Objective space for multiple objectives linear fractional programming with equal denominators

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Abstract:
In this paper we give the construction of the objective space of multiple objectives linear fractional programming (MOLFP) with equal denominators under the linear fractional mapping .In this case the decision space maps to an objective space of less dimension. The important of this study is that the decision-Maker may depend on extreme points of the set of the objective space than that of the decision space since they have fewer extreme points.

Keywords: multiple objective linear fractional programming, efficient solution, decision space, objective space, non-dominated solution.

1-Introduction:

Multiple objectives linear fractional programming (MOLFP) arises when several linear fractional objectives (i.e. ratio objectives that have linear numerator and denominator) are to be maximized over a convex constraints polytope X . This kind of mathematical programming problems have attracted considerable research and interest , since they are useful in production planning , financial planning and corporate planning , health care ,hospital planning. How ever for a single objective linear fractional programming the Charnes and Cooper transformation can be used to transform the problem into a linear programming problem .Few approaches have been reported for solving the multiple objective linear fractional programming (MOLFP) problems .Kormbluth and Steuer considered this problem and presented a simplex –based solution procedure to find all weakly efficient vertices of the augmented feasible region. Also Benson in his article showed that the procedure suggested by Kormbluth and Steuer for computing the numbers to find break points may not work all the time and he proposed afail safe method for computing these numbers. Geoffrion in his article introduced the notion of proper efficiency for(MOLFP) and Choo later on proved that every efficient solution for (MOLFP) is proper efficient In this paper we give a construction of the objective space of multiple objectives linear fractional programming (MOLFP) with equal denominators under the linear fractional mapping, our method is the extension of objective space in multiple objective linear programming (MOLP) problems suggested by Dauer and also by Dauer and Saleh .Then in our case the decision space maps to an objective space of less dimension. The important of this study is that the decision-Maker may depend on extreme points of the set of objective space than that of the decision space since they have fewer extreme points.

2-Definietions and Notations:

Multiple objectives linear fractional programming arises when several linear fractional objectives (i.e. ratio objectives that have linear numerator and denominator) are to be maximized over a convex constraints polytope X, this problem can be formulated as

$$\text{Maximize } Z(x) = (z_1(x), z_2(x)\ldots z_k(x))$$

Subject to

$$x \in X = \{x \in R^n, Ax \leq b\} \quad (2.1)$$

Where $z_i(x) = \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i}$, $i = 1, 2....k$

Here c_i, d_i are vectors in R^n, α_i and β_i are scalar, A is an (m +n) x n matrix and $b \in R^{m+n}$, we point out that the non negativity condition is added to the set of constraints and we also assume that X is compact set and $d_i^T x + \beta_i > 0$, $i = 1, 2...k$ for every $x \in X$. Solving multiple objectives linear fractional programming seeks for the set of solutions called the efficient set every member of this set has following definition [1, 3]

Definition 2-1:

A solution x^0 is said to be efficient for multiple objective linear fractional programming (MOLFP) if $x^0 \in X$ and these is no $x \in X$ such that $\frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} \geq \frac{c_i^T x^0 + \alpha_i}{d_i^T x^0 + \beta_i}$, we also define $z^0 = Z(x^0)$ to be non-dominated if there is no $z = Z(x)$ such that $Z(x) \geq Z(x^0)$ and $Z(x) \neq Z(x^0)$.

A Characterization of an efficient solution of (MOLFP) has been made through the following lemma [1]

Lemma: let $x^0 \in X$, then x^0 is an efficient solution of (MOLFP) if and only if there exist $w_i > 0$ for $i = 1, 2... k$ and $u_i \geq 0$ for all $i \in I$ such that

$$\sum_{i=1}^{k} w_i \nabla z_i(x^0) = \sum_{i \varepsilon I} u_i A_i \quad (2.2)$$

Where I is the set of indices of the binding constrains at x^0.

We are interesting in the construction of the objective space of the above (MOLFP) when the denominators are all equal; in this case the (MOLFP) takes the form:

Maximize $Z(x) = (z_1(x), z_2(x)..., z_k(x))$

Subject to

$$x \in X = \{x \in R^n, Ax \leq b\} \quad (2.3)$$

Where $z_i(x) = \frac{c_i^T x + \alpha_i}{d^T x + \beta}-$, $i = 1, 2....k$

In (2.3) we shall assume that $\beta \neq 0$, then an equivalent form of (2.3) can be formulated as

Maximize $Z(x) = (c_i^t - \frac{\alpha_i}{\beta} d^t)\frac{x}{d^t x + \beta} + \frac{\alpha_i}{\beta}$, $i = 1, 2...k$

Subject to

$$(A + \frac{b}{\beta} d^t)\frac{x}{d^t x + \beta} \leq \frac{b}{\beta} \quad (2.4)$$

If we define $y = \frac{x}{d^t x + \beta} \geq 0$ then (2.4) can be written in the form

Maximize $Z(y) = (c_i^t - \frac{\alpha_i}{\beta} d^t) y + \frac{\alpha_i}{\beta}$, $i = 1, 2... k$

Subject to $(A + \frac{b}{\beta} d^t) y \leq \frac{b}{\beta}$ (2.5)

Simply (2.5) can be written in multiple objectives linear programming (MOLP)

Maximize $Z(y) = C\, y + \frac{\alpha_i}{\beta}$, $i = 1, 2 \ldots k$

Subject to

$Gy \le g$ (2.6)

Where C is k x n matrix whose rows are those represented by ($c_i^t - \frac{\alpha_i}{\beta} d^t$), the constraint matrix G is given by $G = (A + \frac{b}{\beta} d^t)$, and $g = \frac{b}{\beta}$

3-**Objective space of multiple objectives linear fractional programming (MOLFP) with equal denominators**

In this section we shall give the objective space representation of a special case of (MOLFP) when the denominators of the objective functions are all equals, hence due to the well known theorem for (MOLP) [6, 10] that a feasible point y is efficient for multiple objective linear program (2.6) if and only if there is a k vector $\lambda > 0$ (weights) such that y is an optimal solution for the linear program

$$\text{Maximize } \lambda^T C\, y \quad \text{Subject to } G\, y \le g \qquad (3.1)$$

Consider the dual of the linear program of (2.7) in the form

$$\text{Minimize } u^T g \quad \text{Subject to } u^T G = \lambda^T C,\ u \ge 0 \qquad (3.2)$$

On multiply the set of constraints of this dual problem by $T = (T_1 \mid T_2)$, where $T_1 = C^T (C\, C^T)^{-1}$, and the column of the matrix T_2 constitute the bases of $N(C) = \{v;\ C\, v = 0\}$.We have $u^T G\, T_1 = \lambda^T$, $u^T G T_2 = 0$ and $u \ge 0$. In the case when k=n, we have $GT_2 = 0$, and the dual of (2.8) takes the form

$$\text{Maximize } \lambda^T w \quad \text{Subject to } G\, T_1\, w \le b \qquad (3.3)$$

Where T_1 is the inverse matrix of the given matrix C. on the other hand if $AT_2 \ne 0$, an r x (m+n) matrix P of non-negative entries is defined such that $PAT_2 = 0$, this matrix will play an important role for the construction of the set of objective space Z to be in the form $Z = \{w \in R^k \mid PGT_1 w \le P\, g\}$ or simply can be written as $Z = \{w \in R^k \mid Q\, w \le q\}$,where $Q = PGT_1$ and $q = P\, g$ since in our case the dual of (2.8) will take the form

$$\text{Maximize } \lambda^T w \quad \text{Subject to } Q\, w \le q \qquad (3.4)$$

The above results guarantee the algebraic construction of the set of objective space given in [5, 6].Accor dining to the objective space point of view we have the following proposition

Proposition 3-1

A point w^0 is non dominated point of (3.4) if there exists a solution $u \ge 0$, $\lambda > 0$ satisfying $u^T \overline{Q} = \lambda^T$ where $\overline{Q}$ represents the set of active constraint at the given point y^0

Proof: straight forward.

Correlly: If Q'' represents the set of non active constraints at the given point w^0 then $u^T Q'' = 0$ has only the zero solution.

Remark 3-1: The matrix P of non-negative entries such that $PAT_2 = 0$, this matrix can be defined through the ordinary phase one of linear programming problem with objective function zero and with constraints takes the form $u^T GT_2 = 0$, $e^T u = 1$ and $u \geq 0$.

Remark 3-2: In our objective space analysis to find the non dominated points for (MOLFP) in objective space, we have to solve Maximize

$$z(w) = w + \frac{\alpha_i}{\beta} \qquad , i = 1, 2\ldots k$$

Subject to

$$Q\,w \leq q, \qquad \text{and } w \in R^k.$$

An illustrative example

Consider the following (MOLFP)

$$\text{Maximize} \quad z_1 = \frac{x_1 + x_3 + 2}{x_1 + 2}$$

$$\text{Maximize} \quad z_2 = \frac{-x_1 + 2x_2 + 4}{x_1 + 2}$$

Subject to $x_1 + x_2 + x_3 \leq 1$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

For this (MOLFP) $c_1^t = (1\ 0\ 1)$, $c_2^t = (-1\ 2\ 0)$, $d^t = (1\ 0\ 0)$, also $\alpha_1 = 2$, $\alpha_2 = 4$ and $\beta = 2$, then **we have**

$$GT_1 = \begin{pmatrix} 1 & -5/26 \\ 0 & 3/13 \\ 0 & -2/13 \\ -1 & 0 \end{pmatrix} \qquad . \text{ Since } P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence the representation of the above (MOLFP) in the objective space takes the form

$$\text{Maximize } z_1 = w_1 + 1$$

$$\text{Maximize } z_2 = w_2 + 2$$

Subject to:

$$\begin{pmatrix} 1/4 & 1/8 \\ 1/3 & -1/6 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \leq \begin{pmatrix} 1/8 \\ 1/6 \\ 0 \end{pmatrix}$$

Then our objective space constraint polytope in R^2 may be represented by

$$Z = \{(z_1, z_2);\ 2z_1 + z_2 \leq 5,\ 2z_1 - z_2 \leq 1,\ z_1 \geq 1\}$$

4- Conclusion

In this paper we give a representation of the constraint polytope in the objective space of multiple objectives linear fractional programming (MOLFP) with equal denominators under the linear fractional mapping, our method is the extension of objective space in multiple objective linear programming (MOLP) problems suggested by [5, 6] .In this case the decision space maps to an objective space of less dimension. The important of this study is that the decision-Maker may depend on extreme points of the set of the objective space than that of the decision space since they have fewer extreme points.

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