replicating portfolio. Under the no-arbitrage assumption, this bound is shown to be tight when the joint risk-neutral distributions for the smaller subsets of assets are known but the complete risk-neutral distribution is unknown. Using a simulation-based optimization approach, we obtain new price bounds for the basket option, an option on the maximum of several assets and an option on the spread between the maximum and minimum of assets. Extensions to markets where only a finite set of options are traded on smaller subsets of assets is also provided. The paper thus extends some of the recent results in Aspremont and Ghaoui [1] and Hobson et al. [15] to a larger class of options under more general assumptions.

Keywords: Option pricing, Multiple assets, Super-replication, Simulation-based optimization

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1 Introduction

Let $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)$ denotes the random non-negative prices of N different risky assets at time T. The traditional Black-Scholes model [5] assumes a correlated geometric Brownian motion for the asset prices. Under the assumption of no-arbitrage, the price of an option on these assets is given by the discounted expected payoff under the risk-neutral distribution. Let $f(\tilde{\boldsymbol{x}})$ denote the payoff of the European option at maturity time T. The price of this multiasset option at time 0 is then given as:

$$E_{\boldsymbol{\pi}}\left[f(\tilde{\boldsymbol{x}})\right] = \int_{\boldsymbol{x} \geq \boldsymbol{0}} f(\boldsymbol{x}) d\boldsymbol{\pi}(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{\pi}$ is the risk-neutral distribution. For simplicity, we assume that the risk-free rate r is zero. Examples for which the prices can be computed in closed form in the Black-Scholes setting include standard call and put options (Black and Scholes [5]), exchange options (Margrabe [21]) and options on the maximum of several assets (Johnson [13]). Evaluating the option price in (1) for general payoffs in high dimensions is however a challenging numerical problem. Typically, practitioners use simulations, bounds or approximations to estimate these prices.

In this paper, we study the pricing problem for the class of multiasset European options with piecewise linear convex payoff given as:

$$f(\tilde{\boldsymbol{x}}) = \max_{p \in \mathcal{P}} (\boldsymbol{w_p} \cdot \tilde{\boldsymbol{x}} - k_p) = \max\left(\boldsymbol{w_1} \cdot \tilde{\boldsymbol{x}} - k_1, \boldsymbol{w_2} \cdot \tilde{\boldsymbol{x}} - k_2, \ldots, \boldsymbol{w_P} \cdot \tilde{\boldsymbol{x}} - k_P\right). \tag{2}$$

The vector $\boldsymbol{w_p} = \left(w_p^{(1)}, w_p^{(2)}, \ldots, w_p^{(N)}\right)$ denotes the asset weights in the pth term in the option payoff . This option allows the holder to trade a portfolio of assets at maturity among the following choices: the first portfolio of assets (given by vector $\boldsymbol{w_1}$) at k_1 or the second portfolio of assets (given by vector $\boldsymbol{w_2}$) at k_2 and so on. At maturity, the holder trades the portfolio that gives the largest positive payoff [1]. Special cases of this class of options include:

(i) Basket/multiple spread option with payoff $(\boldsymbol{w} \cdot \tilde{\boldsymbol{x}} - k)^+$,

(ii) Option on the maximum of assets with payoff $(\max_i \tilde{x}_i - k)^+$,

(iii) Option on the spread between maximum and minimum with payoff $(\max_i \tilde{x}_i - \min_i \tilde{x}_i - k)^+$.

[1] Typically, the holder also has the option of not exercising the option. This is incorporated by setting $\boldsymbol{w_P} = \boldsymbol{0}$ and $k_P = 0$.

For the basket option with weights $\boldsymbol{w} \geq \mathbf{0}$ and a payoff that depends on a single weighted linear combination, there is no explicit analytical formula in the Black-Scholes setting. The difficulty arises due to the lack of the availability of the distribution of the sum of correlated lognormal distributions. Instead, we focus on a simple upper bound that has been developed in Deelstra et al. [8] and Hobson et al. [15] using call option prices. The upper bound on the basket option price therein is obtained by solving the N-variable minimization problem:

$$E_{\boldsymbol{\pi}}\left[\boldsymbol{w} \cdot \tilde{\boldsymbol{x}} - k\right]^+ \quad \leq \quad \min_{\lambda_i \geq 0, \sum_i \lambda_i = 1} \left(w_i \sum_{i=1}^{N} E_{\pi_i}\left[\tilde{x}_i - \frac{\lambda_i k}{w_i}\right]^+ \right), \tag{3}$$

where π_i is the marginal risk-neutral distribution for asset price $\tilde{x}_i$. Formulation (3) is based on the construction of a static portfolio of call options that super-replicates the payoff of the basket option with probability one. The optimal strike prices for each of these call options are determined by constructing this portfolio at minimum cost. This provides the tightest possible upper bound on the basket option price with known marginal distributions π_i, but without knowledge of the complete joint distribution $\boldsymbol{\pi}$. This formulation can also be obtained from a result in the paper by Meilijson and Nadas [22], albeit in a different context. The goal in [22] is to find a bound on the expected project tardiness with $\tilde{\boldsymbol{x}}$ denoting random activity times and k denoting the target project duration. A similar bound for the option on the maximum of the asset prices can also be obtained from the results in Lai and Robbins [17] and Ross [25] for order statistics. The upper bound using call option prices is therein obtained by solving the single-variable minimization problem:

$$E_{\boldsymbol{\pi}}\left[\max_i x_i - k\right]^+ \quad \leq \quad \min_z \left((z-k)^+ + \sum_{i=1}^{n} E_{\pi_i}\left[\tilde{x}_i - z\right]^+ \right). \tag{4}$$

These results have been extended to the setting where the exact marginal distribution π_i is unknown, rather it lies in a set $\pi_i \in \Pi_i$. The model in Hobson et al. [14], Hobson et al. [15] and Aspremont and Ghaoui [1], describes Π_i by a known set of call option prices for a finite number of strike prices.

In this paper, we extend this idea to derive a price bound that works for a larger class of options under more general distributional assumptions. Our key contributions are summarized below:

(i) In Section 2, we super-replicate the payoff of the multiasset option in (2) with a portfolio of cash and options on smaller subsets of assets. The strike prices in this static super-replicating portfolio are determined by constructing this portfolio in the cheapest possible manner. Such an approach is useful when the dependence among assets in smaller financial markets are well understood but the dependence across markets is not clearly understood. Using options prices in these smaller markets as data, we can find a tight arbitrage-free upper bound on the price of the more complicated options that depend on the entire set of assets. This approach hence naturally leads to a hierarchy of relaxations that provide better and better estimates on the actual price of the option.

(ii) In Section 3, we consider applications of these price bounds to specific multiasset options (i)-(iii). Using a simulation-based optimization approach, we indicate the potential of this method in pricing and hedging multi-asset options.

(iii) In Section 4, we extend the results to the finite market case where only the prices of a finite set of financial options that traded on the smaller sets of assets are known. We then provide two new polynomial sized bounds for this class of multiasset options that can be computed using linear and semidefinite programming.

2 Pricing in Incomplete Markets

The price of the multiasset European option with piecewise linear convex payoff is given by the N dimensional integral:

$$E_{\boldsymbol{\pi}}\left[\max_{p\in\mathcal{P}}(\boldsymbol{w_p}\cdot\tilde{\boldsymbol{x}}-k_p)\right]=\int_{\boldsymbol{x}\geq\boldsymbol{0}}\max_{p\in\mathcal{P}}(\boldsymbol{w_p}\cdot\boldsymbol{x}-k_p)d\boldsymbol{\pi}(\boldsymbol{x}). \tag{5}$$

Consider the following market setup: We partition the entire set of N assets into R disjoint subsets:

$$\mathcal{N}=\{1,2,\ldots,N\}=\mathcal{N}_1\cup\mathcal{N}_2\cup\cdots\cup\mathcal{N}_R$$

such that each $\mathcal{N}_r\subseteq\mathcal{N}$ and $\mathcal{N}_i\cap\mathcal{N}_j=\emptyset$ for all $i\neq j$. Given a vector $\tilde{\boldsymbol{x}}\in\Re^N_+$, we let $\tilde{\boldsymbol{x}}_{\boldsymbol{r}}\in\Re^{N_r}_+$ denote the subvector formed with entries $\tilde{x}_i$ for $i\in\mathcal{N}_r$. Hence $\tilde{\boldsymbol{x}}=(\tilde{\boldsymbol{x}}_{\boldsymbol{r}})_r$. Suppose, we know the risk-neutral distributions $\boldsymbol{\pi_r}$ for each of the subsets of asset prices $\tilde{\boldsymbol{x}}_{\boldsymbol{r}}$. Let $\mathbb{M}(\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_R)$ denote the set of distributions[2] $\boldsymbol{\pi}$ for $\tilde{\boldsymbol{x}}$ with the known distributions $\boldsymbol{\pi_r}$ for each $\tilde{\boldsymbol{x}}_{\boldsymbol{r}}$. Since the exact distribution $\boldsymbol{\pi}\in\mathbb{M}(\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_R)$ is not known, the market is incomplete. The option price in (5) is not uniquely defined and hence an exact replication is not possible. We consider finding the largest

[2]For simplicity, we focus only on continuous distributions from this point onwards. The results can be extended to more general distributions.

possible price for the multiasset option using a static super-replicating portfolio.[3] This brings us to our main result.

Theorem 1 *Let $\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\pi_1}, \ldots, \boldsymbol{\pi_R})$. A tight upper bound on the price of a multiasset option with piecewise linear convex payoff is found by solving the convex minimization problem:*

$$\max_{\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\pi_1}, \ldots, \boldsymbol{\pi_R})} E_{\boldsymbol{\pi}} \left[\max_{p \in \mathcal{P}} (\boldsymbol{w_p} \cdot \tilde{\boldsymbol{x}} - k_p) \right] \;=\; \min_{\boldsymbol{z_1}, \ldots, \boldsymbol{z_P}} \left(\max_{p \in \mathcal{P}} (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}} \left[\max_{p \in \mathcal{P}} (\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}) \right] \right),$$

where $\boldsymbol{e}$ is a vector of ones of dimension R. The optimal variables $\boldsymbol{z_p} = (z_{pr})_r$ are the strike prices in the cheapest static super-replicating portfolio that consists of:

(i) Cash worth $\max_{p \in \mathcal{P}} (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p)$

(ii) A set of $r = 1, \ldots, R$ options each with payoff $\max_{p \in \mathcal{P}} (\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr})$.

Proof. We first show that the right-hand side minimization problem provides an upper bound on the option price. For any term $p \in \mathcal{P}$ and for arbitrary $\boldsymbol{z_p}$, we have:

$$\begin{aligned} \boldsymbol{w_p} \cdot \tilde{\boldsymbol{x}} - k_p &= (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) + (\boldsymbol{w_p} \cdot \tilde{\boldsymbol{x}} - \boldsymbol{e} \cdot \boldsymbol{z_p}), \\ &= (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} (\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}), \\ &\leq \max_{p \in \mathcal{P}} (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} \max_{p \in \mathcal{P}} (\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}). \end{aligned}$$

Taking the maximum of the left hand side over $p \in \mathcal{P}$, we obtain the inequality:

$$\max_{p \in \mathcal{P}} (\boldsymbol{w_p} \cdot \tilde{\boldsymbol{x}} - k_p) \leq \max_{p \in \mathcal{P}} (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} \max_{p \in \mathcal{P}} (\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}).$$

This inequality can be interpreted as the construction of a static portfolio that consists of cash and smaller options that super-replicates the payoff of the multiasset option. The variables z_{pr} corresponds to the strike prices in this portfolio. Taking expectations on both sides of this inequality and minimizing over all $\boldsymbol{z_p}$, we obtain:

$$E_{\boldsymbol{\pi}} \left[\max_{p \in \mathcal{P}} (\boldsymbol{w_p} \cdot \tilde{\boldsymbol{x}} - k_p) \right] \;\leq\; \min_{\boldsymbol{z_1}, \ldots, \boldsymbol{z_P}} \left(\max_{p \in \mathcal{P}} (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}} \left[\max_{p \in \mathcal{P}} (\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}) \right] \right).$$

[3] Finding the best lower bound is also an interesting problem which we do not consider in this paper. A simple bound can be obtained by using Jensen's inequality.

Since this inequality is valid for every risk neutral distribution $\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_R)$, we obtain the upper bound:

$$\max_{\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_R)} E_{\boldsymbol{\pi}}\left[\max_{p\in\mathcal{P}}(\boldsymbol{w_p}\cdot\tilde{\boldsymbol{x}} - k_p)\right] \leq \min_{\boldsymbol{z_1},\ldots,\boldsymbol{z_P}} \left(\max_{p\in\mathcal{P}}(\boldsymbol{e}\cdot\boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}}\left[\max_{p\in\mathcal{P}}\left(\boldsymbol{w_{pr}}\cdot\tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}\right)\right]\right).$$

The proof of the tightness is provided in the Appendix. ■

In Theorem 1, the decision variable z_{pr} can be interpreted as the strike price for the pth term of the option payoff for the rth subset of assets. The cash payoff term links the strike prices among the different subsets while the option payoff terms depends on the individual subsets only. Theorem 1 thus generates the seller's price of the multiasset option by constructing such a buy-and-hold super-replicating portfolio in the cheapest possible manner. This approach to pricing the option is useful when the dependence structure among the asset prices in the rth subset is well-understood but the relation between assets in two different subsets is not well-understood. Each of the R smaller options can then potentially be priced with high accuracy using simulation or any other technique. For the appropriately identified strike prices, the seller can compute an arbitrage-free tight upper bound on the multiasset option price.

For a fixed number of assets N, Theorem 1 can be used to construct a hierarchy of relaxations that provides increasingly tighter price bounds by reducing the number of subsets R. For $R = N$, the super-replicating portfolio consists only of cash and options on individual assets. This is the simplest possible relaxation that has been developed in Deelstra [8], Hobson et al. [15], Lai and Robbins [17], Ross [25] (see Formulations (3)-(4)). Therein, the basket and maximum option price is obtained using a portfolio of cash and simple call options. By decreasing R below N, Theorem 1 allow for the possibility of using more sophisticated options that depends on combinations of assets in the super-replicating portfolio. Setting $R = 1$, of course we recover the exact price of the option with the decision variables set to k_p. Decreasing R however comes with a possible increase in computational complexity that arises from the need to evaluate more complicated option prices. For example, assume that each of the N assets are divided into M price levels. Then to compute the exact price of the option in (5) with a lattice based approach, one would need to need to construct a multidimensional lattice with $O(M^N)$ nodes. Assuming that each of the R subsets contains exactly N/R assets, solving the formulation in Theorem 1 would need the construction of a multidimensional lattice of $O(RM^{N/R})$ nodes. It is clear from Table 1 that for large R this can be significantly smaller as the prices of the options on the smaller lattices can be evaluated independently for a set of strike prices.

R/N	1	2	4	8	16	32
1	1×10^1	1×10^2	1×10^4	1×10^8	1×10^{16}	1×10^{32}
2	-	2×10^1	2×10^2	2×10^4	2×10^8	2×10^{16}
4	-	-	4×10^1	4×10^2	4×10^4	4×10^8
8	-	-	-	8×10^1	8×10^2	8×10^4
16	-	-	-	-	16×10^1	16×10^2
32	-	-	-	-	-	32×10^1

Table 1: Lattice sizes need to solve Theorem 1 for M = 10 price levels.

2.1 Numerical Methods

An alternative interpretation of Theorem 1 is as a two-stage stochastic program with recourse. The strike prices z_{pr} need to be determined in the first stage taking into account the deterministic cash flow and the future (second stage) option payoffs with respect to these strikes. One possible approach to solve the formulation in Theorem 1 is to use a Monte Carlo simulation-based optimization approach as proposed in Shapiro and Homem-de-Mello [26]. Let $\{\boldsymbol{x_{r1}}, \boldsymbol{x_{r2}}, \dots, \boldsymbol{x_{rM_r}}\}$ be i.i.d (independent identically distributed) random samples generated from the distribution $\boldsymbol{\pi_r}$ for $\boldsymbol{\tilde{x}_r}$. The samples for the different subsets r can be generated independently in this case. The two-stage stochastic program can then be approximated as:

$$\min_{\boldsymbol{z_1},\dots,\boldsymbol{z_P}} \left(\max_{p \in \mathcal{P}} (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} \sum_{i=1}^{M_r} \frac{1}{M_r} \max_{p \in \mathcal{P}} \left(\boldsymbol{w_{pr}} \cdot \boldsymbol{x_{ri}} - z_{pr} \right) \right),$$

or equivalently the linear program:

$$\begin{aligned} \min_{\boldsymbol{z_p}, y, y_{ri}} \quad & \left(y + \sum_{r=1}^{R} \sum_{i=1}^{M_r} \frac{1}{M_r} y_{ri} \right) \\ \text{s.t.} \quad & y \geq \boldsymbol{e} \cdot \boldsymbol{z_p} - k_p, \qquad p \in \mathcal{P} \\ & y_{ri} \geq \boldsymbol{w_{pr}} \cdot \boldsymbol{x_{ri}} - z_{pr}, \quad p \in \mathcal{P},\ i = 1, \dots, M_r,\ r = 1, \dots, R. \end{aligned}$$

However this can be a very large-sized linear program if the total sum of samples $\sum_r M_r$ is large. A more practical approach in this case is to use a simulation-based subgradient method. The update step in the subgradient method is:

$$\left(z_{pr}^{(k+1)}\right)_{pr} = \left(z_{pr}^{(k)}\right)_{pr} - \alpha_k \left(g_{pr}^{(k)}\right)_{pr},$$

where $z_{pr}^{(k)}$ is the kth iterate, $\alpha_k > 0$ is the step-size and $g_{pr}^{(k)}$ is the subgradient of the objective function with respect to z_{pr} at the current iterate. In Theorem 1, the subgradient can be approximated using a simulation based approach as:

$$g_{pr}^{(k)} = \mathbb{I}\left(\boldsymbol{e} \cdot \boldsymbol{z_p^{(k)}} - k_p \text{ is max}\right) - \sum_{i=1}^{M_r} \mathbb{I}\left(\boldsymbol{w_{pr}} \cdot \boldsymbol{x_r^{(k)}} - z_{pr}^{(k)} \text{ is max}\right),$$

where $\mathbb{I}(\cdot)$ is the indicator function. Detailed convergence results for this approach are discussed in Shor [27] and Bertsekas [2].

3 Examples of Multiasset Options

In this section, we apply the pricing bounds to some popular multiasset options that are traded in the market or have been considered in literature (cf. Zhang [28]). The three options we consider are:

(i) Basket/multiple spread option with payoff: $(\sum_i w_i \tilde{x}_i - k)^+$,

(ii) Option on the maximum of assets with payoff: $(\max_i \tilde{x}_i - k)^+$,

(iii) Option on the spread between maximum and minimum with payoff: $(\max_i \tilde{x}_i - \min_i \tilde{x}_i - k)^+$.

The general formulation in Theorem 1 has $O(P \times R)$ strike prices that need to be determined where P is the number of terms in the option payoff and R is the number of disjoint subsets. This can however be often significantly reduced using the structural properties of the option payoff. We indicate this simplification for the three options considered above.

Proposition 1 *Let* $\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\pi_1}, \ldots, \boldsymbol{\pi_R})$.

(i) For the basket/multiple spread option, the tightest upper bound on the price is given as:

$$\sum_{r=1}^{R} E_{\boldsymbol{\pi_r}} \left[\sum_{i \in \mathcal{N}_r} w_i \tilde{x}_i - z_r\right]^+,$$

where the optimal strikes $\boldsymbol{z}$ and the dual variable λ solves the system of equations:

$$\begin{aligned} P_{\boldsymbol{\pi_r}}\left(\sum_{i\in\mathcal{N}_r} w_i\tilde{x}_i \geq z_r\right) &= \lambda, \quad r = 1,\ldots,R, \\ \sum_{r=1}^{R} z_r &= k. \end{aligned}$$

(ii) For the option on the maximum of assets, the tightest upper bound on the price is given as:

$$\left((z-k)^+ + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}}\left[\max_{i\in\mathcal{N}_r}\tilde{x}_i - \max(z,k)\right]^+\right),$$

where the optimal strike z solves the equation:

$$\sum_{r=1}^{R} P_{\boldsymbol{\pi_r}}\left(\max_{i\in\mathcal{N}_r}\tilde{x}_i \geq z\right) = 1.$$

(iii) For the option on the spread between maximum and minimum with $R > 1$, the tightest upper bound on the price is given as:

$$\left((z_1 - z_2 - k)^+ + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}}\left[\max_{i\in\mathcal{N}_r}\tilde{x}_i - z_1\right]^+ + E_{\boldsymbol{\pi_r}}\left[z_2 - \min_{i\in\mathcal{N}_r}\tilde{x}_i\right]^+\right).$$

where the optimal strikes z_1 and z_2 can be determined in the following manner:

1. *Solve the two equations:*

$$\sum_{r=1}^{R} P_{\boldsymbol{\pi_r}}\left(\max_{i\in\mathcal{N}_r}\tilde{x}_i \geq z_1\right) = 1 \text{ and } \sum_{r=1}^{R} P_{\boldsymbol{\pi_r}}\left(\min_{i\in\mathcal{N}_r}\tilde{x}_i \leq z_2\right) = 1,$$

If $z_1 \geq z_2 + k$, these are the optimal strikes, else go to 2.

2. *Solve the equation:*

$$-\sum_{r=1}^{R} P_{\boldsymbol{\pi_r}}\left(\max_{i\in\mathcal{N}_r}\tilde{x}_i \geq z_1\right) + \sum_{r=1}^{R} P_{\boldsymbol{\pi_r}}\left(\min_{i\in\mathcal{N}_r}\tilde{x}_i \leq z_1 - k\right) = 1,$$

and set $z_2 = z_1 - k$.

Proof.

(i) Using Theorem 1, the tight upper bound on the price of the basket/multiple spread option is obtained by solving:

$$\min_{z_1,\ldots,z_R}\left(\left(\sum_{r=1}^{R} z_r - k\right)^+ + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}}\left[\sum_{i\in\mathcal{N}_r} w_i\tilde{x}_i - z_r\right]^+\right).$$

The cash term $(\sum_r z_r - k)^+$ is non-decreasing in z_r for any r. Say, the term $(\sum_r z_r - k)$ is strictly positive $\epsilon > 0$. We can then decrease at least one of the z_r's by ϵ such that the cash term decreases by ϵ while one of the option price terms would increase by atmost ϵ. Using a similar argument for a strictly negative ϵ, we can verify that there exists an optimal solution which satisfies $\sum_r z_r = k$. The optimality conditions to the problem:

$$\begin{aligned} \min_{z_1,\ldots,z_R} \quad & \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}} \left(\sum_{i \in \mathcal{N}_r} w_i \tilde{x}_i - z_r \right)^+ \\ \text{s.t.} \quad & \sum_{r=1}^{R} z_r = k, \end{aligned}$$

provides (i) where λ is the Lagrange multiplier for the equation $\sum_r z_r = k$.

(ii) Using Theorem 1, the tight upper bound on the price of the option on the maximum of several assets is obtained by solving:

$$\min_{z_1,\ldots,z_R} \left(\left(\max_{r=1,\ldots,R} z_r - k \right)^+ + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}} \left[\max_{i \in \mathcal{N}_r} \tilde{x}_i - z_r \right]^+ \right).$$

This can be simplified to a single variable problem. Without loss of generality, let $z_1 \geq z_2 \geq \ldots \geq z_R$ denote an optimal solution to the problem. For any $r > 1$, by increasing z_r up to z_1, the first cash term remains unaffected while the second option price term is non-increasing in z_r. Hence there exists an optimal solution with all the z_r values equal. Thus we need to solve the single variable optimization problem:

$$\min_{z} \left((z - k)^+ + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}} \left[\max_{i \in \mathcal{N}_r} \tilde{x}_i - z \right]^+ \right).$$

Clearly, the optimal $z \geq k$. Else, we can increase the z, without increasing the objective but possibly decreasing it. The optimality conditions then provides (ii).

(iii) Using Theorem 1 for $R > 1$, the tight upper bound on the price of the option on the spread between the maximum and minimum of assets is obtained by solving:

$$\min_{\boldsymbol{z_1},\boldsymbol{z_2}} \left(\left(\max_{i=1,\ldots,n} z_{i1} - \min_{i=1,\ldots,n} z_{i2} - k \right)^+ + \sum_{r=1}^{R} \left(E_{\boldsymbol{\pi_r}} \left[\max_{i \in \mathcal{N}_r} (\tilde{x}_i - z_{i1}) \right]^+ + E_{\boldsymbol{\pi_r}} \left[\max_{i \in \mathcal{N}_r} (z_{i2} - \tilde{x}_i) \right]^+ \right) \right).$$

Using an argument as in (ii), it can be checked that the optimal values of z_{i1} are all equal and likewise the optimal values of z_{i2} are all equal. Thus we need to solve the two-variable minimization problem:

$$\min_{z_1,z_2}\left((z_1-z_2-k)^+ + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}}\left(\left[\max_{i\in\mathcal{N}_r}\tilde{x}_i - z_1\right]^+ + E_{\boldsymbol{\pi_r}}\left[z_2 - \min_{i\in\mathcal{N}_r}\tilde{x}_i\right]^+\right)\right).$$

The optimality conditions to this problem provides (iii). ■

Formulation (i) in Proposition 1 can be solved by searching for single variate optimal Lagrange multiplier λ. Formulations (ii)-(iii) can be solved using a bisection search method. Simulations are used to estimate the probabilities and expectations required to solve the formulations in Proposition 1.

3.1 Numerical Example

The example is based on a multivariate Black-Scholes model using parameters from Carmon and Durrleman [7]. A set of $N = 16$ assets, all with initial values of \$100 and the same volatility of $\sigma = 10\%$ are considered. The correlation between any two assets in $\rho = 30\%$. The interest rate is zero and time to maturity is one year. For comparison, we use five partitions with $R = 16, 8, 4, 2, 1$ to obtain price estimates on the multiasset options (see Figure 1).

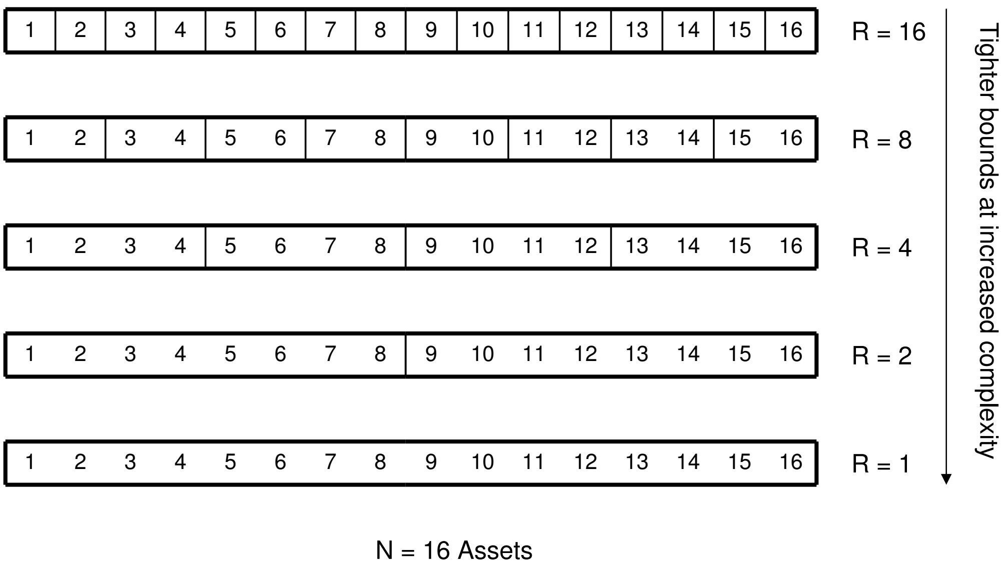

Figure 1: Partition of assets into subsets.

(i) Consider an equally weighted basket option with payoff $\left[\sum_{i=1}^{N} \tilde{x}_i/N - k\right]^+$. Due to symmetry of the data, the optimal decision variables are $z_r = k/R$ with R subsets. The upper bound is given as:

$$\text{Basket option price bound} \quad = \quad RE\left[\sum_{i=1}^{N/R} \tilde{x}_i/N - k/R\right]^+.$$

(ii-iii) For the option on the maximum of asset prices, the upper bound is given as:

$$\text{Max option price bound} \quad = \quad \left((z-k)^+ + RE\left[\max_{i=1,\ldots,N/R} \tilde{x}_i - \max(z,k)\right]^+\right).$$

We use the closed form result from Johnson [13] to price the smaller options on the maximum of N/R assets:

$$E\left[\max_{i=1,\ldots,N/R} \tilde{x}_i - \max(z,k)\right]^+ = \frac{N}{R}\mathbb{N}_{N/R}\left(\boldsymbol{d_1}, \boldsymbol{Q_1}\right) - \max(z,k)\left(1 - \mathbb{N}_{N/R}\left(-\boldsymbol{d_2}, \boldsymbol{Q_2}\right)\right).$$

Here $\mathbb{N}_{N/R}$ is the cumulative distribution function of a standard N/R-variate normal with the vector entries given as:

$$d_{11} = \left(\frac{\log\left(\frac{x}{\max(z,k)}\right) + \frac{\sigma^2}{2}}{\sigma}\right) \text{ and } d_{1j} = \left(\sigma\sqrt{\frac{1-\rho}{2}}\right), \quad j = 2,\ldots,N/R,$$

and

$$d_{2j} = \left(\frac{\log\left(\frac{x}{\max(z,k)}\right) - \frac{\sigma^2}{2}}{\sigma}\right), \quad j = 1,\ldots,N/R.$$

The correlation matrices are given as:

$$Q_{ij1} = \begin{cases} 1, & \text{if } i = j, \\ \sqrt{\frac{1-r}{2}}, & \text{if } i = 1, j \neq 1 \text{ or } j = 1, i \neq 1, \\ \sqrt{\frac{1}{2}}, & \text{otherwise.} \end{cases}$$

and

$$Q_{ij2} = \begin{cases} 1, & \text{if } i = j, \\ \rho, & \text{otherwise.} \end{cases}$$

The optimal decision variable z is found by the solution to the equation:

$$\mathbb{N}_{N/R}\left(-\boldsymbol{d_2}, \boldsymbol{Q_2}\right) = \frac{R-1}{R}.$$

A similar approach can be used to find the bound in (iii), using the formulas for the option on the minimum of the assets from Johnson [13].

The prices for the three options for different strike prices are provided in Figure 2. A total of 500000 simulations with an error tolerance of 10^{-4} was used in the optimization models. In this example, the optimal strike prices are known in closed form for the basket option and identified using bisection search for the other two options. Clearly, from Figure 2 the bounds become tighter as the number of subsets decreases. For the basket and maximum option, our bounds for $R < N$ seem to be new. For the spread option between the maximum and minimum of asset prices, the bounds seem to be new even for $R = N$. Figure 3 plots the optimal strike prices in the super-replicating portfolios for these three options. Figure 4 plots the distribution of the minimum cost super-replicating portfolios. The improvement in the fit is clearly observable as R decreases. A summary of the result is provided in Table 2. Clearly, the computational time (CPU sec) increases rapidly as R decreases. The hedging errors in the optimal super-replicating portfolios exhibits a positive skewness with a small possibility of having large errors. The magnitude of these errors however become lesser as R decreases.

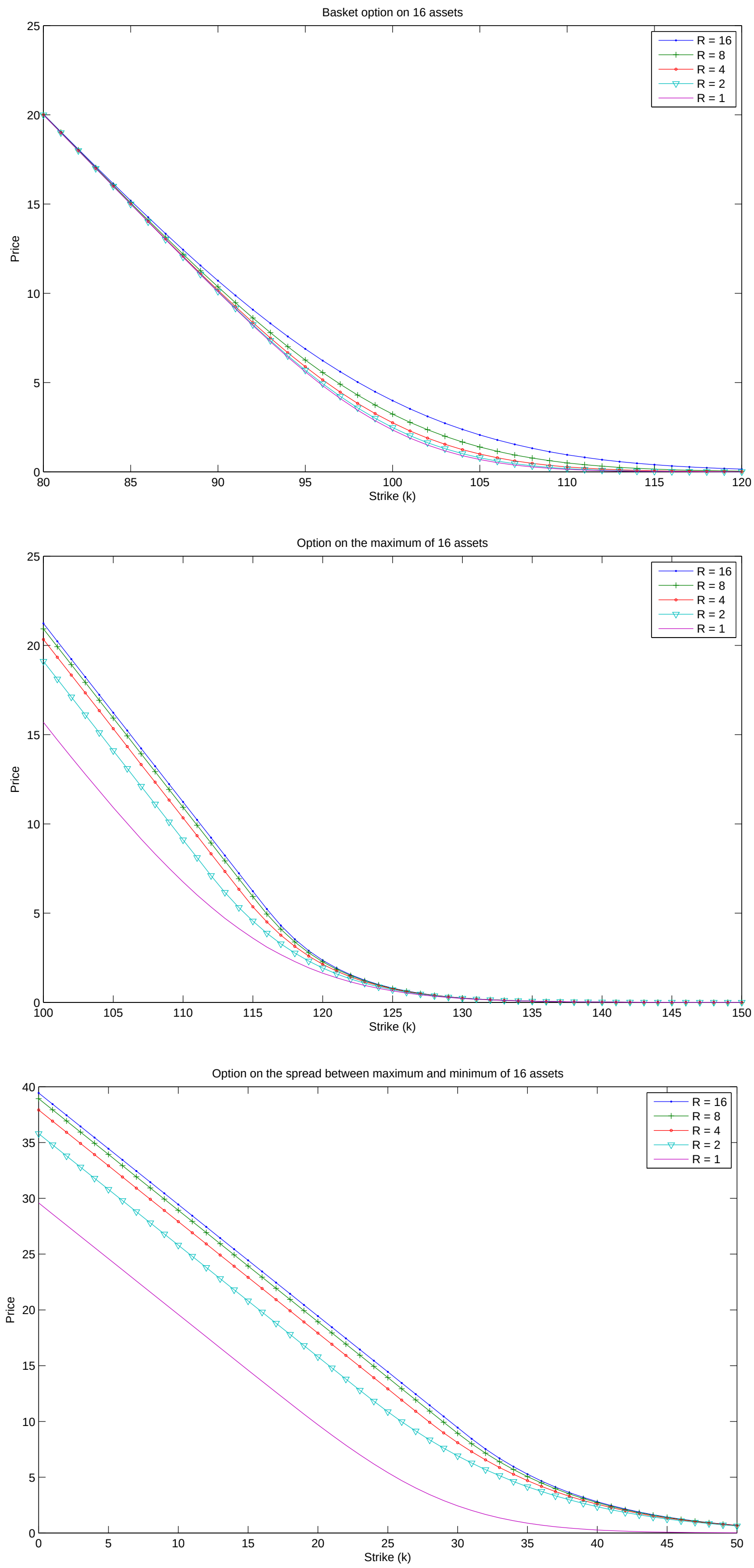

Figure 2: Option price bounds for $N = 16$ assets.

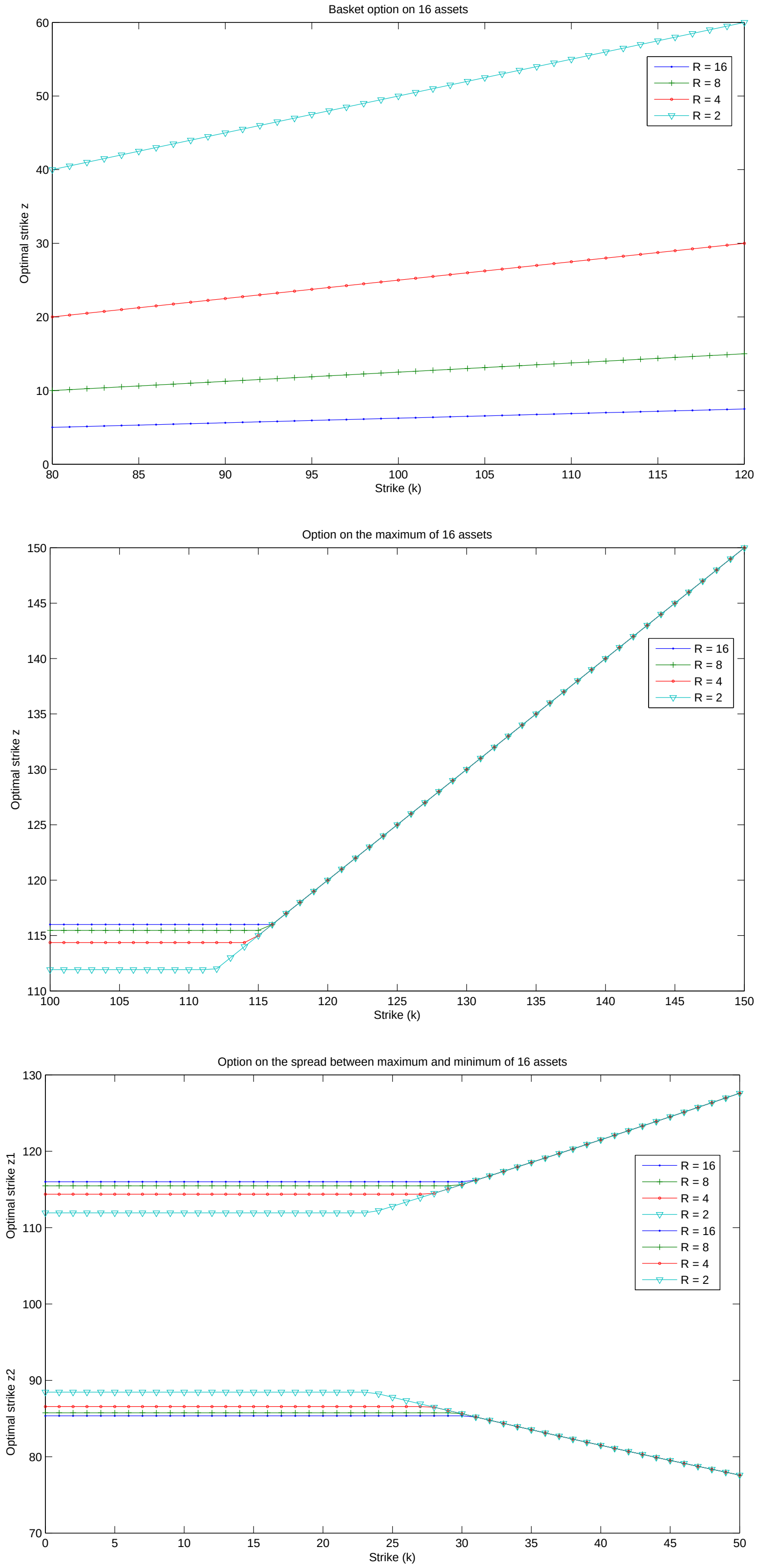

Figure 3: Optimal strike prices for $N = 16$ assets.

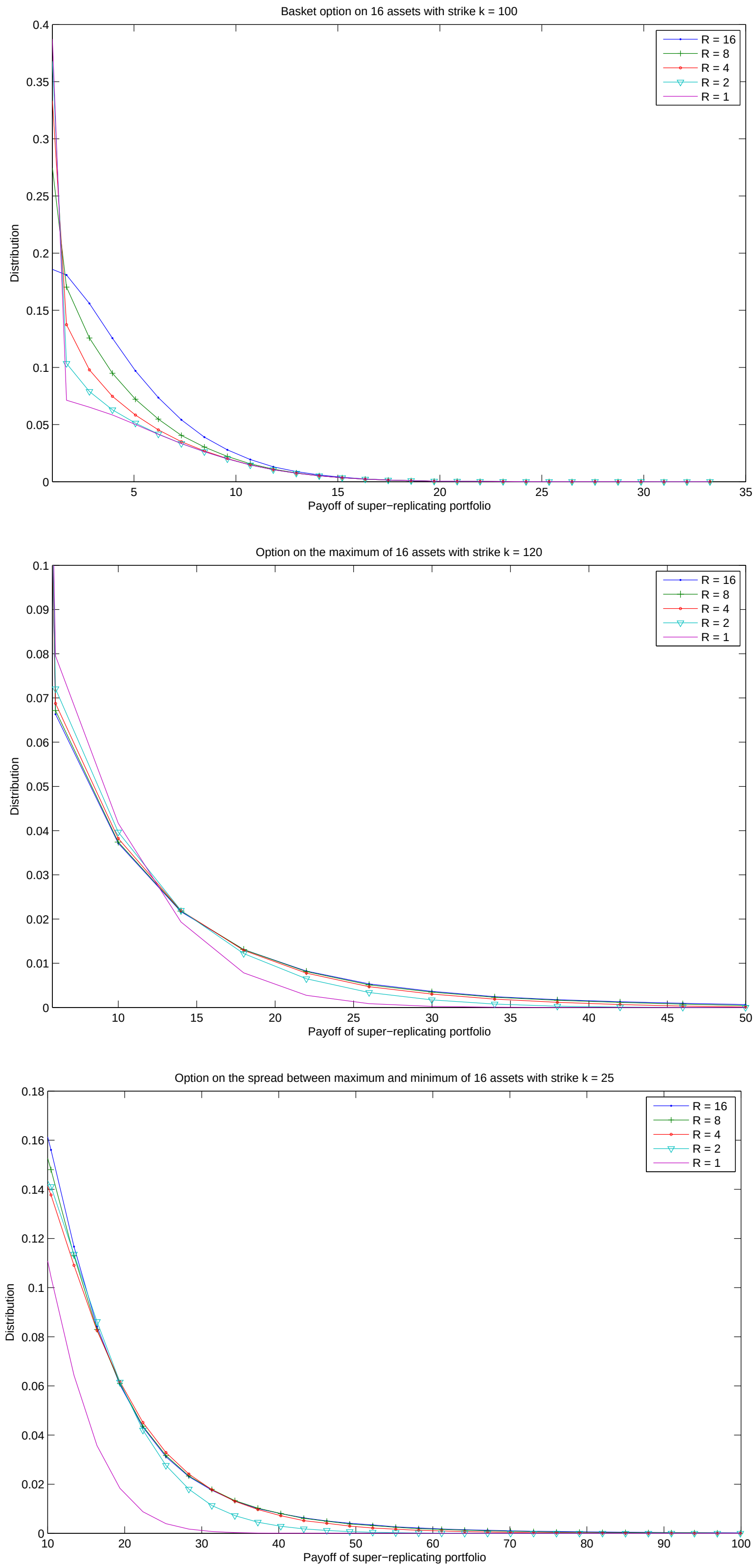

Figure 4: Distribution of super-replicating portfolios for $N = 16$ assets.

	Basket option with strike $k = 100$ Actual price = 2.3421			
No. of subsets	16	8	4	2
Price of option	3.9858	3.2292	2.7470	2.4791
CPU Time (sec)	0.14	0.29	0.60	1.44
Average super-replication error	1.6480	0.8772	0.4108	0.1443
Std. dev of super-replication error	1.0295	0.8083	0.5878	0.3749
Maximum super-replication error	6.5672	5.9426	4.5618	4.3224
	Max option with strike $k = 120$ Actual price = 1.6370			
No. of subsets	16	8	4	2
Price of option	2.3573	2.2739	2.1357	1.9286
CPU Time (sec)	0.30	0.53	4.00	42.0
Average super-replication error	0.7227	0.6405	0.5011	0.2870
Std. dev of super-replication error	3.6240	3.1272	2.3721	1.3766
Maximum super-replication error	197.5051	139.2040	99.5015	37.9512
	Max-min option with strike $k = 25$ Actual price = 5.4147			
No. of subsets	16	8	4	2
Price of option	14.4435	13.9326	12.9156	10.8642
CPU Time (sec)	0.35	0.60	5.00	101
Average super-replication error	9.0506	8.5397	7.5200	5.4629
Std. dev of super-replication error	10.5945	9.8080	8.4822	6.0876
Maximum super-replication error	273.3000	195.2905	139.9110	67.4794

Table 2: Prices of the options and super-replication errors.

4 Pricing in Incomplete Markets with Other Options Being Traded

In this section, we extend the results to find semi-parametric price bounds in incomplete markets that are valid under even weaker distributional assumptions (cf. Lo [20], Grundy [9], Boyle and Lin [6], Bertsimas and Popescu [3], Aspremont and Ghaoui [1], Hobson et al. [15]). The central problem addressed in these papers is: Suppose we are interested in finding bounds on the price of a multiasset option with payoff $f(\tilde{\boldsymbol{x}})$, consistent with known traded option prices $E_{\boldsymbol{\pi}}\left[f_j(\tilde{\boldsymbol{x}})\right] = q_j, j = 1, \ldots, M$ in the market. Under no-arbitrage, the tightest possible upper bound on the price of this option is found by solving the optimization problem:

$$\begin{array}{ll} \max\limits_{\boldsymbol{\pi}} & E_{\boldsymbol{\pi}}\left[f(\tilde{\boldsymbol{x}})\right] \\ \text{s.t.} & E_{\boldsymbol{\pi}}\left[f_j(\tilde{\boldsymbol{x}})\right] = q_j, \quad j = 1, \ldots, M \\ & \boldsymbol{\pi} \in \mathcal{M}(\Re_+^N), \end{array} \tag{6}$$

where $\boldsymbol{\pi}$ is the risk-neutral measure defined on $\mathcal{M}(\Re_+^N)$ (the set of probability measures supported on $\Re_+^N$).

Isii [12] and Bertsimas and Popescu [3] solved the problem in (6) by using a dual approach. Introducing the dual variable y_0 for the probability mass constraint and y_j for the jth constraint, the dual problem can be formulated as:

$$\begin{array}{ll} \min\limits_{y_0, y_j} & y_0 + \sum\limits_{j=1}^{M} y_j q_j \\ \text{s.t.} & y_0 + \sum\limits_{j=1}^{M} y_j f_j(\boldsymbol{x}) \geq f(\boldsymbol{x}), \quad \forall \boldsymbol{x} \geq \mathbf{0}. \end{array} \tag{7}$$

This is a semi-infinite optimization problem where the constraints are valid for every non-negative $\boldsymbol{x}$. Under standard constraint qualification conditions, strong duality holds and the two formulations are equivalent. Bertsimas and Popescu [3] showed that for single-asset options ($N = 1$) with piecewise polynomials payoffs $f_j(\cdot)$ and $f(\cdot)$, the dual problem can be solved in polynomial time using semidefinite programming. However as a negative result, they showed that finding tight bounds for multiasset options is often much more difficult (NP-Hard in general). As a result, weaker semidefinite programming relaxations have been proposed in Boyle and Lin [6], Zuluaga and Pena [29], Han et al. [11] and Lasserre et al. [19]. The dual problem (7) has a natural financial interpretation as the problem of finding a minimum cost buy-and-hold portfolio consisting of cash and options that are traded in the market that dominates the payoff of the option to be priced with probability one (see Nishihara [23] et al.).

In this section, we extend Theorem 1 to find a new class of multiasset option pricing models for which tight semi-parametric bounds can be found in polynomial time. Suppose the exact risk-neutral distributions for each of the subsets of asset prices $\tilde{\boldsymbol{x}}_{\boldsymbol{r}}$ are not known. Rather there exists a finite set of options that are traded on each of these assets for which the prices are known. We denote the set of risk-neutral distributions for each set of assets $\tilde{\boldsymbol{x}}_{\boldsymbol{r}}$ as:

$$\boldsymbol{\Pi}_{\boldsymbol{r}} = \left\{ \boldsymbol{\pi}_{\boldsymbol{r}} \in \mathcal{M}\left(\Re_{+}^{N_r}\right) \;\middle|\; E_{\boldsymbol{\pi}_{\boldsymbol{r}}}\left[\boldsymbol{f}_{\boldsymbol{r}}(\tilde{\boldsymbol{x}}_{\boldsymbol{r}})\right] = \boldsymbol{q}_{\boldsymbol{r}} \right\},$$

where $\boldsymbol{q}_{\boldsymbol{r}} = (q_{r1}, \ldots, q_{rM_r})$ is the vector of option prices with payoffs $\boldsymbol{f}_{\boldsymbol{r}}(\tilde{\boldsymbol{x}}_{\boldsymbol{r}}) = (f_1(\tilde{\boldsymbol{x}}_{\boldsymbol{r}}), \ldots, f_{M_r}(\tilde{\boldsymbol{x}}_{\boldsymbol{r}}))$. Let $\mathbb{M}(\boldsymbol{\Pi}_{\boldsymbol{1}}, \ldots, \boldsymbol{\Pi}_{\boldsymbol{R}})$ denote the set of all joint risk-neutral distributions $\boldsymbol{\pi}$ for asset prices $\tilde{\boldsymbol{x}}$ with the risk-neutral distributions $\boldsymbol{\pi}_{\boldsymbol{r}} \in \boldsymbol{\Pi}_{\boldsymbol{r}}$ for each $\tilde{\boldsymbol{x}}_{\boldsymbol{r}}$. We need the following assumption that is key to the results in this section.

Assumption 1 *Suppose that:*

$$\boldsymbol{q}_{\boldsymbol{r}} \in int\left\{ E_{\boldsymbol{\pi}_{\boldsymbol{r}}}\left[\boldsymbol{f}(\tilde{\boldsymbol{x}}_{\boldsymbol{r}})\right] \;\middle|\; \boldsymbol{\pi}_{\boldsymbol{r}} \text{ is a distribution in } \Re_{+}^{N_r} \right\}, \quad r = 1, \ldots, R,$$

where $int\{S\}$ denotes the interior of the set S.

Assumption 1 ensures that strong duality holds in this model. However checking this assumption for arbitrary subset sizes N_r and payoffs $\boldsymbol{f}(\tilde{\boldsymbol{x}}_{\boldsymbol{r}})$ might not always be easy (cf. Bertsimas and Popescu [3]). We now provide the extension of Theorem 1 to the incomplete market case with options being traded.

Theorem 2 *Let $\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\Pi}_{\boldsymbol{1}}, \ldots, \boldsymbol{\Pi}_{\boldsymbol{R}})$. Under Assumption 1, the tight upper bound on the price of a multiasset option with piecewise linear convex payoff is found by solving the convex minimization problem:*

$$\max_{\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\Pi}_{\boldsymbol{1}}, \ldots, \boldsymbol{\Pi}_{\boldsymbol{R}})} E_{\boldsymbol{\pi}}\left[\max_{p \in \mathcal{P}}(\boldsymbol{w}_{\boldsymbol{p}} \cdot \tilde{\boldsymbol{x}} - k_p)\right] = \min_{\boldsymbol{z}_{\boldsymbol{1}}, \ldots, \boldsymbol{z}_{\boldsymbol{P}}} \left[\max_{p \in \mathcal{P}}(\boldsymbol{e} \cdot \boldsymbol{z}_{\boldsymbol{p}} - k_p) + \sum_{r=1}^{R} \max_{\boldsymbol{\pi}_{\boldsymbol{r}} \in \boldsymbol{\Pi}_{\boldsymbol{r}}} E_{\boldsymbol{\pi}_{\boldsymbol{r}}}\left[\max_{p \in \mathcal{P}}(\boldsymbol{w}_{\boldsymbol{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr})\right]\right].$$

Outline of Proof. Since the exact distribution $\boldsymbol{\pi}_{\boldsymbol{r}} \in \boldsymbol{\Pi}_{\boldsymbol{r}}$ in Theorem 1 is not known, we find the best upper bound by solving:

$$\max_{\boldsymbol{\pi} \in \mathbb{M}(\boldsymbol{\Pi}_{\boldsymbol{1}}, \ldots, \boldsymbol{\Pi}_{\boldsymbol{R}})} E_{\boldsymbol{\pi}}\left[\max_{p \in \mathcal{P}}(\boldsymbol{w}_{\boldsymbol{p}} \cdot \tilde{\boldsymbol{x}} - k_p)\right] = \max_{\boldsymbol{\pi}_{\boldsymbol{r}} \in \boldsymbol{\Pi}_{\boldsymbol{r}} \forall r} \; \min_{\boldsymbol{z}_{\boldsymbol{p}} \forall p \in \mathcal{P}} \left[\max_{p \in \mathcal{P}}(\boldsymbol{e} \cdot \boldsymbol{z}_{\boldsymbol{p}} - k_p) + \sum_{r=1}^{R} E_{\boldsymbol{\pi}_{\boldsymbol{r}}}\left[\max_{p \in \mathcal{P}}(\boldsymbol{w}_{\boldsymbol{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr})\right]\right].$$

By interchanging the order of the maximum and minimum, we obtain the inequality:

$$\max_{\boldsymbol{\pi}\in\mathbb{M}(\boldsymbol{\Pi_1},\ldots,\boldsymbol{\Pi_R})} E_{\boldsymbol{\pi}}\left[\max_{p\in\mathcal{P}}(\boldsymbol{w_p}\cdot\tilde{\boldsymbol{x}}-k_p)\right] \leq \min_{\boldsymbol{z_p}}\left[\max_{p\in\mathcal{P}}(\boldsymbol{e}\cdot\boldsymbol{z_p}-k_p)+\sum_{r=1}^{R}\max_{\boldsymbol{\pi_r}\in\boldsymbol{\Pi_r}} E_{\boldsymbol{\pi_r}}\left[\max_{p\in\mathcal{P}}(\boldsymbol{w_{pr}}\cdot\tilde{\boldsymbol{x}}_{\boldsymbol{r}}-z_{pr})\right]\right].$$

Under Assumption 1, tightness can be proved using a duality-based approach as in Theorem 3.1 in Bertsimas, Natarajan and Teo [4]. Due to similarity of the proof, we skip it. ■

Substituting the inner maximization problems with the dual problem, we can reformulate Theorem 2 as:

$$\begin{aligned}
\min_{y,\boldsymbol{z_p},\boldsymbol{y_r}} \quad & \left(y+\sum_{r=1}^{R}\boldsymbol{y_r}\cdot\boldsymbol{q_r}\right) \\
\text{s.t.} \quad & y \geq \boldsymbol{e}\cdot\boldsymbol{z_p}-k_p, \qquad\qquad p\in\mathcal{P} \\
& \boldsymbol{y_r}\cdot\boldsymbol{f_r}(\boldsymbol{x_r})-\boldsymbol{w_{pr}}\cdot\boldsymbol{x_r}+z_{pr}\geq 0, \quad \forall\boldsymbol{x_r}\geq\boldsymbol{0},\ p\in\mathcal{P},\ r=1,\ldots,R.
\end{aligned} \tag{8}$$

This is a semi-infinite optimization problem, where the constraints are valid for every $\boldsymbol{x_r}$ in the nonnegative orthant. The variable y denotes the units of cash and $\boldsymbol{y_r}$ denotes the units in the options with payoffs $\boldsymbol{f_r}(\cdot)$ that are held in the minimum cost buy-and-hold super-replicating portfolio. The variable $\boldsymbol{z_p}$ denotes the strike prices of the smaller options that need to be super-replicated. In general, when the functions $\boldsymbol{f_r}(\cdot)$ are piecewise-linear or piecewise-polynomial, Formulation (8) can be approximated using linear or semidefinite programs. This follows from the well-known relationship between non-negative polynomials and semidefinite programs (see Parillo [24], Lasserre [18]). However these formulations can be exponentially large in the multivariate setting. We focus on two models wherein Formulation (8) can be solved in polynomial time.

Proposition 2

(i) Suppose each subset r consists of at most three assets ($N_r \leq 3$) with known mean $\boldsymbol{\mu_r}$ and covariance matrix $\boldsymbol{Q_r}$:

$$\Pi_r = \left\{\boldsymbol{\pi_r}\in\mathcal{M}(\Re_+^{N_r}) \;\middle|\; E_{\boldsymbol{\pi_r}}(\tilde{\boldsymbol{x}}_{\boldsymbol{r}}'\tilde{\boldsymbol{x}}_{\boldsymbol{r}})=\boldsymbol{Q_r}+\boldsymbol{\mu_r}\boldsymbol{\mu_r}', E_{\boldsymbol{\pi_r}}(\tilde{\boldsymbol{x}}_{\boldsymbol{r}})=\boldsymbol{\mu_r}\right\}.$$

Then the tight upper bound on the price of a multiasset option with piecewise linear convex *payoff is obtained by solving the semidefinite program:*

$$\begin{aligned}
\min_{y,\boldsymbol{z_p},\boldsymbol{y_r},\boldsymbol{S_{pr}},\boldsymbol{N_{pr}}} \quad & \left(y + \sum_{r=1}^{R} \boldsymbol{Y_r} \cdot (\boldsymbol{Q_r} + \boldsymbol{\mu_r}\boldsymbol{\mu_r}') + \sum_{r=1}^{R} \boldsymbol{y_r} \cdot \boldsymbol{\mu_r} + \sum_{r=1}^{R} y_r \right) & \\
s.t. \quad & y \geq \boldsymbol{e} \cdot \boldsymbol{z_p} - k_p, & p \in \mathcal{P} \\
& \begin{pmatrix} \boldsymbol{Y_r} & (\boldsymbol{y_r} - \boldsymbol{w_{pr}})/2 \\ (\boldsymbol{y_r'} - \boldsymbol{w_{pr}'})/2 & y_r + z_{pr} \end{pmatrix} = \boldsymbol{S_{pr}} + \boldsymbol{N_{pr}}, & p \in \mathcal{P},\ r = 1,\ldots,R \\
& \boldsymbol{S_{pr}} \succeq \boldsymbol{0},\ \boldsymbol{N_{pr}} \geq \boldsymbol{0}, & p \in \mathcal{P},\ r = 1,\ldots,R.
\end{aligned}$$

(ii) *Suppose, for each subset r, we have:*

$$\Pi_r = \left\{ \boldsymbol{\pi_r} \in \mathcal{M}(\Re_+^{N_r}) \ \Big|\ E_{\boldsymbol{\pi_r}} \left[\max_{t=1,\ldots,T} (\boldsymbol{A_{rt}}\boldsymbol{x_r} - \boldsymbol{b_{rt}}) \right] = \boldsymbol{q_r} \right\},$$

where the maximum is taken row-wise. Then the tight upper bound on the price of a multiasset option with piecewise linear convex payoff is obtained by solving the linear program:

$$\begin{aligned}
\min_{y,\boldsymbol{z_p},\boldsymbol{y_r}} \quad & \left(y + \sum_{r=1}^{R} \boldsymbol{y_r} \cdot \boldsymbol{q_r} \right) & \\
s.t. \quad & y \geq \boldsymbol{e} \cdot \boldsymbol{z_p} - k_p, & p \in \mathcal{P} \\
& -\sum_{t=1}^{T} \boldsymbol{\lambda_{prt}} \cdot \boldsymbol{b_{rt}} + z_{pr} \geq \boldsymbol{0}, & p \in \mathcal{P},\ r = 1,\ldots,R, \\
& \sum_{t=1}^{T} \boldsymbol{A_{rt}}\boldsymbol{\lambda_{prt}} - \boldsymbol{w_{pr}} \geq \boldsymbol{0}, & p \in \mathcal{P},\ r = 1,\ldots,R, \\
& \sum_{t=1}^{T} \boldsymbol{\lambda_{prt}} = \boldsymbol{y_r}, & p \in \mathcal{P},\ r = 1,\ldots,R.
\end{aligned}$$

Proof.

(i) The second constraint in Formulation (8) is equivalent to enforcing a quadratic polynomial to be nonnegative over the nonnegative orthant:

$$\boldsymbol{x_r'}\boldsymbol{Y_r}\boldsymbol{x_r} + \boldsymbol{y_r} \cdot \boldsymbol{x_r} + y_r - \boldsymbol{w_{pr}} \cdot \boldsymbol{x_r} + z_{pr} \geq 0, \quad \forall \boldsymbol{x_r} \geq \boldsymbol{0}.$$

The variables $\boldsymbol{Y_r}$, $\boldsymbol{y_r}$ and y_r are the dual variables corresponding to the second moment, first moment and probability mass constraint in Π_r. This nonnegativity constraint is equivalent to enforcing the following matrix to lie in the cone of copositive matrices (see Kabadi and Murty

[16]):

$$\begin{pmatrix} \boldsymbol{Y_r} & (\boldsymbol{y_r} - \boldsymbol{w_{pr}})/2 \\ (\boldsymbol{y_r'} - \boldsymbol{w_{pr}'})/2 & y_r + z_{pr} \end{pmatrix} \in \mathbb{C}_{N_r+1},$$

where $\mathbb{C}_{N_r+1}$ is the cone of copositive matrices of dimension N_r+1. For $N_r \leq 3$, the copositive cone is exactly characterizable as the sum of a semidefinite cone ($\mathbb{S}_{N_r+1}$) and nonnegative cone ($\mathbb{N}_{N_r+1}$) (see Diananda [10]):

$$\mathbb{C}_{N_r+1} = \mathbb{S}_{N_r+1} + \mathbb{N}_{N_r+1} \quad \text{ for } N_r \leq 3.$$

Checking Assumption 1 in this case is equivalent to:

$$\begin{pmatrix} \boldsymbol{Q_r} + \boldsymbol{\mu_r}'\boldsymbol{\mu_r} & \boldsymbol{\mu_r} \\ \boldsymbol{\mu_r'} & 1 \end{pmatrix} \in \text{int}(\mathbb{S}_{N_r+1}) \cup \text{int}(\mathbb{N}_{N_r+1}).$$

Thus with each subset $\mathcal{N}_r$ having at most 3 assets, we obtain an exact polynomial sized semidefinite formulation that can be solved efficiently. For larger subsets of assets, this provides an upper bound that is not necessarily tight. In this case, it is possible to obtain a sequence of semidefinite approximations for the copositive cone using results from Parillo [24]. These are however not polynomial sized formulations.

(ii) The payoffs $\boldsymbol{f_r}(\boldsymbol{x_r}) = \max_t(\boldsymbol{A_{rt}}\boldsymbol{x_r} - \boldsymbol{b_{rt}})$ are piecewise linear and convex. The second constraint in Formulation (8) is equivalent to:

$$\min_{\boldsymbol{x_r} \geq \boldsymbol{0}} \left\{ \boldsymbol{y_r} \cdot \max_{t=1,\ldots,T} (\boldsymbol{A_{rt}}\boldsymbol{x_r} - \boldsymbol{b_{rt}}) - \boldsymbol{w_{pr}} \cdot \boldsymbol{x_r} + z_{pr} \right\} \geq 0.$$

Introducing the decision vector $\boldsymbol{s_r}$, we have:

$$\min_{\boldsymbol{s_r},\boldsymbol{x_r}} \left\{ \boldsymbol{y_r} \cdot \boldsymbol{s_r} - \boldsymbol{w_{pr}} \cdot \boldsymbol{x_r} + z_{pr} \;\middle|\; \boldsymbol{s_r} \geq \boldsymbol{A_{rt}}\boldsymbol{x_r} - \boldsymbol{b_{rt}} \ \forall t, \ \boldsymbol{x_r} \geq \boldsymbol{0} \right\} \geq 0$$

Using linear programming duality, we obtain:

$$\max_{\boldsymbol{\lambda_{prt}}} \left\{ -\sum_t \boldsymbol{\lambda_{prt}} \cdot \boldsymbol{b_{rt}} + z_{pr} \;\middle|\; \sum_t \boldsymbol{\lambda_{prt}} = \boldsymbol{y_r}, \ \sum_t \boldsymbol{A_{rt}}\boldsymbol{\lambda_{prt}} \geq \boldsymbol{w_{pr}} \right\} \geq \boldsymbol{0}$$

This is equivalent to the set of linear constraints:

$$\left\{ -\sum_t \boldsymbol{\lambda_{prt}} \cdot \boldsymbol{b_{rt}} + z_{pr} \geq \boldsymbol{0}, \ \sum_t \boldsymbol{\lambda_{prt}} = \boldsymbol{y_r}, \ \sum_t \boldsymbol{A_{rt}}\boldsymbol{\lambda_{prt}} \geq \boldsymbol{w_{pr}} \right\}$$

By incorporating these constraints for each p and r, we obtain the desired linear program. ■

4.1 Numerical Example

We consider an extension of the example from Boyle and Lin [6] with an option on the maximum of $N = 6$ assets. Each of the assets has an initial price of \$40 and a volatility of $\sigma = 30\%$. The interest rate is $r = 10\%$ with a time to maturity of the option of one year. The correlation matrix is given as:

$$\boldsymbol{Q} = \left(\begin{array}{ccc|ccc} 1.0 & 0.9 & 0.9 & \rho & \rho & \rho \\ 0.9 & 1.0 & 0.9 & \rho & \rho & \rho \\ 0.9 & 0.9 & 1.0 & \rho & \rho & \rho \\ \hline \rho & \rho & \rho & 1 & 0.9 & 0.9 \\ \rho & \rho & \rho & 0.9 & 1.0 & 0.9 \\ \rho & \rho & \rho & 0.9 & 0.9 & 1.0 \end{array}\right),$$

with the exact value of ρ unknown. The first three assets and the last three assets are strongly positively correlated to each other. However the exact correlation between these two subsets of assets is not known. For simplicity, we assume that these correlations are the same and equal to ρ. In this setting, we can use the semidefinite program in Proposition 2 (i) with three assets in each subset to compute an upper bound on the option price. This price bound is valid under all risk-neutral distributions with the given mean and covariance matrix and for all feasible values of ρ. The semidefinite programs were solved using SeDuMi 1.1 in a MATLAB 7.3.0 environment. The results are summarized in Table 3 and Figure 5. The quality of the bound clearly improves as the two subsets of asset are more negatively correlated.

Strike price	Exact Black-Scholes price					Upper bound
k	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0.0$	$\rho = 0.5$	$\rho = 0.9$	
0	53.1012	51.9954	50.3710	48.1851	44.9749	58.1985
40	16.9078	15.8899	14.5511	12.8166	10.3196	22.0043
50	8.4607	8.2044	7.6126	6.6338	5.0889	13.1139
60	3.5616	3.5364	3.4035	3.0073	2.2365	7.7827
70	1.3934	1.3874	1.3667	1.2385	0.9095	5.2839

Table 3: Black-Scholes price and upper bound.

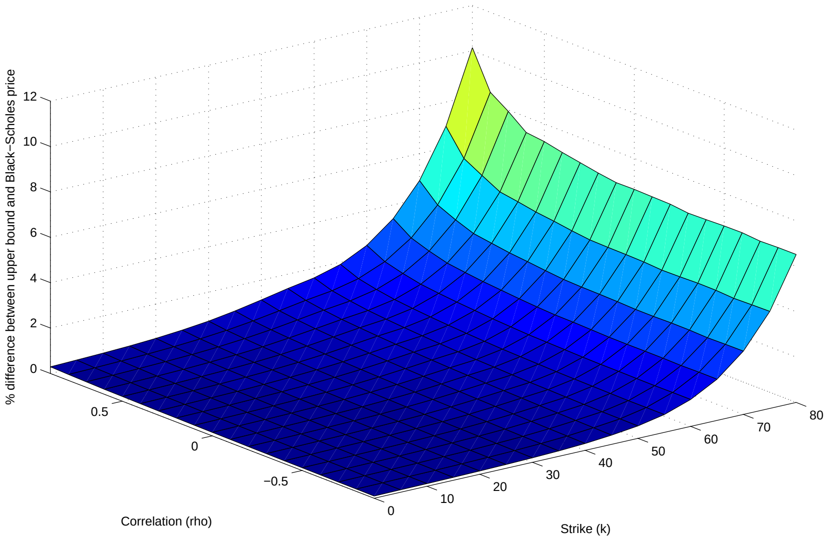

Figure 5: Percentage difference between upper bound and Black-Scholes price as k and ρ varies.

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5 Appendix

Theorem 1 (Proof Continued). To prove the tightness of the bound, we construct a distribution $\boldsymbol{\pi}^*$ from the optimal solution to the minimization problem:

$$\min_{\boldsymbol{z_1},\ldots,\boldsymbol{z_P}} \left(\max_{p\in\mathcal{P}} (\boldsymbol{e}\cdot\boldsymbol{z_p} - k_p) + \sum_{r=1}^{R} E_{\boldsymbol{\pi_r}} \left[\max_{p\in\mathcal{P}} \left(\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr} \right) \right] \right)$$

The Karush-Kuhn-Tucker conditions provide the necessary and sufficient optimality conditions for this convex minimization problem. We define $\mathcal{S}_{pr}$ to be the event:

$$\mathcal{S}_{pr} = \left\{ \boldsymbol{x_r} \;\middle|\; \boldsymbol{w_{pr}}\cdot\boldsymbol{x_r} - z_{pr} \ \geq \max_{q\in\mathcal{P}:q\neq p} \boldsymbol{w_{qr}}\cdot\boldsymbol{x_r} - z_{qr} \right\}.$$

The optimality conditions can then be expressed as:

$$
\begin{array}{ll}
\text{(i)} & \lambda_p \geq 0 \text{ for all } p \in \mathcal{P}, \\
\text{(ii)} & \displaystyle\sum_{p \in \mathcal{P}} \lambda_p = 1, \\
\text{(iii)} & \lambda_p \left(\displaystyle\max_{p \in \mathcal{P}} (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) - (\boldsymbol{e} \cdot \boldsymbol{z_p} - k_p) \right) = 0 \text{ for all } p \in \mathcal{P}, \\
\text{(iv)} & \lambda_p = P\left(\mathcal{S}_{pr}\right) \text{ for all } p \in \mathcal{P} \text{ and } r = 1, \ldots, R.
\end{array}
$$

Consider an optimal solution $\boldsymbol{z_1^*}, \ldots, \boldsymbol{z_P^*}$ and $\lambda_1^*, \ldots, \lambda_P^*$ that satisfies conditions (i)-(iv). Construct the multivariate distribution $\boldsymbol{\pi}^*$ as follows:

$$
\begin{array}{ll}
\text{(a)} & \text{Choose } p \in \mathcal{P} \text{ with probability } \lambda_p^*, \\
\text{(b)} & \text{For each } r = 1, \ldots, R, \text{ generate } \boldsymbol{\tilde{x}_r} \sim \boldsymbol{\pi_r}(\boldsymbol{x_r}) \mathbb{I}(\mathcal{S}_{pr})/\lambda_p^*,
\end{array}
$$

where $\mathbb{I}(S)$ is the indicator function of the set S. Note that the cross dependency between $\boldsymbol{\tilde{x}_{r_1}}$ and $\boldsymbol{\tilde{x}_{r_2}}$ for $r_1 \neq r_2$ is not important in this construction. For a fixed p, the distributions for each $\boldsymbol{\tilde{x}_r}$ can hence be generated independently. Under this construction, if $\boldsymbol{\pi'_r}(\cdot)$ denotes the joint distribution of $\boldsymbol{\tilde{x}_r}$, then we have:

$$
\begin{aligned}
\boldsymbol{\pi'_r}(\boldsymbol{x_r}) &= \sum_{p \in \mathcal{P}'} \lambda_p^* \left(\frac{\boldsymbol{\pi_r}(\boldsymbol{x_r}) \mathbb{I}(\mathcal{S}_{pr})}{\lambda_p^*} \right), \\
&= \boldsymbol{\pi_r}(\boldsymbol{x_r}) \sum_{p \in \mathcal{P}'} \mathbb{I}(\mathcal{S}_{pr}), \\
&= \boldsymbol{\pi_r}(\boldsymbol{x_r}).
\end{aligned}
$$

Hence, the joint distribution $\boldsymbol{\pi}^* \in \mathbb{M}(\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_R)$. Furthermore, under this distribution, we have

$$\begin{aligned}
E_{\boldsymbol{\pi}^*}\left[\max_p(\boldsymbol{w_p} \cdot \tilde{\boldsymbol{x}} - k_p)\right] \geq & \sum_{p\in\mathcal{P}} \lambda_p^* \left(\sum_r \left(\frac{\int(\boldsymbol{w_{pr}} \cdot \boldsymbol{x_r})\mathbb{I}(\mathcal{S}_{pr})\boldsymbol{\pi_r}(d\boldsymbol{x_r})}{\lambda_p^*}\right) - k_p\right), \\
= & \sum_{p\in\mathcal{P}} \sum_r \left(\int\left(\boldsymbol{w_{pr}} \cdot \boldsymbol{x_r} - z_{pr}^* + z_{pr}^*\right) \mathbb{I}(\mathcal{S}_{pr})\boldsymbol{\pi_r}(d\boldsymbol{x_r})\right) - \sum_{p\in\mathcal{P}} \lambda_p^* k_p, \\
= & \sum_r \sum_{p\in\mathcal{P}} \int\left(\boldsymbol{w_{pr}} \cdot \boldsymbol{x_r} - z_{pr}^*\right) \mathbb{I}(\mathcal{S}_{pr})\boldsymbol{\pi_r}(d\boldsymbol{x_r}) + \sum_{p\in\mathcal{P}} \sum_r z_{pr}^* \int \mathbb{I}(\mathcal{S}_{pr})\boldsymbol{\pi_r}(d\boldsymbol{x_r}) \\
& - \sum_{p\in\mathcal{P}} \lambda_p^* k_p, \\
= & \sum_r E_{\boldsymbol{\pi_r}}\left[\max_{p\in\mathcal{P}}\left(\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}^*\right)\right] + \sum_{p\in\mathcal{P}} \lambda_p^*(\boldsymbol{e} \cdot \boldsymbol{z_p^*} - k_p), \\
= & \sum_r E_{\boldsymbol{\pi_r}}\left[\max_{p\in\mathcal{P}}\left(\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}^*\right)\right] + \left(\sum_{p\in\mathcal{P}} \lambda_p^*\right)\left(\max_{p\in\mathcal{P}}(\boldsymbol{e} \cdot \boldsymbol{z_p^*} - k_p)\right), \\
= & \sum_r E_{\boldsymbol{\pi_r}}\left[\max_{p\in\mathcal{P}}\left(\boldsymbol{w_{pr}} \cdot \tilde{\boldsymbol{x}}_{\boldsymbol{r}} - z_{pr}^*\right)\right] + \max_{p\in\mathcal{P}}(\boldsymbol{e} \cdot \boldsymbol{z_p^*} - k_p).
\end{aligned}$$

Since $\boldsymbol{\pi}^*$ generates an expected objective value that is greater than or equal to the optimal solution from the minimization problem, the bound is tight.