SIMPLEX METHODS MEETING THE CONJECTURED HIRSCH BOUND *

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Abstract. In 1957, Hirsch conjectured that every $d-$polytopes with n facets has edge-diameter at most $n-d$. Recently Holt and Klee constructed polytopes which meet this bound for a number of (d,n) pairs with $d \leq 13$ and for all pairs $(14,n)$. The main purpose of this paper is to present linear Hirsch bound $\lfloor \frac{n}{2} \rfloor + d - 2$ for all pairs (d,n) and to review mathematical ideas behind simplex methods, investigating the duality formation implying in them.

Key words. Linear programming, Duality gap, Simplex method, Pivot rule, Hirsch conjecture

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1. Introduction. Linear programming is the problem of minimizing a linear objective function over a polyhedron $P \subset R^d$ given by a system of n linear inequalities, and as such has received considerable attention in the last five decades. Many algorithms have been proposed for its solution, starting with the simplex algorithm and its relatives, proceeding through the polynomial-time solution of Khachiyan and Kamarkar, and continuing with several more recent techniques (for example, see [26] and its references therein). Recently, an excellent overview of the history of linear programming and its applications was given by Todd in [29].

The simplex algorithm is the most popularly method used for linear programming for performing sufficiently well in practice, particularly on linear problems of small or medium size. To the knowledge of the author, virtually all of the pivot rules in modern literature are so-called combinatorial pivot rules [3, 4, 6, 11, 17, 34], ratio test type rules [1, 7, 13, 24] and other rules [12, 30], in which the main role of the pivot rules is to iteratively drive the current point towards to the optimum solution. Prominent variants (in the huge set of possible variants) are the rule of greatest improvement, the rule of steepest ascent, the Dantzig rule, the parametric rule and the rule of random choice. The use of either the Bland rule or the lexicographic rule can avoid the cycling effectively when the degeneracy occurs. They are guaranteed termination in exact arithmetic but are often prohibitively expensive to implement for the revised simplex method and do not address the problem of inexact arithmetic. For a suite of large programs, the dual simplex method using the steepest-edge pivot rule is the fastest [10, 29]. Recently, Chen et al. [5], Paparrizos et al. [25] and Yan Zizong et al. [33] developed a primal-dual simplex algorithm for the general linear programming problem.

For two vertices x and y of a polytope P, the *distance* $\delta_p(x,y)$ is defined as the smallest number of edges of P that can be used to form a path from x and y. The *edge diameter* $\delta(P)$ of P is the maximum over all pairs (x,y) of P's vertices. For $n > d \geq 2$, let $\Delta(d,n)$ be defined as the edge diameter of convex polytopes P in R^d with n facets. Hirsch asserted that

$$\Delta(d,n) \leq n-d, \quad \forall n > d \geq 1. \tag{1.1}$$

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The $d-$*step conjecture* is a special case of the Hirsch conjecture and asserts that

$$\Delta(d, 2d) = d, \quad \forall d \geq 1. \tag{1.2}$$

Arising from the desire to understand better the computational complexity of edges-following algorithms in linear programming, it is one of the fundamental open problems concerning the structure of convex polytopes (= convex compact polyhedra) and the theory of linear programming.

Our purpose in writing this paper is twofold. First, it is to acquire a linear bound of $\Delta(d, n)$ for all n and d:

$$\Delta(d, n) \leq \lfloor \frac{n}{2} \rfloor + d - 2, \tag{1.3}$$

which is a weaker linear upper bound estimation than the Hirsch conjecture if $1 \leq d < n < 4(d-1)$. Our second purpose is to review simplex methods for linear programs by the use of to a current equivalent facet technique.

This paper is organized as follows. In Section 2 we recall the basic theory and some well-known facts of LP. In Section 3 we acquire a linear Hirsch bound for polytopes. In Section 4 we present a novel version of primal simplex algorithms that use the current equivalent facet technique. An example is given in Section 5 and duality gaps is studied in Section 6. Finally, we give our remarks and discuss the possibility of on the extension of a short admissible pivot path to an optimal basis.

2. Background. Let us introduce some notations and state some results for LPs. Letting $c, x \in R^n, b \in R^m$ and $A \in R^{m \times n}$, the primal linear problem in standard format is given by

$$\min\ c^T x \quad s.t. \quad Ax = b,\ x \geq 0. \tag{2.1}$$

The associated dual problem reads

$$\max\ b^T w \quad s.t. \quad A^T w \leq c. \tag{2.2}$$

The sets of feasible solutions of (2.1) and (2.2) are denoted by the notations P and D, respectively. If the set P is nonempty the problem (2.1) is called feasible; otherwise (2.1) is infeasible. A solution $x \in R^n$ is primal feasible if $x \in P$. If there is a sequence of feasible solutions for which the objective value goes to minus infinity the (2.1) is said to be unbounded. Analogous statements hold for (2.2).

If (2.1) is feasible, P is the intersection of $d := n - m$ dimensional affine subspace with the nonnegative orthant in R^n, which can be parametrized by d variables such that P is the intersection of n halfspaces in R^d. A current point $w \in D$ if and only if $r \geq 0$, where the vector $r = c - A^T w \in R^n$ is called to be the reduced cost vector. It is easy check that for any primal feasible x and dual feasible w it holds $b^T w \leq c^T x$, which is *weak duality*. The product $x^T r = c^T x - b^T w$ denote the *duality gap* of (2.1) and (2.2). The first theorem is the main result in the theory of LP.

THEOREM 2.1. *For (2.1) and (2.2) one of the following alternatives holds.*

1. *Both (2.1) and (2.2) are feasible and there exist* $x^* \in P$ *and* $w^* \in D$ *such that* $c^T x^* = b^T w^*$*;*
2. *(2.1) is infeasible and (2.2) is unbounded;*
3. *(2.2) is infeasible and (2.1) is unbounded;*
4. *Both (2.1) and (2.2) are infeasible.*

An alternative way of writing the optimality condition in Theorem 2.1 (i) is by using the *complementary slackness condition*

$$x_i^* r_i^* = 0, \quad i = 1, 2, \cdots, n. \tag{2.3}$$

Goldman and Tucker [15] presented the following strict complement condition in 1956, which is involved in theoretical analysis, in sensitivity analysis as well as in the development and analysis of polynomial time interior point methods.

THEOREM 2.2. *If (2.1) and (2.2) are feasible then there exist $x^* \in P$ and $w^* \in D$ such that $r^* = c - A^T w^* \geq 0$ and*

$$x_i^* + r_i^* > 0, \quad i = 1, 2, \cdots, n \tag{2.4}$$

and (2.3) holds.

The pair (x^*, r^*) satisfying (2.3) and (2.4) is called strictly complementary, which implies that for each index i exactly of x_i^* and r_i^* is zero, while the other is positive.

An elementary proof of the above fundamental theorems based on interior point ideas can be found in [26].

The simplex method employs two phases to such an LP with a similar manner. It starts at the initial feasible vertex to generate a sequence of vertices such that successive vertices are adjacent and that the objective values of these vertices decreases. The sequence ends at a vertex which is either the optimal vertex or a vertex exhibiting the information that no optimal vertex can exist. The precise rules for choosing the successor-vertex in the sequence determines a variant of the simplex algorithms.

Given a linear programming problem, a *basis* B is a maximal subset of indices $\{1, 2, \cdots, n\}$ such that the corresponding column vectors of matrix A are independent. The subset of nonbasic indices N is defined by $N = \{1, 2, \cdots, n\} - B$. The variables x is partitioned into basic variables x_B and nonbasic variables x_N, indexed by the sets B and N, respectively. Similarly A is partitioned as $A = [A_B | A_N]$ where A_B is an $m \times m$ invertible matrix. The current value of $\hat{x}$ is called a basic feasible solution if its partition satisfies $\hat{x}_B = A_B^{-1} b \geq 0$ and $\hat{x}_N = 0$. By convention the system (2.1) can be represented by the tableau

(2.5)

	x_N	rhs
	$c_N - c_B^T A_B^{-1} A_N$	$c_B^T A_B^{-1} b$
x_B	$A_B^{-1} A_N$	$A_B^{-1} b$

and it is convenient to refer to rows and columns of the tableau by the variables to which they correspond. If a variable $q \in B$ is interchanged with a variable $p \in N$, there are standard rules for updating the entries in the tableau, i.e., x_p enters basis and x_q leaves basis.

THEOREM 2.3. *Suppose that $x^* \in P$ is a basic feasible solution of (2.1) with the basis B and the nonbasis N. If $c_N - c_B^T A_B^{-1} A_N \geq 0$, then x^* is an optimum solution of (2.1).*

3. Hirsch conjecture. Unfortunately, for most all known pivoting rules (see [2, 13, 14, 19, 22, 23, 27]) sequences of examples have been constructed, such that the number of the iterations is exponential in n or d. Up to now, no pivoting rule is known to give rise to a polynomial time method. Constructing such a pivoting rule and settling the related *Hirsch conjecture* remain the most challenging open problem in the theory of LP.

Klee and Walkup [20] showed that the truth of the $d-$step conjecture for all d implies the truth of Hirsch conjecture for all n and d. The $d-$step conjecture has been proved for all $d \leq 5$. The Hirsch conjecture have been proved for $d \leq 3$ and all n, and also for all pairs (d, n) with $n - d \leq 5$. These results and others are described in the comprehensive reviews in [16, 18, 21].

To estimate Hirsch bound, let us recall some facts of the active set method for solving an LP. The *active set* of (2.2) denote by J is a selection of n constraint indices such that the matrix A_J^T is nonsinglar. The current point $\hat{w}$ is a vertex of the feasible region, obtained by solving the equation $(A^T w)_J = c_J$. The general approach to solving an LP by an *active set method* has been outlined for example in [8].

The theoretical fundament of the active sets method for solving an LP depends on the following known fact:

Theorem 3.1. *Let J be a active set of an optimal solution w^* of (2.2). Then w^* is also an optimal solution of the following problem*

$$\max\ b^T w \quad s.t. \quad (A^T w)_J \leq c_J, \tag{3.1}$$

and vice versa.

Theorem 3.2. *Let x^* be an optimal objective value of (2.1). For each basic feasible point $\hat{x}$ of (2.1) with the basis B and the nonbasis N, if $c^T\hat{x}$ is not equal to $c^T x^*$ such that the index set*

$$I(\hat{x}) = \{i | c_N - c_B^T A_B^{-1} A_N < 0\}$$

is not empty, then $I(\hat{x})$ and the basis of x^* have an nonempty intersection.

Proof. Suppose that $w^* \in D$ is an optimal solution of (2.2) with $r^* = c - A^T w^*$, and $I(\hat{x})$ and the basis of x^* have an empty intersection. Then, for any $i \in I(\hat{x})$, i belong to the nonbasis of optimal solution x^* such that $r_i^* > 0$ by Theorem 2.2, which implies that i does not belong to the active set of w^*. Theorem 3.1 shows that the optimal objective value of (2.2) can not be changed if we delete these non-active constraints, i.e., the the optimal objective value of (2.1) can not be changed if we delete all components of x in $I(\hat{x})$, which is a conflict with the assumption $c^T\hat{x} \neq c^T x^*$. The proof is finished. □

Theorem 3.2 is interesting and useful for which assures that there is a pivot rule such that each entering variable is a basic feasible component of an optimal basis.

Theorem 3.3. *Assume that the feasible region $P \subset R^d$ of an LP given by a system of n linear inequalities is nonempty bounded polyhedron. There exists a pivot rule such that the number of the iterations starting with a basic feasible point is equal to or less than $\lfloor \frac{n}{2} \rfloor + d - 2$.*

Proof. Without loss of generality, we consider the following linear programming

$$\min\ c_N^T x_N \quad s.t.\ A_N x_N \leq b,\ x_N \geq 0, \tag{3.2}$$

where $c_N \in R^d, A_N \in R^{(n-d)\times d}$, and b lies in the nonnegative orthant in R^{n-d}. Adding $n - d$ slack variables, (3.2) becomes the equality form (2.1), in which there are d nonbasic variables with $|N| = d$ by the assumption.

We proceed by induction over the dimension d. One can easily check that $\Delta(d, n)$ is the integer part of $\frac{n}{2}$ for the case $d = 2$.

Assume, as an inductive hypothesis, that the theorem is true when d is less than some positive integer k, and consider the case $d = k$.

It may happen that there is a component x_i of x_N such that i belongs to the nonbasic N of the optimal solution. By Theorem 3.2, there is a pivot rule such that

x_i has never been pivoted in as a basic variable and the projection of the LP in the hyperplane $\{x_N \in R^d | x_i = 0\}$ has the same optimal objective value with it. Then, since the dimension of this hyperplane is $k-1$, by the inductive hypothesis, there exists a pivot rule such that the number of the iterations is equal to or less than $\lfloor \frac{n}{2} \rfloor + (k-1) - 2 = \lfloor \frac{n}{2} \rfloor + k - 3$.

In the remaining case, after pivoting a step, there is a component x_j of x_N being pivot out such that j belongs to the nonbasic N of the optimal solution. As the above proof, there exists a pivot rule such that the number of the iterations is equal to or less than $1 + \left(\lfloor \frac{n}{2} \rfloor + k - 3\right) = \lfloor \frac{n}{2} \rfloor + k - 2$.

Thus the induction succeeds and the theorem is established. □

4. Algorithm. We proceed by describing the current equivalent facet technique for solving (2.1) in [31, 32], which is related to the *One-at-a-time method* of Fletcher [9]. Following this idea, we refer the increment of the objective value (replaced by $-x_0$) to as a variable x_0 and rewriter the problem (2.1) in equality form

$$\begin{array}{rl} \max & x_0 \\ s.t. & x_0 + c^T x = 0, \quad Ax = b, \quad x \geq 0 \end{array} \tag{4.1}$$

A row corresponding to the constraint $x_0 + c^T x = 0$ is adjoined to the simplex tableau and the target variable to increase is x_0. The problem (4.1) is called the increment model or the current equivalent facet model of the problem (2.1).

The current equivalent facet formulation of LP considers a system of constraints of the form

$$\mathcal{A}\mathbb{x} = \mathbb{b}, \ \mathbb{x} \geq 0 \tag{4.2}$$

where

$$\mathcal{A} = \begin{pmatrix} 1 & c^T \\ 0 & A \end{pmatrix}, \quad \mathbb{x} = \begin{pmatrix} x_0 \\ x \end{pmatrix}, \quad \mathbb{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

To distinguish successive pivotal transforms of simplex methods, we shall use the superscript ν as an iteration counter. The initial value of ν will be 0. The vector x^ν and r^ν represent the current basic solution and the current reduced cost vector, respectively.

In general, after ν principal pivots, the system (4.2) will be $\mathcal{A}^\nu \mathbb{x}^\nu = \mathbb{b}^\nu$. Its can also be represented in the tableau form

$$\begin{array}{cccc|c} x_0^\nu & x_1^\nu & \cdots & x_d^\nu & rhs \\ \hline \mathbb{a}_{00}^\nu & \mathbb{a}_{01}^\nu & \cdots & \mathbb{a}_{0d}^\nu & \mathbb{b}_0^\nu \\ \mathbb{a}_{10}^\nu & \mathbb{a}_{11}^\nu & \cdots & \mathbb{a}_{1d}^\nu & \mathbb{b}_1^\nu \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbb{a}_{m0}^\nu & \mathbb{a}_{m1}^\nu & \cdots & \mathbb{a}_{md}^\nu & \mathbb{b}_m^\nu \end{array} \tag{4.3}$$

All coefficients will become $\mathbb{a}_{ij}^{\nu+1}$ and $\mathbb{b}_i^{\nu+1}$ after one pivot. For brevity, we sometimes only display the nonbasic columns of $\mathcal{A}$ in the left hand side of (4.3).

Let the natation $\mathbb{a}_0^\nu$ denote the x_0^ν column and $\mathbb{a}_{pq}^\nu$ denote the pivot principle element in (4.3), i.e., the column q be called the pivot column and the row p be called the pivot row. A formal description of the current equivalent facet method is given below.

Algorithm *Current equivalent facet method*

1. *Start with an initial feasible basis and an index* l *with* $\mathbb{a}_{lj} \geq 0, j = 0, 1, \cdots, n$. *Set* $p = 0$ *and* $\nu = 0$.

2. *If* $\mathbb{a}_{pj} \geq 0$ *for* $j = 0, 1, \cdots, n$, *STOP.*

3. *Generate new coefficients* $\mathbb{a}_{ij}^{\nu+1}$ *and* $\mathbb{b}_i^{\nu+1}$ *by pivoting on* $\mathbb{a}_{pq}^{\nu}(< 0)$ *with a new column index*

$$q = \arg\min_j \{-\frac{\mathbb{a}_{lj}^{\nu}}{\mathbb{a}_{pj}^{\nu}} | \mathbb{a}_{lj}^{\nu} \geq 0, \mathbb{a}_{pj}^{\nu} < 0\}. \tag{4.4}$$

4. *Choose an index* p *such that*

$$p = \arg\min \{\frac{\mathbb{b}_i^{\nu+1}}{\mathbb{a}_{i0}^{\nu+1}} | \mathbb{a}_{i0}^{\nu+1} > 0\}, \tag{4.5}$$

and

$$\lambda_{\nu+1} = \frac{\mathbb{b}_p^{\nu+1}}{\mathbb{a}_{p0}^{\nu+1}}. \tag{4.6}$$

5. *Update* $\mathbb{b}^{\nu+1} \leftarrow \mathbb{b}^{\nu+1} - \lambda_{\nu+1}\mathbb{a}_0^{\nu+1}$ *and set* $\nu \leftarrow \nu + 1$, *return to step 2.*

There is an interesting property of this algorithm. There are two ratio tests in every pivot steps - both the leaving and entering variables are selected by performing their ratio tests, respectively, which were usual used in the primal or dual methods independently.

All simplex methods can use the current equivalent facet technique. We can always pivot on primal test rows in place of constraint rows in the modern version of simplex methods by the use of the increment model (4.2). If we replace the ratio test (4.4) by the other pivot rule, such as Dantzig's rule, this algorithm is a primal simplex method. The dual simplex method can also be carried out according to this algorithm if we replace (4.6) and (4.5) by the following step size formula

$$\lambda_{\nu+1} = \frac{\mathbb{b}_l^{\nu+1}}{\mathbb{a}_{l0}^{\nu+1}} \tag{4.7}$$

and other pivot rules, respectively. Both primal and dual simplex method are essentially equivalent to each other, while the unique different is the choice of the moving stepsizes if we pivot on primal test rows in the increment model (4.2).

The $l-$th row in (4.3) is call to be dual feasible if $\mathbb{a}_{lj} \geq 0$ for each $j = 0, 1, \cdots, n$. If there is no any dual feasible constraint in (4.3), we might add an artificial dual feasible constraint (as the dual simplex method), while an initial feasible basis can be found by Phase I (as the primal simplex method). On the assumption of nondegeneracy, a dual feasible constraint decides an unique search path of the current equivalent facet method, which is called to be a *algebraic path*.

5. Introductory example. We solve the four variable, four constraint problem (5.1) by the current equivalent facet method.

$$\begin{array}{lllllll}
\min & -2x_1 & -8x_2 & -7x_3 & -19x_4 & & \\
s.t. & 10x_1 & -6x_2 & +10x_3 & -4x_4 & \leq & 18 \\
 & -x_1 & 5x_2 & +6x_3 & +7x_4 & \leq & 19 \\
 & -6x_1 & +8x_2 & -x_3 & +12x_4 & \leq & 13 \\
 & 3x_1 & +8x_2 & +13x_3 & +18x_4 & \leq & 57 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0.
\end{array} \tag{5.1}$$

This example will be used throughout this paper to illustrate the various concepts and computational techniques.

Slack variables x_5, x_6, x_7, x_8 are added to the first four constraints and x_0 is the target variable for the objective function of the problem (5.1). The initial tableau is

(5.2)

	x_1	x_2	x_3	x_4	rhs
x_0	-2	-8	-7	-19	0
x_5	10	-6	10	-4	18
x_6	-1	5	6	7	19
x_7	-6	8	-1	12	13
x_8	3	8	13	18	57

and is primal feasible so that no Phase I is required. The x_8 row is dual feasible such that the current equivalent facet method can be carried out. The first step of the process is to make x_0 nonbasic which is achieved by pivoting on x_4 defined by (4.4) in (5.2), yielding the tableau

(5.3)

	x_0	x_1	x_2	x_3	rhs
x_4	$-\frac{1}{19}$	$\frac{2}{19}$	$\frac{8}{19}$	$\frac{7}{19}$	$\frac{13}{12}$
x_5	$-\frac{4}{19}$	$\frac{198}{19}$	$-\frac{82}{19}$	$\frac{218}{19}$	$\frac{67}{3}$
x_6	$\frac{7}{19}$	$-\frac{33}{19}$	$\frac{39}{19}$	$\frac{65}{19}$	$\frac{137}{12}$
x_7	$\frac{12}{19}$	$-\frac{138}{19}$	$\frac{56}{19}$	$-\frac{103}{19}$	0
x_8	$\frac{18}{19}$	$\frac{21}{19}$	$\frac{8}{19}$	$\frac{121}{19}$	$\frac{75}{2}$

and a primal basic feasible solution $(0, 0, 0, \frac{67}{3}, 0, \frac{137}{12}, \frac{13}{12}, \frac{75}{2})^T$. The right hand side coefficients update as

$$\left(\frac{13}{12}, \frac{67}{3}, \frac{137}{12}, 0, \frac{75}{2}\right)^T = (0, 18, 19, 13, 57)^T - \lambda_1 \left(-\frac{1}{19}, -\frac{4}{19}, \frac{7}{19}, \frac{12}{19}, \frac{18}{19}\right)^T,$$

where λ_1 denotes the first moving stepsize as defined (4.5) by

$$\lambda_1 = \min\left\{\frac{19}{7/19}, \frac{13}{12/19}, \frac{57}{2/19}\right\} = \frac{247}{12}.$$

The next step is to solve the level 2 problem by using the current equivalent facet technique to restore primal and dual feasibility to the level 2 in (5.4). The x_7 row is a current equivalent facet such that x_7 must leave basis. The entering basis variable x_1 and the updating of right hand side coefficients are decided by the minimum ratio test (4.5) and (4.6), respectively. This gives rise to the tableau

(5.4)

	x_0	x_2	x_3	x_7	rhs
x_4	$-\frac{1}{23}$	$\frac{32}{69}$	$\frac{20}{69}$	$\frac{1}{69}$	$\frac{119}{48}$
x_5	$\frac{16}{23}$	$-\frac{2}{23}$	$\frac{85}{23}$	$\frac{33}{23}$	0
x_6	$\frac{5}{23}$	$\frac{31}{23}$	$\frac{217}{46}$	$-\frac{11}{46}$	$\frac{71}{16}$
x_1	$-\frac{2}{23}$	$-\frac{28}{69}$	$\frac{103}{138}$	$-\frac{49}{138}$	$\frac{67}{24}$
x_8	$\frac{24}{23}$	$\frac{20}{23}$	$\frac{255}{46}$	$\frac{7}{46}$	4

Analogous, we can solve the level 3 problem and acquire the tableau

(5.5)

	x_0	x_3	x_5	x_7	rhs
x_4	$\frac{11}{3}$	20	$\frac{16}{3}$	$-\frac{23}{3}$	1
x_2	-8	$-\frac{85}{2}$	$-\frac{23}{2}$	$-\frac{33}{2}$	$\frac{71}{22}$
x_6	11	62	$\frac{31}{2}$	22	0
x_1	$-\frac{10}{3}$	$-\frac{33}{2}$	$-\frac{14}{3}$	$-\frac{41}{6}$	$\frac{91}{22}$
x_8	8	$\frac{85}{2}$	10	$\frac{29}{2}$	$\frac{17}{22}$

The x_6 row shows that x_0 can not be increased and the optimal solution be obtained.

6. Duality gaps. As mentioned in the paper [33], if $a^\nu_{i0} \neq 0$ for each $i = 0, 1, \cdots, m$, the model (4.2) offers us $m+1$ vectors of reduced costs

$$(\frac{a^\nu_{i1}}{a^\nu_{i0}}, \frac{a^\nu_{i2}}{a^\nu_{i0}}, \cdots, \frac{a^\nu_{is}}{a^\nu_{i0}})^T \tag{6.1}$$

for each current basis x^ν at most in our algorithm. For example, in the tableau (5.3), there are five different reduced cost vectors

$$\begin{gathered}(-2, -8, -7, -19, 0, 0, 0, 0)^T,\\ (-\frac{99}{2}, \frac{41}{2}, -\frac{109}{2}, 0, -\frac{19}{4}, 0, 0, 0)^T,\\ (-\frac{33}{7}, \frac{39}{7}, \frac{65}{7}, 0, 0, \frac{19}{7}, 0, 0)^T,\\ (-\frac{23}{2}, \frac{14}{3}, -\frac{103}{12}, 0, 0, 0, \frac{19}{12}, 0)^T,\\ (\frac{7}{6}, \frac{4}{9}, \frac{121}{18}, 0, 0, 0, 0, \frac{19}{18})^T.\end{gathered}$$

Two of them are very critical in the above current equivalent facet method: one is called to be the current primal reduced cost vector $r^\nu_P = (\frac{a^\nu_{p1}}{a^\nu_{p0}}, \frac{a^\nu_{p2}}{a^\nu_{p0}}, \cdots, \frac{a^\nu_{pn}}{a^\nu_{p0}})^T$ for each p; another is called to be the fixed dual reduced cost vector $r^\nu_D = (\frac{a^\nu_{l1}}{a^\nu_{l0}}, \frac{a^\nu_{l2}}{a^\nu_{l0}}, \cdots, \frac{a^\nu_{ln}}{a^\nu_{l0}})^T$ for a given l.

The products $(x^\nu)^T r^\nu_P$ and $(x^\nu)^T r^\nu_D$ denote two different duality gaps. A general detail of statements for the duality gaps in the theory of the linear system is given.

THEOREM 6.1. *Suppose that a sequence $\{(x^\nu, w^\nu, r^\nu)\}^k_{\nu=1}$ satisfies the linear system*

$$Ax = b, \quad A^T w + r = c. \tag{6.2}$$

Then, for any $1 \leq \nu \leq k$,

$$(x^{\nu+1})^T r^{\nu+1} - (x^\nu)^T r^\nu = (c^T x^{\nu+1} - c^T x^\nu) + (b^T w^\nu - b^T w^{\nu+1}). \tag{6.3}$$

Proof. Since the sequence $\{(x^\nu, w^\nu, r^\nu)\}$ satisfies the linear system (6.2), then for any $1 \leq \nu \leq k$, $Ax^\nu = Ax^{\mu+1} = b$, $r^\nu = c - A^T w^\nu$ and $r^{\nu+1} = c - A^T w^{\nu+1}$, which imply that

$$\begin{aligned}(x^{\nu+1})^T r^{\nu+1} - (x^\nu)^T r^\nu &= (c^T x^{\nu+1} - (x^{\nu+1})^T A^T w^{\nu+1}) - (c^T x^\nu - (x^\nu)^T A^T w^\nu)\\ &= (c^T x^{\nu+1} - b^T w^{\nu+1}) - (c^T x^\nu - b^T w^\nu)\\ &= (c^T x^{\nu+1} - c^T x^\nu) + (b^T w^\nu - b^T w^{\nu+1}).\end{aligned}$$

The proof is finished. □

(6.3) shows that the change of duality gaps is the sum of the changes of both the primal and the dual objective value, which form the theoretical foundation of the current equivalent facet method. Two very interesting and important special cases of Theorem 6.1 are

$$(x^{\nu+1})^T r^{\nu+1} - (x^{\nu})^T r^{\nu+1} = c^T x^{\nu+1} - c^T x^{\nu}, \tag{6.4}$$

$$(x^{\nu+1})^T r^{\nu} - (x^{\nu+1})^T r^{\nu+1} = b^T w^{\nu+1} - b^T w^{\nu}, \tag{6.5}$$

in which (6.4) and (6.5) denote the changes of the primal and the dual objective value, i.e., the (primal or dual) moving stepsizes in the current equivalent facet method, respectively.

The following analysis was used to construct the example in Section 5. All current basic feasible solutions and their dual reduced costs in the current equivalent facet method are displayed as follows:

ν	x^{ν}	r_P^{ν}	r_D^{ν}
0	$(0,0,0,0,18,19,13,57)^T$	$(-2,-8,-7,-19,0,0,0,0)^T$	
1	$(0,0,0,\frac{13}{12},\frac{67}{3},\frac{137}{12},0,\frac{75}{2})^T$	$(-\frac{23}{2},\frac{14}{3},-\frac{103}{12},0,0,0,\frac{19}{12},0)^T$	$(\frac{7}{6},\frac{4}{9},\frac{121}{18},0,0,0,0,\frac{19}{18})^T$
2	$(\frac{67}{24},0,0,\frac{119}{48},0,\frac{71}{16},0,4)^T$	$(0,-\frac{1}{8},\frac{85}{16},0,\frac{23}{16},0,\frac{33}{16},0)^T$	$(0,\frac{5}{6},\frac{85}{16},0,0,0,\frac{7}{48}\frac{23}{24})^T$
3	$(\frac{91}{22},\frac{71}{22},0,1,0,0,0,\frac{17}{22})^T$	$(0,0,\frac{62}{11},0,\frac{31}{22},\frac{1}{11},2,0)^T$	$(0,0,\frac{85}{16},0,\frac{5}{4},0,\frac{29}{16},\frac{1}{8})^T$

whose duality gaps are

$(x^i)^T r^j$	r_P^0	r_P^1	r_P^2	r_P^3	r_D^1	r_D^2	r_D^3
x^0	0	$\frac{247}{12}$	$\frac{843}{16}$	$\frac{584}{11}$	$\frac{361}{6}$	$\frac{2713}{48}$	$\frac{851}{16}$
x^1	$-\frac{247}{12}$	0	$\frac{1541}{48}$	$\frac{4291}{132}$	$\frac{475}{12}$	$\frac{575}{16}$	$\frac{1565}{48}$
x^2	$-\frac{843}{16}$	$-\frac{1541}{48}$	0	$\frac{71}{176}$	$\frac{359}{48}$	$\frac{23}{6}$	$\frac{1}{2}$
x^3	$-\frac{584}{11}$	$-\frac{4291}{132}$	$-\frac{71}{176}$	0	$\frac{467}{66}$	$\frac{1811}{528}$	$\frac{17}{176}$

(6.4) implies that three move stepsizes are equal to

$$\begin{aligned} \lambda_1 = (x^1)^T r_P^1 - (x^0)^T r_P^1 &= \frac{247}{12}, \\ \lambda_2 = (x^2)^T r_P^2 - (x^1)^T r_P^2 &= \frac{1541}{48}, \\ \lambda_3 = (x^3)^T r_P^3 - (x^2)^T r_P^3 &= \frac{71}{176}, \end{aligned}$$

which result in the decrement of the primal objective value is

$$\frac{584}{11} = \frac{247}{12} + \frac{1541}{48} + \frac{71}{176}.$$

The increment of the dual objective value is

$$\frac{335}{48} = \frac{175}{48} + \frac{10}{3},$$

which, from the formula (6.5), is the sum of two dual move stepsizes

$$\begin{aligned} (x^1)^T r_D^1 - (x^1)^T r_D^2 = \frac{475}{12} - \frac{575}{16} &= \frac{175}{48}, \\ (x^2)^T r_D^2 - (x^2)^T r_D^3 = \frac{23}{6} - \frac{1}{2} &= \frac{10}{3}. \end{aligned}$$

In fact, the above two stepsizes can also be obtained if we use (4.7) in place of (4.6).

The total decrement of duality gaps is the sum of the decrement of primal objective and the increment of the dual objective, i.e.,

$$(x^0)^T r_D^1 - (x^3)^T r_D^3 = \frac{361}{6} - \frac{17}{176} = \frac{9431}{157} = \frac{584}{11} + \frac{335}{48}.$$

Since the ratio test (4.4) assures $r_D^\nu \geq 0$ at each pivot step, the current primal feasible basis x^ν always maintains dual feasibility for r_D^ν in our algorithm, although it is dual infeasibility for r_P^ν except for the last step. Because of this, this algorithm is also called to be the primal dual simplex method. Both primal and dual feasibilities are preserved during the algorithm due to two ratio tests (4.5) and (4.4), respectively - while the complementary slackness condition is lost for the duality gap $(x^\nu)^T r_D^\nu$, besides recovered in the last step. The move and the pivot are two basic operations in the algorithm, which result in the decreases of the duality gap $(x^\nu)^T r_D^\nu$. The algorithm reaches two goals: the primal objective value is decreased because of the move and the dual objective value is increased because of the pivot.

THEOREM 6.2. *On the assumption of Theorem 6.1, if the sequence* $\{(x^\nu, r^\nu)\}_{\nu=0}^k$ *satisfies the complementary slackness conditions*

$$(r^\nu)^T x^\nu = 0, \quad \nu = 0, 1, \cdots, k, \tag{6.6}$$

then

$$c^T x^{\nu+1} - c^T x^\nu = b^T w^{\nu+1} - b^T w^\nu. \tag{6.7}$$

Proof. This is a directly result of (6.3) and (6.6). □

The idea of the classical dual method is to find an optimal dual feasible solution, accompanying some basis B, which is optimal for the problem (2.1), by using the primal presentation. In the iterations of the dual method the sequences of bases B and the corresponding vectors of reduced costs r are constructed. Theorem 6.2 reveals an important fact and gives an explanation for simplex methods: when we carry out one simplex method to solve an LP, we are executing another *exterior simplex method* to solve its dual problem. The dual method is essentially an exterior simplex method.

7. Further remarks. The goal of the present paper is to provide a good motivation to study a class of pivot algorithms which is more general than the class of simplex algorithms. This broader view on pivot methods resulted in a family of simple methods, it allows us to explore the geometric structures of LPs. New results on the existence of a short admissible pivot path is presented.

The above discussion has shown that all simplex methods for solving an LP can be used with caution in the current equivalent facet technique. To illustrate the worst behaviors of Dantzig's rule, like most other authors, we have used in the Klee-Schrijver example

$$\begin{array}{lll}
\max & 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + x_n & \\
s.t. & x_1 \qquad\qquad\qquad\qquad\qquad +x_{n+1} & = 5, \\
& 2^2 x_1 + x_2 \qquad\qquad\qquad\quad +x_{n+2} & = 5^2, \\
& \cdots \quad \cdots \quad \cdots & \\
& 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + x_{n-1} \quad +x_{2n-1} & = 5^{n-1}, \\
& 2^n x_1 + 2^{n-1}x_2 + \cdots + 2^2 x_{n-1} + x_n \quad +x_{2n} & = 5^n.
\end{array}$$

Suppose that we carry out our algorithm starting with an initial basis $x^0 = (0, \cdots, 0, 5, \cdots, 5^n)^T \in R^{2n}$ and an artificial dual feasible constraint

$$2^n x_1 + (2^{n-1} + \alpha_2)x_2 + \cdots + (2^2 + \alpha_{n-1})x_{n-1} + (2 + \alpha_n)x_n + x_{2n+1} = M, \tag{7.1}$$

where M is an enough big positive number and $\{\alpha_2, \frac{\alpha_3}{2}, \cdots, \frac{\alpha_n}{2^{n-2}}\}$ is a strictly monotone increasing positive real sequence. If we always remain the dual feasibility of this additional constraint, we will acquire the same search path as the Dantzig rule in the primal simplex method. After $2^k(3 \leq k \leq n)$ pivot steps, the additional dual feasible constraint becomes

$$(2+\frac{\alpha_k}{2^{n-k-1}})x_0 + \sum_{j=2}^{k-1}\alpha_j x_j + \sum_{j=k+1}^{n}(\alpha_j - \frac{\alpha_k}{2^{j-k-1}})x_j + \alpha_k x_{n+k} + x_{2n+1} = M_{2^k-1}$$

and the nonbasic variable x_k is pivoted in place of the basic variable x_{n+k}, where M_{2^k-1} is a suitable positive number. An optimum solution of this linear programming relation is $x_n = 5^n$ and $x_j = 0$ for all $j \in \{1, 2, \cdots, 2n\} - \{n\}$ after 2^n pivot steps.

The number of the iterations depends on the choice of algebraic paths. If the search path is decide by the final constraint, the optimal solution of the Klee-Schrijver example can be obtained after a pivoting step. We propose the following problem:

For a given LP, is there an (nondegeneracy) algebraic path for the current equivalent facet method which terminates in a number of pivots bounded by a polynomial in m and n?

If c lies in the negative orthant in R^n for the problem (2.1), a possible choice of algebraic paths seems to be

$$(A^T w^*)^T x + x_{n+1} = (w^*)^T b + 1,$$

where w^* is an optimal solution of (2.2). Perhaps the lexicographic rule must be used in order to avoid (dual) degeneracy.

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