The Variational Inequality Approach for Solving Spatial Auction Problems with Joint Constraints

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Abstract. We consider a problem of managing a system of spatially distributed markets under capacity and balance constraints and show that solutions of a variational inequality enjoy auction principle properties implicitly. This enables us to develop efficient tools both for derivation of existence and uniqueness results and for creation of solution methods.

Keywords. Spatial market problem, capacity constraints, variational inequality, auction principle.

MS Classification. 49J40, 91B26, 91B72, 65K10

1 Introduction

Complex systems with spatially distributed elements arise in various fields of applications, such as Engineering, Energy and Economics, and their management requires special methods which take into account essential features of associated graphs, unlike those for general unstructured models. Recently, such problems drew more attention due to the necessity to handle many problems arising from restructuring large energy systems; see e.g. [1]–[4] and references therein. Usually, the models are based on either optimization or variational inequality approaches and yield rather complicated mathematical formulations, such as two-level (sequential) optimization problems or so-called MPEC problems, or mixed integer optimization problems, which create certain difficulties in dealing with high-dimensional problems arising in applications.

Recently, very simple variational inequality formulations for auction market problems without binding constraints were proposed in [5]–[7]. Being based on this approach, we now consider an essentially more general problem of managing spatially separated markets with capacity and balance constraints. We propose a variational inequality model whose solutions possess implicitly auction market properties, thus eliminating the corresponding conditions in the initial formulation. This enables us to develop efficient tools both for derivation of existence and uniqueness results and for creation of solution methods.

2 Model and its properties

We consider a system of n markets of a homogeneous commodity, which are joined by links (transmission lines) in a network. Denote by I_k and J_k respectively, the index sets of sellers and buyers of the k-th local market associated with the k-th node. Next, the i-th seller chooses his offer value x_i in the segment $[\alpha'_i, \beta'_i]$, $i \in I_k$ and the j-th buyer chooses his bid value in the segment $[\alpha''_j, \beta''_j]$, $j \in J_k$. Given the volume vectors $x_{(k)} = (x_i)_{i\in I_k}$ and $y_{(k)} = (y_j)_{j\in J_k}$, the i-th seller (j-th buyer) determines his price $g_i = g_i(x_i)$ (respectively, $h_j = h_j(y_j)$). Set

$$X_{(k)} = \prod_{i\in I_k} [\alpha'_i, \beta'_i], \qquad Y_{(k)} = \prod_{j\in J_k} [\alpha''_j, \beta''_j].$$

If the auctioneer intends to maximize his profit, he should solve the problem: Find $(x^*_{(k)}, y^*_{(k)}) \in X_{(k)} \times Y_{(k)}, k = 1, \dots, n$ such that

$$\sum_{k=1}^{n} \left[\sum_{i\in I_k} g_i(x^*_i)(x_i - x^*_i) - \sum_{j\in J_k} h_j(y^*_j)(y_j - y^*_j) \right] \geq 0$$
$$\forall (x_{(k)}, y_{(k)}) \in X_{(k)} \times Y_{(k)}$$

for $k = 1, \ldots, n$. However, this formulation does not reflect natural constraints which must be included. First of all, the solution $(x^*_{(k)}, y^*_{(k)})$ must satisfy the local auction market conditions:

$$g_i(x_i^*) \begin{cases} \geq p_k^* & \text{if} \quad x_i^* = \alpha_i', \\ = p_k^* & \text{if} \quad x_i^* \in (\alpha_i', \beta_i'), \quad i \in I_k, \\ \leq p_k^* & \text{if} \quad x_i^* = \beta_i', \end{cases} \tag{1}$$

and

$$h_j(y_j^*) \begin{cases} \leq p_k^* & \text{if} \quad y_j^* = \alpha_j'', \\ = p_k^* & \text{if} \quad y_j^* \in (\alpha_j'', \beta_j''), \quad j \in J_k, \\ \geq p_k^* & \text{if} \quad y_j^* = \beta_j'', \end{cases} \tag{2}$$

for some numbers p_k^*, $k = 1, \ldots, n$ which are treated as auction clearing prices. Also, the total volume balance must be fulfilled:

$$\sum_{k=1}^{n} \left(\sum_{i \in I_k} x_i - \sum_{j \in J_k} y_j \right) = 0. \tag{3}$$

Let f_{kl} denote the commodity flow from node k to node l and let $c_{kl} = c_{kl}(f_{kl})$ denote the cost of shipment of one unit of the commodity between these nodes. The flows satisfy the capacity constraints, i.e. $f_{kl} \in [0, a_{kl}]$ where $a_{kl} \geq 0$. By setting $a_{kl} = 0$ for any absent link (k, l) we will consider the network as a full graph. We now add the flow balance and capacity constraints:

$$\left(\sum_{l=1}^{n} f_{kl} - \sum_{l=1}^{n} f_{lk} \right) - \left(\sum_{i \in I_k} x_i - \sum_{j \in J_k} y_j \right) = 0 \quad k = 1, \ldots, n; \tag{4}$$

$$f_{kl} \in [0, a_{kl}], \quad k, l = 1, \ldots, n; \tag{5}$$

$$x_i \in [\alpha_i', \beta_i'], i \in I_k, y_i \in [\alpha_j'', \beta_j''], j \in J_k, k = 1, \ldots, n. \tag{6}$$

Note that the index sets I_k and J_k can be empty for some k and the case $I_k = J_k = \emptyset$ corresponds to an intermediate node. Set

$$x = (x_{(k)})_{k=1,\ldots,n}, y = (y_{(k)})_{k=1,\ldots,n}, f = (f_{kl})_{k,l=1,\ldots,n}$$

and denote by Z the set of points (x, y, f) satisfying conditions (4)–(6).

Thus, the solution (x^*, y^*, f^*) of the auctioneer problem should satisfy conditions (1)–(6), maximize the profit from all the markets and minimize the total transportation costs. Taking into account all these conditions leads to very difficult mathematical problems even in the simplest case when all the functions g_i, h_j and c_{kl} are constant. For this reason, we now consider a reduced variational inequality formulation. Namely, the problem is to find $(x^*, y^*, f^*) \in Z$ such that

$$\begin{array}{c} \displaystyle\sum_{k=1}^{n} \left[\sum_{i \in I_k} g_i(x_i^*)(x_i - x_i^*) - \sum_{j \in J_k} h_j(y_j^*)(y_j - y_j^*) \right] \\ + \sum_{k=1}^{n} \sum_{l=1}^{n} c_{kl}(f_{kl}^*)(f_{kl} - f_{kl}^*) \geq 0 \quad \forall (x, y, f) \in Z. \end{array} \tag{7}$$

We intend to show that each solution of problem (7) satisfies conditions (1), (2).

Theorem 2.1 *If (x^*, y^*, f^*) is a solution to problem (7), there exist numbers $p_k^*, k = 1, \ldots, n$ such that (1)–(3) hold.*

Proof. Let (x^*, y^*, f^*) be a solution to (7). We first observe that summing (4) over $k = 1, \ldots, n$ yields (3). For brevity, set

$$\begin{array}{c} b_i = g_i(x_i^*), i \in I_k, d_j = h_j(y_j^*), j \in J_k, \\ q_{kl} = c_{kl}(f_{kl}^*), k, l = 1, \ldots, n. \end{array}$$

Then (x^*, y^*, f^*) also solves the optimization problem

$$\text{minimize} \quad \sum_{k=1}^{n} \left[\sum_{i \in I_k} b_i x_i - \sum_{j \in J_k} d_j y_j \right] + \sum_{k=1}^{n} \sum_{l=1}^{n} q_{kl} f_{kl} \tag{8}$$

subject to (4)–(6). Clearly, (4)–(6), (8) is a linear programming problem. By using the usual duality results (see e.g. [8] or [9, Chapter 4]), there exist numbers μ_k^*, $k = 1, \ldots, n$ such that the point $(x^*, y^*, f^*, \mu^*) \in X \times Y \times F \times \mathbb{R}^n$ constitutes a saddle point of the Lagrangian

$$\begin{array}{l} L(x, y, f, \mu) = \displaystyle\sum_{k=1}^{n} \left(\sum_{i \in I_k} b_i x_i - \sum_{j \in J_k} d_j y_j \right) + \sum_{k=1}^{n} \sum_{l=1}^{n} q_{kl} f_{kl} \\ + \displaystyle\sum_{k=1}^{n} \mu_k \left[\left(\sum_{l=1}^{n} f_{kl} - \sum_{l=1}^{n} f_{lk} \right) - \left(\sum_{i \in I_k} x_i - \sum_{j \in J_k} y_j \right) \right], \end{array}$$

i.e.

$$\begin{array}{l} L(x^*, y^*, f^*, \mu) \leq L(x^*, y^*, f^*, \mu^*) \leq L(x, y, f, \mu^*) \\ \forall \mu \in \mathbb{R}^n \text{ and } \forall (x, y, f) \in X \times Y \times F, \end{array} \tag{9}$$

where

$$X = \prod_{k=1}^{n} X_{(k)}, Y = \prod_{k=1}^{n} Y_{(k)}, F = \prod_{k=1}^{n} \prod_{l=1}^{n} [0, a_{kl}].$$

The right inequality in (9) is equivalent to the following system of inequalities:

$$\begin{array}{l} (b_i - \mu_k^*)(x_i - x_i^*) \geq 0 \ \forall x_i \in [\alpha_i', \beta_i'], i \in I_k, k = 1, \dots, n; \\ (d_j - \mu_k^*)(y_j^* - y_j) \geq 0 \ \forall y_j \in [\alpha_j'', \beta_j''], j \in J_k, k = 1, \dots, n; \\ (q_{kl} + \mu_k^* - \mu_l^*)(f_{kl} - f_{kl}^*) \geq 0 \ \forall f_{kl} \in [0, a_{kl}], k, l = 1, \dots, n. \end{array} \tag{10}$$

Setting $p_k^* = \mu_k^*$ for $k = 1, \dots, n$, we see that the first and second rows in (10) yield (1) and (2), respectively, as desired. □

Thus, solutions of the variational inequality (7) with binding constraints can be utilized for setting auction clearing prices for local auctions, i.e. conditions (1) and (2) can be dropped in the initial formulation.

Observe that we do not impose any conditions on the functions g_i, h_j and c_{kl}, however, it would be reasonable to suppose that they be continuous and have non-negative (positive) values. For instance, we can even set $c_{kl} \equiv 0$, which leads to maximizing the pure auction markets profit. The third row in (10) reflects equilibrium conditions for arc flows and transmission costs.

The above result enables us to utilize efficient tools from the theory of variational inequalities and optimization problems for investigation and solution of the constrained spatial market problems. For example, we now give some existence and uniqueness results.

Theorem 2.2 *Suppose that the set Z is nonempty and bounded and that all the functions $g_i, i \in I_k, h_j, j \in J_k, k = 1, \dots, n,$ and $c_{kl}, k, l = 1, \dots, n$ are continuous. Then problem (7) has a solution.*

Proof. Clearly, Z is convex and closed. Hence (7) is a variational inequality whose cost mapping is continuous and feasible set is nonempty, convex, and compact. The existence result follows now e.g. from Theorem 3.1 in [10], Chapter 1. □

Now, for simplicity, we define the composite mapping

$$(x, y, f) \mapsto (g(x), -h(y), c(f))$$

where

$$g(x) = (g_i(x_i)), \quad i \in I_k, k = 1, \dots, n;$$
$$h(y) = (h_j(y_j)), \quad j \in J_k, k = 1, \dots, n;$$

and

$$c(f) = (c_{kl}(f_{kl})), \quad k, l = 1, \dots, n.$$

Theorem 2.3 *Suppose that the set Z is nonempty and bounded, the mapping $(x, y, f) \mapsto (g(x), -h(y), c(f))$ is continuous and strictly monotone. Then problem (7) has a unique solution.*

Proof. The solvability of problem (7) follows from Theorem 2.2. By assumption, the cost mapping in (7) is now strictly monotone, but this yields the uniqueness; see e.g. [10, Chapter 1]. □

Observe that the mappings g, h, and c are diagonal, hence they are integrable, i.e. there exist functions φ_i, $i \in I_k$, ψ_j, $j \in J_k$, $k = 1, \dots, n$, and σ_{kl}, $k, l = 1, \dots, n$ such that $\varphi_i' = g_i$, $\psi_j' = h_j$, and $\sigma_{kl}' = c_{kl}$. Therefore, the monotonicity of g_i, $-h_j$, and c_{kl} is equivalent to the convexity of φ_i, $-\psi_j$, and σ_{kl}, respectively. Also, the optimization problem

$$\text{minimize} \quad \sum_{k=1}^{n} \left[\sum_{i \in I_k} \varphi_i(x_i) - \sum_{j \in J_k} \psi_j(y_j) \right] + \sum_{k=1}^{n} \sum_{l=1}^{n} \sigma_{kl}(f_{kl})$$

subject to (4)–(6), which minimizes the total diseconomies in the system, then implies (7), and the reverse assertion is true if all the functions φ_i, $i \in I_k$, $-\psi_j$, $j \in J_k$, $k = 1, \dots, n$, and σ_{kl}, $k, l = 1, \dots, n$ are convex.

3 Solution methods

Being based on the above results, we can propose various efficient methods for solving problem (1)–(7), which are adjusted for its essential features. In the general variational inequality case, we can apply the corresponding iterative methods; see e.g. [11], [12]. In the case when the mappings g and h are integrable, we can apply various optimization methods; see e.g. [13]. Note that the constant mappings g, h, and c yield the usual linear programming problem (see (8)) and the features of the problem admit various decomposition techniques (see e.g. [14]), which can be utilized for high-dimensional ones arising in applications.

To illustrate this assertion, we describe now only one of the possible iterative methods applicable to (7) under additional monotonicity assumptions.

We recall that a mapping $A : \mathbb{R}^m \to \mathbb{R}^m$ is said to be co-coercive if there exists a constant $\nu > 0$ such that

$$\langle A(u') - A(u''), u' - u'' \rangle \geq \nu \|A(u') - A(u'')\|^2 \qquad \forall u', u''.$$

In general, this property is stronger than monotonicity, but it is well-known that each monotone integrable and Lipschitz continous mapping is co-coercive; see e.g. [12] for more details.

The simplest projection method applied to problem (7) can be described as follows.

Method (PM). Choose a point $(x^0, y^0, f^0) \in Z$ and a number $\theta > 0$.

At the s-th iteration, $s = 0, 1, \ldots$, we have a point $(x^s, y^s, f^s) \in Z$ and find the next iterate $(x^{s+1}, y^{s+1}, f^{s+1}) \in Z$ such that

$$\begin{array}{c} \displaystyle\sum_{k=1}^{n} \left[\sum_{i \in I_k} \left(g_i(x_i^s) + \theta^{-1}(x_i^{s+1} - x_i^s) \right) (x_i - x_i^{s+1}) \right. \\ \left. - \sum_{j \in J_k}^{k} \left(h_j(y_j^s) - \theta^{-1}(y_j^{s+1} - y_j^s) \right) (y_j - y_j^{s+1}) \right] \\ + \displaystyle\sum_{k=1}^{n} \sum_{l=1}^{n} \left(c_{kl}(f_{kl}^s) + \theta^{-1}(f_{kl}^{s+1} - f_{kl}^s) \right) (f_{kl} - f_{kl}^{s+1}) \geq 0 \\ \forall (x, y, f) \in Z. \end{array} \tag{11}$$

Observe that (11) is a quadratic programming problem, which always has a unique solution under the assumptions of Theorem 2.2.

Theorem 3.1 *Suppose that the set Z is nonempty and bounded, the mapping $(x, y, f) \mapsto (g(x), -h(y), c(f))$ is co-coercive. Then there exists a number $\theta > 0$ such that the sequence $\{(x^s, y^s, f^s)\}$ generated by Method (PM) converges to a solution of problem (7).*

From the assumptions we have that the cost mapping in (7) is co-coercive, so the result follows e.g. from Theorem 12.1.8 in [12].

Together with this simplest variant with fixed stepsize, the descent versions of the method can be also applied. If the cost mapping is only monotone, we can utilize combined proximal point and descent procedures. In general, there exist a great number of efficient iterative methods for finding a solution.

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