

Efficiency of Maximum Likelihood Estimators under Different Censored Sampling Schemes for Rayleigh Distribution

Abd-Elfattah, A. M. *

Amal S. Hassan *

Ziedan, D. M. *

Abstract

The objective of this article is to study the effect of different types of censored sampling schemes on the estimation of the unknown parameter for Rayleigh distribution. The censored sampling schemes namely; type-I, type-II and progressive type-II censored sampling are to be considered. The comparisons made between the samples are based on the Fisher information, expected duration of the life test and the mean squared error of the maximum likelihood estimators. A numerical study is carried out to assess these effects. The results indicate that, if the experimenter would reduce the required time to conclude the test, then he should prefer type II censored sampling than type-I. Consequently, type II is more efficient than type I and type II in multistage is more efficient than type II in one stage.

Keywords: Type-I, type-II and progressive type-II censored sampling; Maximum likelihood estimator; Mean square error; Fisher information; Expected duration.

1. Introduction

Life testing experiments often deal with censored sample in order to estimate the parameters involved in the life distribution. Two types of censoring are generally recognized, type-I and type-II censoring. In type-I censoring scheme, the experiment continues until a pre-assigned time T , and failures that occur after T are not observed. In contrast, in type-II censoring scheme the experiment decides to terminate the test after a pre-assigned number of failures observed, say k , $k \leq n$. In either case, the advantage is that it takes less time to complete the experiment. In many cases, the experimenter may be at liberty to choose between the two censoring schemes. There are many reasons affect in the choice of the two types of censoring schemes, for example, the time required to complete the experiment, the amount of Fisher information contained in a sample about the unknown parameters.

Progressively censored samples arise when at various stages of an experiment, some though not all of the surviving specimens remaining are eliminated from further observations. The sample specimens remaining after each stage of censoring are continued under observation until failure or until a subsequent stage of censoring. This type of censoring is useful in both industrial life testing applications and clinical setting, it allows for the removal of survival experiment units at points other than the

*Institute of Statistical Studies & Research, Cairo University

termination of the experiment. Progressive type-II carried out as follows: suppose that n independent items are placed on a test at time zero, with k failures to be observed. When the first failure is observed, r_1 of the surviving items are randomly selected and removed. At the second observed failure, r_2 of the surviving items are randomly selected and removed. This process stops at time when the k -th failure is observed and the remaining $r_k = n - \sum_{i=1}^{k-1} r_i - k$ surviving items are all removed.

The experimenter specified some conditions to choose an appropriate sampling plan, so the comparisons between different sampling schemes are made. Tse and Tso (1996) studied the effects of type-I and type-II censored sampling schemes on the estimation of the unknown parameter assuming that the data being to exponential distribution. Epstein and Sobel (1953) presented the ratio of the expected experiment times of type-II censoring scheme and complete sampling when the lifetimes are exponential distributed. Using Bayesian approach, Brooks (1982) compared the expected gain in information about the scale parameter for exponential distribution under type-I censoring and complete sampling. Vasudeva Rao *et al* (1991) studied the efficiency of maximum likelihood estimate of the parameters of Weibull distribution under two cases, grouped and ungrouped type-I censored sampling. Tse and Yuen (1996) compared the expected duration to complete the experiment under progressive type-II censoring with random removals (PCR) with that under type-II censoring when the lifetimes are Weibull distribution. Also, Tse and Yuen (1998) are compared the expected time required to complete the experiment under PCR with the expected time required under complete sampling when the experimental units are Weibull distributed.

The Rayleigh distribution is a suitable model for life testing studies. Polovko (1968), Dyer and Whisenand (1973), demonstrated the importance of this distribution in electro vacuum devices and communication engineering. In this article, three censoring schemes are considered, namely: type-I, type-II and progressive Type-II censoring. The Fisher information and the mean square error (MSE) for the maximum likelihood estimators are discussed, in addition the expected duration of the tests are obtained in **Sections** 2, 3, 4 and 6. In **Section** 5, we study the effect of type I and type II on estimation of the unknown parameter with respect to complete sample. In **Section** 7 the comparison between the estimators are performed under type-II censored sampling in case of one stage and multistage. These comparisons of estimators are performed for different samples using maximum likelihood estimation (MLE) procedure. Furthermore, a numerical study is performed to illustrate these comparisons using Mathcad (2001) Package. Tables and some graphs for these numerical results are displayed at the appendix.

2. Complete Sampling

Consider a life testing experiment in which n units put on test and successive failure times are recorded. Assume that the lifetimes are independent and identically distributed Rayleigh random variables with probability density function

$$f(x, \lambda) = \frac{2x}{\lambda} e^{-\frac{x^2}{\lambda}}, \quad x > 0, \lambda > 0 \quad (1)$$

and the corresponding cumulative distribution function

$$F(x, \lambda) = 1 - e^{-\frac{x^2}{\lambda}}.$$

Suppose that X_1, X_2, \dots, X_n are the n complete observed in an experiment, then the likelihood function is

$$L(\lambda) = \left(\frac{2}{\lambda}\right)^n \prod_{i=1}^n x_i e^{-\frac{x_i^2}{\lambda}}.$$

And the natural logarithm of the likelihood function is

$$\ln L(\lambda) = n \ln 2 - n \ln \lambda + \sum_{i=1}^n \ln x_i - \frac{\sum_{i=1}^n x_i^2}{\lambda}.$$

The MLE of the parameter λ is

$$\hat{\lambda} = \frac{\sum_{i=1}^n x_i^2}{n}.$$

The MSE of $\hat{\lambda}$ is given by

$$\begin{aligned} MSE(\hat{\lambda}) &= \text{var}(\hat{\lambda}) + (\text{bias})^2 \\ &= \text{var}(\hat{\lambda}) + [E(\hat{\lambda}) - \lambda]^2 = \frac{\lambda^2}{n}, \end{aligned} \quad (2)$$

where $E(\hat{\lambda}) = \lambda$, and $\text{var}(\hat{\lambda}) = \frac{\lambda^2}{n}$

The Fisher information matrix corresponding to the complete sampling, denoted by $I(\lambda)$, is given by

$$I(\lambda) = E\left[-\frac{d^2 \ln L}{d\lambda^2}\right] = \frac{n}{\lambda^2}. \quad (3)$$

The time required to complete the experiment equal to $X_{(n)}$, where $X_{(n)}$ is the largest observation. Therefore, the expected time of the experiment is the expected value of $X_{(n)}$. The probability density function of $X_{(n)}$ is

$$f_n(x_{(n)}) = \frac{2n x_{(n)}}{\lambda} e^{-\frac{x_{(n)}^2}{\lambda}} \left[1 - e^{-\frac{x_{(n)}^2}{\lambda}}\right]^{n-1}.$$

So, the expected value of $X_{(n)}$ is

$$E(X_{(n)}) = n \lambda^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \left(\frac{1}{j+1}\right)^{\frac{3}{2}}. \quad (4)$$

3. Type-I Censored Sampling

In this **Section**, the exact Fisher information, expected time duration, where the sample is censored at a fixed length of time are obtained. In particular, properties of maximum likelihood estimator, such as bias and mean square error are also obtained. In type-I censored sample, suppose that a random sample of n units is tested until a predetermined time T at which time the test is terminate. Times to failure for k observations are observed where k is random. Thus the lifetimes of X_i , $i = 1, 2, \dots, n$ observed only if $x_i < T$. Therefore

$$\delta_i = \begin{cases} 1 & \text{if } x_i \leq T \\ 0 & \text{if } x_i > T \end{cases}$$

Therefore $k = \sum_{i=1}^n \delta_i$, and the likelihood function in this case is given by

$$L(\lambda) = \prod_{i=1}^n \left[\frac{2x_i}{\lambda} e^{-\frac{x_i^2}{\lambda}} \right]^{\delta_i} \left[e^{-\frac{T^2}{\lambda}} \right]^{1-\delta_i}.$$

The logarithm of the likelihood function is given by

$$\ln L(\lambda) = k \ln 2 - k \ln \lambda + \sum_{i=1}^n \delta_i \ln x_i - \frac{1}{\lambda} \sum_{i=1}^n \delta_i x_i^2 - (n-k) \frac{T^2}{\lambda}$$

The MLE of λ is a solution of the following equation with respect to λ

$$\frac{d \ln L(\lambda)}{d \lambda} = -\frac{k}{\lambda} + \frac{1}{\lambda^2} \sum_{i=1}^n \delta_i x_i^2 + \frac{1}{\lambda^2} (n-k) T^2. \quad (5)$$

The MLE of λ under type-I censored sampling, say $\hat{\lambda}_l$, can be obtained by putting equation (5) equal to zero

$$\hat{\lambda}_l = \frac{1}{k} \left[\sum_{i=1}^n \delta_i x_i^2 + (n-k) T^2 \right]. \quad (6)$$

From equation (6) when $k = 0$, the maximum likelihood estimate does not exist. So, in finding the mean and MSE of $\hat{\lambda}_l$, the results will restrict to the set of estimates for which $k > 0$. From Mendenhall and Lehman (1960), the expected value and the variance of $\hat{\lambda}_l$ are given by

$$E(\hat{\lambda}_l) = \lambda - \frac{T^2 q}{p} + n T^2 E\left(\frac{1}{k}\right) - T^2,$$

and

$$\text{var}(\hat{\lambda}_l) = n^2 T^4 \text{var}\left(\frac{1}{k}\right) + \left(\lambda^2 - \frac{T^4 q}{p^2}\right) E\left(\frac{1}{k}\right),$$

where, $p = P(X < T) = 1 - e^{-\frac{T^2}{\lambda}}$ and $q = 1 - p$.

In particular, k denote the number of non-censored test items, which is a truncated binomial distribution with parameters n and p . Thus, the expected value and the

variance of $\hat{\lambda}_l$ are functions of the negative moments of the positive binomial, i.e., functions of $E\left(1/k^r\right)$,

$$p(k) = \frac{\binom{n}{k} p^k q^{n-k}}{1-q^n}, \quad k = 1, 2, \dots, n$$

Since there is no exact form of these negative moments, Mendenhall and Lehman (1960) derived an approximation to overcome this difficulty. This approximation to negative moments is obtained for $r \geq 1$, and it is called the beta approximation. The accuracy of this approximation can expect more than two place for values of $np > 5$.

The beta approximation to the moments of $1/k$ is

$$E\left(\frac{1}{k^r}\right) \approx \frac{1}{n^r} \frac{\beta(a-r, b)}{\beta(a, b)}, \quad (7)$$

where $a = (n-1)p$, $b = (n-1)q$ and $\beta(a, b)$ is beta function.

By using equation (7), then the approximation of expected value and variance of $\hat{\lambda}_l$ are, respectively

$$E(\hat{\lambda}_l) = \lambda - T^2 \left[\frac{1}{p} - \frac{(n-2)}{(a-1)} \right],$$

and

$$\text{var}(\hat{\lambda}_l) = T^4 \left[\frac{(n-2)(n-a-1)}{(a-1)^2(a-2)} \right] + \left(\lambda^2 - \frac{T^4 q}{p^2} \right) \left[\frac{(n-2)}{n(a-1)} \right].$$

Therefore, the MLE of λ under type-I censoring, $\hat{\lambda}_l$, is biased estimator for λ , where

$$\text{bias}(\hat{\lambda}_l) = -T^2 \left[\frac{1}{p} - \frac{(n-2)}{(a-1)} \right].$$

Also, the MSE of $\hat{\lambda}_l$ is given by

$$\text{MSE}(\hat{\lambda}_l) = T^4 \left[\left[\frac{(n-2)(n-a-1)}{(a-1)^2(a-2)} \right] + \left[\frac{1}{p} - \frac{(n-2)}{(a-1)} \right]^2 - \left[\frac{q(n-2)}{np^2(a-1)} \right] \right] + \lambda^2 \left[\frac{(n-2)}{n(a-1)} \right].$$

Let $\ln L_i(\lambda)$ is the contribution of the i -th observation to the total logarithm of the likelihood function under type-I censoring,

$$\ln L_i(\lambda) = \delta_i \ln[f(x_i)] + (1-\delta_i) \ln[1-F(T)] = \delta_i \ln \left[\frac{2x_i}{\lambda} e^{-\frac{x_i^2}{\lambda}} \right] - (1-\delta_i) \left[\frac{T^2}{\lambda} \right].$$

Nelson (1982) introduced the formula for computing the expected information for item i under type-I censoring,

$$E\left(\frac{-d^2 \ln L_i(\lambda)}{d\lambda^2}\right) = \int_{-\infty}^T \left(\frac{-d^2 \ln f(x_i)}{d\lambda^2} \right) f(x_i) dx_i + \left(\frac{-d^2 \ln[1-F(T)]}{d\lambda^2} \right) [1-F(T)].$$

Therefore, the Fisher information matrix for unit i is given by

$$E\left(-\frac{d^2 \ln L_i(\lambda)}{d\lambda^2}\right) = \frac{1}{\lambda^2} \left[1 - e^{-\frac{T^2}{\lambda}}\right].$$

For a type-I censored sample of independent items, the Fisher information is

$$I_1(\lambda) = \sum_{i=1}^n \frac{1}{\lambda^2} \left[1 - e^{-\frac{T^2}{\lambda}}\right] = \frac{np}{\lambda^2}. \quad (8)$$

In this case, the time required to complete the experiment is the pre-determined time T or the time to observe the largest order statistics $X_{(n)}$. So the time is given by

$$H = \begin{cases} T & \text{if } x_{(n)} \geq T \\ X_{(n)} & \text{if } x_{(n)} < T \end{cases}$$

Hence, the expected time of this censoring scheme is

$$\begin{aligned} E(H) &= E(H | x_{(n)} \geq T) p(X_{(n)} \geq T) + E(H | x_{(n)} < T) p(X_{(n)} < T) \\ &= T p(X_{(n)} \geq T) + E(H | x_{(n)} < T) p(X_{(n)} < T). \end{aligned}$$

But

$$E(H | x_{(n)} < T) = E[X_{(n)} | x_{(n)} < T] = T - \int_0^T \frac{[F(x_{(n)})]^n}{[F(T)]^n} dx_{(n)}.$$

Thus,

$$E(H) = T - \int_0^T \left[1 - e^{-\frac{x_{(n)}^2}{\lambda}}\right]^n dx_{(n)}.$$

Define

$$s_n = \int_0^T \left[1 - e^{-\frac{x_{(n)}^2}{\lambda}}\right]^n dx_{(n)}.$$

s_n can be rewrite as the following

$$s_n = \frac{1}{(n+1)} \left[n s_{n-1} - \frac{\lambda}{2T} \left[1 - e^{-\frac{T^2}{\lambda}}\right]^n \right].$$

Using this relationship recursively, the expected time to terminate the test is given by

$$E(H) = \left(\frac{n}{n+1}\right) T + \frac{\lambda}{2(n+1)T} \sum_{j=1}^n \left[1 - e^{-\frac{T^2}{\lambda}}\right]^j. \quad (9)$$

4. Type-II Censored Sampling

Suppose that only the first k ordered observations in a random sample of size n from one-parameter Rayleigh distribution are available, where k is fixed before the experiment is conducted. The likelihood function of the sample in this case is given by

$$L(\lambda) = \frac{n!}{(n-k)!} \prod_{i=1}^k \frac{2x_{(i)}}{\lambda} e^{-\frac{x_{(i)}}{\lambda}} \left[e^{-\frac{x_{(k)}}{\lambda}} \right]^{(n-k)}.$$

The MLE of the parameter λ under type-II censoring can be shown to be of the form

$$\hat{\lambda}_{II} = \frac{1}{k} \left[\sum_{i=1}^k x_{(i)}^2 + (n-k)x_{(k)}^2 \right].$$

To obtain the expected value and mean squared error of $\hat{\lambda}_{II}$, let us now perform the transformation, $U_{(i)} = X_{(i)}^2$, $i = 1, 2, \dots, k$. This random sample is distributed as exponential distribution from type-II censoring sampling. Also, consider the following transformation:

$$\text{Let, } W_1 = nU_{(1)} \text{ and } W_i = (n-i+1)(U_{(i)} - U_{(i-1)}), \quad i = 2, 3, \dots, k.$$

$$\text{Hence, } \sum_{i=1}^k U_{(i)} + (n-k)U_{(k)} = \sum_{i=1}^k W_i.$$

It can be shown that W_1, W_2, \dots, W_k are independent exponential random variables with parameter λ . Then, the expected value of $\hat{\lambda}_{II}$ is $E(\hat{\lambda}_{II}) = \lambda$.

Thus $\hat{\lambda}_{II}$ is unbiased estimator for λ , and the variance of $\hat{\lambda}_{II}$ is

$$\text{var}(\hat{\lambda}_{II}) = \text{var} \left[\frac{1}{k} \sum_{i=1}^k W_i \right] = \frac{\lambda^2}{k}.$$

So the mean square error of the estimated parameter under type-II censoring, $\hat{\lambda}_{II}$, is given by

$$MSE(\hat{\lambda}_{II}) = \frac{\lambda^2}{k}. \quad (10)$$

The Fisher information matrix associated with λ under type-II censoring is given by

$$\begin{aligned} I_{II}(\lambda) &= E \left(-\frac{d^2 \ln L}{d\lambda^2} \right) = -\frac{k}{\lambda^2} + \frac{2}{\lambda^3} E \left[\sum_{i=1}^k X_{(i)}^2 + (n-k)X_{(k)}^2 \right] \\ &= -\frac{k}{\lambda^2} + \frac{2}{\lambda^3} E \left[\sum_{i=1}^k U_{(i)} + (n-k)U_{(k)} \right] \\ &= -\frac{k}{\lambda^2} + \frac{2}{\lambda^3} E \left[\sum_{i=1}^k W_i \right] = \frac{k}{\lambda^2}. \end{aligned} \quad (11)$$

The time required to complete the experiment under type-II censoring, $E(X_{(k)})$, is given by

$$\begin{aligned} E(X_{(k)}) &= k \binom{n}{k} \int_0^\infty x_{(k)} f(x_{(k)}) [F(x_{(k)})]^{k-1} [1-F(x_{(k)})]^{n-k} dx_{(k)} \\ &= k \binom{n}{k} \lambda^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left(\frac{1}{n-k+j+1} \right)^{\frac{3}{2}}. \end{aligned} \quad (12)$$

5. Comparative Study

The effects of the two censored sampling scheme on the estimation of the unknown parameter λ with respect to the complete sampling are compared. This comparison based on bias, mean square error, expected information and duration of an experiment with n items.

5.1 Comparison of Bias and MSE

The analytically comparisons between different samples based on bias and MSE indicate that the MLE of λ for complete sampling is the best estimator, where is unbiased estimator of λ and have the minimum MSE. The MLE of λ under type-II censoring is also unbiased but gives a large MSE. In the other hand, the MLE of λ under Type-I censoring is biased. So the MLE of λ for type-II censoring is better than the MLE of λ for type-I censoring. These facts will obvious by numerical study. Therefore, following Tse and Tso (1996), for fixed k or p , selected T such that the ratios of the expected information under type-I and type-II censored sampling to the expected information of the complete sampling are equal, namely; $n \left[1 - e^{-T^2/\lambda} \right] = n p = k$. The ratios of the MSE of λ under type-I and type-II censored sampling to the MSE of λ under complete sampling are obtained, namely,

$$\frac{\text{MSE of } \lambda \text{ for censored sampling}}{\text{MSE of } \lambda \text{ for complete sampling}}$$

Using Mathcad (2001) package, several combinations of n and p are considered, where n and p are choose such that $n = 10, 20, 30, 40, 50$ and 100 ; $p = \left[1 - e^{-T^2/\lambda} \right] = 0.9, 0.8, 0.7, 0.6$ and 0.5 . Then, the MSE of λ under the two types of censored sampling can be compared.

Table (1) gives the ratio of the MES of λ for type-I and type-II censoring to the MSE of λ for complete sampling. The tabulated results appear that the MSE of λ under type-I censored sampling is larger than the MSE of λ under type-II censored sampling. However, type-I censored sampling produces much large MSE when p is small, i.e. T is small. For large sample the MSE of λ under type-I censored sampling is near from the MSE of λ under type-II censored sampling, although the MSE of λ under type-II censored sampling does not depends on n .

5.1 Comparison of Expected Information and Duration

In this subsection, the comparisons between the two censored samples and complete sampling based on expected information and duration are performed. From equations (3), (8) and (11), we can see that the amount of Fisher information contained in a sample of fixed size n under either type of censored sampling is less than that contained with the complete sampling. It is obvious from equations (4), (9) and (12) that the analytical comparing of the three expected test times is very difficult, so it is better to calculate them numerically. A numerical study assumes that, T and k selected such that the ratio of the expected information under type-I and type-II

censored sampling to the expected information of the complete sampling are constant (R) for fixed n , namely;

$$R = \frac{\text{Expected information for censored sampling}}{\text{Expected information for complete sampling}}.$$

Table (2) shows the numerical values of the ratios of the expected times to terminate the experiments under type-I and type-II censoring to the expected time to terminate the experiment with complete sampling. This Table summarizes the results of $n = 10, 20$ and 30 with $R = 0.9, 0.8, 0.7, 0.6$ and 0.5 . It is clear from the tabulated results that the expected duration to terminate the experiment for type-I and type-II censored sampling are less than the expected duration to terminate the experiment for complete sampling. The expected durations to conclude the experiment for two types of censored sampling can be reduced by more than 30 % without loss too much of the information when compared with those achieved without censoring. In addition, the expected duration to conclude the experiment for type-II censored sampling is relatively smaller than that of a type-I censored sampling if both experiments are set to yield the same amount of information. When n increase the difference between the expected duration of type-I censored sampling and the expected duration of type-II censored sampling is decrease. Furthermore, for the two types of censored sampling if the amount of information increased, the expected duration to terminate the experiment increased, for more appearance see graph (1).

6. Type II Progressive Censoring Scheme

Let $X_{(1)}, X_{(2)}, \dots, X_{(k)}$ be a progressively type-II right censored sampling from one-parameter Rayleigh distribution, with censoring schemes r_1, \dots, r_k . The likelihood function is given by

$$\begin{aligned} L(\lambda) &= c \prod_{i=1}^k \frac{2x_{(i)}}{\lambda} e^{-\frac{x_{(i)}^2}{\lambda}} \left[e^{-\frac{x_{(i)}^2}{\lambda}} \right]^{r_i} \\ &= c \prod_{i=1}^k \frac{2x_{(i)}}{\lambda} \left[e^{-\frac{x_{(i)}^2}{\lambda}} \right]^{r_i+1}, \end{aligned}$$

where, $c = n(n - r_1 - 1) \dots (n - r_1 - r_2 - \dots - r_{k-1} - k + 1)$. Thus, the logarithm of the likelihood function is given by

$$\ln L(\lambda) = \ln c + k \ln 2 - k \ln \lambda + \sum_{i=1}^k \ln x_{(i)} - \sum_{i=1}^k (r_i + 1) \frac{x_{(i)}^2}{\lambda}$$

The maximum likelihood estimator of λ under progressive type-II censored sampling, say $\hat{\lambda}_{II}^*$, is given by

$$\hat{\lambda}_{II}^* = \frac{1}{k} \sum_{i=1}^k (r_i + 1) x_{(i)}^2.$$

The expected value of $\hat{\lambda}_{II}^*$ is

$$E(\hat{\lambda}_{II}^*) = \frac{1}{k} \sum_{i=1}^k (r_i + 1) E(X_{(i)}^2).$$

Apply the order transformation $U_{(i)} = X_{(i)}^2$, $i = 1, 2, \dots, k$, then $U_{(1)}, U_{(2)}, \dots, U_{(k)}$ is a progressively type-II right censored sample from the exponential distribution, and then using the transformation

$$W_1 = nU_{(1)}, W_2 = (n - r_1 - 1)(U_{(2)} - U_{(1)}) \dots W_k = (n - r_1 - \dots - r_{k-1} - k + 1)(U_{(k)} - U_{(k-1)})$$

Therefore,

$$\sum_{i=1}^k (r_i + 1)U_{(i)} = \sum_{i=1}^k W_i.$$

The Jacobian of this transformation is, therefore $\frac{1}{c}$, and W_i has exponential distribution, therefore

$$E(\hat{\lambda}_{II}^*) = \frac{1}{k} \sum_{i=1}^k E(W_i) = \lambda$$

Thus, $\hat{\lambda}_{II}^*$ is unbiased estimator of λ , and the variance of $\hat{\lambda}_{II}^*$ is

$$\text{var}(\hat{\lambda}_{II}^*) = \text{var}\left(\frac{1}{k} \sum_{i=1}^k W_i\right) = \frac{\lambda^2}{k}$$

The mean square error of the maximum likelihood estimator is given by

$$MSE(\hat{\lambda}_{II}^*) = \text{var}(\hat{\lambda}_{II}^*) + [\text{bias}(\hat{\lambda}_{II}^*)]^2 = \frac{\lambda^2}{k}. \quad (13)$$

The Fisher information associated with λ under this type of censoring, say $I_{II}^*(\lambda)$, is given by

$$I_{II}^*(\lambda) = E\left(\frac{k}{\lambda^2} - \frac{2}{\lambda^3} \sum_{i=1}^k (r_i + 1)X_{(i)}^2\right) = \frac{k}{\lambda^2}. \quad (14)$$

Under this type of censored sampling, the expected time to terminate the experiment is given by $E(X_{(k)}^*)$, hence

$$E(X_{(k)}^*) = c \sum_{l_1=0}^{r_1} \sum_{l_2=0}^{r_2} \dots \sum_{l_k=0}^{r_k} (-1)^{\sum_{i=1}^k l_i} \frac{\binom{r_1}{l_1} \binom{r_2}{l_2} \dots \binom{r_k}{l_k}}{\prod_{i=1}^{k-1} \left(\sum_{j=1}^i l_j + i\right)} \int_0^{\infty} x_{(k)} f(x_{(k)}) [F(x_{(k)})]^{\sum_{i=1}^k l_i + k - 1} dx_{(k)}$$

Therefore

$$E(X_{(k)}^*) = c \lambda^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) \sum_{l_1=0}^{r_1} \sum_{l_2=0}^{r_2} \dots \sum_{l_k=0}^{r_k} \left\{ (-1)^{\sum_{i=1}^k l_i} \frac{\binom{r_1}{l_1} \binom{r_2}{l_2} \dots \binom{r_k}{l_k}}{\prod_{i=1}^{k-1} \left(\sum_{j=1}^i l_j + i\right)} \times \sum_{j=0}^{\sum_{i=1}^k l_i + k - 1} (-1)^j \binom{\sum_{i=1}^k l_i + k - 1}{j} (j+1)^{\frac{-3}{2}} \right\}. \quad (15)$$

(7) Comparisons between Type-II and Progressive Type-II Censoring

From **Section 5**, the comparisons between type-II and type-I censored sampling indicate that, type-II censored sampling is better than type-I censored sampling. Therefore, the aim in this **Section** is to study, under type-II censored sampling, whether one stage censoring or multistage censoring will be more efficient. Consequently, the comparisons between the MLE under one stage type-II and multistage type-II censored sampling are performed. These comparisons are based on the Fisher information, MSE and expected duration to terminate the experiment.

From theoretical results that appear in equations (10), (13), (11), (14), (12) and (15), the MSE and the Fisher information for the MLE under multistage type-II censored sampling are equal those under one stage type-II censored sampling. The expected time to terminate the experiment under multistage type-II censored sampling is difficult to compare with that under one stage type-II censored sampling theoretically. Therefore, the numerical illustration will perform based on the expected duration to terminate the experiment. This study carried out for sample sizes $n = 10, 15, 20, 25$ and 30 , different choices for the number of failures, and different progressive censoring schemes for each choices of n and k are selected. These schemes including the two extreme censoring schemes of $(0, 0, \dots, n - k)$ and $(n - k, 0, 0, \dots, 0)$. For simplicity in notation, denote these schemes by $[(k - 1)^* 0, n - k]$ and $[n - k, (k - 1)^* 0]$, respectively; for example, $(2^* 0, 5)$ denotes the censoring scheme $(0, 0, 5)$ and so on.

Table (3) provides the values of the ratios of the expected times to terminate the experiment under progressive type-II censored sampling to the expected times to terminate the experiment under type-II censored sampling for different choices of n , k and schemes. It is obvious from the tabulated results that:

- (i) For different choices for n , k and schemes, progressive type-II censored sampling has expected duration to terminate the experiment smaller than that under type-II censored sampling in each case.
- (ii) The expected durations to terminate the experiment under progressive type-II censored sampling are equal those under type-II censored sampling for the extreme schemes $[(k - 1)^* 0, n - k]$. That is because when $r_1 = r_2 = \dots = r_{k-1} = 0$ and $r_k = n - k$, progressive type-II censored sampling corresponds to type-II censored sampling.
- (iii) For fixed n and k , when the removal units are increase in the first stages the expected time to terminate the experiment under progressive type-II censored sampling decrease. So, the choice of the extreme scheme $[n - k, (k - 1)^* 0]$ leads to the experiment under progressive type-II censored sampling take smaller time if compared with other schemes.

8. Conclusions

This study illustrates that. The type-II censored sampling requires shorter time than does type-I censored sampling. Although, the MLE under type-II censored sampling has the same MSE and expected information in one stage and multistage, the experimenter should prefer multistage to one stage censored sampling if he wants decrease the time of the test. Furthermore, the MLE of λ derived from type-II censored data enjoys better statistical properties than MLE of λ from type-I censored data. It is unbiased and has a smaller MSE.

Appendix

Table (1)

Ratio of the MSE with censoring to the MSE without censoring

p	n	I	II
0.9	10	1.644	1.111
	20	1.318	1.111
	30	1.239	1.111
	40	1.204	1.111
	50	1.184	1.111
	100	1.146	1.111
0.8	10	2.154	1.125
	20	1.589	1.125
	30	1.458	1.125
	40	1.400	1.125
	50	1.367	1.125
	100	1.306	1.125
0.7	10	2.857	1.429
	20	1.940	1.429
	30	1.739	1.429
	40	1.651	1.429
	50	1.602	1.429
	100	1.511	1.429
0.6	10	3.988	1.667
	20	2.445	1.667
	30	2.131	1.667
	40	1.998	1.667
	50	1.924	1.667
	100	1.788	1.667
0.5	10	6.129	2.000
	20	3.248	2.000
	30	2.729	2.000
	40	2.514	2.000
	50	2.397	2.000
	100	2.185	2.000

Table (2)

Ratio of expected time with censoring to expected time without censoring

R	n = 10		n = 20		n = 30	
	I	II	I	II	I	II
0.9	0.928	0.814	0.839	0.767	0.790	0.739
0.8	0.765	0.701	0.686	0.652	0.647	0.625
0.7	0.651	0.613	0.586	0.567	0.555	0.543
0.6	0.562	0.537	0.508	0.496	0.482	0.475
0.5	0.484	0.467	0.439	0.432	0.418	0.413

Note that: columns I and II give the comparison results for type I and type II censored sampling to complete sampling respectively

Table (3)

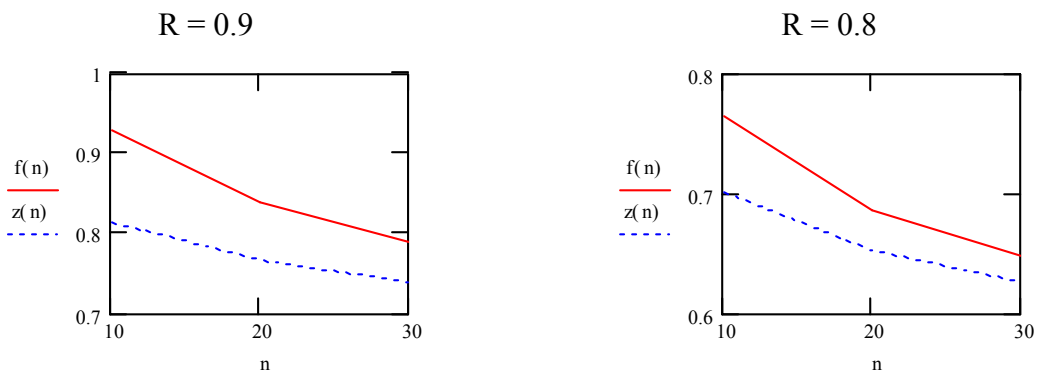
Ratio of expected duration for progressive type-II censoring with respect to type-II censoring

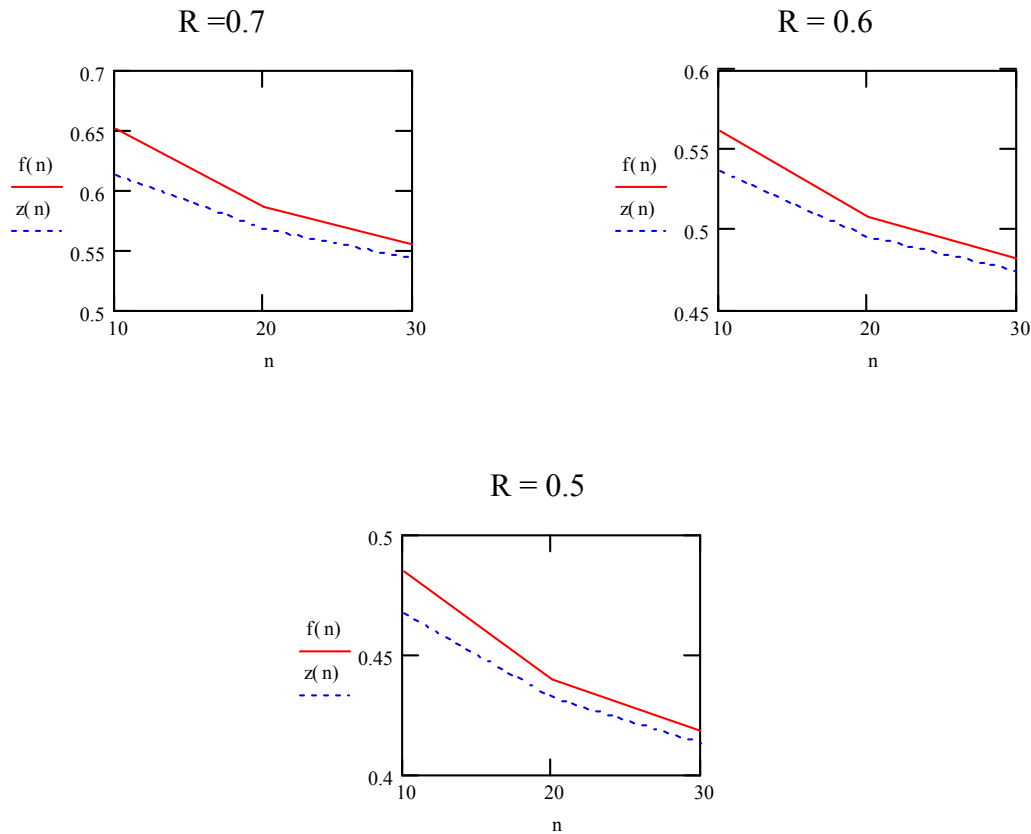
n	k	scheme	$E(X_{(k)}^*)/E(X_{(k)})$
10	2	(0, 8)	1.000
		(8, 0)	0.453
	3	(2*0, 7)	1.000
		(7, 2*0)	0.466
		(1, 0, 6)	0.956
	4	(3*0, 6)	1.000
		(6, 3*0)	0.504
		(2, 1, 1, 2)	0.789
		(2*0, 2, 4)	0.946
	5	(4*0, 5)	1.000
(5, 4*0)		0.549	
(2*0, 2, 0, 3)		0.908	
(3*0, 1, 4)		0.976	
15	2	(0, 13)	1.000
		(13, 0)	0.374
	3	(2*0, 12)	1.000
		(12, 2*0)	0.377
		(6, 0, 6)	0.804
	4	(3*0, 11)	1.000
		(11, 3*0)	0.402
		(2, 2, 2, 5)	0.838
	5	(4*0, 10)	1.000
		(10, 4*0)	0.430
(2, 2, 2, 2, 2)		0.721	
10	(9*0, 5)	1.000	
(5, 9*0)	0.603		

Continued Table (3)

n	k	scheme	$E(X_{(k)}^*)/E(X_{(k)})$
20	2	(0, 18)	1.000
		(18, 0)	0.326
	3	(2*0, 17)	1.000
		(17, 2*0)	0.326
		(0, 17, 0)	0.399
	4	(3*0, 16)	1.000
(16, 3*0)		0.344	
(4, 4, 4, 4)		0.727	
5	(4*0, 15)	1.000	
	(15, 4*0)	0.366	
10	(2*0, 5, 5, 5)	0.832	
25	2	(9*0, 10)	1.000
		(10, 9*0)	0.603
	3	(0, 23)	1.000
		(23, 0)	0.293
		(2*0, 22)	1.000
	4	(22, 2*0)	0.291
(11, 0, 11)		0.793	
(3*0, 21)		1.000	
5	(21, 3*0)	0.306	
	(4*0, 20)	1.000	
	(20, 4*0)	0.325	
10	(2*0, 20, 2*0)	0.378	
	(4, 4, 4, 4, 4)	0.697	
30	2	(9*0, 15)	1.000
		(15, 9*0)	0.420
	3	(4*0, 5, 5, 5, 3*0)	0.482
		(0, 28)	1.000
		(28, 0)	0.269
	5	(2*0, 27)	1.000
(27, 2*0)		0.266	
(0, 27, 0)		0.330	
10	(4*0, 25)	1.000	
	(25, 4*0)	0.294	
10	(5, 5, 5, 5, 5)	0.691	
	(9*0, 20)	1.000	
10	(20, 9*0)	0.376	
	(4*0, 10, 10, 4*0)	0.421	

Graph (1)





Note that: $f(n)$ and $z(n)$ are the ratios of expected time under type-I censoring and type-II censoring to expected time under complete sampling

References

1. Brooks, R. J. (1982). On the Loss of Information through Censoring. *Biometrika*, **69**, 137-144.
2. Dyer, D. D. and Whisenand, C. W. (1973). Best Linear Unbiased Estimator of the Parameter of the Rayleigh Distribution. *IEEE Transactions Reliability*. **22**, 27-34.
3. Epstein, B. and Sobel, M. (1953). Life Testing. *JASA*, **48**, 486-502.
4. Kendell, P. J. and Anderson, R. L. (1971). An Estimation Problem in Life Testing. *Technometrics*. **13**, 289-301
5. Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*. John Wiley, New York.
6. Mathcad (2001) Professional (1986-2000) Mathsoft, Inc.
7. Mendenhall, W. and Lehman, E. H. (1960). An Approximation to the Negative Moments of the Positive Binomial Useful in Life Testing. *Technometrics*. **2**, 227-242.
8. Nelson, W. B. (1982) *Applied Life Data Analysis*. John Wiley. New York.
9. Polovko, A. M. (1968). *Fundamentals of Reliability Theory*. Academic press.
10. Tse, S. k. and Tso, G. (1996). Efficiencies of Maximum Likelihood Estimators under Censoring in Life Testing. *Journal of Applied Statistics*. **23**, 515-524.
11. Tse, S. k. and Yuen, H. K. (1996). Parameters Estimation for Weibull Distribution Lifetimes under Progressive Censoring with Random Removals. *J. Statis. Comput. Simul.* **55**, 57-71.

12. Tse, S. k. and Yuen, H. K. (1998). Expected Experiment Times for the Weibull Distribution under Progressive Censoring with Random Removals. *Journal of Applied Statistics*. **25**, 75-83.
13. Vasudeva Rao, A., Dattatreya Rao, A. V. and Narasimham, V. L. (1991). Asymptotic Relative Efficiencies of the ML Estimates of the Weibull Parameters in Grouped Type-I Right Censored Samples. Reports of Statistical Application Research, *Union of Japanese Scientists and Engineers*. **4**, 1-12.