Polynomial interior point algorithms for general LCPs

Tibor Illés*, Marianna Nagy*, and Tamás Terlaky†

April 10, 2007

Abstract

Linear Complementarity Problems (*LCP*s) belong to the class of $\mathbb{NP}$-complete problems. Therefore we can not expect a polynomial time solution method for *LCP*s without requiring some special property of the matrix coefficient matrix. Our aim is to construct some interior point algorithms which, according to the duality theorem in EP form, gives a solution of the original problem or detects the lack of property $\mathcal{P}_*(\tilde{\kappa})$ (with arbitrary large, but apriori fixed $\tilde{\kappa}$) and gives a polynomial size certificate of it in polynomial time (depending on parameter $\tilde{\kappa}$, the initial interior point and the dimension of the *LCP*). We give the general idea of a modification of interior point algorithms and present three concrete methods: affine scaling, long-step path-following and predictor-corrector interior point algorithm.

Keywords: linear complementarity problem, sufficient matrix, $\mathcal{P}_*$-matrix, interior point method, long step method, affine scaling method, predictor-corrector algorithm.

1 Introduction

Consider the *linear complementarity problem* (*LCP*): find vectors $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ that satisfy

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}\mathbf{s} = \mathbf{0}, \quad \mathbf{x},\ \mathbf{s} \geq \mathbf{0}, \tag{1}$$

where $M \in \mathbb{R}^{n \times n}$ and $\mathbf{q} \in \mathbb{R}^n$, and the notation $\mathbf{xs}$ is used for the coordinatewise (Hadamard) product of the vectors $\mathbf{x}$ and $\mathbf{s}$.

The *LCP* belongs to the class of $\mathbb{NP}$-complete problems, since the feasibility problem of linear equations with binary variables can be described as an *LCP* [13]. Therefore we can not expect an efficient (polynomial time) solution method for *LCP*s without requiring some special property of the matrix M.

There are known polynomial time algorithms for solving an *LCP* if the matrix M is a positive semidefinite matrix (see e.g., [11, 12, 18, 23]). Furthermore, an *LCP* can

*Tibor Illés and Marianna Nagy acknowledge the Research Fellowship of MOL, the Hungarian Oil Company. The authors research is supported by the Hungarian National Research Fund OTKA No. T 049789 and by the Hungarian Science and Technology Foundation TÉT SLO-4/2005.

†Supported by an NSERC Discovery grant, MITACS and the Canada Research Chair program.

be solved in polynomial time if the matrix M is a $\mathcal{P}_*(\kappa)$-matrix[1], however in this case the computational complexity of the algorithm depends on κ too (see e.g., [10, 16, 18]). Positive semidefiniteness of a matrix can be checked in strongly polynomial time [15], but no polynomial time algorithm is known for checking whether a matrix is $\mathcal{P}_*(\kappa)$ or not. The best known test for the $\mathcal{P}_*(\kappa)$ property, introduced by Väliaho [22], is not polynomial.

For applying an interior point method (IPM) to an LCP with a $\mathcal{P}_*(\kappa)$-matrix, we need an initial interior point (or use an infeasible IPM) and one need to know apriori the κ value of the matrix M. An initial interior point can be found by using an embedding model [19], but the apriori knowledge of κ is a too strong assumption. Potra et al. [18] softened this assumption, they modified their IPM in such a way, that we need to know only the sufficiency of the matrix. However, this is still a condition, that can not be verified in polynomial time. Consequently, there is a need to design such an algorithm, that can handle any LCP with an arbitrary matrix. Therefore, in this paper interior point algorithms for $\mathcal{P}_*(\kappa)$-matrix LCPs are appropriately modified. The new algorithms either solve the LCP, or give a polynomial size certificate in polynomial time that the matrix M is not a $\mathcal{P}_*(\tilde{\kappa})$-matrix with arbitrary large, but apriori fixed $\tilde{\kappa}$, and the polynomiality depends on the parameter $\tilde{\kappa}$, the initial interior point and the dimension of the LCP. Through the paper, except in Section 4, we assume that a feasible interior point of the LCP is known.

Let now consider the *dual linear complementarity problem* ($DLCP$) [4]: find vectors $\mathbf{u}$, $\mathbf{z} \in \mathbb{R}^n$ which satisfy the constraints

$$\mathbf{u} + M^T\mathbf{z} = \mathbf{0}, \quad \mathbf{q}^T\mathbf{z} = -1, \quad \mathbf{u}\,\mathbf{z} = \mathbf{0}, \quad \mathbf{u},\, \mathbf{z} \geq \mathbf{0}. \tag{2}$$

An EP (existentially polynomial-time) theorem [2] is a theorem of the form:

$$[\forall x : F_1(x), F_2(x), \dots, F_k(x)],$$

where $F_i(x)$ is a predicate formula which has the form

$$F_i(x) = [\exists y_i \text{ such that } \|y_i\| \leq \|x\|^{n_i} \text{ and } f_i(x, y_i)].$$

Here $n_i \in \mathbb{Z}^+$, $\|z\|$ denotes the encoding length of z and $f_i(x, y_i)$ is a predicate for which there is a polynomial-size certificate.

The LCP duality theorem in EP form [5] is as follows:

Theorem 1 *Let matrix $M \in \mathbb{Q}^{n\times n}$ and vector $\mathbf{q} \in \mathbb{Q}^n$. At least one of the following statements holds:*

(1) problem LCP has a complementary feasible solution $(\mathbf{x}, \mathbf{s})$, whose encoding size is polynomially bounded.

(2) problem $DLCP$ has a complementary feasible solution $(\mathbf{u}, \mathbf{z})$, whose encoding size is polynomially bounded.

(3) matrix M is not sufficient and there is a certificate whose encoding size is polynomially bounded.

[1] The definition of matrix classes is given in the next section

The criss-cross algorithm for sufficient matrices was introduced by Hertog, Roos and Terlaky [8]. The first criss-cross type pivot algorithm in EP form, which does not use apriori knowledge of sufficiency of the matrix M, was given by Fukuda, Namiki and Tamura [5]. They utilized the LCP duality theorem of Fukuda and Terlaky [6]. Csizmadia and Illés [4] extended this method to several flexible pivot rules. These variants of the criss-cross method solve LCPs with an arbitrary matrix. They either solve the primal LCP or give a dual solution, or detect that the algorithm may begin cycling (due to lack of sufficiency) and in this case they give a polynomial size certificate of the lack of sufficiency. Such an EP form interior point algorithm does not existed yet.

Summarizing, our aim is to construct interior point algorithms, that according to the duality theorem of LCP in EP form either give a solution of the original LCP or for the dual LCP, or detect the lack of property $\mathcal{P}_*(\tilde{\kappa})$, and give a polynomial certificate in all cases.

The rest of the paper is organized as follows. The following section deals with the fundamental properties of $\mathcal{P}_*(\kappa)$-matrices and with some well-known results. In Section 3 we describe the general idea of modified IPMs and than present the modification of three popular interior point algorithms: affine scaling algorithms, long-step path-following algorithms and predictor-corrector algorithms. Section 4 addresses the question, how the interior point assumption can be eliminated, where we present the technique of embedding. For ease of understanding and self containedness we collect the necessary results of the papers [10, 16, 18] in the Appendix.

Notations:
We use the following notations throughout the paper. Scalars and indices are denoted by lowercase Latin letters, vectors by lowercase boldface Latin letters, matrices by capital Latin letters, and finally sets by capital calligraphic letters. $\mathbb{R}^n_\oplus$ ($\mathbb{R}^n_+$) is the nonnegative (positive) orthant of $\mathbb{R}^n$. Further, X is the diagonal matrix whose diagonal elements are the coordinates of the vector $\mathbf{x}$, so $X = \mathrm{diag}(\mathbf{x})$, and I denotes the identity matrix of appropriate dimension. The vector $\mathbf{x}\,\mathbf{s} = X\mathbf{s}$ is the componentwise product (Hadamard product) of the vectors $\mathbf{x}$ and $\mathbf{s}$, and for $\alpha \in \mathbb{R}$ the vector $\mathbf{x}^\alpha$ denotes the vector whose ith component is x_i^α. We denote the vector of ones by $\mathbf{e}$. Furthermore, for vector $\mathbf{x}$ we define the sets $\mathcal{I}_+(\mathbf{x}) = \{1 \le i \le n :\ x_i(Mx)_i > 0\}$ and $\mathcal{I}_-(\mathbf{x}) = \{1 \le i\ \le n :\ x_i(Mx)_i < 0\}$, which are used in the definition of $\mathcal{P}_*(\kappa)$ matrices.

Let the current point be $(\mathbf{x}, \mathbf{s})$ and $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the current Newton direction[2]. The new point with step length θ is given by $(\mathbf{x}(\theta), \mathbf{s}(\theta)) = (\mathbf{x} + \theta\Delta\mathbf{x}, \mathbf{s} + \theta\Delta\mathbf{s})$. We use the following notations for scaling

$$\mathbf{v} = \sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d^x} = \frac{\mathbf{v}\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{d^s} = \frac{\mathbf{v}\Delta\mathbf{s}}{\mathbf{s}}, \quad \mathbf{g} = \mathbf{d^x}\mathbf{d^s}, \quad \mathbf{p} = \mathbf{d^x} + \mathbf{d^s}, \tag{3}$$

where in the affine scaling algorithm for the purpose of scaling we have $\mu \equiv 1$. In the affine scaling algorithm we use the δ_a, and in two other algorithms the δ_c centrality measures, where

$$\delta_a(\mathbf{x}\mathbf{s}) = \frac{\max(\sqrt{\mathbf{x}\mathbf{s}}\,)}{\min(\sqrt{\mathbf{x}\mathbf{s}}\,)}, \qquad \delta_c(\mathbf{x}\mathbf{s}, \mu) = \left\| \sqrt{\frac{\mathbf{x}\mathbf{s}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{x}\mathbf{s}}} \right\|.$$

[2] Generally the Newton direction is the unique solution of system (5), see page 5. We will discuss how to define the actual Newton directions for the various algorithms in Section 3.

Furthermore, the so-called negative infinity neighborhood, which is defined by Potra in [18], is used in the predictor-corrector algorithm:

$$\mathcal{D}(\beta) := \left\{ (\mathbf{x},\, \mathbf{s}) \in \mathcal{F}^0 :\ \mathbf{x}\,\mathbf{s} \geq \beta\, \frac{\mathbf{x}^T\mathbf{s}}{n} \right\},$$

where $\mathcal{F}^0 := \left\{(\mathbf{x},\, \mathbf{s}) \in \mathbb{R}^{2n}_+ : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\right\}$ denotes the set of strictly feasible solutions of the LCP. The $\mathcal{D}(\beta)$ neighborhood is considered to be a "wide neighborhood".

2 Matrix classes and the Newton step

The class of $P_*(\kappa)$-matrices were introduced by Kojima et al. [13], and it can be considered as a generalization of the class of positive semidefinite matrices.

Definition 2 *Let $\kappa \geq 0$ be a nonnegative number. A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_*(\kappa)$*-matrix *if for all $\mathbf{x} \in \mathbb{R}^n$*

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}_+(\mathbf{x})} x_i(Mx)_i + \sum_{i \in \mathcal{I}_-(\mathbf{x})} x_i(Mx)_i \geq 0, \tag{4}$$

where $\mathcal{I}_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i > 0\}$ and $\mathcal{I}_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i < 0\}$.

The nonnegative real number κ denotes the weight that need to be used at the positive terms so that the weighted 'scalar product' is nonnegative for each vector $\mathbf{x} \in \mathbb{R}^n$. Therefore, naturally $\mathcal{P}_*(0)$ is the class of positive semidefinite matrices (if we set aside the symmetry of the matrix M).

Definition 3 *A matrix $M \in \mathbb{R}^{n\times n}$ is called a $\mathcal{P}_*$*-matrix *if it is a $\mathcal{P}_*(\kappa)$-matrix for some $\kappa \geq 0$, i.e.*

$$\mathcal{P}_* = \bigcup_{\kappa \geq 0} \mathcal{P}_*(\kappa).$$

The class of sufficient matrices was introduced by Cottle, Pang and Venkateswaran [3].

Definition 4 *A matrix $M \in \mathbb{R}^{n\times n}$ is a* column sufficient matrix *if for all $\mathbf{x} \in \mathbb{R}^n$*

$$X(M\mathbf{x}) \leq 0 \ \ \textit{implies} \ \ X(M\mathbf{x}) = 0,$$

and row sufficient *if M^T is column sufficient. Matrix M is* sufficient *if it is both row and column sufficient.*

Kojima et al. [13] proved that any $\mathcal{P}_*$-matrix is column sufficient and Guu and Cottle [7] proved that it is row sufficient, too. Therefore, each $\mathcal{P}_*$-matrix is sufficient. Väliaho proved the other direction of inclusion [21], thus the class of $\mathcal{P}_*$-matrices coincides with the class of sufficient matrices.

Definition 5 *A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_0$*-matrix*, if all of its principal minors are nonnegative.*

For further use we recall some well-known results. The reader may consult the book of Kojima et al. [13, Lemma 4.1 p. 35] for the proof of the following Proposition.

Proposition 6 *A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_0$-matrix if and only if*

$$M' = \begin{bmatrix} -M & I \\ S & X \end{bmatrix} \text{ is a nonsingular matrix}$$

for any positive diagonal matrices $X, S \in \mathbb{R}^{n\times n}$. □

Proposition 6 enables us to check whether matrix M is $\mathcal{P}_0$ or not. The next statement is used to guarantee the existence and uniqueness of Newton directions that are the solution of system (5) for various values of vector $\mathbf{a} \in \mathbb{R}^n$, where $\mathbf{a}$ depends on the particular interior point algorithm.

Corollary 7 *Let $M \in \mathbb{R}^{n\times n}$ be a $\mathcal{P}_0$-matrix, $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$. Then for all $\mathbf{a} \in \mathbb{R}^n$ the system*

$$\begin{array}{rcrcl} -M\boldsymbol{\Delta}\mathbf{x} & + & \boldsymbol{\Delta}\mathbf{s} & = & \mathbf{0} \\ \mathbf{s}\boldsymbol{\Delta}\mathbf{x} & + & \mathbf{x}\boldsymbol{\Delta}\mathbf{s} & = & \mathbf{a} \end{array} \tag{5}$$

has a unique solution $(\boldsymbol{\Delta}\mathbf{x}, \boldsymbol{\Delta}\mathbf{s})$. □

The following estimations for the Newton direction are used in the complexity analysis of interior point methods. The next lemma is proved by Potra in [17].

Lemma 8 *Let M be an arbitrary $n \times n$ real matrix and $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be a solution of system (5). Then*

$$\sum_{i\in\mathcal{I}_+} \Delta x_i \Delta s_i \leq \frac{1}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

□

Lemma 9 *Let matrix M be a $\mathcal{P}_*(\kappa)$-matrix and $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$, $\mathbf{a} \in \mathbb{R}^n$. Let $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the solution of system (5). Then*

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty \leq \left(\frac{1}{4}+\kappa\right)\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2, \qquad \|\Delta\mathbf{x}\Delta\mathbf{s}\|_1 \leq \left(\frac{1}{2}+\kappa\right)\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_2 \leq \sqrt{\left(\frac{1}{4}+\kappa\right)\left(\frac{1}{2}+\kappa\right)}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

□

The first statement's proof in the lemma is similar to the proof of Lemma 5.1 by Illés, Roos and Terlaky [10]. The second estimation follows from the previous lemma by using some properties of $\mathcal{P}_*(\kappa)$-matrices, and the last estimation is a corollary of the first and second statements using some properties of norms.

3 Interior point algorithms in EP form

Our aim is to modify interior point algorithms in such a way, that they solve the *LCP* with any arbitrary matrix, or give a certificate, that the matrix is not $\mathcal{P}_*(\tilde{\kappa})$, where $\tilde{\kappa}$ is a given (arbitrary big) number. Potra et al. [18] gave the first interior point algorithm, where we do not need to know apriori the value of κ, it is enough to know that the matrix is $\mathcal{P}_*$. Their algorithm initially assumes that the matrix is $\mathcal{P}_*(1)$. In each iteration they check whether the new point is in the appropriate neighborhood of the central path, or not. In the latter case they double the value of κ. We use this idea in a modified way. Because the larger κ is, the worse the iteration complexity is, we take only the necessary enlargement of κ (until it reaches $\tilde{\kappa}$). The inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices gives the following lower bound on κ for any vector $\mathbf{x} \in \mathbb{R}^n$:

$$\kappa \geq \kappa(\mathbf{x}) = -\frac{1}{4}\frac{\mathbf{x}^T M\mathbf{x}}{\sum_{i\in\mathcal{I}_+} x_i(Mx)_i}.$$

In IPMs the $\mathcal{P}_*(\kappa)$ property need to be true only for the actual Newton direction $\Delta\mathbf{x}$ in various ways, for example this property ensures that with a certain step size the new iterate is in an appropriate neighborhood of the central path and/or the complementarity gap is sufficiently reduced. Consequently, if the desired results do not hold with the current κ value, we update κ to the lower bound determined by the Newton direction $\Delta\mathbf{x}$, i.e.,

$$\kappa(\Delta\mathbf{x}) = -\frac{1}{4}\frac{\Delta\mathbf{x}^T\Delta\mathbf{s}}{\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i} \qquad\qquad (\Delta\mathbf{s} = M\Delta\mathbf{x}). \tag{6}$$

The following two lemmas are immediate consequences of the definition of $\mathcal{P}_*(\kappa)$ and $\mathcal{P}_*$-matrices.

Lemma 10 *Let M be a real $n\times n$ matrix. If there exists a vector $\mathbf{x}\in\mathbb{R}^n$ such that $\kappa(\mathbf{x}) > \tilde{\kappa}$, then the matrix M is not $\mathcal{P}_*(\tilde{\kappa})$ and $\mathbf{x}$ is a certificate for this fact.*

Lemma 11 *Let M be a real $n\times n$ matrix. If there exists a vector $\mathbf{x}\in\mathbb{R}^n$ such that $\mathcal{I}_+(\mathbf{x}) = \{i\in\mathcal{I} : x_i(Mx)_i > 0\} = \emptyset$, then the matrix M is not $\mathcal{P}_*$ and $\mathbf{x}$ is a certificate for this fact.*

Therefore, if there exists such a vector $\Delta\mathbf{x}$ for which $\mathcal{I}_+ = \emptyset$, and thus $\kappa(\Delta\mathbf{x})$ is not defined, then the matrix M of the *LCP* is not a $\mathcal{P}_*$-matrix. In this case we stop the algorithm, and the output will be $\Delta\mathbf{x}$ as a certificate to prove that M is not a $\mathcal{P}_*$-matrix.

There is another point where IPMs may fail if the matrix of the *LCP* is not $\mathcal{P}_*$. If the matrix is not $\mathcal{P}_0$, then the Newton system may not have a solution, or the solution may not be unique (see Corollary 7). In this case the actual point $(\mathbf{x},\mathbf{s})$ is a certificate which proves that the matrix is not $\mathcal{P}_0$, so it is not $\mathcal{P}_*$ either.

Summarizing, we make three tests in our algorithms. In each tests the property of the *LCP* matrix M is examined indirectly. When we inquire about the existence and uniqueness of the solution of the Newton system, we check whether the matrix is $\mathcal{P}_0$, or not. When we test some of properties of the new point, for example whether it is in the appropriate neighborhood of the central path, we examine the $\mathcal{P}_*(\kappa)$ property for the current value of κ. Finally, if the $\kappa(\Delta\mathbf{x})$ value given by (6) is not defined, then the matrix is not $\mathcal{P}_*$. We note that at each step all properties are checked only locally, only for one vector of $\mathbb{R}^n$. Consequently, it is possible, that the matrix is not a $\mathcal{P}_0$ or $\mathcal{P}_*$-matrix, but the algorithm does not discover it, because those properties were true for the vectors $\mathbf{x}$ and $\Delta\mathbf{x}$ that were generated by the algorithm. It may also occur, that the matrix is not $\mathcal{P}_*$, but the algorithm does not detect it. It only increases the value of κ if $\kappa < \kappa(\Delta\mathbf{x})$ and then it proceeds to the next iterate. This is the reason why we need the threshold $\tilde{\kappa}$ parameter that enables us to get a finite algorithm. In practice this is not a real restriction, because for big values of κ the step length might be smaller than the machine precision.

Hereafter we modify the following three popular IPMs:

- A family of affine scaling algorithms [10]:
  The Newton direction is the solution of system (5) with $\mu = 1$ and $\mathbf{a} = -\frac{\mathbf{v}^{2r+2}}{\|\mathbf{v}^{2r}\|}$, where $r \geq 0$ is the degree of the algorithm.
  The choice of $\mathbf{a}$ implies $\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = \frac{\|\mathbf{v}^{2r+1}\|^2}{\|\mathbf{v}^{2r}\|^2}$.
- Long-step path-following algorithms [16]:
  The Newton direction is the solution of system (5) with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$, where μ is a fraction of $\frac{\mathbf{x}^T\mathbf{s}}{n}$.
  The choice of $\mathbf{a}$ implies $\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = \mu\delta_c^2(\mathbf{xs}, \mu)$.
- Predictor-corrector algorithm [18]:
  The predictor Newton direction is the solution of system (5) with $\mathbf{a} = -\mathbf{xs}$ (affine with $r = 0$).
  The choice of $\mathbf{a}$ implies $\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = \mathbf{x}^T\mathbf{s}$;
  The corrector Newton direction is the solution of system (5) with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$, where $\mu = \frac{\mathbf{x}^T\mathbf{s}}{n}$.
  The choice of $\mathbf{a}$ implies $\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = \mu\delta_c^2(\mathbf{xs}, \mu)$.

The following lemma is our main tool to verify when the $\mathcal{P}_*(\kappa)$ property does not hold. Furthermore, the concerned vector $\Delta\mathbf{x}$ is a certificate, whose encoding size is polynomial when it is computed as the solution of the Newton system (5). We use this lemma during the analysis of the aforementioned IPMs. The first statement is simply the negation of the definition. We show in Lemma 13 that if Lemma 4.3 of [11] does not hold, then the second statement is realized, and we point out in Lemma 16 that if Theorem 10.5 of [16] does not hold, then the second or the third statement is realized. Finally, we show in Lemma 19 and Lemma 20 that if Theorem 3.3 of [18] does not hold then the second, the third or the last statement is realized.

Lemma 12 *If one of the following statements holds then the matrix M is not a $\mathcal{P}_*(\kappa)$-matrix.*

1. *There exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that*

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}_+(\mathbf{y})} y_i w_i + \sum_{i \in \mathcal{I}_-(\mathbf{y})} y_i w_i < 0,$$

where $\mathbf{w} = M\mathbf{y}$ *and* $\mathcal{I}_+(\mathbf{y}) = \{i \in I : y_i w_i > 0\}$, $\mathcal{I}_-(\mathbf{y}) = \{i \in I : y_i w_i < 0\}$.

2. *There exists a solution* $(\Delta\mathbf{x}, \Delta\mathbf{s})$ *of system (5) such that*

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty > \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

3. *There exists a solution* $(\Delta\mathbf{x}, \Delta\mathbf{s})$ *of system (5) such that*

$$\max\left(\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i,\ -\sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i\right) > \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

4. *There exists a solution* $(\Delta\mathbf{x}, \Delta\mathbf{s})$ *of system (5) such that*

$$\Delta x^T\Delta s < -\kappa\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

Proof: The first statement is the negation of the definition of $\mathcal{P}_*(\kappa)$ matrices. Now we prove that the first statement follows from the others.

By Lemma 8, one has

$$\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i \le \frac{1}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2, \tag{7}$$

so $\Delta x_i\Delta s_i \le \|\mathbf{a}/\sqrt{\mathbf{xs}}\|^2/4$ for all $i \in \mathcal{I}_+$. Accordingly, if the inequality of the second statement holds, let $\|\Delta\mathbf{x}\,\Delta\mathbf{s}\|_\infty = |\Delta x_j\,\Delta s_j|$, then $j \in \mathcal{I}_-$, i.e., $\Delta x_j\Delta s_j < 0$. Therefore

$$\begin{aligned}(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i &\le (1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \Delta x_j\Delta s_j \\ &< (1+4\kappa)\frac{1}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 - \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = 0. \quad (8)\end{aligned}$$

This is the same as the first statement with $\mathbf{y} = \Delta\mathbf{x},\ \mathbf{w} = \Delta\mathbf{s}$.

From the assumption of statement 3 and inequality (7), the second term is greater in the maximum, hence one has

$$\sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i < -\frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

Therefore $(\Delta\mathbf{x}, \Delta\mathbf{s})$ satisfies inequality (8), so $\mathbf{y} = \Delta\mathbf{x},\ \mathbf{w} = \Delta\mathbf{s}$ proves that the first statement holds.

The proof of the last statement, by using inequality (7) follows from the following inequality

$$\begin{aligned}(1+4\kappa)\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i &= 4\kappa\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i + \sum_{i\in I}\Delta x_i\Delta s_i \\ &< \kappa\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 - \kappa\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = 0,\end{aligned}$$

where we can use $\mathbf{y} = \Delta\mathbf{x},\ \mathbf{w} = \Delta\mathbf{s}$ again to get the first statement. ■

3.1 Affine scaling interior point algorithm

First we deal with affine scaling IPMs. We modify the algorithm proposed in [10]. Let us note that here we consider a family of affine scaling algorithms corresponding to the degree $r \geq 0$ of the algorithm. Further, there is a step length parameter ν, that depends on the degree r (defined among the inputs of the algorithm), and $\mu \equiv 1$ in scaling (3). Note that $r = 0$ gives the classical primal-dual affine scaling algorithm, while $r = 1$ gives the primal-dual Dikin affine scaling algorithm [1].

We do not only check the solvability and uniqueness of the Newton system, but also the decrease of the complementarity gap after a step. For the actual value of κ we determine θ_a^*, which is a theoretical lower bound for the feasible step length in the specified neighborhood if matrix M satisfies the $\mathcal{P}_*(\kappa)$ property. Therefore, if after a step the decrease of the complementarity gap is not large enough, it means, that matrix M is not $\mathcal{P}_*(\kappa)$ with the actual value of κ, so we update κ or exit the algorithm with a proper certificate. If the new value of κ can not be defined by (6), then matrix M is not $\mathcal{P}_*$, so we stop and the Newton direction $\Delta\mathbf{x}$ is a certificate. If the new value of κ is larger than $\tilde{\kappa}$, then the matrix is not $\mathcal{P}_*(\tilde{\kappa})$, therefore the algorithm stops as well and $\Delta\mathbf{x}$ is a certificate. In the rest of this subsection we consider the case $r > 0$. The modified algorithm is as follows.

Affine scaling algorithm

Input:

an upper bound $\tilde{\kappa} > 0$ on the value of κ;
an accuracy parameter $\varepsilon > 0$;
a centrality parameter τ;
the degree of scaling $r > 0$;
a strictly feasible initial point $(\mathbf{x}^0, \mathbf{s}^0) > \mathbf{0}$ such that $\delta_a(\mathbf{x}^0\,\mathbf{s}^0) \leq \tau$;

$$\nu := \begin{cases} 2/\sqrt{n}, & \text{if } 0 < r \leq 1 \\ 2\tau^{2-2r}/\sqrt{n}, & \text{if } 1 \leq r; \end{cases}$$

$$\theta_a^* := \min\left\{ \frac{2}{(1+4\kappa)\tau}\left(\sqrt{1+4\kappa+\frac{1}{\tau^2\, n}} - \frac{1}{\tau\sqrt{n}}\right),\ \frac{\sqrt{n}}{(r+1)\tau^{2r}},\ \frac{4(\tau^{2r}-1)}{(1+4\kappa)(1+\tau^2)\tau^{2r}\sqrt{n}},\ \nu \right\}.$$

begin
  $\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\kappa := 0$;
  while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
    calculate the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$ with $\mathbf{a} = -\mathbf{v}^{2r+2}/\|\mathbf{v}^{2r}\|$;
    if (the Newton direction does not exist or it is not unique) **then**
      return the matrix is not $\mathcal{P}_0$;   *% see Corollary 7*
    $\bar{\theta} = \text{argmin}\left\{\mathbf{x}(\theta)^{\mathrm{T}}\mathbf{s}(\theta) :\ \delta_{\mathrm{a}}\left(\mathbf{x}(\theta), \mathbf{s}(\theta)\right) \leq \tau,\ \left(\mathbf{x}(\theta), \mathbf{s}(\theta)\right) \geq \mathbf{0}\right\}$;
    if $\left(\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) > (1 - 0.25\,\nu\,\theta_a^*)\,\mathbf{x}^T\mathbf{s}\right)$ **then**
      calculate $\kappa(\Delta\mathbf{x})$;   *% see (6)*
      if ($\kappa(\Delta\mathbf{x})$ is not defined) **then**
        return the matrix is not $\mathcal{P}_*$;   *% see Lemma 11*
      if ($\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$) **then**

return the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 10*
$\kappa = \kappa(\Delta\mathbf{x})$;
update θ_a^*; *% it depends on κ*
$\mathbf{x} = \mathbf{x}(\bar{\theta}),\ \mathbf{s} = \mathbf{s}(\bar{\theta})$;
end
end.

Illés et al. proved [10], that if matrix M is a $\mathcal{P}_*(\kappa)$-matrix, then the step length θ_a^* is feasible, with that step size the new iterate stays within the specified neighborhood and it provides the required decrease of the complementarity gap. The following lemma shows, if the decrease of the complementarity gap is not sufficient, then matrix M does not belong to the class of $\mathcal{P}_*(\kappa)$-matrices.

Lemma 13 *If $\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) > (1 - 0.25\,\nu\,\theta_a^*)\,\mathbf{x}^T\mathbf{s}$, that is, the decrease of the complementarity gap within the $\delta_a \leq \tau$ neighborhood is not sufficient, then matrix M of the LCP is not $\mathcal{P}_*(\kappa)$ with the actual value of κ. The Newton direction $\Delta\mathbf{x}$ serves as a certificate.*

Proof: Based on Lemma 28 (see the Appendix) the complementarity gap at θ_a^* is smaller than $(1 - 0.25\,\nu\,\theta_a^*)\,\mathbf{x}^T\mathbf{s}$, furthermore by Theorem 29, if M is a $\mathcal{P}_*(\kappa)$-matrix, then the point $(\mathbf{x}^*, \mathbf{s}^*) = (\mathbf{x}(\theta_a^*), \mathbf{s}(\theta_a^*))$ is feasible. Therefore, if $\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) > (1 - 0.25\,\nu\,\theta_a^*)\,\mathbf{x}^T\mathbf{s}$, then because the step length θ_a^* is not considered in definition of $\bar{\theta}$ (see the affine scaling algorithm), so either $(\mathbf{x}_a^*, \mathbf{s}_a^*)$ is not feasible, or this point is not in the τ neighborhood of the central path, namely $\delta_a(\mathbf{x}^*\mathbf{s}^*) > \tau$. We show that both cases imply, that matrix M is not $\mathcal{P}_*(\kappa)$ with the actual κ value.

Let us denote the first three term in the definition of θ_a^* by θ_1, θ_2, θ_3, respectively. We follow the proof of Theorem 6.1 in [10] (see Theorem 29 in the Appendix). We need to reconsider only the expressions depending on κ. Therefore the function $\varphi(t) = t - \theta\frac{t^{r+1}}{\|\mathbf{v}^{2r}\|}$ remains monotonically increasing for $\theta \leq \theta_2$, and there exist positive constants α and β such that $\frac{\beta}{\alpha} = \tau^2$ and $\alpha\mathbf{e} \leq \mathbf{v}^2 \leq \beta\mathbf{e}$. Additionally, inequalities (17) in [10] hold for $\theta \leq \theta_2$, thus for θ_a^* too:

$$\min(\mathbf{v}^{*2}) \geq \alpha - \theta_a^*\frac{\alpha^{r+1}}{\|\mathbf{v}^{2r}\|} - (\theta_a^*)^2\|\mathbf{g}\|_\infty, \tag{9}$$

$$\max(\mathbf{v}^{*2}) \leq \beta - \theta_a^*\frac{\beta^{r+1}}{\|\mathbf{v}^{2r}\|} + (\theta_a^*)^2\|\mathbf{g}\|_\infty, \tag{10}$$

where $\mathbf{g}$ is defined by (3) (see p.3).

Let us first consider the case $\delta_a(\mathbf{x}^*\mathbf{s}^*) > \tau$, namely $\max(\mathbf{x}^*\mathbf{s}^*) > \tau^2\min(\mathbf{x}^*\mathbf{s}^*)$. From inequalities (9) and (10) one has

$$\tau^2\left(\alpha - \theta_a^*\frac{\alpha^{r+1}}{\|\mathbf{v}^{2r}\|} - (\theta_a^*)^2\|\mathbf{g}\|_\infty\right) < \beta - \theta_a^*\frac{\beta^{r+1}}{\|\mathbf{v}^{2r}\|} + (\theta_a^*)^2\|\mathbf{g}\|_\infty,$$

$$\frac{\beta^r - \alpha^r}{\|\mathbf{v}^{2r}\|} < \theta_a^*\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\|\mathbf{g}\|_\infty. \tag{11}$$

If θ_a^* is substituted by $\theta_3 = \frac{4(\tau^{2r}-1)}{(1+4\kappa)(1+\tau^2)\tau^{2r}\sqrt{n}}$, the right hand side of inequality (11) increases, so the inequality is still true. After substitution one has

$$\frac{1+4\kappa}{4}\,\beta < \|\mathbf{g}\|_\infty. \tag{12}$$

Since $\mathbf{v}^2 \le \beta\mathbf{e}$ and $\mathbf{g} = \Delta\mathbf{x}\Delta\mathbf{s}$ (see notations given by (3)), inequality (12) gives

$$\|\mathbf{v}\|_\infty^2 \le \beta < \frac{4}{1+4\kappa}\,\|\mathbf{g}\|_\infty = \frac{4}{1+4\kappa}\,\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty. \tag{13}$$

One can easy check, that

$$\|\mathbf{v}^{2r+1}\|^2 \le \|\mathbf{v}\|_\infty^2 \|\mathbf{v}^{2r}\|^2. \tag{14}$$

Since $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is a solution of system (5) with $\mathbf{a} = -\mathbf{v}^{2r+2}/\|\mathbf{v}^{2r}\|$, by inequalities (13) and (14) we have

$$\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 = \left\|\frac{\mathbf{v}^{2r+1}}{\|\mathbf{v}^{2r}\|}\right\|^2 \le \|\mathbf{v}\|_\infty^2.$$

Therefore, by the second statement of Lemma 12, we get that inequality (13) contradicts to the $\mathcal{P}_*(\kappa)$ property and vector $\Delta\mathbf{x}$ is a certificate for this fact.

Now we consider the case $(\mathbf{x}^*, \mathbf{s}^*)$ is not feasible, so there exists such an index i, that either $x_i^* < 0$ or $s_i^* < 0$. Let us consider the maximum feasible step size $\hat\theta < \theta_a^*$, for which $(\mathbf{x}(\hat\theta), \mathbf{s}(\hat\theta)) = (\hat{\mathbf{x}}, \hat{\mathbf{s}}) \ge \mathbf{0}$ holds and at least one of its coordinate is 0. For this point $\hat{\mathbf{x}}\,\hat{\mathbf{s}} \ne \mathbf{0}$, else $\bar\theta = \hat\theta$ by definition of $\bar\theta$, and the new point would be an exact solution, so the decrease of complementarity gap would be $\mathbf{x}^T\mathbf{s}$ contradicting with the assumption of the lemma. Therefore $0 \ne \max(\hat{\mathbf{x}}\,\hat{\mathbf{s}}) > \tau^2 \min(\hat{\mathbf{x}}\,\hat{\mathbf{s}}) = 0$, so inequality (11) holds with $\hat\theta$. Because of $\theta_3 \ge \theta_a^* > \hat\theta$, inequality (12) holds as well, and as we have already seen this means that the matrix M is not $\mathcal{P}_*(\kappa)$ and the vector $\Delta\mathbf{x}$ is a certificate for this fact. ■

The following lemma proves, that the algorithm is well defined.

Lemma 14 *At each iteration, when the value of κ is updated, then the new value of θ_a^* satisfies the inequality $\mathbf{x}(\bar\theta)^T\mathbf{s}(\bar\theta) \le (1 - 0.25\,\nu\,\theta_a^*)\mathbf{x}^T\mathbf{s}$.*

Proof: In the proof of Theorem 29 we use the $\mathcal{P}_*(\kappa)$ property only for the vector $\Delta\mathbf{x}$. When parameter κ is updated, then we choose the new value in such a way, that the inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices (4) would hold for vector $\Delta\mathbf{x}$. Therefore the new point defined by the updated value of step size θ_a^* is feasible and it is in the τ-neighborhood of the central path. Thus the new value of θ_a^* was considered in the definition of $\bar\theta$, so $\mathbf{x}(\bar\theta)^T\mathbf{s}(\bar\theta) \le (1 - 0.25\,\nu\,\theta_a^*)\,\mathbf{x}^T\mathbf{s}$. ■

Now we are ready to state the complexity result for the modified affine scaling algorithm for general LCPs in case an initial interior point is given.

Theorem 15 *Let $(\mathbf{x}^0, \mathbf{s}^0)$ be a feasible interior point such that $\delta_a(\mathbf{x}^0\mathbf{s}^0) \le \tau = \sqrt{2}$. Then after at most*

$$\begin{cases} \mathcal{O}\left(\frac{n(1+4\hat\kappa)}{1-2^{-r}}\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right), & \text{if } 0 < r \le 1 \text{ and } n \ge 4 \\ \mathcal{O}\left(n(1+4\hat\kappa)\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right), & \text{if } r = 1 \text{ and } n \ge 4 \\ \mathcal{O}\left(2^{2r-2}n(1+4\hat\kappa)\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right), & \text{if } 1 < r \text{ and } n \text{ sufficiently large} \end{cases}$$

iterations either the affine scaling algorithm yields a vector $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$*, such that* $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$ *and* $\delta_a(\hat{\mathbf{x}}\hat{\mathbf{s}}) \leq \tau$*, or it gives a polynomial size certificate that the matrix is not* $\mathcal{P}_*(\tilde{\kappa})$*, where* $\hat{\kappa} \leq \tilde{\kappa}$ *is the largest value of parameter* κ *throughout the algorithm.*

Proof: The algorithm at each iteration either takes a step, or detects, that the matrix is not $\mathcal{P}_*(\tilde{\kappa})$ and stops. If we take a Newton step, then by the definition of the algorithm and by Lemma 14 the decrease of the complementarity gap is at least $0.25\,\nu\,\theta_a^*\,\mathbf{x}^T\mathbf{s}$. One can see from the definition of θ_a^* that larger κ means smaller θ_a^*, so smaller lower bound on the decrease of the complementarity gap. Therefore, if the algorithm stops with an ε-optimal solution, then each Newton step decreases the complementarity gap with more than $0.25\,\nu\,\hat{\theta}_a^*\,\mathbf{x}^T\mathbf{s}$, where $\hat{\theta}_a^*$ is determined by $\hat{\kappa}$. It means that after at most as many steps as in the original method the complementarity gap decreases below ε in case for each vector during the algorithm sufficient decrease of the complementarity gap is realized according to the $\mathcal{P}_*(\hat{\kappa})$ property or at an earlier iteration the lack of $\mathcal{P}_*(\tilde{\kappa})$-property is detected. This observation, combined with the complexity theorem of the original algorithm (see Theorem 30 in the Appendix) proves our statement. ■

At the end of this subsection let us note that the case $r = 0$ can be treated in a similar way.

3.2 Long-step path-following interior point algorithm

In this section we deal with the algorithm proposed in [16]. The long-step algorithm has two loops, in the inner loop one take steps towards the central path and in the outer loop the parameter μ is updated. In this algorithm we check the decrease of the centrality measure after one inner step and if it is too small, then κ is updated by (6). Similarly to the modified algorithm stated in the previous subsection, if $\kappa(\Delta\mathbf{x})$ is not defined, then the matrix is not $\mathcal{P}_*$ and $\Delta\mathbf{x}$ is a certificate of it. Furthermore if $\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$, then matrix M is not $\mathcal{P}_*(\tilde{\kappa})$ and the Newton direction $\Delta\mathbf{x}$ is a certificate for this fact. The modified algorithm is as follows:

Long-step path-following interior point algorithm

Input:

an upper bound $\tilde{\kappa} > 0$ on the value of κ;
a proximity parameter $\tau \geq 2$;
an accuracy parameter $\varepsilon > 0$;
a fix barrier update parameter $\gamma \in (0, 1)$;
an initial point $(\mathbf{x}^0, \mathbf{s}^0)$, and $\mu^0 > 0$ such that $\delta_c(\mathbf{x}^0\mathbf{s}^0, \mu^0) < \tau$.

begin
$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\mu := \mu^0$, $\kappa := 0$;
while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
$\mu = (1 - \gamma)\mu$;

while $\delta_c(\mathbf{xs}, \mu) \geq \tau$ **do**
calculate the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$ with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$;
if (the Newton direction does not exist or it is not unique) **then**
return the matrix is not $\mathcal{P}_0$; *% see Corollary 7*
$\bar{\theta} = \text{argmin}\{\delta_c(\mathbf{x}(\theta)\mathbf{s}(\theta), \mu) : \ (\mathbf{x}(\theta), \mathbf{s}(\theta)) > \mathbf{0}\}$;
if $\left(\delta_c^2(\mathbf{xs}, \mu) - \delta_c^2(\mathbf{x}(\bar{\theta})\mathbf{s}(\bar{\theta}), \mu) < \frac{5}{3(1+4\kappa)}\right)$ **then**
calculate $\kappa(\Delta\mathbf{x})$; *% see (6)*
if ($\kappa(\Delta\mathbf{x})$ is not defined) **then**
return the matrix is not $\mathcal{P}_*$; *% see Lemma 11*
if ($\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$) **then**
return the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 10*
$\kappa = \kappa(\Delta\mathbf{x})$;
$\mathbf{x} = \mathbf{x}(\bar{\theta}),\ \mathbf{s} = \mathbf{s}(\bar{\theta})$;
end
end
end.

We use the notations of [16]:

$$\sigma_+ = \sum_{i\in\mathcal{I}_+} g_i = \frac{1}{\mu}\sum_{i\in\mathcal{I}_+} \Delta x_i \Delta s_i, \quad \sigma_- = -\sum_{i\in\mathcal{I}_-} g_i = -\frac{1}{\mu}\sum_{i\in\mathcal{I}_-} \Delta x_i \Delta s_i, \quad \sigma = \max(\sigma_+, \sigma_-).$$

Further, let

$$\theta_\ell^* := \frac{2}{(1+4\kappa)\delta_c^2(\mathbf{xs}, \mu)}.$$

To simplify the notation we write δ and δ^* instead of $\delta_c(\mathbf{xs}, \mu)$, $\delta_c(\mathbf{x}(\theta_\ell^*)\mathbf{x}(\theta_\ell^*), \mu)$, respectively.

Peng et al. [16] proved, that for $\mathcal{P}_*(\kappa)$ LCP's the step length θ_ℓ^* is feasible, and taking this step the decrease of the proximity measure is sufficient to ensure polynomiality of the algorithm. The following lemma shows if the decrease of the proximity is not sufficient, then the matrix of the problem is not $\mathcal{P}_*(\kappa)$.

Lemma 16 *If after an inner iteration the decrease of the proximity is not sufficient, i.e.,* $\delta^2(\mathbf{xs}, \mu) - \delta^2(\mathbf{x}(\bar{\theta})\mathbf{s}(\bar{\theta}), \mu) < \frac{5}{3(1+4\kappa)}$*, then the matrix of the LCP is not* $\mathcal{P}_*(\kappa)$ *with the actual* κ *value, and the Newton direction* $\Delta\mathbf{x}$ *is a certificate for this fact.*

Proof: By Lemma 32 (see the Appendix), if the matrix is $\mathcal{P}_*(\kappa)$ we achieve the sufficient decrease of the centrality measure with step length θ_ℓ^*. Therefore, if the maximum decrease is smaller, then either $(\mathbf{x}^*, \mathbf{s}^*)$ is not feasible or the decrease of the proximity with step size θ_ℓ^* is not sufficient, i.e., $\delta^2(\mathbf{xs}, \mu) - \delta^2(\mathbf{x}^*\mathbf{s}^*, \mu) < \frac{5}{3(1+4\kappa)}$. We prove in both cases that the matrix of the problem is not $\mathcal{P}_*(\kappa)$ and $\Delta\mathbf{x}$ is a certificate of it.

If the point $(\mathbf{x}^*, \mathbf{s}^*)$ is not feasible, then $\mathbf{e} + \theta_\ell^* \mathbf{g} \not\geq 0$, so there exists such an index k, that $1 + \theta_\ell^* g_k \leq 0$. It means, that $g_k \leq -1/\theta_\ell^* = -\frac{1}{2}(1+4\kappa)\delta^2 < 0$. Since $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is a solution of system (5) with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$, therefore

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty = \mu\|\mathbf{g}\|_\infty > \frac{1+4\kappa}{4}\,\mu\,\delta^2 = \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

but this contradicts to the $\mathcal{P}_*(\kappa)$ property by the second statement of Lemma 12.

Now let us analyze the case when the decrease of the proximity measure is not sufficient with step length θ_l^*. According to the condition of Theorem 31 (see the Appendix), let us consider the cases $\theta_\ell^* < 1/\sigma$ and $\theta_\ell^* \geq 1/\sigma$ separately. If $\theta_\ell^* < 1/\sigma$, by the definition of θ_ℓ^* and Theorem 31, one has

$$(\delta^*)^2 - \delta^2 \leq -\frac{2}{1+4\kappa} + \frac{2(\theta_\ell^*)^3\sigma^2}{1-(\theta_\ell^*)^2\sigma^2}. \tag{15}$$

Since $\delta \geq 2$, we can write

$$-\frac{2}{1+4\kappa} + \frac{4}{3(1+4\kappa)\delta^2} \leq -\frac{5}{3(1+4\kappa)}. \tag{16}$$

Therefore, by inequalities (15) and (16) and by the assumption of the lemma, the following inequality holds

$$-\frac{2}{1+4\kappa} + \frac{4}{3(1+4\kappa)\delta^2} \leq -\frac{5}{3(1+4\kappa)} < (\delta^*)^2 - \delta^2 \leq -\frac{2}{1+4\kappa} + \frac{2(\theta_\ell^*)^3\sigma^2}{1-(\theta_\ell^*)^2\sigma^2}.$$

Using the definition of θ_ℓ^* we get

$$\frac{4}{3(1+4\kappa)\delta^2} < \frac{2(\theta_\ell^*)^2\sigma^2}{1-(\theta_\ell^*)^2\sigma^2}\,\frac{2}{(1+4\kappa)\delta^2}.$$

After reordering one has $\frac{1}{2} < \theta_\ell^*\sigma$. Substituting the definition of θ_ℓ^* we get the following lower bound on σ

$$\max(\sigma_+, \sigma_-) = \sigma > \frac{1+4\kappa}{4}\,\delta^2. \tag{17}$$

By the definitions of σ and the proximity measure, one has

$$\max\left(\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i, -\sum_{i\in\mathcal{I}_-}\Delta x_i\Delta s_i\right) > \frac{1+4\kappa}{4}\,\mu\,\delta^2 = \frac{1+4\kappa}{4}\left\|\frac{\mu\mathbf{e}-\mathbf{xs}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

By the third statement of Lemma 12 this implies that matrix M is not $\mathcal{P}_*(\kappa)$ and the vector $\Delta\mathbf{x}$ is a certificate of it.

If $\theta_\ell^* \geq 1/\sigma$, then by the definition of θ_ℓ^* one has $\sigma \geq (1+4\kappa)\delta^2/2$, therefore inequality (17) holds, so the lemma is true in this case, too. ∎

The following lemma proves, that the long-step path-following IPM is well defined.

Lemma 17 *At each iteration when the value of κ is updated, then the new value of κ satisfies the inequality* $\delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}(\bar\theta)\mathbf{s}(\bar\theta),\mu) \ge \frac{5}{3(1+4\kappa)}$.

Proof: In the proof of Theorem 32 we use the $\mathcal{P}_*(\kappa)$ property only for the vector $\Delta\mathbf{x}$. When parameter κ is updated, then we choose the new value such a way that the inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices (4) would hold for the vector $\Delta\mathbf{x}$. Therefore the new point defined by the updated value of step size θ_ℓ^* is strictly feasible and $\delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}^*\mathbf{s}^*,\mu)) \ge \frac{5}{3(1+4\kappa)}$. Thus the new value of θ_ℓ^* was considered in the definition of $\bar\theta$ as $\delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}(\bar\theta)\mathbf{s}(\bar\theta),\mu) \ge \delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}^*\mathbf{s}^*,\mu) \ge \frac{5}{3(1+4\kappa)}$. ∎

Now we are ready to state the complexity result for the modified long-step path-following interior point algorithm for general LCP in case an initial interior point is given.

Theorem 18 *Let $\tau = 2$, $\gamma = 1/2$ and $(\mathbf{x}^0,\mathbf{s}^0)$ be a feasible interior point such that $\delta_c(\mathbf{x}^0\mathbf{s}^0,\mu^0) \le \tau$. Then after at most $\mathcal{O}\left((1+4\hat\kappa)n\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$ steps, where $\hat\kappa \le \tilde\kappa$ is the largest value of parameter κ throughout the algorithm, the long-step path-following interior point algorithm either produces a point $(\hat{\mathbf{x}},\hat{\mathbf{s}})$ such that $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \le \varepsilon$ and $\delta_c(\hat{\mathbf{x}}\hat{\mathbf{s}},\hat\mu) \le \tau$ or it gives a certificate that the matrix of the LCP is not $\mathcal{P}_*(\tilde\kappa)$.*

Proof: We follow the proof of Theorem 15 that gives analogous complexity result for the affine scaling algorithm in the previous subsection. By the construction of the algorithm and by Lemma 17, if we take a Newton step, the decrease of the squared proximity measure is at least $5/[3(1+4\kappa)]$. We can see, that larger κ means smaller lower bound on decrease of the proximity measure. Therefore, after each Newton step the decrease of the squared proximity measure is at least $5/[3(1+4\hat\kappa)]$. Thus at each outer iteration we take at most as many inner iterations as in the original long-step algorithm with a $\mathcal{P}_*(\hat\kappa)$-matrix do, or the algorithm stops earlier with a certificate that M is not $\mathcal{P}_*(\tilde\kappa)$-matrix. By the complexity theorem of the original algorithm (see Theorem 33 in the Appendix) we proved our statement. ∎

3.3 Predictor-corrector interior point algorithm

In this section we modify the algorithm proposed in [18]. In this predictor-corrector algorithm we take affine and centering steps alternately. In a predictor step θ_p^* (see the definition in Lemma 19) is a theoretical feasible step length if the matrix M is $\mathcal{P}_*(\kappa)$. Therefore if the maximal feasible step length is smaller than θ_p^*, then the matrix is not $\mathcal{P}_*(\kappa)$ with the actual value of κ, so κ should be increased. In a corrector step we return to the smaller $\mathcal{D}(\beta)$ neighborhood with step size θ_c^* (see the definition in Lemma 20) if the matrix is $\mathcal{P}_*(\kappa)$. Accordingly, if the new point with step length θ_c^* is not in $\mathcal{D}(\beta)$, then the matrix M is not $\mathcal{P}_*(\kappa)$ with actual value of κ, so κ should be updated. Similarly to the previous two algorithms, if in a predictor or corrector step the new value of κ is not defined by (6), then the matrix is not $\mathcal{P}_*$ and the current Newton direction is a certificate of it. Furthermore, if the new value of κ is larger than $\tilde\kappa$, then the matrix is not $\mathcal{P}_*(\tilde\kappa)$ and the Newton direction is a certificate for it. The modified algorithm is as follows:

Predictor-corrector algorithm

Input:

an upper bound $\tilde{\kappa} > 0$ on the value of κ;
an accuracy parameter $\varepsilon > 0$;
a proximity parameter $\beta \in (0,1)$;
an initial point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{D}(\beta)$;

begin
$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\mu := (\mathbf{x}^0)^T\mathbf{s}^0/n$, $\kappa := 0$;
while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
Predictor step
$\gamma = \frac{1-\beta}{(1+4\kappa)n+1}$;
calculate the affine Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$ with $\mathbf{a} = -\mathbf{xs}$;
if (the Newton direction does not exist or it is not unique) **then**
return the matrix is not $\mathcal{P}_0$; *% see Corollary 7*
$\bar{\theta} = \sup\left\{\hat{\theta} > 0 : \ (\mathbf{x}(\theta), \mathbf{s}(\theta)) \in \mathcal{D}\big((1-\gamma)\beta\big), \ \forall\, \theta \in [0, \hat{\theta}]\right\}$;
if $\left(\bar{\theta} < \theta_p^*\right)$ **then**
calculate $\kappa(\Delta\mathbf{x})$; *% see (6)*
if ($\kappa(\Delta\mathbf{x})$ is not defined) **then**
return the matrix is not $\mathcal{P}_*$; *% see Lemma 11*
if ($\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$) **then**
return the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 10*
$\kappa = \kappa(\Delta\mathbf{x})$;
$\bar{\mathbf{x}} = \mathbf{x}(\bar{\theta}), \ \bar{\mathbf{s}} = \mathbf{s}(\bar{\theta}), \ \bar{\mu} = \bar{\mathbf{x}}^T\bar{\mathbf{s}}/n$;
Corrector step
calculate the centering Newton direction $(\Delta\bar{\mathbf{x}}, \Delta\bar{\mathbf{s}})$ with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$;
if (the Newton direction does not exist or it is not unique) **then**
return the matrix is not $\mathcal{P}_0$; *% see Corollary 7*
if $\big(\ (\bar{\mathbf{x}}(\theta_c^*), \bar{\mathbf{x}}(\theta_c^*)) \notin \mathcal{D}(\beta)\ \big)$
calculate $\kappa(\Delta\bar{\mathbf{x}})$; *% see (6)*
if ($\kappa(\Delta\bar{\mathbf{x}})$ is not defined) **then**
return the matrix is not $\mathcal{P}_*$; *% see Lemma 11*
if ($\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$) **then**
return the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 10*
$\kappa = \kappa(\Delta\bar{\mathbf{x}})$;
$\theta^+ = \operatorname{argmin}\{\bar{\mu}(\theta) : \ (\bar{\mathbf{x}}(\theta), \bar{\mathbf{s}}(\theta)) \in \mathcal{D}(\beta)\}$;
$\mathbf{x}^+ = \bar{\mathbf{x}} + \theta^+\Delta\bar{\mathbf{x}}, \ \mathbf{s}^+ = \bar{\mathbf{s}} + \theta^+\Delta\bar{\mathbf{s}}, \ \mu^+ = (\mathbf{x}^+)^T\mathbf{s}^+/n$;
$\mathbf{x} = \mathbf{x}^+, \ \mathbf{s} = \mathbf{s}^+, \ \mu = \mu^+$;
end
end.

Potra et al. [18] determined the maximum predictor step length as the minimum of $n+1$ number ($\bar{\theta} = \min\{\bar{\theta}_i : 0 \leq i \leq n\}$ see Lemma 34 in the Appendix). Furthermore, they proved, that if matrix M is a $\mathcal{P}_*(\kappa)$-matrix, than θ_p^* and θ_c^* (defined in the following lemmas) give a feasible predictor and corrector step length pair. The following lemmas show that if θ_p^* or the θ_c^* is not a feasible step length, than the matrix is not a $\mathcal{P}_*(\kappa)$-matrix.

Lemma 19 *If there exists an index i $(0 \leq i \leq n)$ such that*

$$\bar{\theta}_i \leq \theta_p^* := \frac{2\sqrt{(1-\beta)\beta}}{(1+4\kappa)n+2},$$

then matrix M is not a $\mathcal{P}_*(\kappa)$-matrix and the affine Newton direction is a certificate for this.

Proof: For any $\kappa \geq 0$ and $n \geq 1$

$$\theta_p^* < \frac{2}{1+\sqrt{1+4\kappa}},$$

therefore if $\bar{\theta}_0 \leq \theta_p^*$, then by the definition of $\bar{\theta}_0$ one has

$$\frac{2}{1+\sqrt{1-4\mathbf{e}^T\mathbf{g}/n}} = \bar{\theta}_0 < \frac{2}{1+\sqrt{1+4\kappa}},$$

implying $\mathbf{e}^T\mathbf{g}/n < -\kappa$, thus $\sum_{i\in I} \Delta x_i \Delta s_i < -\kappa n\mu = -\kappa\, \mathbf{x}^T\mathbf{s}$. Therefore, by Lemma 12 matrix M is not a $\mathcal{P}_*(\kappa)$-matrix and the affine Newton direction $\Delta\mathbf{x}$ is a certificate for this.

If $\bar{\theta}_i \leq \theta_p^*$, where $0 < i \leq n$, then let consider the following inequality, which was proved by Potra et al. in [18] on p.158:

$$\sqrt{(1-\beta)\beta} + \sqrt{((1+4\kappa)n+1)^2 + \beta(1-\beta)} < (1+4\kappa)n+2. \tag{18}$$

Using Lemma 35, Lemma 9 and the definition of γ, one has

$$\begin{aligned}
\frac{2\sqrt{(1-\beta)\beta}}{(1+4\kappa)n+2} = \theta_p^* \quad &\geq \quad \bar{\theta}_i \geq \frac{2}{1+\sqrt{1+(\beta\gamma)^{-1}(4\|\mathbf{g}\|_\infty + 4\mathbf{e}^T\mathbf{g}/n)}} \\
&\geq \quad \frac{2}{1+\sqrt{1+(\beta\gamma)^{-1}(4\|\mathbf{g}\|_\infty+1)}} \\
&= \quad \frac{2\sqrt{(1-\beta)\beta}}{\sqrt{(1-\beta)\beta}+\sqrt{((1+4\kappa)n+1)(4\|\mathbf{g}\|_\infty+1)+\beta(1-\beta)}}.
\end{aligned} \tag{19}$$

From inequality (19) and (18) we get

$$4\|\mathbf{g}\|_\infty + 1 > (1+4\kappa)n+1. \tag{20}$$

Since $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is a solution of system (5) with $\mathbf{a} = -\mathbf{xs}$, and by inequality (20) ($\mu\, n = \mathbf{x}^T\mathbf{s}$), one has

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty > \frac{(1+4\kappa)}{4}\,\mathbf{x}^T\mathbf{s} = \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

so by the second statement of Lemma 12, $M \notin \mathcal{P}_*(\kappa)$ and $\Delta\mathbf{x}$ is a certificate. ∎

Now let us analyze the corrector step.

Lemma 20 *If*

$$\theta_c^* := \frac{2\beta}{(1+4\kappa)n+1}$$

is such a corrector step length that $(\bar{\mathbf{x}}(\theta_c^*), \bar{\mathbf{s}}(\theta_c^*)) \notin \mathcal{D}(\beta)$*, then matrix* M *is not a* $\mathcal{P}_*(\kappa)$*-matrix and the corrector Newton direction is a certificate for this.*

Proof: Notice that

$$\bar{\mathbf{x}}(\theta)\bar{\mathbf{s}}(\theta) = (1-\theta)\bar{\mathbf{x}}\bar{\mathbf{s}} + \theta\bar{\mu}\,\mathbf{e} + \theta^2\Delta\bar{\mathbf{x}}\Delta\bar{\mathbf{s}}$$

and

$$\bar{\mu}(\theta) = \bar{\mu} + \theta^2\frac{\Delta\bar{\mathbf{x}}^T\Delta\bar{\mathbf{s}}}{n}.$$

From Lemma 9 and Lemma 36 we get

$$\Delta\bar{\mathbf{x}}^T\Delta\bar{\mathbf{s}} \le \sum_{\mathcal{I}_+}\Delta\bar{x}_i\Delta\bar{s}_i \le \frac{1}{4}\left\|\frac{\bar{\mu}\mathbf{e}-\bar{\mathbf{x}}\bar{\mathbf{s}}}{\sqrt{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 = \frac{1}{4}\,\bar{\mu}\left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar{\mu}}}-\sqrt{\frac{\bar{\mu}}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 \le \frac{1}{4}\,\bar{\mu}\,\frac{1-(1-\gamma)\beta}{(1-\gamma)\beta}\,n,$$

therefore

$$\bar{\mu}(\theta) \le \left(1+\frac{1-(1-\gamma)\beta}{4(1-\gamma)\beta}\,\theta^2\right)\bar{\mu}. \tag{21}$$

Since θ_c^* is an infeasible step length, there exists index i such that

$$\bar{x}(\theta_c^*)_i\;\bar{s}(\theta_c^*)_i < \beta\bar{\mu}(\theta_c^*), \quad \text{namely}$$

$$(1-\theta_c^*)\bar{x}_i\bar{s}_i + \theta_c^*\bar{\mu}\; + (\theta_c^*)^2\Delta\bar{x}_i\Delta\bar{s}_i < \beta\bar{\mu}(\theta_c^*).$$

The predictor point $(\bar{\mathbf{x}},\bar{\mathbf{s}}) \in \mathcal{D}((1-\gamma)\beta)$, so $\bar{x}_i\bar{s}_i \ge (1-\gamma)\beta\bar{\mu}$. Furthermore, by inequality (21) one has

$$(1-\theta_c^*)(1-\gamma)\beta\bar{\mu} + \theta_c^*\bar{\mu} + (\theta_c^*)^2\Delta\bar{x}_i\Delta\bar{s}_i < \beta\left(1+\frac{1-(1-\gamma)\beta}{4(1-\gamma)\beta}\,(\theta_c^*)^2\right)\bar{\mu},$$

which implies

$$(\theta_c^*)^2\frac{\Delta\bar{x}_i\Delta\bar{s}_i}{\bar{\mu}} < \gamma\beta - \theta_c^*(1-(1-\gamma)\beta) + \frac{1-(1-\gamma)\beta}{4(1-\gamma)}(\theta_c^*)^2. \tag{22}$$

One can check, the following equality by substituting the values of γ and θ_c^*

$$0 \le \frac{(1-\beta)\beta^2}{((1+4\kappa)n+1)^2} = -\gamma\beta + \theta_c^*(1-(1-\gamma)\beta) - \frac{1-(1-\gamma)\beta}{4(1-\gamma)\beta}\big[(1+4\kappa)n+\beta\big](\theta_c^*)^2.$$

Therefore

$$-\frac{1-(1-\gamma)\beta}{4(1-\gamma)\beta}\,(1+4\kappa)\,n\,(\theta_c^*)^2 \ge \gamma\beta - \theta_c^*(1-(1-\gamma)\beta) + \frac{1-(1-\gamma)\beta}{4(1-\gamma)}\,(\theta_c^*)^2.$$

Combining this with inequality (22) and then considering Lemma 36, we get

$$\Delta\bar{x}_i\Delta\bar{s}_i < -\frac{1-(1-\gamma)\beta}{4(1-\gamma)\beta}\,(1+4\kappa)\,n\bar{\mu} \le -\frac{(1+4\kappa)\bar{\mu}}{4}\left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar{\mu}}}-\sqrt{\frac{\bar{\mu}}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2. \tag{23}$$

Since $(\Delta\bar{\mathbf{x}}, \Delta\bar{\mathbf{s}})$ is a solution of system (5) with $\mathbf{a} = \bar{\mu}\mathbf{e} - \bar{\mathbf{x}}\bar{\mathbf{s}}$ and from inequality (23), one get

$$\|\Delta\bar{\mathbf{x}}\Delta\bar{\mathbf{s}}\|_\infty > \frac{(1+4\kappa)\bar{\mu}}{4}\left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar{\mu}}} - \sqrt{\frac{\bar{\mu}}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 = \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2.$$

Thus, by the second statement of Lemma 12, matrix M is not a $\mathcal{P}_*(\kappa)$-matrix and the corrector Newton direction $\Delta\bar{\mathbf{x}}$ is a certificate for this. ■

The following lemma proves, that the predictor-corrector algorithm is well defined.

Lemma 21 *At each iteration when the value of κ is updated, the new value of θ_p^* satisfies the inequality $\bar{\theta} \geq \theta_p^*$, and the new point $(\bar{\mathbf{x}}(\theta_c^*), \bar{\mathbf{s}}(\theta_c^*))$ determined by the new value of the corrector step size θ_c^*, is in the $\mathcal{D}(\beta)$ neighborhood.*

Proof: In the proof of Lemma 37 we use the $\mathcal{P}_*(\kappa)$ property only for the vector $\Delta\mathbf{x}$ or $\Delta\bar{\mathbf{x}}$. When parameter κ is updated, then we choose the new value in such a way that the inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices (4) holds for the vectors $\Delta\mathbf{x}$ and $\Delta\bar{\mathbf{x}}$. Therefore the new value of θ_p^* satisfies the inequality $\bar{\theta} \geq \theta_p^*$, and the new value of θ_c^* determines a point in the $\mathcal{D}(\beta)$ neighborhood. ■

Now we are ready to state the complexity result for the modified predictor-corrector algorithm for general LCPs in case an initial interior point is available.

Theorem 22 *Let $(\mathbf{x}^0, \mathbf{s}^0)$ be a feasible interior point such that $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{D}(\beta)$. Then after at most $\mathcal{O}\left((1+\hat{\kappa})n \log \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$ steps, where $\hat{\kappa} \leq \tilde{\kappa}$ is the largest value of parameter κ throughout the algorithm, the predictor-corrector algorithm generate a point $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$, such that $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$ and $(\hat{\mathbf{x}}, \hat{\mathbf{s}}) \in \mathcal{D}(\beta)$, or provides a certificate that the matrix is not $\mathcal{P}_*(\tilde{\kappa})$.*

Proof: We follow the proof of the previous two complexity theorems (see Theorem 15 and Theorem 18). If we take a predictor and a corrector step, then by Theorem 38 and Lemma 21 the decrease of the complementarity gap is at least

$$\frac{3\sqrt{(1-\beta)\beta}}{2((1+4\kappa)n+2)}\frac{\mathbf{x}^T\mathbf{s}}{n}.$$

This expression is a decreasing function of κ, so at each iteration, when we make a predictor and a corrector step, the complementarity gap decreases at least by

$$\frac{3\sqrt{(1-\beta)\beta}}{2((1+4\tilde{\kappa})n+2)}\frac{\mathbf{x}^T\mathbf{s}}{n}.$$

We take at most as many iterations as in the original predictor-corrector IPM with a $\mathcal{P}_*(\hat{\kappa})$-matrix. Thus referring to the complexity theorem of the original algorithm (see Theorem 39 in the Appendix) we have proved the theorem. ■

3.4 An EP theorem for LCPs based on interior point algorithms

It is known from the literature [13] that assuming $\mathcal{F}^0 \neq \emptyset$ and the matrix of the LCP is sufficient, then the LCP has a solution. According to this result and making use of the complexity theorems of the previous subsections (Theorem 15, 18 and 22), we can now present the following EP type theorem. We assume that the data are rational (solving problems with computer this is a reasonable assumption), ensuring polynomial encoding size of certificates and polynomial complexity of the algorithms.

Theorem 23 *Let an arbitrary matrix $M \in \mathbb{Q}^{n\times n}$, a vector $\mathbf{q} \in \mathbb{Q}^n$ and a point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^0$ be given. Then one can verify in polynomial time that at least one of the following statements holds*

(1) problem LCP has a feasible complementary solution $(\mathbf{x}, \mathbf{s})$ whose encoding size is polynomially bounded.

(2) matrix M is not in the class of $\mathcal{P}_*(\tilde{\kappa})$ and there is a certificate whose encoding size is polynomially bounded.

4 Solving general LCP without having an initial interior point

When for an *LCP* no initial interior point is known then we have two possibilities: (i) apply an infeasible interior point algorithm, or (ii) use the embedding technique of Kojima et al. [13]. In this section we discuss the solution method based on the embedding technique.

4.1 Embedded model for general LCPs

In this section we deal with a technique that allows us to handle the initialization problem of IPMs for *LCP*s, i.e., how to get a well centered initial interior point. The embedding model discussed in this section was introduced by Kojima et al. [13]. The following lemma plays a crucial role in this model.

Lemma 24 (Lemma 5.3 in [13]) *Let M be a real matrix. The matrix $M' = \begin{pmatrix} M & I \\ -I & O \end{pmatrix}$ belongs to the class $\mathcal{P}_0$, column sufficient, $\mathcal{P}_*$, $\mathcal{P}_*(\kappa)$, positive semidefinite or skew symmetric if and only if M belongs to the same matrix class.*

Let us consider the *LCP* as given by (1). We assume that all the entries of matrix M and vector $\mathbf{q}$ are integral. The input length L of problem *LCP* is defined as

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} \log_2(|m_{ij} + 1|) + \sum_{i=1}^{n} \log_2(|q_i| + 1) + 2\log_2 n,$$

and let

$$\tilde{\mathbf{q}} = \frac{2^{L+1}}{n^2}\,\mathbf{e}.$$

The embedding problem of Kojima et al. [13] is as follows:

$$\left.\begin{aligned} -M'\mathbf{x}' + \mathbf{s}' &= \mathbf{q}' \\ \mathbf{x}'\,\mathbf{s}' &= \mathbf{0} \\ \mathbf{x}',\, \mathbf{s}' &\geq \mathbf{0} \end{aligned}\right\}\ \textit{(LCP')}$$

where

$$\mathbf{x}' = \begin{pmatrix} \mathbf{x} \\ \tilde{\mathbf{x}} \end{pmatrix}, \ \mathbf{s}' = \begin{pmatrix} \mathbf{s} \\ \tilde{\mathbf{s}} \end{pmatrix}, \ \mathbf{q}' = \begin{pmatrix} \mathbf{q} \\ \tilde{\mathbf{q}} \end{pmatrix}, \ M' = \begin{pmatrix} M & I \\ -I & O \end{pmatrix}.$$

An initial interior point for the embedding model (LCP') is readily available:

$$\mathbf{x} = \frac{2^L}{n^2}\mathbf{e}, \quad \tilde{\mathbf{x}} = \frac{2^{2L}}{n^3}\mathbf{e}, \quad \mathbf{s} = \frac{2^L}{n^2}M\mathbf{e} + \frac{2^{2L}}{n^3}\mathbf{e} + \mathbf{q}, \quad \tilde{\mathbf{s}} = \frac{2^L}{n^2}\mathbf{e}.$$

The following lemma indicates the connection between the solutions of the embedding problem and the solutions of the original LCP.

Lemma 25 (Lemma 5.4 in [13]) *Let* $(\mathbf{x}', \mathbf{s}') = \left(\begin{pmatrix} \mathbf{x} \\ \tilde{\mathbf{x}} \end{pmatrix}, \begin{pmatrix} \mathbf{s} \\ \tilde{\mathbf{s}} \end{pmatrix} \right)$ *be a solution of problem* (LCP').

(i) If $\tilde{\mathbf{x}} = \mathbf{0}$, *then* $(\mathbf{x}, \mathbf{s})$ *is a solution of the original* LCP.

(ii) If M *is column sufficient and* $\tilde{\mathbf{x}} \neq \mathbf{0}$, *then the original* LCP *has no solution.*

4.2 Using dual information

We deal with the dual of the LCP in our paper [9]. Let us denote the set of the dual feasible solutions by $\mathcal{F}_D := \left\{(\mathbf{u}, \mathbf{z}) \geq \mathbf{0} : \ \mathbf{u} + M^T\mathbf{z} = \mathbf{0}, \ \mathbf{q}^T\mathbf{z} = -1\right\}$. The following result is proved:

Lemma 26 *Let matrix* M *be row sufficient. If* $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D$, *then* $(\mathbf{u}, \mathbf{z})$ *is a solution of* $DLCP$.

Based on Lemma 26 let us approach the problem from the dual side. First try to solve the feasibility problem of $DLCP$. It is a linear programming problem, therefore we can solve it in polynomial time. We have the following cases:

(a) $\mathcal{F}_D \neq \emptyset$ and for the computed $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D : \mathbf{uz} = \mathbf{0}$ holds, therefore we solved problem $DLCP$.

(b) $\mathcal{F}_D \neq \emptyset$ and for the computed $(\mathbf{u}, \mathbf{z}) \in \mathcal{F}_D : \mathbf{uz} \neq \mathbf{0}$ holds, then by Lemma 26 we know that M is not a row sufficient matrix, therefore it is not a sufficient matrix either, and vector $\mathbf{z}$ is a certificate for this.

(c) $\mathcal{F}_D = \emptyset$ then problem $DLCP$ has no solution.

In cases (a) and (b) we have solved the LCP in the sense of Theorem 1. In case (c) we try to solve the embedded problem (LCP') using one of the modified algorithms. Any of the modified algorithms either shows that the matrix is not in the class of $\mathcal{P}_*(\tilde{\kappa})$ or solves problem (LCP'). In the latter case we have two subcases:

(i) $\tilde{\mathbf{x}} = \mathbf{0}$ then by Lemma 25 LCP has a solution.

(ii) $\tilde{\mathbf{x}} \neq \mathbf{0}$.

When $\tilde{\mathbf{x}} \neq \mathbf{0}$ and $\mathcal{F}_D = \emptyset$, then if matrix M is sufficient, then it is also column sufficient so the LCP has no solution by Lemma 25. But this contradicts to the Fukuda-Terlaky LCP duality theorem [4, 5, 6], therefore in this case matrix M can not be sufficient and the vector $\tilde{\mathbf{x}}$ is an indirect certificate for this.

The dual side approach combining with the complexity result Theorem 23 (an interior point of problem (LCP') is known by construction) we can state our main result.

Theorem 27 *Let an arbitrary matrix $M \in \mathbb{Q}^{n\times n}$ and a vector $\mathbf{q} \in \mathbb{Q}^n$ be given. Then one can verify in polynomial time that at least one of the following statements hold*

(1) the LCP problem (1) has a feasible complementary solution $(\mathbf{x}, \mathbf{s})$ whose encoding size is polynomially bounded.

(2) problem DLCP has a feasible complementary solution $(\mathbf{u}, \mathbf{z})$ whose encoding size is polynomially bounded.

(3) matrix M is not in the class $\mathcal{P}_*(\tilde{\kappa})$.

Theorem 27 is a generalization of Theorem 23. Since the interior point assumption is eliminated, it can occur that the LCP has no solution while matrix M is sufficient. This is the second statement of Theorem 27. On the other hand, as we have seen in (see page 21) case (ii) in the dual side approach, when the matrix is not sufficient, but we have only an indirect certificate $\tilde{\mathbf{x}}$. This is the reason why in the last case of Theorem 27 we can not ensure an explicit certificate. Therefore, Theorem 27 is stronger than Theorem 23, because the interior point assumption is eliminated, however only an indirect certificate is provided in the last case.

It is interesting to note that Theorem 27 and Theorem 1 (a result of Fukuda et al. [5]) are different in two aspects: first, our statement (3) is weaker in some cases then theirs (there is no direct certificate in one case), but on the other hand our constructive proof is based on polynomial time algorithms and a polynomial size certificate is provided in all other cases in polynomial time.

5 Summary

Cameron and Edmonds' [2] EP theorem and its LCP form proven by Fukuda, Namiki and Tamura [5] motivated our research. The use of the LCP-duality theorem (Theorem 1) in EP form is a novel idea in the interior point literature.

Among others, Potra et al. [18] extended some IPMs for sufficient matrix LCPs. Our aim was to modify IPMs in such a way that they may be applied to LCPs without any restriction or knowledge about the properties of the coefficient matrices. We have shown that LCPs with arbitrary matrix can be solved in polynomial time in the following extended manner: in polynomial time we either solve the problem up to ε-optimality or show that the matrix does not belong to the class of $\mathcal{P}_*(\tilde{\kappa})$ matrices, for which a polynomial size certificate is provided by the algorithm.

References

[1] I.I. Dikin, Iterative solution of problems of linear and quadratic programming, *Doklady Academii Nauk SSSR*, 174 (1967) 747-748. Translated in: Soviet Mathematics Doklady, 8 (1967) 674-675.

[2] K. Cameron and J. Edmonds, Existentially polytime theorems, *Polyhedral combinatorics* (Morristown, NJ, 1989), 83–100, DIMACS Series in Discrete Mathematics and Theoretical Computer Science Discrete 1, American Mathematical Society, Providence, RI, 1990.

[3] R.W. Cottle, J.-S. Pang, and V. Venkateswaran, Sufficient matrices and the linear complementarity problem, *Linear Algebra and Its Applications* 114/115 (1989), 231-249.

[4] Zs. Csizmadia and T. Illés, New criss-cross type algorithms for linear complementarity problems with sufficient matrices, *Optimization Methods and Software* 21 (2006), 247-266.

[5] K. Fukuda, M. Namiki, and A. Tamura, EP theorems and linear complementarity problems, *Discrete Applied Mathematics* 84 (1998), 107-119.

[6] K. Fukuda and T. Terlaky, Linear complementarity and oriented matroids, *Journal of the Operations Research Society of Japan* 35 (1992), 45-61.

[7] S.-M. Guu and R.W. Cottle, On a subclass of P_0, *Linear Algebra and Its Applications* 223/224 (1995), 325-335.

[8] D. den Hertog, C. Roos, and T. Terlaky, The Linear Comlementarity Problem, Sufficient Matrices and the Criss–Cross Method, *Linear Algebra and Its Applications*, 187. (1993) 1–14.

[9] T. Illés, M. Nagy, and T. Terlaky, An EP theorem for dual linear complementarity problems, *Operation Research Reports*, ORR 2007-2, Eötvös Loránd University of Sciences, Department of Operations Research, Budapest, Hungary.

[10] T. Illés, C. Roos, and T. Terlaky, Polynomial affine-scaling algorithms for $P_*(\kappa)$ linear complementarity problems, in *Recent Advances in Optimization*, Eds.: P. Gritzmann, R. Horst, E. Sachs, and R. Tichatschke, Proceedings of the 8^{th} French-German Conference on Optimization, Trier, July 21-26, 1996, Lecture Notes in Economics and Mathematical Systems 452, pp. 119-137, Springer Verlag, 1997.

[11] B. Jansen, C. Roos, and T. Terlaky, A family of polynomial affine scaling algorithms for positive semi-definite linear complementarity problems, *SIAM Journal on Optimization* 7 (1996), 126-140.

[12] M. Kojima, S. Mizuno, and A. Yoshise, A polynomial-time algorithm for a class of linear complementarity problems, *Mathematical Programming* 44 (1989), 1-26.

[13] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, volume 538 of Lecture Notes in Computer Science. Springer Verlag, Berlin, Germany, 1991.

[14] S. Mizuno, M.J. Todd, and Y. Ye, On adaptive-step primal-dual interior-point algorithms for linear programming, *Mathematics of Operations Research*, 18 (1993), 964-981.

[15] K.G. Murty, *Linear and Combinatorial Programming*, John Wiley & Sons, Inc., New York-London-Sydney, 1976.

[16] J. Peng, C. Roos, and T. Terlaky, New complexity analysis of primal-dual Newton methods for $P_*(\kappa)$ linear complementarity problems, in *High Performance Optimization Techniques* 245-266. Eds.: J.B.G. Frenk, C. Roos, T. Terlaky, and S. Zhang, Kluwer Academic Publishers, Dordrecht, 1999.

[17] F.A. Potra, The Mizuno-Todd-Ye algorithm in a larger neighborhood of the central path, *European Journal of Operational Research* 143 (2002), 257-267.

[18] F.A. Potra and X. Liu, Predictor-corrector methods for sufficient linear complementarity problems in a wide neighborhood of the central path, *Optimization Methods and Software* 20(1) (2005), 145-168.

[19] C. Roos, T. Terlaky, and J.-Ph. Vial, *Theory and Algorithms for Linear Optimization, An Interior Point Approach*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, USA, 1997. (Second edition: *Interior Point Methods for Linear Optimization*, Springer, New York, 2006.)

[20] Gy. Sonnevend, J. Stoer, and G. Zhao, On the complexity of following the central path of linear programs by linear extrapolation, *Methods of Operations Research* 63 (1989), 19-31.

[21] H. Väliaho, P_*-matrices are just sufficient, *Linear Algebra and Its Applications* 239 (1996), 103-108.

[22] H. Väliaho, Determining the handicap of a suffcient matrix, *Linear Algebra and its Applications* 253 (1997), 279-298.

[23] Y. Ye and K. Anstreicher, On quadratic and $O(\sqrt{n}L)$ convergence of a predictor-corrector algorithm for LCP, *Mathematical Programming* 62 (1993), 537-551.

Tibor Illés, Marianna Nagy
Eötvös Loránd University of Science
Department of Operation Research
Budapest, Hungary
E-mail: illes@math.elte.hu, nmariann@cs.elte.hu

Tamás Terlaky
McMaster University
Department of Computing and Software
Hamilton, Ontario, Canada
E-mail: terlaky@mcmaster.ca

6 Appendix

To make our paper self contained, here we present those results from [10, 16, 18] that are needed for our developments. All lemmas, theorems are converted to our notations.

Lemma 28 *(Lemma 4.3 in [11]) Let M be an arbitrary matrix, $\delta_a(\mathbf{xs}) < \tau$ and $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the affine scaling direction.*
(i) If $0 \le r \le 1$ and $\theta \le \frac{2}{\sqrt{n}}$ then

$$\mathbf{x}(\theta)^T\mathbf{s}(\theta) \le \left(1 - \frac{\theta}{2\sqrt{n}}\right)\|\mathbf{v}\|^2.$$

(ii) If $1 \le r$ and $\theta \le \frac{2\tau^{2-2r}}{\sqrt{n}}$ then

$$\mathbf{x}(\theta)^T\mathbf{s}(\theta) \le \left(1 - \frac{\theta\tau^{2-2r}}{2\sqrt{n}}\right)\|\mathbf{v}\|^2.$$

Theorem 29 *(Theorem 6.1 in [10]) Let M be a $\mathcal{P}_*(\kappa)$-matrix, $r > 0$, $\tau > 1$ and $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the affine scaling direction. If $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^0$, $\delta_a(\mathbf{xs}) \le \tau$ and*

$$0 \le \theta \le \min\left(\frac{2}{(1+4\kappa)\tau}\left(\sqrt{1+4\kappa+\frac{1}{\tau^2\, n}} - \frac{1}{\tau\sqrt{n}}\right),\ \frac{\sqrt{n}}{(r+1)\tau^{2r}},\ \frac{4(\tau^{2r}-1)}{(1+4\kappa)(1+\tau^2)\tau^{2r}\sqrt{n}}\right)$$

then $(\mathbf{x}(\theta), \mathbf{s}(\theta))$ is strictly feasible and $\delta_a(\mathbf{x}(\theta)\mathbf{s}(\theta)) \le \tau$.

Theorem 30 *(Corollary 6.1 in [10])*
Let $M \in \mathcal{P}_*(\kappa)$ and $(\mathbf{x}^0, \mathbf{s}^0)$ be a feasible interior point such that $\delta(\mathbf{x}^0\mathbf{s}^0) \le \tau = \sqrt{2}$.

- *If $0 < r \le 1$ and $n \ge 4$ then we may choose $\theta = \frac{4(1-2^{-r})}{3(1+4\kappa)\sqrt{n}}$, hence the complexity of the affine scaling algorithm is $\mathcal{O}\left(\frac{n(1+4\kappa)}{1-2^{-r}}\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$.*
- *If $r = 1$ and $n \ge 4$ then we may choose $\theta = \frac{1}{2(1+4\kappa)\sqrt{n}}$, hence the complexity of the affine scaling algorithm is $\mathcal{O}\left(n(1+4\kappa)\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$.*
- *If $r > 1$ and n is sufficiently large then we may choose $\theta = \frac{1}{2^r(1+4\kappa)\sqrt{n}}$, hence the complexity of the affine scaling algorithm is $\mathcal{O}\left(2^{2r-2}n(1+4\kappa)\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$.*

Theorem 31 *(Theorem 10.2 in [16]) Let M be an arbitrary matrix, $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^0$, $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the Newton direction of the long-step path-following algorithm, $\delta := \delta_c(\mathbf{xs}, \mu)$ and $\delta^+ := \delta_c(\mathbf{x}(\theta)\mathbf{s}(\theta), \mu)$. Then for all $0 \le \theta \le 1/\sigma$, one has*

$$\delta^+ \le (1-\theta)\delta^2 + \frac{2\,\theta^3\sigma^2}{1-\theta^2\sigma^2}.$$

Theorem 32 *(Theorem 10.5 in [16]) If $M \in \mathcal{P}_*(\kappa)$, $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^0$, $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the Newton direction of the long-step path-following algorithm, $\delta := \delta_c(\mathbf{xs}, \mu) \geq 2$ and $\delta^* := \delta_c(\mathbf{x}(\theta^*)\mathbf{s}(\theta^*), \mu)$, where $\theta^* = \frac{2}{(1+4\kappa)\delta^2}$. Then*

$$(\delta^*)^2 - \delta^2 \leq -\frac{5}{3(1+4\kappa)}. \tag{24}$$

Theorem 33 *(From Theorem 10.10 and the subsequent remarks in [16])*
Let matrix $M \in \mathcal{P}_*(\kappa)$, $\tau = 2$, $\gamma = 1/2$ and $(\mathbf{x}^0, \mathbf{s}^0)$ be a feasible interior point such that $\delta_c(\mathbf{x}^0\mathbf{s}^0, \mu^0) \leq \tau$. Then the long-step path-following algorithm produces a point $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$ such that $\delta_c(\hat{\mathbf{x}}\hat{\mathbf{s}}, \hat{\mu}) \leq \tau$ and $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$ in at most $\mathcal{O}\left((1+4\kappa)n \log \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$ iterations.

Lemma 34 *(From expressions (3.16), (3.17) in [18])*
Let M be an arbitrary matrix, $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}(\beta)$, $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the predictor direction in the predictor-corrector algorithm and let the predictor step length be

$$\bar{\theta} = \sup\left\{\hat{\theta} > 0 : \ (\mathbf{x}(\theta), \mathbf{s}(\theta)) \in \mathcal{D}\big((1-\gamma)\beta\big), \ \forall\, \theta \in [0, \hat{\theta}]\right\}.$$

Furthermore let $\bar{\theta}_0 = \frac{2}{1+\sqrt{1-4\mathbf{e}^T\mathbf{g}/n}}$ and

$$\bar{\theta}_i = \begin{cases} \infty & \text{if } \Delta_i \leq 0 \\ 1 & \text{if } q_i - (1-\gamma)\beta\mathbf{e}^T\mathbf{g}/n = 0 \\ \frac{2(p_i-(1-\gamma)\beta)}{p_i-(1-\gamma)\beta+\sqrt{\Delta_i}} & \text{if } \Delta_i > 0 \text{ and } q_i - (1-\gamma)\beta\mathbf{e}^T\mathbf{g}/n \neq 0, \end{cases}$$

where

$$\Delta_i = (p_i - (1-\gamma)\beta)^2 - 4(p_i - (1-\gamma)\beta)\left(q_i - (1-\gamma)\beta\mathbf{e}^T\mathbf{g}/n\right).$$

Then we have

$$\bar{\theta} = \min\left\{\bar{\theta}_i : \ 0 \leq i \leq n\right\}.$$

Lemma 35 *(From the proof of Theorem 3.3 in [18])*
Let the assumptions of Lemma 34 hold, and $\bar{\theta}_i$, $1 \leq i \leq n$ be as it is given in Lemma 34, then

$$\bar{\theta}_i \geq \frac{2}{1+\sqrt{1+(\beta\gamma)^{-1}(4\|\mathbf{g}\|_\infty + 4\mathbf{e}^T\mathbf{g}/n)}}.$$

Lemma 36 *(From the proof of Theorem 3.3 in [18])*
Let M be an arbitrary matrix and let the point after the predictor step in the predictor-corrector algorithm satisfy $(\bar{\mathbf{x}}, \bar{\mathbf{s}}) \in \mathcal{D}((1-\gamma)\beta)$. Then

$$\left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar{\mu}}} - \sqrt{\frac{\bar{\mu}}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 \leq \frac{1-(1-\gamma)\beta}{(1-\gamma)\beta}\, n.$$

Lemma 37 *(From Theorem 3.3 in [18])*
Let M be a $\mathcal{P}_*(\kappa)$-matrix and $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}(\beta)$. Then the predictor step length

$$\theta_p^* := \frac{2\sqrt{(1-\beta)\beta}}{(1+4\kappa)n+2} \leq \sup\left\{\hat{\theta} > 0 : \ (\mathbf{x}(\theta), \mathbf{s}(\theta)) \in \mathcal{D}\big((1-\gamma)\beta\big), \ \forall\, \theta \in [0, \hat{\theta}]\right\}$$

and the corrector step length

$$\theta_c^* := \frac{2\beta}{(1+4\kappa)n+1}$$

determines a point in the $\mathcal{D}(\beta)$ *neighborhood, namely,* $(\bar{\mathbf{x}}(\theta_c^*), \bar{\mathbf{s}}(\theta_c^*)) \in \mathcal{D}(\beta)$, *where* $(\bar{\mathbf{x}}, \bar{\mathbf{s}}) = (\mathbf{x}(\theta_p^*), \mathbf{s}(\theta_p^*)) \in \mathcal{D}((1-\gamma)\beta)$.

Lemma 38 *(From Theorem 3.3 in [18])*
Let M *be an arbitrary matrix,* $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}(\beta)$, $\mu_g = \mathbf{x}^T\mathbf{s}/n$, *the definition of parameters* θ_p^* *and* θ_c^* *be the same as in Lemma 37,* $\bar{\theta}$ *be the predictor and* θ^+ *be the corrector step length,* $(\Delta\mathbf{x}, \Delta\mathbf{s})$ *be the predictor and* $(\Delta\bar{\mathbf{x}}, \Delta\bar{\mathbf{s}})$ *the corrector Newton direction in the predictor-corrector algorithm. If* $\bar{\theta} \geq \theta_p^*$ *and the step length* θ_c^* *determines a point in the* $\mathcal{D}(\beta)$ *neighborhood, i.e.,* $(\bar{\mathbf{x}}(\theta_c^*), \bar{\mathbf{s}}(\theta_c^*)) \in \mathcal{D}(\beta)$, *where* $(\bar{\mathbf{x}}, \bar{\mathbf{s}}) = (\mathbf{x}(\bar{\theta}), \mathbf{s}(\bar{\theta}))$, *then*

$$\mu_g^+ \leq \left(1 - \frac{3\sqrt{(1-\beta)\beta}}{2((1+4\kappa)n+2)}\right)\mu_g,$$

where $\mu_g^+ = (\mathbf{x}^+)^T\mathbf{s}^+/n = \bar{\mathbf{x}}(\theta^+)^T\bar{\mathbf{s}}(\theta^+)/n$.

Theorem 39 *(Corollary 3.4 in [18])*
Let M *be a* $\mathcal{P}_*(\kappa)$*-matrix and* $(\mathbf{x}^0, \mathbf{s}^0)$ *be a feasible interior point such that* $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{D}(\beta)$. *Then in at most* $\mathcal{O}\left((1+\kappa)n\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$ *steps the predictor-corrector algorithm produces a point* $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$ *such that* $(\hat{\mathbf{x}}, \hat{\mathbf{s}}) \in \mathcal{D}(\beta)$ *and* $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$.