Hyperplane Arrangements with Large Average Diameter

Antoine Deza and Feng Xie

September 23, 2007

McMaster University
Hamilton, Ontario, Canada
deza, xief @mcmaster.ca

Abstract: The largest possible average diameter of a bounded cell of a simple hyperplane arrangement is conjectured to be not greater than the dimension. We prove that this conjecture holds in dimension 2, and is asymptotically tight in fixed dimension. We give the exact value of the largest possible average diameter for all simple arrangements in dimension 2, for arrangements having at most the dimension plus 2 hyperplanes, and for arrangements having 6 hyperplanes in dimension 3. In dimension 3, we give lower and upper bounds which are both asymptotically equal to the dimension.

Keywords: hyperplane arrangements, bounded cell, average diameter

1 Introduction

Let $\mathcal{A}$ be a simple arrangement formed by n hyperplanes in dimension d. We recall that an arrangement is called simple if $n \geq d+1$ and any d hyperplanes intersect at a unique distinct point. The number of bounded cells (closures of the bounded connected components of the complement) of $\mathcal{A}$ is $I = \binom{n-1}{d}$. Let $\delta(\mathcal{A})$ denote the average diameter of a bounded cell P_i of $\mathcal{A}$; that is,

$$\delta(\mathcal{A}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I}$$

where $\delta(P_i)$ denotes the diameter of P_i, i.e., the smallest number such that any two vertices of P_i can be connected by a path with at most $\delta(P_i)$ edges. Let $\Delta_{\mathcal{A}}(d,n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d. Deza, Terlaky and Zinchenko conjectured that $\Delta_{\mathcal{A}}(d,n) \leq d$.

Conjecture 1 [5] *The average diameter of a bounded cell of a simple arrangement defined by m inequalities in dimension n is not greater than n.*

It was showed in [5] that if the conjecture of Hirsch holds for polytopes in dimension d, then $\Delta_{\mathcal{A}}(d,n)$ would satisfy $\Delta_{\mathcal{A}}(d,n) \leq d + \frac{2d}{n-1}$. In dimension 2 and 3, we have $\Delta_{\mathcal{A}}(2,n) \leq 2 + \frac{2}{n-1}$ and $\Delta_{\mathcal{A}}(3,n) \leq 3 + \frac{4}{n-1}$. We recall that a polytope is a bounded polyhedron and that the conjecture of Hirsch, formulated in 1957 and reported in [1], states that the diameter of a polyhedron defined by n inequalities in dimension d is not greater than $n-d$. The conjecture does not hold for unbounded polyhedra.

Conjecture 1 can be regarded a discrete analogue of a result of Dedieu, Malajovich and Shub [4] on the average total curvature of the central path associated to a bounded cell of a simple arrangement. We first recall the definitions of the central path and of the total curvature. For a polytope $P = \{x : Ax \geq b\}$ with $A \in \Re^{n\times d}$, the central path corresponding to $\min\{c^T x : x \in P\}$ is a set of minimizers of $\min\{c^T x + \mu f(x) : x \in P\}$ for $\mu \in (0, \infty)$ where $f(x) = -\sum_{i=1}^{n} \ln(A_i x - b_i)$ – the standard logarithmic barrier function [12]. Intuitively, the total curvature [14] is a measure of how far off a certain curve is from being a straight line. Let $\psi : [\alpha, \beta] \to \Re^d$ be a $C^2((\alpha-\varepsilon, \beta+\varepsilon))$ map for some $\varepsilon > 0$ with a non-zero derivative in $[\alpha, \beta]$. Denote its arc length by $l(t) = \int_\alpha^t \|\dot\psi(\tau)\| d\tau$, its parametrization by the arc length by $\psi_{\text{arc}} = \psi \circ l^{-1} : [0, l(\beta)] \to \Re^d$, and its curvature at the point t by $\kappa(t) = \ddot\psi_{\text{arc}}(t)$. The total curvature is defined as $\int_0^{l(\beta)} \|\kappa(t)\| dt$. The requirement $\dot\psi \neq 0$ insures that any given segment of the curve is traversed only once and allows to define a curvature at any point on the curve. Let $\lambda^c(\mathcal{A})$ denote the average associated total curvature of a bounded cell P_i of a simple arrangement $\mathcal{A}$; that is,

$$\lambda^c(\mathcal{A}) = \sum_{i=1}^{i=I} \frac{\lambda^c(P_i)}{I}$$

where $\lambda^c(P)$ denotes the total curvature of the central path corresponding to the linear optimization problem $\min\{c^T x : x \in P\}$. Dedieu, Malajovich and Shub [4] demonstrated that $\lambda^c(\mathcal{A}) \leq 2\pi d$ for any fixed c. Keeping the linear optimization approach but replacing central path following interior point methods by simplex methods, Haimovich's probabilistic analysis of the shadow-vertex simplex algorithm, see [2, Section 0.7], showed that the expected number of pivots is bounded by d. Note that while Dedieu, Malajovich and Shub consider only the bounded cells (the central path may not be defined over some unbounded ones), Haimovich considers the average over bounded and unbounded cells. While the result of Haimovich and Conjecture 1 are similar in nature, they differ in some aspects: Conjecture 1 considers the average over bounded cells, and the number of pivots could be smaller than the diameter for some cells.

In Section 4 we consider a simple hyperplane arrangement $\mathcal{A}^*_{d,n}$ combinatorially equivalent to the cyclic hyperplane arrangement which is dual to the cyclic polytope, see [8] for some combinatorial properties of the (projective) cyclic hyperplane arrangement. We show that the bounded cells of $\mathcal{A}^*_{d,n}$ are mainly combinatorial cubes and, therefore, that the dimension d is an asymptotic lower bound for $\Delta_{\mathcal{A}}(d, n)$ for fixed d. In Section 2, we consider the arrangement $\mathcal{A}^o_{2,n}$ resulting from the addition of one hyperplane to $\mathcal{A}^*_{2,n-1}$ such that all the vertices are on one side of the added hyperplane. We show that the arrangement $\mathcal{A}^o_{2,n}$ maximizes the average diameter and, thus, Conjecture 1 holds in dimension 2. In Section 3, considering a 3-dimensional analogue, we give lower and upper bounds asymptotically equal to 3 for $\Delta_{\mathcal{A}}(3, n)$. The combinatorics of the addition of a (pseudo) hyperplane to the cyclic hyperplane arrangement is studied in details in [16]. For example, the arrangements $\mathcal{A}^*_{2,6}$ and $\mathcal{A}^o_{2,6}$ correspond to the top and bottom elements of the higher Bruhat order $B(5, 2)$ given in Figure 3 of [16]. For polytopes and arrangements, we refer to the books of Edelsbrunner [6], Grünbaum [10] and Ziegler [17].

2 Line Arrangements with Maximal Average Diameter

For $n \geq 4$, we consider the simple line arrangement $\mathcal{A}^o_{2,n}$ made of the 2 lines h_1 and h_2 forming, respectively, the x_1 and x_2 axis, and the $(n-2)$ lines defined by their intersections with h_1 and h_2. We have $h_k \cap h_1 = \{1 + (k-3)\varepsilon, 0\}$ and $h_k \cap h_2 = \{0, 1-(k-3)\varepsilon\}$ for $k = 3, 4, \ldots, n-1$, and $h_n \cap h_1 = \{2, 0\}$ and $h_n \cap h_1 = \{0, 2+\varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < 1/(n-3)$. See Figure 1 for an arrangement combinatorially equivalent to $\mathcal{A}^o_{2,7}$.

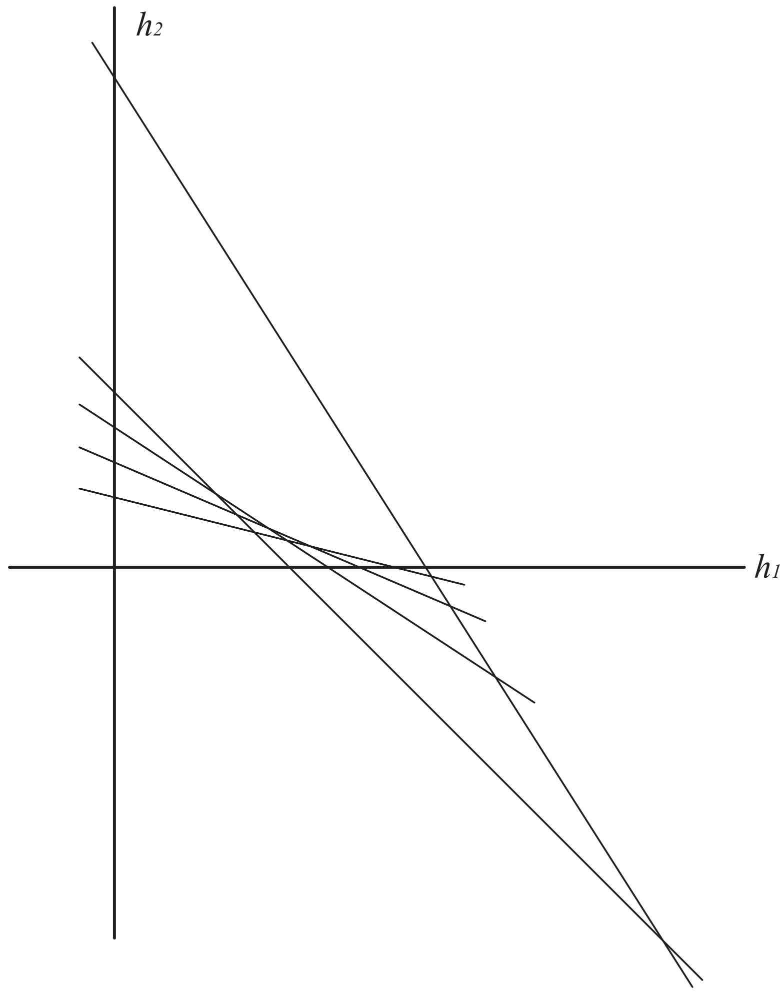

Figure 1: An arrangement combinatorially equivalent to $\mathcal{A}^o_{2,7}$

Proposition 2 *For $n \geq 4$, the bounded cells of the arrangement $\mathcal{A}^o_{2,n}$ consist of $(n-2)$ triangles, $\frac{(n-1)(n-4)}{2}$ 4-gons, and 1 n-gon. We have $\delta(\mathcal{A}^o_{2,n}) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$ for $n \geq 4$.*

Proof: The first $(n-1)$ lines of $\mathcal{A}^o_{2,n}$ clearly form a simple line arrangement $\mathcal{A}^*_{2,n-1}$ which bounded cells are $(n-3)$ triangles and $\binom{n-3}{2}$ 4-gons. The last line h_n adds 1 n-gon, 1 triangle and $(n-4)$ 4-gons. Since the diameter of a k-gon is $\lfloor \frac{k}{2} \rfloor$, we have $\delta(\mathcal{A}^o_{2,n}) = 2 - 2\frac{(n-2)-(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)} = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$. $\square$

Exploiting the fact that a line arrangement contains at least $n-2$ triangles (at least $n-d$ simplices for a simple hyperplane arrangement [13]) and a bound on the number of facets on the boundary of the union of the bounded cells, we can show that $\mathcal{A}^o_{2,n}$ attains the largest possible average diameter of a simple line arrangement.

Proposition 3 *For $n \geq 4$, the largest possible average diameter of a bounded cell of a simple line arrangement satisfies $\Delta_{\mathcal{A}}(2,n) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$.*

PROOF: Let $f_1(\mathcal{A})$ denote the number of bounded edges of a simple arrangement $\mathcal{A}$ of n lines, and let $f_1(P_i)$ denote the number of edges of a bounded cell P_i of $\mathcal{A}$. Let call an edge of $\mathcal{A}$ *external* if it belongs to exactly one bounded cell, and let $f_1^0(\mathcal{A})$ denote the number of external edges of $\mathcal{A}$. Let $p_{odd}(\mathcal{A})$ be the number of bounded cells having an odd number of edges. We have:

$$I \times \delta(\mathcal{A}) = \sum_{i=1}^{I} \delta(P_i) = \sum_{i=1}^{I} \left\lfloor \frac{f_1(P_i)}{2} \right\rfloor = \sum_{i=1}^{I} \frac{f_1(P_i)}{2} - \frac{p_{odd}(\mathcal{A})}{2} = \frac{2f_1(\mathcal{A}) - f_1^0(\mathcal{A}) - p_{odd}(\mathcal{A})}{2}.$$

Since $f_1(\mathcal{A}) = n(n-2)$, to maximize $\delta(\mathcal{A})$ is equivalent to minimize $f_1^0(\mathcal{A}) + p_{odd}(\mathcal{A})$. We clearly have $f_1^0(\mathcal{A}^o_{2,n}) = 2(n-1)$, and this is the best possible as the number of external edges $f_1^0(\mathcal{A})$ is at least $2(n-1)$, see [3]. We have have $p_{odd}(\mathcal{A}^o_{2,n}) = n-2$ for even n, and this is the best possible since at least $n-2$ bounded cells of a simple line arrangement are triangles. If $p_{odd}(\mathcal{A})$ is odd, $\sum_{i=1}^{I} f_1(P_i)$ is odd. If $f_1^0(\mathcal{A}^o_{2,n}) = 2(n-1)$, $\sum_{i=1}^{I} f_1(P_i) = 2f_1(\mathcal{A}) - f_1^0(\mathcal{A})$ is even. Thus, for odd n, $f_1^0(\mathcal{A}) + p_{odd}(\mathcal{A})$ is at least $2(n-1) + (n-2) + 1$ which is achieved by $\mathcal{A}^o_{2,n}$. Thus $\mathcal{A}^o_{2,n}$ minimizes $f_1^0(\mathcal{A}) + p_{odd}(\mathcal{A})$; that is, maximizes $\delta(\mathcal{A})$. $\square$

3 Plane Arrangements with Large Average Diameter

For $n \geq 5$, we consider the simple plane arrangement $\mathcal{A}^o_{3,n}$ made of the the 3 planes h_1, h_2 and h_3 corresponding, respectively, to $x_3 = 0$, $x_2 = 0$ and $x_1 = 0$, and $(n-3)$ planes defined by their intersections with the x_1, x_2 and x_3 axis. We have $h_k \cap h_1 \cap h_2 = \{1 + 2(k-4)\varepsilon, 0, 0\}$, $h_k \cap h_1 \cap h_3 = \{0, 1 + (k-4)\varepsilon, 0\}$ and $h_k \cap h_2 \cap h_3 = \{0, 0, 1 - (k-4)\varepsilon\}$ for $k = 4, 5, \ldots, n-1$, and $h_n \cap h_1 \cap h_2 = \{3, 0, 0\}$, $h_n \cap h_1 \cap h_3 = \{0, 2, 0\}$ and $h_n \cap h_2 \cap h_3 = \{0, 0, 3 + \varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < 1/(n-4)$. See Figure 2 for an illustration of an arrangement combinatorially equivalent to $\mathcal{A}^o_{3,7}$ where, for clarity, only the bounded cells belonging to the positive orthant are drawn.

Proposition 4 *For $n \geq 5$, the bounded cells of the arrangement $\mathcal{A}^o_{3,n}$ consist of $(n-3)$ tetrahedra, $(n-3)(n-4) - 1$ cells combinatorially equivalent to a prism with a triangular base, $\binom{n-3}{3}$ cells combinatorially equivalent to a cube, and 1 cell combinatorially equivalent to a shell S_n with n facets and $2(n-2)$ vertices. See Figure 3 for an illustration of S_7. We have $\delta(\mathcal{A}^o_{3,n}) = 3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)}$ for $n \geq 5$.*

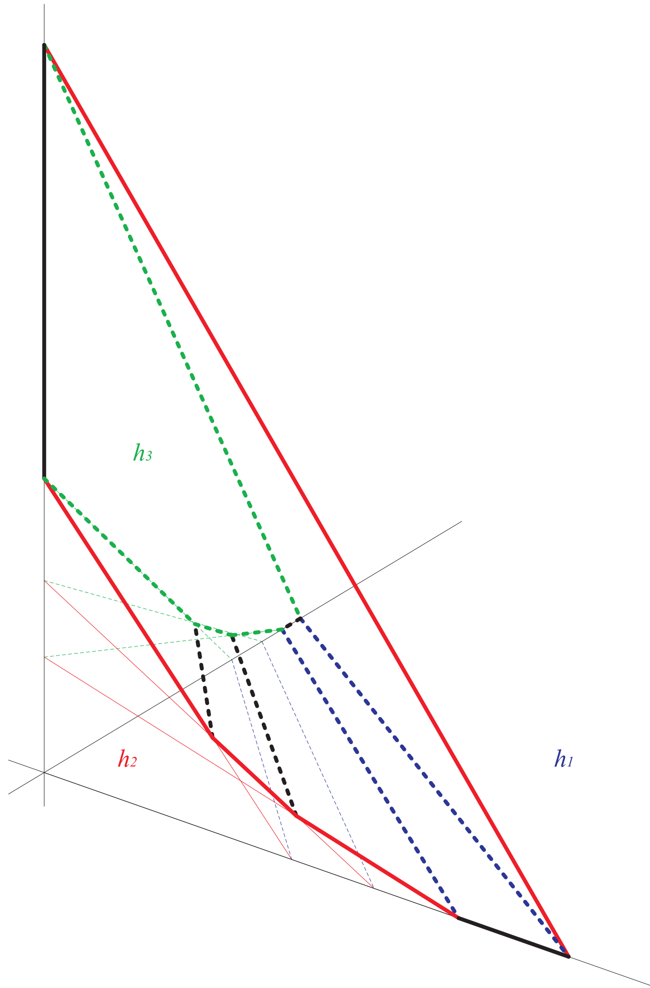

Figure 2: An arrangement combinatorially equivalent to $\mathcal{A}^o_{3,7}$

PROOF: For $4 \leq k \leq n-1$, let $\mathcal{A}^*_{3,k}$ denote the arrangement formed by the first k planes of $\mathcal{A}^o_{3,n}$. See Figure 4 for an arrangement combinatorially equivalent to $\mathcal{A}^*_{3,6}$. We first show by induction that the bounded cells of the arrangement $\mathcal{A}^*_{3,n-1}$ consist of $(n-4)$ tetrahedra, $(n-4)(n-5)$ combinatorial triangular prisms and $\binom{n-4}{3}$ combinatorial cubes. We use the following notation to describe the bounded cells of $\mathcal{A}^*_{3,k-1}$: $T_\triangle$ for a tetrahedron with a facet on h_1 ; $P_\triangle$, respectively $P_\diamond$, for a combinatorial triangular prism with a triangular, respectively square, facet on h_1; $C_\diamond$ for a combinatorial cube with a square facet on h_1; and C, respectively T and P, for a combinatorial cube, respectively tetrahedron and triangular prism, not touching h_1. When the plane h_k is added, the cells $T_\triangle$, $P_\triangle$, $P_\diamond$, and $C_\diamond$ are sliced, respectively, into T and $P_\triangle$, P and $P_\triangle$, P and $C_\diamond$, and C and $C_\diamond$. In addition, one $T_\triangle$ cell and $(k-4)$ $P_\diamond$ cells are created by bounding $(k-3)$ unbounded cells of $\mathcal{A}^*_{3,k-1}$. Let $c(k)$ denotes the number of C cells of $\mathcal{A}^*_{3,k}$, similarly for $C_\diamond$, T, $T_\triangle$, P, $P_\triangle$ and $P_\diamond$. For $\mathcal{A}^*_{3,4}$ we have $t_\triangle(4) = 1$ and $t(4) = p(4) = p_\triangle(4) = p_\diamond(4) = c(4) = c(4) = 0$. The addition of h_k removes and adds one $T_\triangle$, thus, $t_\triangle(k) = 1$. Similarly, all $P_\diamond$ are removed and $(k-4)$ are added, thus, $p_\diamond(k) = (k-4)$.

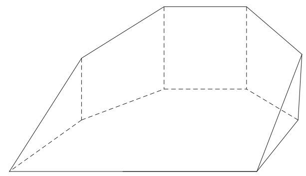

Figure 3: A polytope combinatorially equivalent to the shell S_7

Since $t(k) = t(k-1) + t_\triangle(k-1)$ and $p_\triangle(k) = p_\triangle(k-1) + t_\triangle(k-1)$, we have $t(k) = p_\triangle(k) = (k-4)$. Since $p(k) = p(k-1) + p_\triangle(k-1) + p_\diamond(k-1)$, we have $p(k) = (k-4)(k-5)$. Since $c_\diamond(k) = c_\diamond(k-1) + p_\diamond(k-1)$, we have $c_\diamond(k) = \binom{k-4}{2}$. Since $c(k) = c(k-1) + c_\diamond(k-1)$, we have $c(k) = \binom{k-4}{3}$. Therefore the bounded cells of $\mathcal{A}^*_{3,n-1}$ consist of $t(n-1) + t_\triangle(n-1) = (n-4)$ tetrahedra, $p(n-1) + p_\triangle(n-1) + p_\diamond(n-1) = (n-4)(n-5)$ combinatorial triangular prisms, and $c(n-1) + c_\diamond(n-1) = \binom{n-4}{3}$ combinatorial cubes. The addition of h_n to $\mathcal{A}^*_{3,n-1}$ creates 1 shell S_n with 2 triangular facets belonging to h_2 and h_3 and 1 square facet belonging to h_1. Besides S_n, all the bounded cells created by the addition of h_n are below h_1. One $P_\diamond$ and $n-5$ combinatorial cubes are created between h_2 and h_3. The other bounded cells are on the negative side of h_3: $n-5$ $P_\diamond$ and 1 $T_\triangle$ between h_n and h_{n-1}, and $n-k-5$ $C_\diamond$ and 1 $P_\triangle$ between h_{n-k} and h_{n-k-1} for $k = 1, \ldots, n-5$. In total, we have 1 tetrahedron, $\binom{n-4}{2}$ combinatorial cubes and $(2n-9)$ combinatorial triangular prisms below h_1. Since the diameter of a tetrahedron, triangular prism, cube and n-shell is, respectively, $1, 2, 3$ and $\lfloor \frac{n}{2} \rfloor$, we have $\delta(\mathcal{A}^o_{3,n}) = 3 - 6\frac{2(n-3)+(n-3)(n-4)-1-(\lfloor \frac{n}{2} \rfloor - 3)}{(n-1)(n-2)(n-3)} = 3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)}$. $\square$

Remark 5 *There is only one combinatorial type of simple arrangement of* 5 *planes, and we have* $\Delta_{\mathcal{A}}(3,5) = \delta(\mathcal{A}^o_{3,5}) = \frac{3}{2}$. *Among the* 43 *simple combinatorial types of arrangements formed by* 6 *planes [7], the maximum average diameter is* 2 *while* $\delta(\mathcal{A}^o_{3,6}) = 1.8$. *See Figure 5 for an illustration of the combinatorial type of one of the two simple arrangements with* 6 *planes maximizing the average diameter. The far away vertex on the right and* 3 *bounded edges incident to it are cut off (same for the far away vertex on the left) so the* 10 *bounded cells of the arrangement (*3 *tetrahedra,* 4 *simplex prisms, and* 3 *6-shells) appear not too small.*

Proposition 6 *For* $n \geq 4$, *the largest possible average diameter of a bounded cell of a simple arrangement of* n *planes satisfies* $3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)} \leq \Delta_{\mathcal{A}}(3,n) \leq 3 + \frac{4(2n^2-16n+21)}{3(n-1)(n-2)(n-3)}$.

Proof: Let $f_2(\mathcal{A})$ denote the number of bounded facets of a simple arrangement $\mathcal{A}$ of n planes, and let $f_2(P_i)$ denote the number of facets of a bounded cell P_i of $\mathcal{A}$. Let call a facet of $\mathcal{A}$ *external* if it belongs to exactly one bounded cell, and let $f_2^0(\mathcal{A})$ denote the number of external facets of $\mathcal{A}$. We have: $I \times \delta(\mathcal{A}) =$

$$\sum_{i=1}^{I} \delta(P_i) \leq \sum_{i=1}^{I} \left(\left\lfloor \frac{2f_2(P_i)}{3} \right\rfloor - 1 \right) \leq \sum_{i=1}^{I} \frac{2f_2(P_i)}{3} - \frac{n-3}{3} - I = \frac{4f_2(\mathcal{A}) - 2f_2^0(\mathcal{A}) - n + 3 - 3I}{3}$$

where the second inequality holds since at least $(n-3)$ bounded cells of $\mathcal{A}$ are simplices [13]. Since $f_2(\mathcal{A}) = n\binom{n-2}{2}$ and $f_2^0(\mathcal{A})$ is at least $\frac{n(n-2)}{3} + 2$, see [3], we have $\delta(\mathcal{A}) \leq 3 + 4(2n^2 - 16n + 21)/3(n-1)(n-2)(n-3)$. $\square$

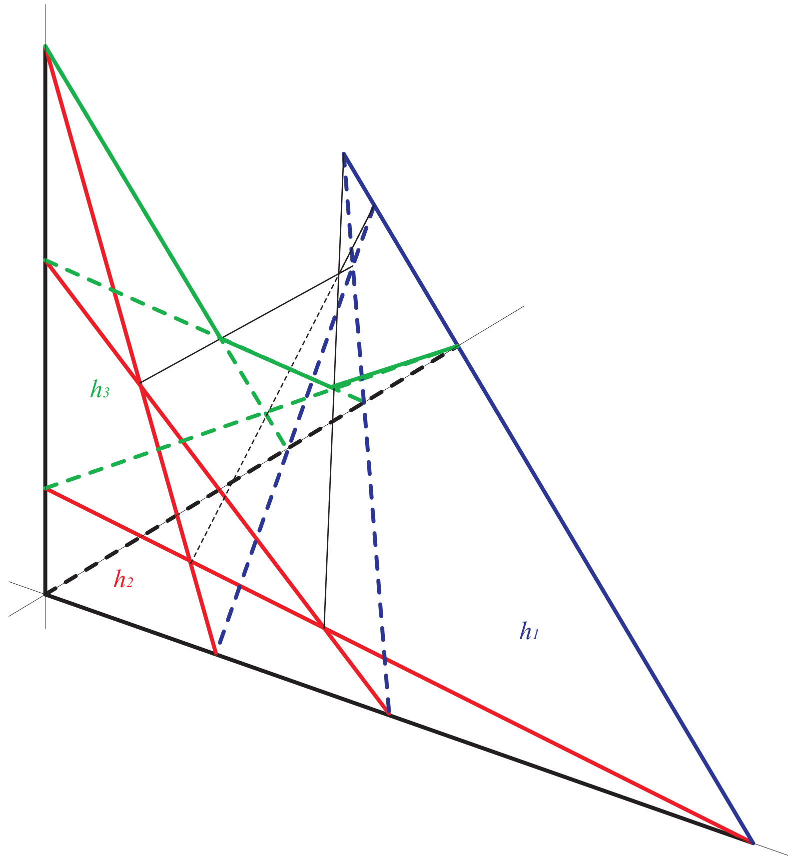

Figure 4: An arrangement combinatorially equivalent to $\mathcal{A}^*_{3,6}$

4 Hyperplane Arrangements with Large Average Diameter

After recalling in Section 4.1 the unique combinatorial structure of a simple arrangement formed by $d+2$ hyperplanes in dimension d, we show in Section 4.2 that the cyclic hyperplane arrangement $\mathcal{A}^*_{d,n}$ contains $\binom{n-d}{d}$ cubical cells for $n \geq 2d$. It implies that the average diameter $\delta(\mathcal{A}^*_{d,n})$ is arbitrarily close to d for n large enough. Thus, the dimension d is an asymptotic lower bound for $\Delta_{\mathcal{A}}(d,n)$ for fixed d.

4.1 The average diameter of a simple arrangement with $d+2$ hyperplanes

Let $\mathcal{A}_{d,d+2}$ be a simple arrangement formed by $d+2$ hyperplanes in dimension d. Besides simplices, the bounded cells of $\mathcal{A}_{d,d+2}$ are simple polytopes with $d+2$ facets corresponding to the product of a k-simplex with a $(d-k)$-simplex for $k = 1, \dots, \lfloor \frac{d}{2} \rfloor$, see for example [10]. We recall one way to show that the combinatorial type of the arrangement of $d+2$ hyperplanes in dimension d is unique. The affine Gale dual, see [16, Chapter 6], of the $d+3$ vectors in dimension $d+1$ corresponding to the linear arrangement associated to $\mathcal{A}_{d,d+2}$ (and the hyperplane at infinity) forms a configuration of $d+3$ distinct signed points on a line; i.e., is unique up to relabeling and reorientation. We also recall the combinatorial structure of $\mathcal{A}_{d,d+2}$ as some of the

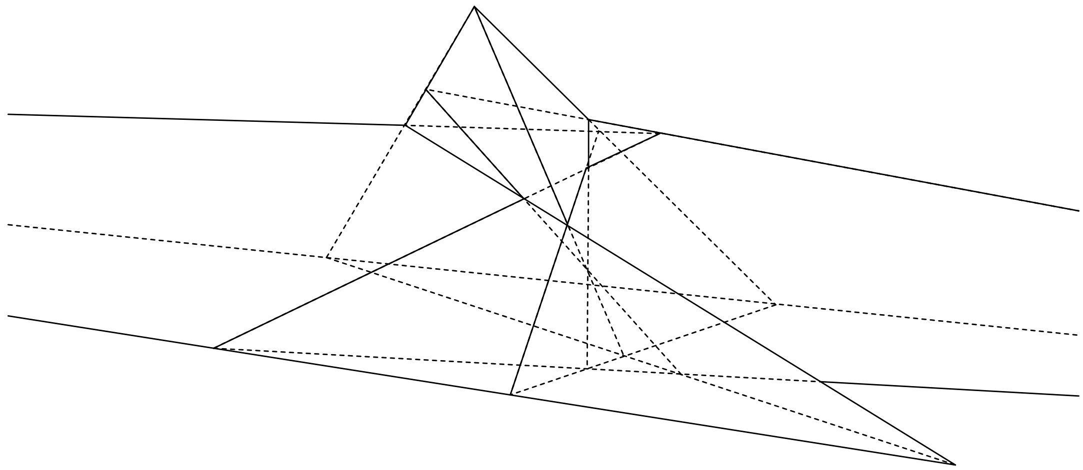

Figure 5: An arrangement formed by 6 planes maximizing the average diameter

notions presented are used in Section 4.2. Since there is only one combinatorial type of simple arrangement with $d+2$ hyperplanes, the arrangement $\mathcal{A}_{d,d+2}$ can be obtained from the simplex $\mathcal{A}_{d,d+1}$ by cutting off one its vertices v with the hyperplane h_{d+2}. As a result, a prism P with a simplex base is created. Let us call *top base* the base of P which belongs to h_{d+2} and assume, without loss of generality, that the hyperplane containing the bottom base of P is h_{d+1}. Besides the simplex defined by v and the vertices of the top base of P, the remaining d bounded cells of $\mathcal{A}_{d,d+2}$ are between h_{d+2} and h_{d+1}. See Figure 6 for an illustration the combinatorial structure of $\mathcal{A}_{3,5}$. As the projection of $\mathcal{A}_{d,d+2}$ on h_{d+1} is combinatorially equivalent to $\mathcal{A}_{d-1,d+1}$, the d bounded cells between h_{d+2} and h_{d+1} can be obtained from the d bounded cells of $\mathcal{A}_{d-1,d+1}$ by the *shell-lifting* of $\mathcal{A}_{d-1,d+1}$ over the ridge $h_{d+1} \cap h_{d+2}$; that is, besides the vertices belonging to $h_{d+1} \cap h_{d+2}$, all the vertices in h_{d+1} (forming $\mathcal{A}_{d-1,d+1}$) are lifted. See Figure 7 where the skeletons of the $d+1$ bounded cells of $\mathcal{A}_{d,d+2}$ are given for $d = 2, 3, \ldots, 6$, and the shell-lifting of the bounded cells is indicated by an arrow. The vertices not belonging to h_{d+1} are represented in black in Figure 7, e.g., the simplex cell containing v is the one made of black vertices. The bounded cells of $\mathcal{A}_{d,d+2}$ are 2 simplices and a pair of product of a k-simplex with a $(d-k)$-simplex for $k = 1, \ldots, \lfloor \frac{d}{2} \rfloor$ for odd d. For even d the product of the $\frac{d}{2}$-simplex with itself is present only once. Since all the bounded cells, besides the 2 simplices, have diameter 2, we have $\delta(\mathcal{A}_{d,d+2}) = \frac{2+2(d-1)}{d+1}$.

Proposition 7 *We have* $\Delta_{\mathcal{A}}(d, d+2) = \delta(\mathcal{A}_{d,d+2}) = \frac{2d}{d+1}$.

4.2 Hyperplane Arrangements with Large Average Diameter

We consider the simple hyperplane arrangement $\mathcal{A}^*_{d,n}$ combinatorially equivalent to the cyclic hyperplane and formed by the following n hyperplanes h^d_k for $k = 1, 2, \ldots, n$. The hyperplanes $h^d_k = \{x : x_{d+1-k} = 0\}$ for $k = 1, 2, \ldots, d$ form the positive orthant, and the hyperplanes h^d_k for $k = d+1, \ldots, n$ are defined by their intersections with the axes $\bar{x}_i$ of the positive orthant. We have $h^d_k \cap \bar{x}_i = \{0, \ldots, 0, 1 + (d-i)(k-d-1)\varepsilon, 0, \ldots, 0\}$ for $i = 1, 2 \ldots, d-1$ and

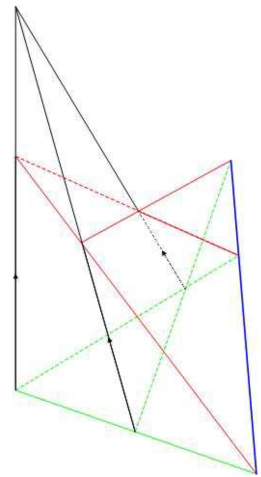

Figure 6: An arrangement combinatorially equivalent to $\mathcal{A}_{3,5}$

$h_k^d \cap \bar{x}_d = \{0, \ldots, 0, 1-(k-d-1)\varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < 1/(n-d-1)$. The combinatorial structure of $\mathcal{A}^*_{d,n}$ can be derived inductively. All the bounded cells of $\mathcal{A}^*_{d,n}$ are on the positive side of h_1^d and h_2^d with the bounded cells between h_2^d and h_3^d being obtained by the shell-lifting of a combinatorial equivalent of $\mathcal{A}^*_{d-1,n-1}$ over the ridge $h_2^d \cap h_3^d$, and the bounded cells on the other side of h_3^d forming a combinatorial equivalent of $\mathcal{A}^*_{d,n-1}$. The intersection $\mathcal{A}^*_{d,n} \cap h_k^d$ is combinatorially equivalent to $\mathcal{A}^*_{d-1,n-1}$ for $k = 2, 3, \ldots, d$ and removing h_2^d from $\mathcal{A}^*_{d,n}$ yields an arrangement combinatorially equivalent to $\mathcal{A}^*_{d,n-1}$. See Figure 4 for an arrangement combinatorially equivalent to $\mathcal{A}^*_{3,6}$.

Proposition 8 *The arrangement $\mathcal{A}^*_{d,n}$ contains $\binom{n-d}{d}$ cubical cells for $n \geq 2d$. We have $\delta(\mathcal{A}^*_{d,n}) \geq d\binom{n-d}{d}/\binom{n-1}{d}$ for $n \geq 2d$. It implies that for d fixed, $\Delta_{\mathcal{A}}(d,n)$ is arbitrarily close to d for n large enough.*

Proof: The arrangements $\mathcal{A}^*_{n,2}$ and $\mathcal{A}^*_{n,3}$ contain, respectively, $\binom{n-2}{2}$ and $\binom{n-3}{3}$ cubical cells. The arrangement $\mathcal{A}^*_{d,2d}$ has 1 cubical cell. Since $\mathcal{A}^*_{d,n}$ is obtained inductively from $\mathcal{A}^*_{d,n-1}$ by lifting $\mathcal{A}^*_{d-1,n-1}$ over the ridge $h_2^d \cap h_3^d$, we count separately the cubical cells between h_2^d and h_3^d and the ones on the other side of h_3^d. The ridge $h_2^d \cap h_3^d$ is an hyperplane of the arrangements $\mathcal{A}^*_{d,n} \cap h_2^d$ and $\mathcal{A}^*_{d,n} \cap h_3^d$ which are both combinatorially equivalent to $\mathcal{A}^*_{d-1,n-1}$. Removing h_2^{d-1} from $\mathcal{A}^*_{d,n} \cap h_2^d$ yields an arrangement combinatorially equivalent to $\mathcal{A}^*_{d-1,n-2}$. It implies that $\binom{(n-2)-(d-1)}{d-1}$ cubical cells of $\mathcal{A}^*_{d,n} \cap h_2^d$ are not incident to the ridge $h_2^d \cap h_3^d$. The shell-lifting of these $\binom{n-d-1}{d-1}$ cubical cells (of dimension $d-1$) creates $\binom{n-d-1}{d-1}$ cubical cells between h_2^d and h_3^d. As removing h_2^d from $\mathcal{A}^*_{d,n}$ yields an arrangement combinatorial equivalent to $\mathcal{A}^*_{d,n-1}$, there are $\binom{n-1-d}{d}$ cubical cells on the other side of h_3^d. Thus, $\mathcal{A}^*_{d,n}$ contains $\binom{n-d-1}{d-1} + \binom{n-d-1}{d} = \binom{n-d}{d}$ cubical cells. □

Proposition 8 can be slightly strengthened to the following proposition.

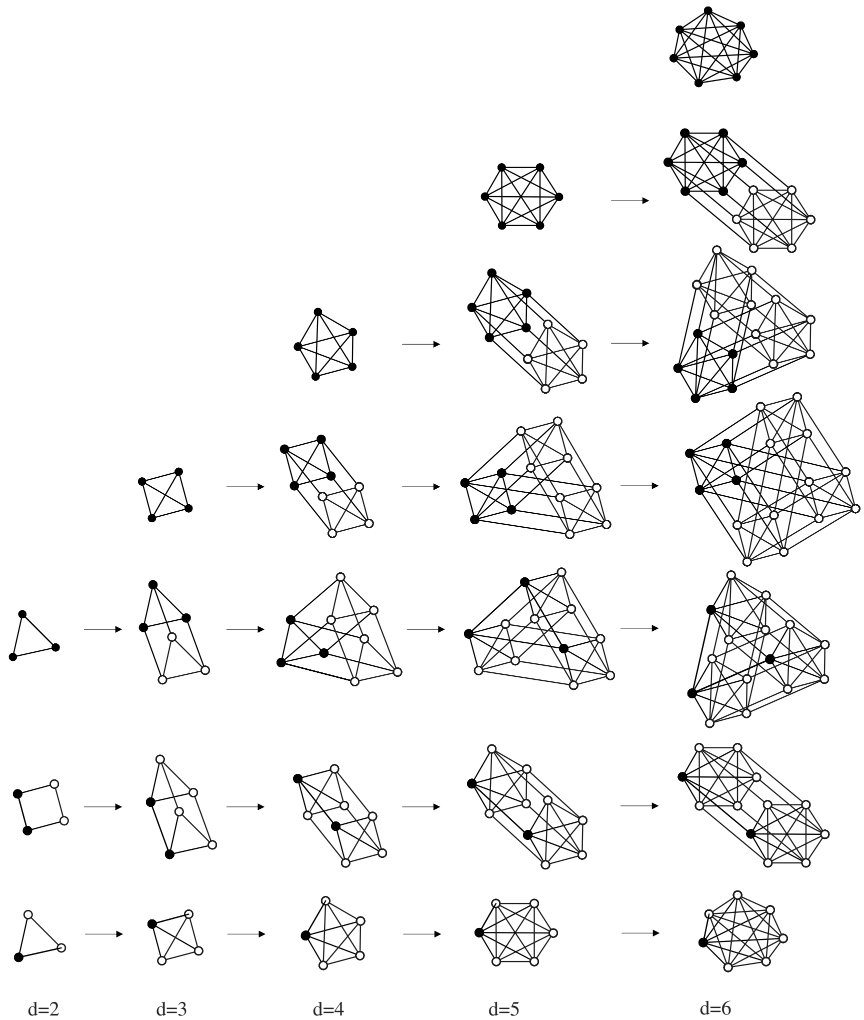

Figure 7: The skeletons of the $d+1$ bounded cells of $\mathcal{A}_{d,d+2}$ for $d = 2, 3, \ldots, 6$.

Proposition 9 *Besides $\binom{n-d}{d}$ cubical cells, the arrangement $\mathcal{A}^*_{d,n}$ contains $(n-d)$ simplices and $(n-d)(n-d-1)$ bounded cells combinatorially equivalent to a prism with a simplex base for $n \geq 2d$. We have $\Delta_{\mathcal{A}}(d,n) \geq 1 + \frac{(d-1)\binom{n-d}{d}+(n-d)(n-d-1)}{\binom{n-1}{d}}$ for $n \geq 2d$.*

Proof: Similarly to Proposition 8, we can inductively count $(n-d)$ simplices and $(n-d)(n-d-1)$ bounded cells of $\mathcal{A}^*_{d,n}$ combinatorially equivalent to a prism with a simplex base. We have $(n-1)-(d-1)$ simplices in $\mathcal{A}^*_{d,n} \cap h_2^d$ and, since removing h_2^{d-1} from $\mathcal{A}^*_{d,n} \cap h_2^d$ yields an arrangement combinatorially equivalent to $\mathcal{A}^*_{d-1,n-2}$, only one of these $(n-d)$ simplices of $\mathcal{A}^*_{d,n} \cap h_2^d$ is incident to the ridge $h_2^d \cap h_3^d$. Thus, between h_2^d and h_3^d, we have 1 simplex incident to the ridge $h_2^d \cap h_3^d$ and $(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base not incident to the ridge $h_2^d \cap h_3^d$. In addition, $(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base are incident to the ridge $h_2^d \cap h_3^d$ and between h_2^d and h_3^d. These $(n-d-1)$

cells correspond to the truncations of the simplex $\mathcal{A}^*_{d,d+1}$ by h^d_k for $k = d+2, d+3, \dots, n$. Thus, we have $2(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base between h^d_2 and h^d_3. Since the other side of h^d_3 is combinatorially equivalent to $\mathcal{A}^*_{n-1,d}$, it contains $(n-1-d)$ simplices and $(n-d-1)(n-d-2)$ bounded cells combinatorially equivalent to a prism with a simplex base. Thus, $\mathcal{A}^*_{d,n}$ has $(n-d-1)(n-d-2)+2(n-d-1) = (n-d)(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base and $(n-d)$ simplices. As a prism with a simplex base has diameter 2 and the diameter of a bounded cell is at least 1, we have $\delta(\mathcal{A}^*_{d,n}) \geq \frac{d\binom{n-d}{d}+2(n-d)(n-d-1)+\binom{n-1}{d}-\binom{n-d}{d}-(n-d)(n-d-1)}{\binom{n-1}{d}}$ for $n \geq 2d$. $\square$

Acknowledgments The authors would like to thank Komei Fukuda, Hiroki Nakayama, and Christophe Weibel for use of their codes [9, 11, 15] which helped to investigate small simple arrangements. The authors are grateful to the anonymous referees for many helpful suggestions and for pointing out relevant papers [2, 16]. Research supported by an NSERC Discovery grant, a MITACS grant and the Canada Research Chair program.

References

[1] G. Dantzig: *Linear Programming and Extensions*. Princeton University Press (1963).

[2] K. H. Borgwardt: *The Simplex Method – A Probabilistic Analysis*. Springer-Verlag (1987).

[3] D. Bremner, A. Deza and F. Xie: The complexity of the envelope of line and plane arrangements. AdvOL-Report 2007/14, McMaster University (2007).

[4] J.-P. Dedieu, G. Malajovich and M. Shub: On the curvature of the central path of linear programming theory. *Foundations of Computational Mathematics* 5 (2005) 145–171.

[5] A. Deza, T. Terlaky and Y. Zinchenko: Polytopes and arrangements: diameter and curvature. *Operations Research Letters* (to appear).

[6] H. Edelsbrunner: *Algorithms in Combinatorial Geometry*. Springer-Verlag (1987).

[7] L. Finschi: *Oriented matroids database* `http://www.om.math.ethz.ch` .

[8] D. Forge and J. L. Ramírez Alfonsín: On counting the k-face cells of cyclic arrangements. *European Journal of Combinatorics* 22 (2001) 307–312.

[9] K. Fukuda: `http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html` .

[10] B. Grünbaum: *Convex Polytopes*. V. Kaibel, V. Klee and G. Ziegler (eds.), Graduate Texts in Mathematics 221, Springer-Verlag (2003).

[11] H. Nakayama: `http://www-imai.is.s.u-tokyo.ac.jp/~nak-den/arr_gen` .

[12] C. Roos, T. Terlaky and J.-Ph. Vial: *Interior Point Methods for Linear Optimization*. Springer (2006).

[13] R. W. Shannon: Simplicial cells in arrangements of hyperplanes. *Geometriae Dedicata* 8 (1979) 179–187.

[14] M. Spivak: *Comprehensive Introduction to Differential Geometry*. Publish or Perish (1990).

[15] C. Weibel: `http://roso.epfl.ch/cw/poly/public.php` .

[16] G. Ziegler: Higher Bruhat orders and cyclic hyperplane arrangements. *Topology* 32 (1993) 259–279.

[17] G. Ziegler: *Lectures on Polytopes*. Graduate Texts in Mathematics 152, Springer-Verlag (1995).