Locating a semi-obnoxious facility with repelling polygonal regions *

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Abstract

In this work, a semi-obnoxious facility must be located in the Euclidean plane to give service to a group of customers. Simultaneously, a set of populated areas, with shapes approximated via polygons, must be protected from the negative effects derived from that facility. The problem is formulated as a margin maximization model, following a strategy successfully used in Support Vector Machines. Necessary optimality conditions are studied and a finite dominating set of solutions is obtained, leading to a polynomial algorithm.

Keywords: Continuous Location, Computational Geometry, Semi-Obnoxious Facilities, Optimization.

1 Introduction

For the last years, the location of semi-desirable facilities has been a widely studied topic by the researchers in location theory (see [1, 2, 3, 7, 10, 11, 12, 15]). A facility is said to be semi-desirable (or semi-obnoxious) when it gives service to certain customers in the neighborhood but, on the other hand, is felt as obnoxious to its environment. For instance, hospitals, airports or train stations are examples of semi-obnoxious facilities, since they are useful and necessary for the community, but they are a source of negative effects, such as noise, and therefore, they are considered as NIMBY (*not in my backyard*) facilities.
In our problem, a semi-obnoxious facility must be located in the plane and there are two different groups of customers to be considered. On the one hand, there exists the group of *attracting* points, whose demand must be satisfied by the facility which must be therefore as close as possible to all of them. On the other hand, there exists a set of *repelling* regions,

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which represent populated areas (whose shapes will be approximated via convex polygons) to be protected from the noxious effects coming from the facility, and hence, they must be as far as possible from the facility.
In the next section, we introduce a formulation of this problem as a margin maximization model similar to Support Vector Machine methods in Machine Learning. In section 3, several structural properties are proved leading to a finite dominating set. This allows to obtain a polynomial solution method in section 4, which will be tested on several artificial databases in section 5.

2 The model

2.1 The basic aim

Consider G_+ and G_- two groups of objects in the Euclidean plane, where G_+ is a finite set of points $G_+ = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$, and G_- is a set of convex polygonal areas $G_- = \{S_1, \ldots, S_m\} \subset \mathbb{R}^2$ (with $n, m \geq 3$). The points of G_+ represent individual customers to be serviced by the facility, while the polygons represent areas to be protected from the inconveniences of the semi-obnoxious facility to be located. The points of G_+ are assumed not to be contained in any element of G_-. Also the polygons in G_- are assumed to have pairwise disjoint interiors. Note that this is not a restriction because any (possibly disconnected) polygonal region can be decomposed into a finite set G_- which satisfies our assumptions.
Our aim is to locate a single semi-obnoxious facility, $x_0 \in \mathbb{R}^2$, which is as near as possible to the points of G_+ (*attracting* elements) in order to receive a high-quality service, and far from the polygons of G_- (*repelling* elements).
In this work, the location of the facility will be done through the construction of a ball $B(x_0, r)$, with $x_0 \in \mathbb{R}^2$ and $r \in \mathbb{R}_+$, such that every point of G_+ is strictly contained in the ball and every polygon of G_- lies outside the ball.
In Figure 1, an example of the problem is depicted. The black points represent the attracting points of G_+, whereas the grey-coloured areas represent the repelling elements of G_-. Our problem is to build a ball such that it contains all the points and it does not intersect the interior of any polygon.
Different solutions may exist separating the elements in G_+ and G_-. For instance, in Figure 1 two possible circles separating the two groups have been depicted. In order to single out one ball, we follow the strategy successfully used in Support Vector Machines, [4, 16], and maximize a margin as defined in next section. Following this strategy, the smallest circle in Figure 1 will be preferred as a solution.

2.2 The optimization problem

Given the elements of the two groups, G_+ and G_-, the following constraints must be satisfied, if possible,

$$d^2(x_0, x_i) < r^2, \ \forall x_i \in G_+ \quad \leftrightarrow \quad \|x_0 - x_i\|^2 < r^2, \ \forall x_i \in G_+, \tag{1}$$
$$d^2(x_0, S_j) \geq r^2, \ \forall S_j \in G_- \quad \leftrightarrow \quad \min_{x \in S_j} \|x_0 - x\|^2 \geq r^2, \ \forall S_j \in G_-, \tag{2}$$

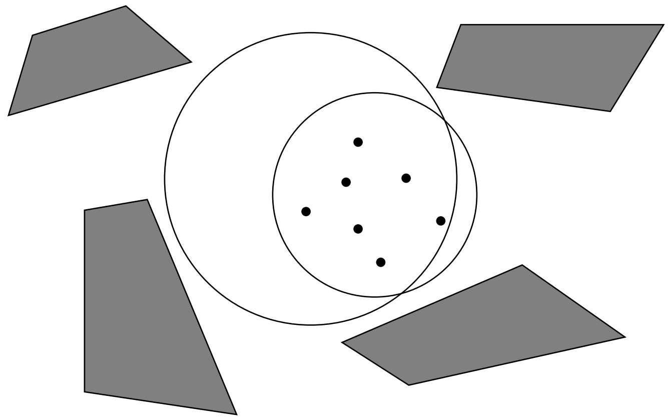

Figure 1: Two possible separating balls

where d and $\|\cdot\|$ are the Euclidean distance and norm, respectively.
Constraints (1)-(2) are equivalent respectively to

$$r^2 - \|x_0 - x_i\|^2 > 0,\ \forall x_i \in G_+ \quad \leftrightarrow \quad \min_{x_i \in G_+} (r^2 - \|x_0 - x_i\|^2) > 0, \tag{3}$$

$$\min_{x \in S_j} (\|x_0 - x\|^2 - r^2) \geq 0,\ \forall S_j \in G_- \quad \leftrightarrow \quad \min_{S_j \in G_-} \min_{x \in S_j} (\|x_0 - x\|^2 - r^2) \geq 0. \tag{4}$$

Following the strategy used in Support Vector Machines implies that we must maximize the minimum of the two positive amounts described in (3)-(4), that is, the optimization problem we want to solve is

$$\max_{x_0, r} \ \min \left\{ \min_{x_i \in G_+} (r^2 - \|x_0 - x_i\|^2),\ \min_{S_j \in G_-} \min_{x \in S_j} (\|x_0 - x\|^2 - r^2) \right\}. \tag{5}$$

Denote by Δ the margin, which is defined as the minimum between the two differences considered in Problem (5),

$$\Delta = \min \left\{ \min_{x_i \in G_+} (r^2 - \|x_0 - x_i\|^2),\ \min_{S_j \in G_-} \min_{x \in S_j} (\|x_0 - x\|^2 - r^2) \right\}. \tag{6}$$

Thus, our margin maximization problem can be written as

$$\begin{array}{ll} \max\limits_{x_0, r, \Delta} & \Delta \\ \text{s.t.} & \Delta \leq \min\limits_{x_i \in G_+} (r^2 - \|x_0 - x_i\|^2) \\ & \Delta \leq \min\limits_{S_j \in G_-} \min\limits_{x \in S_j} (\|x_0 - x\|^2 - r^2) \end{array} \tag{7}$$

or equivalently,

$$\begin{array}{ll} \max\limits_{x_0, r, \Delta} & \Delta \\ \text{s.t.} & \Delta \leq r^2 - \|x_0 - x_i\|^2,\ \forall x_i \in G_+ \\ & \Delta \leq \|x_0 - x\|^2 - r^2,\ \forall x \in S_j,\ \forall S_j \in G_-. \end{array} \tag{8}$$

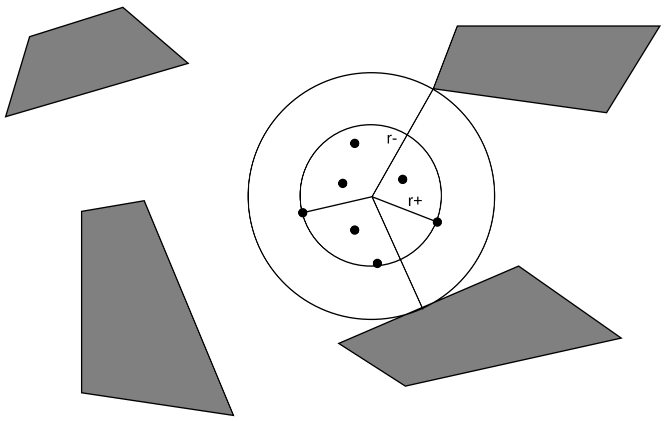

Figure 2: Construction of the two separating concentric balls with maximum margin

If we denote by $r_+^2 = r^2 - \Delta$ and $r_-^2 = r^2 + \Delta$, the objective function of Problem (8) changes into $\Delta = \dfrac{r_-^2 - r_+^2}{2}$, and the problem can be rewritten as

$$\begin{array}{ll} \max\limits_{x_0, r_+, r_-} & r_-^2 - r_+^2 \\ \text{s.t.} & \|x_0 - x_i\|^2 \leq r_+^2, \ \forall x_i \in G_+ \\ & \|x_0 - x\|^2 \geq r_-^2, \ \forall x \in S_j, \ \forall S_j \in G_- \\ & r_+, r_- \geq 0. \end{array} \tag{9}$$

In fact, this Problem (9) is more general since it also allows for situations with negative optimal values, which were unfeasible problems according to (7).
It will follow from Theorem 3.1 that Problem (9) can be reformulated by taking into account that, once the center x_0 is fixed, the optimal radii are fully defined and therefore, the objective depends on x_0 only, as follows

$$\begin{array}{ll} \max\limits_{x_0 \in \mathbb{R}^2} & f(x_0) \\ \text{s.t.} & f(x_0) = r_-^2(x_0) - r_+^2(x_0) \\ & r_+(x_0) = \max\limits_{x_i \in G_+} \|x_0 - x_i\| \\ & r_-(x_0) = \min\limits_{S_j \in G_-} \min\limits_{x \in S_j} \|x_0 - x\|. \end{array} \tag{10}$$

Therefore, our problem can be seen as that of obtaining two concentric balls $B(x_0, r_+)$, $B(x_0, r_-)$, where the ball $B(x_0, r_+)$ contains every point x_i belonging to G_+, the ball $B(x_0, r_-)$ does not contain strictly any points of the polygons of G_- and the difference between the squares of the radii is as large as possible, or geometrically, the area between the two circles is as large as possible. Figure 2 shows the graphical idea of the problem. Our problem is thus related with the so-called largest empty annulus problem [5], in which an annulus of maximal area not containing points in its interior is sought, although

in this problem, there is a unique set of points (instead of two groups) and no regions are considered.
In the following section, we derive some necessary optimality conditions. We will use (x_0, r_+, r_-) to either denote a finite feasible solution to Problem (9) or assume that $r_+ = r_+(x_0)$ and $r_- = r_-(x_0)$, as defined in Problem (10).

3 Necessary conditions for optimality

For deriving the necessary conditions for optimality of a feasible solution, the concept of active element will be necessary.
A point x_i from G_+ is an *active point* for the solution (x_0, r_+, r_-) iff the distance from x_i to the center x_0 is exactly r_+, that is, $d(x_0, x_i) = \|x_0 - x_i\| = r_+$. Thus, the set of active points of G_+, which is denoted by $A_+(x_0)$, is formed by the points lying on the boundary of the ball $B(x_0, r_+)$.
In the same way, a polygon S_j from G_- is an *active polygon* for (x_0, r_+, r_-) iff the distance from S_j to x_0 is exactly r_-, that is, $d(x_0, S_j) = \min_{x \in S_j} \|x_0 - x\| = r_-$. We denote by $A_-(x_0)$ the set of active polygons from G_-.
When x_0 is clear from the context, we will simply write A_+ and A_-.
In the proofs, the way to show that a feasible solution (x_0, r_+, r_-) is not optimal will be by finding another solution (x_0', r_+', r_-') with a better value of the objective function or by exhibiting a direction of increase of f at x_0.

Theorem 3.1. *If (x_0, r_+, r_-) is an optimal solution, there exists at least one active element in each group G_+ and G_-, that is, the sets A_+ and A_- of active elements are non-empty.*

Proof.
Suppose that A_+ is an empty set. Since (x_0, r_+, r_-) is a feasible solution of Problem (9), all the points of the group G_+ must be (due to the emptiness of A_+) contained strictly in the ball $B(x_0, r_+)$, that is, $\|x_0 - x_i\| < r_+$, $\forall x_i \in G_+$.
Then, it is sufficient to take

$$r_+' = r_+(x_0) = \max_{x_i \in G_+} \|x_0 - x_i\|,$$

which is strictly smaller than r_+, and we obtain (x_0, r_+', r_-), a feasible solution improving strictly the value of the objective function.
On the other hand, suppose that A_- is empty. Due to the feasibility of (x_0, r_+, r_-), the distance from x_0 to every polygon of G_- is strictly greater than r_-, that is, $d(x_0, S_j) > r_-$, $\forall S_j \in G_-$. Thus, it is sufficient to consider

$$r_-' = r_-(x_0) = \min_{S_j \in G_-} d(x_0, S_j) = \min_{S_j \in G_-} \min_{x \in S_j} \|x_0 - x\|,$$

strictly greater than r_-, and the solution (x_0, r_+, r_-') improves strictly the objective function.
In both cases, we conclude that the initial solution (x_0, r_+, r_-) cannot be optimal.

□

Theorem 3.2. *Let (x_0, r_+, r_-) be an optimal solution, one has that:*

1. *If $r_+ \leq r_-$, then there must exist at least two active polygons in G_-.*
2. *If $r_+ \geq r_-$, then there must exist at least two active points in G_+.*

Proof.
By Theorem 3.1, if (x_0, r_+, r_-) is an optimal solution, there must exist at least one active point a in G_+ and one active polygon S in G_-. Below, we obtain new conditions about the number of active elements in each case.

1. When $r_+ \leq r_-$, suppose there is only one polygon S in the set A_-. Let y be the projection of x_0 on S, i.e., the point in S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$ and consider the direction $p = x_0 - y$. Our aim is to prove that this vector p represents a direction of improvement for the objective function.

   If we move x_0 an amount $\epsilon > 0$, small enough (for not finding any new active element), in the direction $u = \frac{p}{\|p\|}$, we obtain that $x'_0 = x_0 + \epsilon u$ and $r'_- = r_- + \epsilon$.

   The other radius r'_+ must be measured as the maximum distance from x'_0 to the points belonging to $A_+(x_0)$.

   In case we obtain that $r'_+ \leq r_+$, because the new center is closer to all the points in $A_+(x_0)$, the radii r'_+ and r'_- will have decreased and increased respectively, and consequently the objective function will also have strictly improved.

   Otherwise, the radius r'_+ will be the distance from x'_0 to the point a of $A_+(x_0)$ which is now the furthest one (see Figure 3). Due to the triangle inequality on a, x_0 and x'_0, one has that $r'_+ \leq r_+ + \epsilon$, and the value of the objective function is strictly improved when $r_+ < r_-$, since

   $$\begin{aligned} r'^2_- - r'^2_+ &\geq (r_- + \epsilon)^2 - (r_+ + \epsilon)^2 \\ &= r^2_- - r^2_+ + 2\epsilon(r_- - r_+) > r^2_- - r^2_+. \end{aligned}$$

   In the case that $r_+ = r_-$, two cases can arise: either x'_0, a and y are not collinear, or $a = y$, but this last is contrary to our assumption that no point of G_+ belongs to an element of G_-. Therefore, by strict triangle inequality $r'_+ < r_+ + \epsilon$,

   $$r'^2_- - r'^2_+ > (r_- + \epsilon)^2 - (r_+ + \epsilon)^2 = r^2_- - r^2_+.$$

2. When $r_+ \geq r_-$, suppose there is only one active point a in A_+. Then, the vector $p = a - x_0$ will be proved to represent a direction of improvement.

   If x_0 is moved an amount $\epsilon > 0$, small enough for not having new active elements, in the direction $u = \frac{p}{\|p\|}$, we obtain that $r'_+ = r_+ - \epsilon$. The radius r'_- will be the minimum distance from $x'_0 = x_0 + \epsilon u$ to the polygons in $A_-(x_0)$. If we obtain that $r'_- \geq r_-$, because the new center is further from all the polygons candidates to become active, the two radii r'_+ and r'_- have improved and also the objective function. Otherwise, we denote by S one of the active polygons for the center x_0, which is now also the closest to the new center x'_0 (since there are not any new active

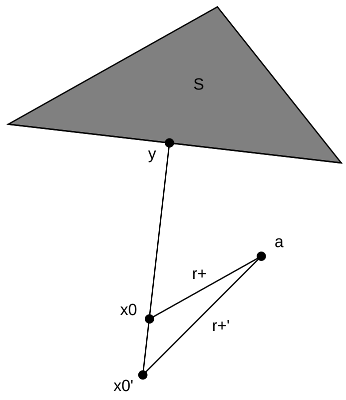

Figure 3: Proof of Theorem 3.2, case $r_+ \leq r_-$

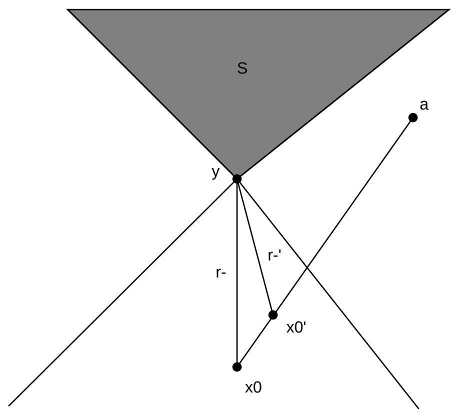

Figure 4: Proof of Theorem 3.2, case $r_+ \geq r_-$: The distance to the polygon is measured in the vertex

polygons in G_-) and by y the projection of x_0 on S, i.e., the point of S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$, and the objective function can be expressed as follows,

$$r_-^2 - r_+^2 \quad = \quad \|x_0 - y\|^2 - \|x_0 - a\|^2.$$

Three different situations must be considered.

- If y is a vertex of the polygon S and x_0 is strictly contained in the normal cone of S in y (denoted by $N_S(y)$), that is, x_0 satisfies that $(x_0 - y)^t(y - s) > 0$, $\forall s \in S$, then, for $\epsilon > 0$ small enough, x_0' will also be contained strictly in this normal cone, and $d(x_0', S) = \min_{x \in S} d(x_0', x) = d(x_0', y)$ (see Figure 4).
  In that case, due to the triangle inequality, one has that $r_- \leq r_-' + \epsilon$ and consequently, $r_-' \geq r_- - \epsilon$, and the value of the objective function is improved

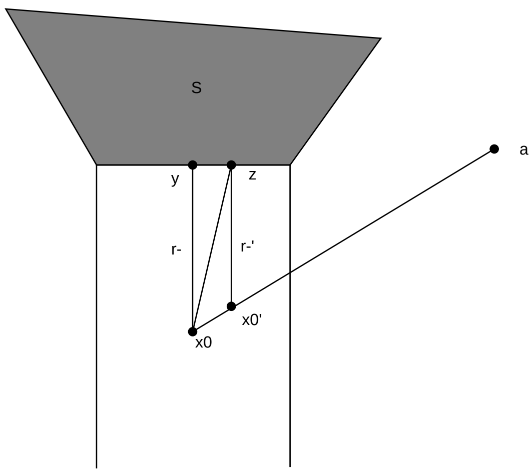

Figure 5: Proof of Theorem 3.2, case $r_+ \geq r_-$: The distance to the polygon is measured in the edge

in case $r_+ > r_-$, because

$$\begin{aligned} r'^2_- - r'^2_+ &\geq (r_- - \epsilon)^2 - (r_+ - \epsilon)^2 \\ &= r^2_- - r^2_+ + 2\epsilon(r_+ - r_-) > r^2_- - r^2_+ \end{aligned} \quad (11)$$

For $r_+ = r_-$, we know that $r_- < r'_- + \epsilon$, except for the case when x'_0, y and a are collinear. But this situation is not possible for $r_+ = r_-$, because it would mean that $a \in S$, which is not allowed by assumption. Therefore,

$$r'^2_- - r'^2_+ > (r_- - \epsilon)^2 - (r_+ - \epsilon)^2 = r^2_- - r^2_+ \quad (12)$$

- If the point y is on an edge of S, then, for an amount $\epsilon > 0$ small enough, to measure the distance from the new center x'_0 to the polygon S, we also have to find the point z (along the same edge of the polygon) which is the projection of x'_0 on S, i.e., the point satisfying $d(x'_0, S) = \min_{x\in S} d(x'_0, x) = d(x'_0, z)$ (see Figure 5).

  In that case, one has that

  $$r_- = \min_{x\in S} d(x_0, x) \leq d(x_0, z) \leq r'_- + \epsilon$$

  by using the definition of r_- and the triangle inequality on x_0, z and x'_0. Thus, we have that $r'_- \geq r_- - \epsilon$ and we can obtain again the same expression as in (11). Then, the objective function is improved for $r_+ > r_-$.

  And for $r_+ = r_-$, since $r'_- > r_- - \epsilon$ (except for the case in which x_0, y and a are collinear, and this situation cannot occur because it would mean that $a \in S$), we obtain again the expression (12), and the objective function is also improved.

- If y is a vertex of S and x_0 is on the boundary of the normal cone of S in y, then the projection of the new center x_0' on S will be either the same vertex y or a point z on an adjacent edge of S, depending on the position of a. Hence, one of the two arguments used previously applies to find a solution with a better value of the objective function.

□

Remark 3.1. *It can be proved that without the assumption that all points of G_+ lie outside the elements of G_-, Theorem 3.2 still holds in case of strict inequalities, but when $r_+ = r_-$, one may only conclude in the existence of an optimal solution with two active elements in G_+, and of an optimal solution (possibly different from the previous) with two active elements in G_-.*

Remark 3.2. *Note that only the first case in Theorem 3.2 is of interest to our original problem.*

Theorem 3.3. *If (x_0, r_+, r_-) is an optimal solution, then the intersection of the convex hulls of the two groups of active elements A_+ and A_- is a non-empty set.*

Proof.
Suppose $CH(A_+) \cap CH(A_-)$, the intersection of the convex hulls of the sets of active elements A_+ and A_-, is empty. In that case, a straight line h of equation $p^t x = c$ can be found which strictly separates these two convex hulls, where p is a vector in $\mathbb{R}^2$ of unit length and $c \in \mathbb{R}$, such that the halfplane containing $CH(A_+)$ is defined by $\{p^t x > c\}$. Consider the straight line $r : \{x = x_0 + \lambda p,\ \lambda \in \mathbb{R}\}$. We show now that the objective function will be improved by moving x_0 along this straight line a certain amount $\epsilon > 0$, small enough, which will terminate the proof.
Denote by S an active polygon from $A_-(x_0)$ which is the closest one to the new center $x_0' = x_0 + \epsilon p$, and by a a point from $A_+(x_0)$ which maximizes the distance from x_0' to $A_+(x_0)$. Denote by a_0 the orthogonal projection of a on r. Let y be the point of S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$ and y_0 its orthogonal projection to the straight line r. With this notation, the objective function can be expressed as follows,

$$\begin{aligned} r_-^2 - r_+^2 &= \|x_0 - y\|^2 - \|x_0 - a\|^2 \\ &= \|x_0 - y_0\|^2 + \|y_0 - y\|^2 - \|x_0 - a_0\|^2 - \|a_0 - a\|^2 \end{aligned}$$

If we move x_0 to x_0' along the straight line r, to measure the new radius r_-', three different situations must be analyzed.

- In case the point y is a vertex of the polygon and x_0 is strictly contained in the normal cone of S in y, then, for an amount $\epsilon > 0$ small enough, the new center x_0' will also be contained strictly in the normal cone, and the distance from x_0' to S will continue being the distance from x_0' to the vertex y, that is, $d(x_0', S) = \min_{x \in S} d(x_0', x) = d(x_0', y)$.

Then, since $p = \frac{a_0 - y_0}{\|a_0 - y_0\|}$ (observe that $a_0 \neq y_0$, because r is orthogonal to the separating hyperplane and hence, a_0 and y_0 are also separated by the straight line h), the following calculation shows that the objective function improves,

$$\begin{aligned}
r_-'^2 - r_+'^2 &= \|x_0 + \epsilon p - y_0\|^2 + \|y_0 - y\|^2 - \|x_0 + \epsilon p - a_0\|^2 - \|a_0 - a\|^2 \\
&= \|x_0 - y_0\|^2 + 2\epsilon(x_0 - y_0)^t p + \|y_0 - y\|^2 \\
&\quad - \|x_0 - a_0\|^2 - 2\epsilon(x_0 - a_0)^t p - \|a_0 - a\|^2 \\
&= r_-^2 - r_+^2 + 2\epsilon(a_0 - y_0)^t \frac{a_0 - y_0}{\|a_0 - y_0\|} \\
&= r_-^2 - r_+^2 + 2\epsilon\|a_0 - y_0\| > r_-^2 - r_+^2
\end{aligned}$$

- In case the point y is on an edge of the polygon S, then, for an amount $\epsilon > 0$ small enough, to measure the distance from the new center x_0' to S, we also have to move along the same edge of the polygon to find the point z such that $d(x_0', S) = \min_{x \in S} d(x_0', x) = d(x_0', z)$, the projection of x_0' on S (see Figure 6).

  Consider z_0 the orthogonal projection of z to the straight line r. Observe that $p = \frac{a_0 - z_0}{\|a_0 - z_0\|}$ ($a_0 \neq z_0$, because r is orthogonal to h and h separates a and z) and observe also that $x_0 - y$ and $z - y$ are orthogonal, because y is the projection of x_0 on the edge containing z. Therefore, by Pythagoras' Theorem, one has that

$$\begin{aligned}
\|x_0 - z_0\|^2 + \|z_0 - z\|^2 &= \|x_0 - z\|^2 = \|x_0 - y\|^2 + \|y - z\|^2 \\
&\geq \|x_0 - y\|^2 = \|x_0 - y_0\|^2 + \|y_0 - y\|^2
\end{aligned} \tag{13}$$

  The objective function remains as follows,

$$\begin{aligned}
r_-'^2 - r_+'^2 &= \|x_0 + \epsilon p - z_0\|^2 + \|z_0 - z\|^2 - \|x_0 + \epsilon p - a_0\|^2 - \|a_0 - a\|^2 \\
&= \|x_0 - z_0\|^2 + 2\epsilon(x_0 - z_0)^t p + \|z_0 - z\|^2 \\
&\quad - \|x_0 - a_0\|^2 - 2\epsilon(x_0 - a_0)^t p - \|a_0 - a\|^2
\end{aligned}$$

  And now, by using inequality (13), we obtain

$$\begin{aligned}
r_-'^2 - r_+'^2 &\geq \|x_0 - y_0\|^2 + \|y_0 - y\|^2 \\
&\quad - \|x_0 - a_0\|^2 - \|a_0 - a\|^2 + 2\epsilon(a_0 - z_0)^t p \\
&= r_-^2 - r_+^2 + 2\epsilon\|a_0 - z_0\| > r_-^2 - r_+^2
\end{aligned}$$

- In case y is a vertex of S and x_0 is on the boundary of the normal cone of S in y, then the projection of x_0' on S will be either the vertex y or a point z on an edge of S, and, depending on the position of a, one of the two previous arguments applies to find a solution which improves the objective function.

□

Remark 3.3. *For the following theorem, we need the additional assumption that the data must be in general position. This means the exceptional situations described below do NOT appear. The aim of introducing this assumption is to avoid situations in which the associated solution has a slightly different behaviour to the general one:*

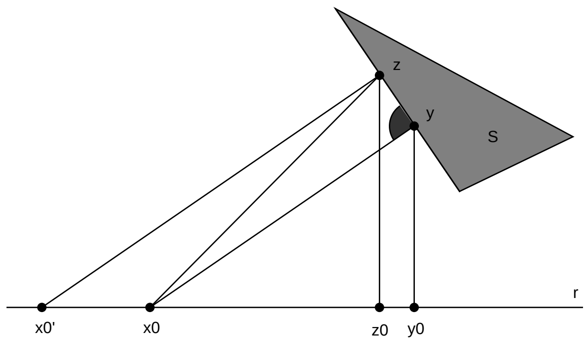

Figure 6: Proof of Theorem 3.3, second part

1. *One point of G_+ and two vertices of two polygons of G_- are collinear.*

2. *One point of G_+ and one vertex of a polygon of G_- define an orthogonal direction to an edge of a polygon of G_-.*

3. *Two points of G_+ and one vertex of a polygon of G_- are collinear.*

4. *Two points of G_+ define an orthogonal direction to an edge of a polygon of G_-.*

Likewise, the concept of bisector for two convex polygons and breakpoints will be necessary for the proof of Theorem 3.4 (see [6, 13] for a detailed description).

Definition 3.1. *The bisector of two convex polygons S_1 and S_2 is the locus of points $x \in \mathbb{R}^2$ satisfying that $d(x, S_1) = d(x, S_2)$. One has that this bisector is a continuous open curve consisting of linear segments and parabolic segments.*
The points at which two such segments meet will be called breakpoints (see Figure 7).

Theorem 3.4. *Under the assumption that the data are in general position, if (x_0, r_+, r_-) is an optimal solution, one of the following situations arises:*

1. *there exist at least four associated active elements ;*

2. *there exist at least three active elements, two polygons $S_1, S_2 \in A_-$ and one point $a \in A_+$, satisfying that y_1, a and y_2 are collinear, with y_i such that $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$;*

3. *there exist at least three active elements, two polygons $S_1, S_2 \in A_-$ and one point $a \in A_+$, and x_0 is a breakpoint.*

Proof.
In the case in which the two radii are equal, we obtain directly the result of having four active elements associated, by Theorem 3.2. Below, we consider the remaining two cases.

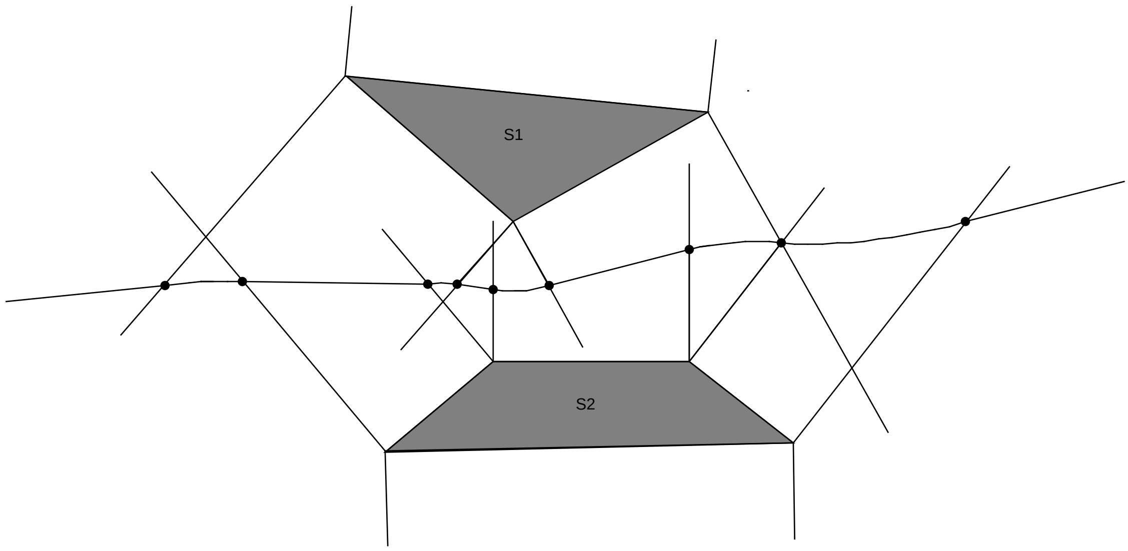

Figure 7: Bisector of two polygons S_1 and S_2. Breakpoints $\bullet$

1. When $r_+ < r_-$, by Theorems 3.1 and 3.2, we know that an optimal solution must have at least two distinct active polygons $S_1, S_2 \in A_-$, and one active point, $a \in A_+$. Suppose an optimal solution (x_0, r_+, r_-) has been obtained with only these three active elements.

   Since x_0 must be at the same distance from the two active polygons of A_-, it must be along their bisector which is composed of line segments and pieces of parabola. So x_0 is either a breakpoint or an 'inner point' of such a segment or piece.

   Then, in this last case, we must still show that a new better solution can be found, and the following different cases must be considered.

   - If there exist two vertices $y_1 \in S_1$ and $y_2 \in S_2$ which satisfy $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$, x_0 lies on the mediatrix r between the vertices y_1 and y_2 (see Figure 8).

     Suppose that the active point $a \in A_+$ is nearer to S_1 than to S_2 (the other case is analogous by symmetry).

     Define R the convex region determined by those points nearer to S_1 than to S_2 which are in the normal cone of S_1 at y_1, that is, $R = \{x : d(x, y_1) \leq d(x, y_2)\} \cap N_{S_1}(y_1)$. In this region, define the following function,

     $$g(x) = \|x - y_1\|^2 - \|x - a\|^2 = 2x^t(a - y_1) + C'. \tag{14}$$

     where $C' = \|y_1\|^2 - \|a\|^2$. One has that $f(x) = g(x)$, $\forall x \in R$, with f the objective function of Problem (10), in particular, $f(x_0) = g(x_0)$.

     In order to find a direction of improvement for the objective function in the neighbourhood of x_0, we study the directional derivatives of the objective function f at this point. Since the function g is differentiable in the region

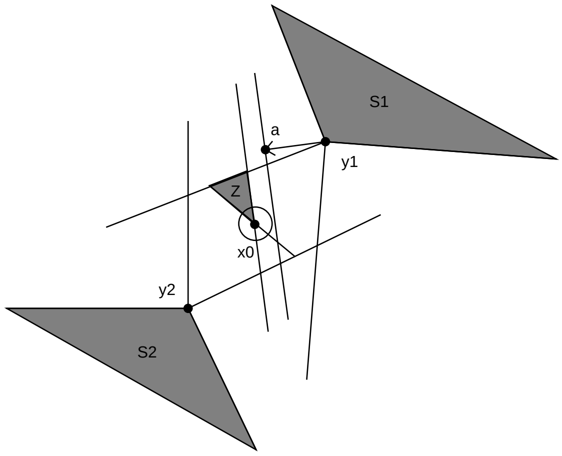

Figure 8: Distances from x_0 to the polygons are the distances to two vertices

R, and $f \equiv g$ in R, we obtain that the gradient of the function at x_0 is $\nabla g(x_0) = 2(a - y_1)$, and the directional derivative along a vector v is

$$\nabla_v f(x_0) = \nabla_v g(x_0) = \nabla g(x_0)^t \cdot v = 2(a - y_1)^t v, \ \forall v = y - x_0, \text{ with } y \in R \quad (15)$$

Hence, to obtain a direction of improvement, it is sufficient to choose a vector v such that the scalar product $(a - y_1)^t v$ is strictly larger than zero.

If we define the straight line orthogonal to the vector $(a - y_1)$ and containing the point x_0, that is, $r : (a - y_1)^t (x - x_0) = 0$, and if we consider the region Z determined by those points in R which are also in the positive halfplane defined by the straight line r, that is, $Z = R \cap \{x : (a - y_1)^t (x - x_0) > 0\}$, then the intersection $Z \cap B(x_0, \epsilon)$, with $\epsilon > 0$ small enough, is not empty, except for the case in which the straight line coincides with the mediatrix.

Then, we can find one point $z \in Z \cap B(x_0, \epsilon)$, and by moving the point x_0 in the direction $v = z - x_0$, the objective function is improved.

The case in which r coincides with the mediatrix is only possible if y_1, a, and y_2 are collinear, which is the exception number 1 in Remark 3.3. Anyway, in this exceptional case, if we move x_0 along the mediatrix, the value of the objective function remains constant (then, the solution is not unique).

- If there exist a vertex $y_1 \in S_1$ and a point y_2 lying on an edge of S_2 such that $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$, x_0 lies on a parabolic piece of the bisector, this parabola being the bisector between the vertex y_1 and the edge of S_2 (see Figure 9).

  Suppose that the active point $a \in A_+$ is nearer to S_1 than to S_2 (for the other situation, see the reasoning described for the following case, with y_1 and y_2 lying on the edges of the polygons).

  Define $R = N_{S_1}(y_1) \cap \{x : d(x, y_1) \le d(x, y_2)\}$ and the function g as in expression (14). One has that $f(x) = g(x)$, $\forall x \in R$, and hence, the expression

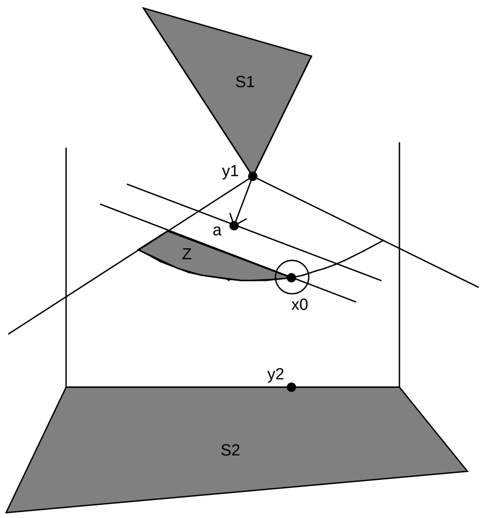

Figure 9: Distances from x_0 to the polygons are the distances to a vertex and to an edge

(15) for the directional derivative of f at x_0 along any vector $v = y - x_0$, with $y \in R$, remains valid.

Hence, to obtain a direction of improvement, it is sufficient to choose a vector v such that the scalar product $(a - y_1)^t v$ is strictly bigger than zero. And if we define $r : (a - y_1)^t(x - x_0) = 0$ (the straight line containing x_0 and orthogonal to $(a - y_1)$) and $Z = R \cap \{x : (a - y_1)^t(x - x_0) > 0\}$, the intersection $Z \cap B(x_0, \epsilon)$, with $\epsilon > 0$ small enough, is not empty, except for the case in which the straight line is tangent to the parabola. Then, a point $z \in Z \cap B(x_0, \epsilon)$ can be found, and by moving the point x_0 in the direction $v = z - x_0$, the objective function is improved.

The case in which r is tangent to the parabola is only possible when y_1, a, x_0 and y_2 are collinear, that is, when $a - y_1$ is orthogonal to the edge containing y_2, which is the exception number 2 of Remark 3.3 (in that case, a local optimum is found).

- If there exist two points y_1 and y_2, with y_i lying on an edge of S_i, such that $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$, x_0 lies on the bisectrix of the angle formed by the two edges, which represents the bisector in this case (see Figure 10).

  Suppose that the active point $a \in A_+$ is nearer to S_1 than to S_2 (by symmetry, the other case is analogous).

  Denote by a_0 and y_0 the orthogonal projections of a and y_i on the bisectrix. Then, the objective function can be written in x_0 as

$$\begin{aligned} r_-^2 - r_+^2 &= \|x_0 - y_1\|^2 - \|x_0 - a\|^2 \\ &= \|x_0 - y_0\|^2 + \|y_0 - y_1\|^2 - \|x_0 - a_0\|^2 - \|a_0 - a\|^2. \end{aligned}$$

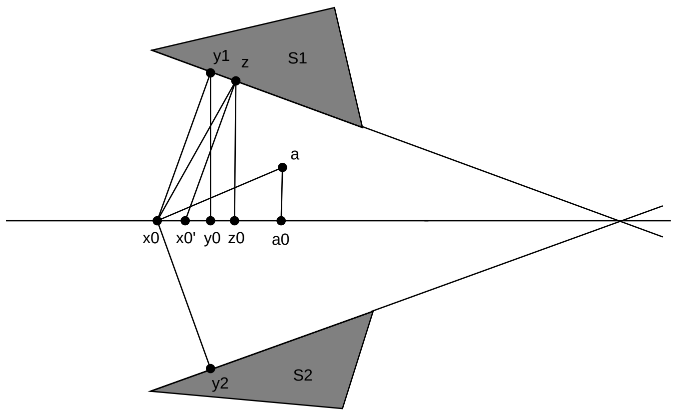

Figure 10: Distances from x_0 to the polygons are the distances to two edges

The vector $p = a_0 - y_0$, if non-zero, will be a direction of improvement of the objective function, for $\epsilon > 0$, small enough.

Let z be the orthogonal projection of the new center $x_0' = x_0 + \epsilon p$ on the edge, and z_0 its orthogonal projection on the bisectrix.

If we move x_0 along the direction p an amount $\epsilon > 0$, the new value of the objective function is

$$\begin{aligned} r_-'^2 - r_+'^2 &= \|x_0 + \epsilon p - z_0\|^2 + \|z_0 - z\|^2 - \|x_0 + \epsilon p - a_0\|^2 - \|a_0 - a\|^2 \\ &= \|x_0 - z_0\|^2 + 2\epsilon(x_0 - z_0)^t p + \|z_0 - z\|^2 \\ &\quad -\|x_0 - a_0\|^2 - 2\epsilon(x_0 - a_0)^t p - \|a_0 - a\|^2. \end{aligned}$$

By Pythagoras' Theorem, we obtain that

$$\begin{aligned} \|x_0 - z_0\|^2 + \|z_0 - z\|^2 &= \|x_0 - z\|^2 = \|x_0 - y_1\|^2 + \|y_1 - z\|^2 \\ &\geq \|x_0 - y_1\|^2 = \|x_0 - y_0\|^2 + \|y_0 - y_1\|^2 \end{aligned}$$

In fact, the inequality is strict, since $y_1 \neq z$. Then, one has that

$$\begin{aligned} r_-'^2 - r_+'^2 &= \|x_0 - z_0\|^2 + \|z_0 - z\|^2 \\ &\quad -\|x_0 - a_0\|^2 - \|a_0 - a\|^2 + 2\epsilon(a_0 - z_0)^t p \\ &> \|x_0 - y_0\|^2 + \|y_0 - y_1\|^2 \\ &\quad -\|x_0 - a_0\|^2 - \|a_0 - a\|^2 + 2\epsilon(a_0 - z_0)^t p \\ &= r_-^2 - r_+^2 + 2\epsilon(a_0 - z_0)^t(a_0 - y_0) \geq r_-^2 - r_+^2 \end{aligned}$$

and the objective function has improved, since the vectors $(a_0 - y_0)$ and $(a_0 - z_0)$ are parallel and in the same sense, for $\epsilon > 0$ small enough.

In case $a_0 = y_0$, the objective function cannot be improved, thus we have obtained a local optimal solution. The result is the situation 2 of Theorem 3.4, that is, three active elements (two polygons S_1, S_2 and one point a) with the points y_1, a and y_2 being collinear.

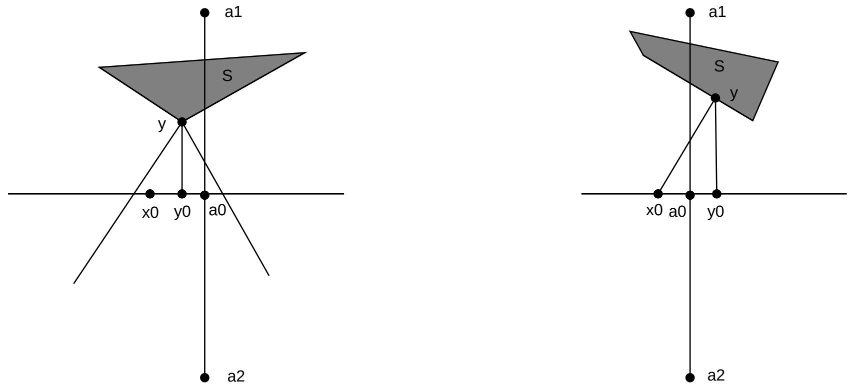

Figure 11: Two situations when $r_+ > r_-$

2. When $r_+ > r_-$, by Theorems 3.1 and 3.2, we know that an optimal solution of the problem must have at least two active points $a_1, a_2 \in A_+$, and one active polygon, $S \in A_-$. Suppose an optimal solution (x_0, r_+, r_-) with only these three active elements has been obtained. A new solution will be found with a better value of the objective function.

   Denote by r the mediatrix between the two active points of A_+, by y the point belonging to S such that $d(x_0, S) = \min_{x \in S} d(x_0, x) = d(x_0, y)$, and by a_0 and y_0 the orthogonal projections of the points a_1 and y on the straight line r. We have to consider two situations (y is a vertex of the polygon or y lies on an edge of the polygon, see Figure 11), which are exactly the same as those described in the proof of Theorem 3.3.

   With a similar reasoning, we derive that a new feasible solution can be obtained which improves the objective function. The exception in this case is when a_1, y, x_0 and a_2 are collinear. This can happen because there are two points a_1, a_2 and a vertex y of a polygon S which are collinear (exception 3 in Remark 3.3) or because there are two points a_1, a_2 defining an orthogonal direction to an edge of a polygon S (exception 4 in Remark 3.3).

□

The concepts of nearest and farthest-point Voronoi diagrams (see [13, 14]) for a set of points or polygons will be necessary for the proof of Theorem 3.5.

Definition 3.2. *Given the set of points $\{x_1, \ldots, x_n\}$ and the set of polygons $\{S_1, \ldots, S_m\}$, the farthest-point (resp. nearest-polygon) Voronoi cell associated to x_k (resp. S_l) denoted* *by V_k (resp. W_l) is defined as follows:*

$$V_k = \bigcap_{i\in\{1,\ldots,n\}\setminus\{k\}} \{x : d(x, x_k) \geq d(x, x_i)\}, \tag{16}$$

$$W_l = \bigcap_{j\in\{1,\ldots,m\}\setminus\{j\}} \{x : d(x, S_l) \leq d(x, S_j)\}. \tag{17}$$

The sets $V = \bigcup_{k=1,\ldots,n} V_k$ and $W = \bigcup_{l=1,\ldots,m} W_l$ are called the farthest-point and the nearest-polygon Voronoi diagrams.

Theorem 3.5. *If the convex hulls of the two groups G_+ and G_- are disjoint, that is, $CH(G_+) \cap CH(G_-) = \emptyset$, then the solution is unbounded and the separating balls are transformed into straight lines.*

Proof.
Since $CH(G_+) \cap CH(G_-) = \emptyset$, a straight line $h : \{p^t x = c\}$, with $p \in \mathbb{R}^2$ and $c \in \mathbb{R}$, separating the two convex hulls can be found, in the same way as done in the proof of Theorem 3.3. Let $l : \{p^t x = c'\}$ be another straight line, parallel to h, such that every point $x_k \in G_+$ satisfies that $p^t x_k > c$ and $p^t x_k < c'$.
Construct the farthest-point and nearest-polygon Voronoi diagrams in the plane for G_+ and G_-, respectively, and the intersection of the two diagrams. Let V be a cell obtained as the intersection of the resulting diagram with the halfplane $\{p^t x > c'\}$, such that there exists a point x_0 inside the cell satisfying that the semi-straight line $r : \{x = x_0 + \lambda p,\ \lambda \geq 0\}$ is completely included in the cell V.
Once x_0 is chosen, since it is inside a cell of the intersection of the two diagrams, the farthest point in G_+, say a, and the nearest polygon in G_-, say S, are known, that is, $a \in A_+$ and $S \in A_-$, and these two elements remain active for all the possible solutions in the cell, in particular for all the possible solutions in r. Then, with a similar reasoning to that done in the proof of Theorem 3.3, one has that if we move x_0 along r, for certain $\lambda' > 0$, the objective function increases linearly, thus, a new feasible solution $(x_0 + \lambda' p, r'_+, r'_-)$ with the same active elements can be found which is strictly better than the original one.
In fact, the larger the value of λ, the better the solution. Therefore, the solution is unbounded and, in that case, the concentric balls are transformed in two straight lines $\{p^t x = b\}$ and $\{p^t x = d\}$, with $b > d$, and such that the closed halfplane $\{p^t x \geq b\}$ contains $CH(G_+)$ whereas $\{p^t x \leq d\}$ contains $CH(G_-)$.

□

4 An algorithm to build the set of optimal solutions

With the necessary optimality conditions studied in the previous section, a finite dominating set of solutions has been obtained. A method to obtain the optimal solution is to perform a complete enumeration of all the candidate solutions, as will be described below. We are going to study all the local optimal solutions, and we will compute the value of the objective value for those points, and the one with the biggest value will be the global optimal solution. According to Theorems 3.1, 3.2 and 3.4, there must exist at least one active element in each set (A_+ and A_-), there must exist at least two active elements in the set associated to the biggest ball (that is, if $r_+ > r_-$, there will exist at least two active points in A_+, and if $r_- > r_+$, there will be at least two active polygons), and one of the situations described in Theorem 3.4 must be reached. That way, the finite dominating set of solutions will be formed by points x_0 whose configuration of associated active elements belongs to one of the following options:

1. three active polygons S_1, S_2, S_3 and one active point a (in this case, $r_- > r_+$);
2. two active polygons S_1, S_2 and two active points a_1, a_2 (no condition on the radii);
3. two active polygons S_1, S_2, one active point a and x_0 is a breakpoint of the bisector defined by S_1 and S_2 (in this case, $r_- > r_+$);
4. two active polygons S_1, S_2, one active point a and x_0 satisfies that y_1, y_2 and a are collinear, with y_i such that $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$ (in this case, $r_- > r_+$);
5. three active points a_1, a_2, a_3, and one active polygon S (in this case, $r_+ > r_-$).

In the algorithm, to describe all the candidates, we will consider all the possible configurations and we will compute the solution x_0 as the intersection of the corresponding bisectors of the sets A_+ and A_-. Since the bisector of two polygons consists of segments (for two vertices, the bisector is their mediatrix, and for two edges, the bisector is the bisectrix) and pieces of parabola (for one vertex and one edge), we will study each vertex and edge of a polygon as different active elements in the algorithm.

4.1 Case 1: card(A_+)=1 and card(A_-)=3

Let S_1, S_2 and S_3 be the three active polygons. As has been said before, every vertex and every edge of a polygon is studied as a possible active element. For a polygon S, considering a vertex v as the active element will mean that the closest point of S to the solution x_0 is v. Analogously, considering an edge e as the active element will mean that the point of S which is the closest one to x_0 lies on this edge e (not being one of the two vertices defining e).
Then, x_0 will be computed by following a different strategy depending on the number of active vertices and edges:

- Three vertices: x_0 is the circumcenter of the triangle defined by these three points (equivalently, x_0 is the intersection of the mediatrices for each pair of points).
- Two vertices and one edge: x_0 is the intersection of the mediatrix of the vertices and the parabola of one vertex and one edge.
- One vertex and two edges: x_0 is the intersection of the bisectrix of the two edges and the parabola of one vertex and one edge.
- Three edges: x_0 is the intersection of two bisectrices.

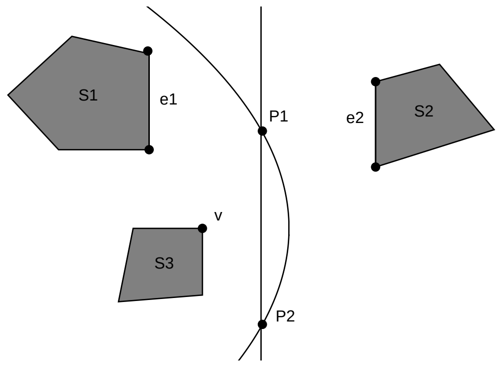

Figure 12: Computing a solution as the intersection of a bisectrix and a parabola

Once x_0 is computed (in some cases, more than one solution can be obtained), next step is to check if this solution is feasible, that is, given the three active elements, we must check if x_0 belongs to the intersection of the normal cones of the polygon S_i at the vertex v_i or the edge e_i respectively, for $i = 1, 2, 3$.

An example of this situation can be seen in Figures 12 and 13. In Figure 12, there are two active edges e_1 and e_2, and one active vertex v (belonging, respectively, to the active polygons S_1, S_2 and S_3). The bisectrix for the two edges is computed, and as well the parabola which represents the bisector of the edge e_2 and the vertex v. There exist two points (P_1 and P_2) as the result of intersecting the bisectrix and the parabola. In Figure 13, we check the feasibility of these two possible solutions, and P_1 is accepted as a solution, because it belongs to the intersection of the normal cones of the three active elements (the shadowed rectangle in the picture) whereas P_2 is outside that rectangle.

If we obtain a solution x_0 with this combination of active elements, we define r_- as the distance from x_0 to any of these active elements. Observe that we must also check that the distance from x_0 to these active polygons S_i, $i = 1, 2, 3$, coincides with the distance to the active vertices or edges which have been considered, that is, the closest points from the polygons to x_0 must be the selected active vertices or must lie on the selected active edges (otherwise, the solution is not feasible).

Afterwards, we compute the distance from x_0 to the rest of polygons of G_-. If the minimum of these distances is bigger than or equal to r_- (if this minimum was smaller, the polygons S_1, S_2 and S_3 could not belong to A_-), we compute r_+ as the maximum distance from x_0 to the points of G_+, and the point a whose distance to x_0 is r_+ will be the fourth active element.

In this case, r_+ must be smaller than r_- to have the guarantee of having obtained a local optimal solution (else, according to Theorem 3.2, a better solution can be found in a neighbourhood of x_0).

4.2 Case 2: card(A_+)=2 and card(A_-)=2

Let a_1 and a_2 be the active points. Let S_1 and S_2 be the active polygons (in this case, we choose directly from the beginning the four active elements). We compute the mediatrix of the two active points, and we compute the bisector of the two active elements in the

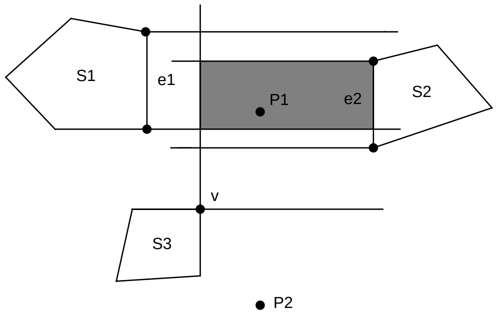

Figure 13: Checking the feasibility of the two points

polygons (it will be a mediatrix if we have two active vertices, a bisectrix if there are two active edges, or a parabola if there are a vertex and an edge). The intersection of the mediatrix and the bisector is computed and we check the feasibility of this solution as done in the previous case (that is, we check if the solution x_0 belongs to the intersection of the normal cones of the polygons at the corresponding vertex or edge, and we also check that each selected vertex or edge is really active, in the sense that it is or it contains the closest point from the corresponding polygon to x_0).
Once a solution x_0 is obtained, we compute r_+ as the distance from x_0 to one of the active points and r_- as the distance to one of the active polygons. Then, we compute the maximum distance from x_0 to the rest of points of G_+ (x_0 is a candidate if this maximum distance is smaller than or equal to r_+) and the minimum distance from x_0 to the rest of polygons of G_- (x_0 is candidate to optimal solution if this minimum distance is bigger than or equal to r_-).

4.3 Case 3: card(A_+)=2, card(A_-)=1 and x_0 is a breakpoint

Let S_1 and S_2 be the two active polygons. In this case, for the first polygon, we can always consider as active elements in the algorithm only the vertices, since a breakpoint is built as the intersection of the bisector of two polygons with the boundary of the normal cone of one of the polygons at some of its vertices (see Figure 7). Then, the option of having two active edges can be ruled out (otherwise, each breakpoint would be studied twice).
Given one active vertex of S_1 and one active element of S_2 (a vertex or an edge), we compute the corresponding bisector (a mediatrix or a parabola, respectively) and we compute the intersection of this bisector with the intersection of the boundaries of the normal cones of the polygons at their active elements. That way, we obtain one (or several) breakpoint and we compute r_- as the distance from x_0 to the active polygons (if the selected vertices or edges are really active elements for the corresponding polygons).
Then, we compute the distances from x_0 to the rest of the polygons, and the minimum of these distances must be bigger than or equal to r_- (otherwise, x_0 is not a candidate optimal solution). We compute r_+ as the maximum distance from x_0 to the points in G_+ (r_+ must be smaller than r_-, otherwise, we could find a better solution in a neighbourhood

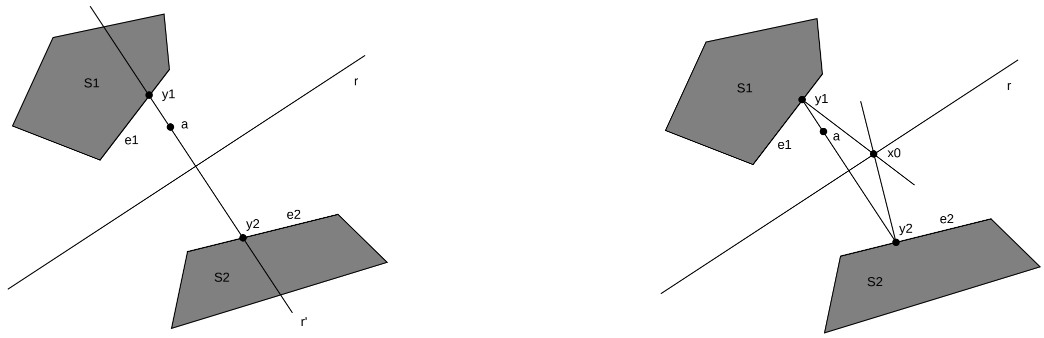

Figure 14: Case 5. Left: Constructing y_1 and y_2. Right: Constructing x_0

of x_0).

4.4 Case 4: card(A_+)=2, card(A_-)=1 and y_1, y_2 and a are collinear

Let a be the active point. Let S_1 and S_2 be the two active polygons. In this case, we only consider the edges of the polygons as possible active elements, since the condition we impose is that a, y_1 and y_2 are collinear, with y_i such that $d(x_0, S_i) = \min_{x \in S_i} d(x_0, x) = d(x_0, y_i)$, $i = 1, 2$, and y_i not being vertices. The case of y_i being vertices cannot happen with data in general position (see Remark 3.3).
Given a and the edges e_1 and e_2, we study if there can exist two points y_1 and y_2 lying on the edges, such that the condition of collinearity is satisfied. If this is possible, we compute the bisectrix r of the two edges and the orthogonal straight line r' to the bisectrix containing the point a (see Figure 14, left). Let y_1 and y_2 be the intersection of r' with e_1 and e_2, respectively. Then, x_0 will be built as the intersection of the bisectrix with the orthogonal straight line to e_1 containing y_1 (see Figure 14, right). Symmetrically, we can do the same with e_2.
Once x_0 is built, we follow the same reasoning to build the radii as in case 2.

4.5 Case 5: card(A_+)=3 and card(A_-)=1

Let a_1, a_2, a_3 be the three active points, we compute x_0 as their circumcenter (equivalently, x_0 is the intersection of the mediatrices between these points), and $r_+ = d(x_0, a_i)$, for any $i = 1, 2, 3$.
Now, we compute the distance from x_0 to the rest of points of G_+. If the maximum of these distances is smaller than or equal to r_+ (if it is bigger than r_+, the points a_i, $i = 1, 2, 3$, cannot be active), we compute r_- as the minimum distance from x_0 to the polygons of G_-. The polygon S whose distance to x_0 is equal to r_- will be the fourth active element.
Finally, r_+ must be bigger than r_- to assure that we have a local optimal solution (otherwise, a better solution can be found in a neighbourhood of x_0, according to Theorem

3.2). This implies that the value of the objective function for a candidate solution with this configuration of active elements will be negative. Hence, if we have already found a candidate solution with positive value, we do not need to compute any candidate of this type, since it cannot be a global optimum.

Let us study now the size of the set of candidate points obtained this way. Denote by n the number of points in G_+, by m the number of polygons in G_- and by k the number of vertices of each polygon (in case of having polygons with different number of vertices, k would be the maximum number of vertices for these polygons).

For the candidate solutions of type 1, we need to study all the possible combinations of three polygons (and all the possible combinations of vertices and edges, which are different active elements). This yields a set of $\mathcal{O}(k^3m^3)$ points.

For the candidates of type 2, we need to select two polygons (and every possible combination of active elements, vertices and edges, of these two polygons) and two vertices. We have then $\mathcal{O}(n^2k^2m^2)$ points.

Two polygons are needed to build each candidate of type 3. We have $\mathcal{O}(k^2m^2)$ such points.

For the candidates of type 4, we need to consider all possible combinations of edges of two polygons and one point, yielding $\mathcal{O}(nk^2m^2)$. Finally, if we need to compute the candidates of type 5, we need to study all the combinations of three active points, and we have $\mathcal{O}(n^3)$ points.

The overall cardinality is then $\mathcal{O}((n + km)^3 + n^2k^2m^2)$.

5 Computational experience

The algorithm described in the previous section to compute the optimal solution of our problem via complete enumeration of all the possible candidates has been implemented by using Matlab 6.5 on a computer with Pentium IV CPU 3.06 GHz.

Different numerical tests have been performed with artificial databases, built at random.

5.1 Small dataset: Comparing areas for all the candidates

The first example is a small dataset (4 points and 4 squares) to show the different types of candidate solutions that one can have in a problem. We have generated 4 points for the group G_+ coming from a uniform distribution (in particular, we have taken the distribution $U(-5, 5)$), and other 4 points also coming from a uniform distribution, $U(-20, 20)$, as the center of the squares (all the squares with the same area) which are the polygons for the group G_-. Our aim is to locate a single semi-obnoxious facility in a point $x_0 \in \mathbb{R}^2$, or equivalently, to compute two concentric balls such that $B(x_0, r_+)$ contains all the points and $B(x_0, r_-)$ does not intersect any squares. Figure 15 (left) shows a picture of the artificial database.

All the candidate optimal solutions have been computed via the method described in Section 4, by taking into account all the possible combinations of active elements. Figure 15 (right) shows this set of candidate locations, represented via stars.

In Figure 16, we show the two candidates with a configuration of type 1 (according to the previous section), that is, there are three active polygons (squares) and one active point. In the picture, the active squares are the black ones, while the active point is inside a

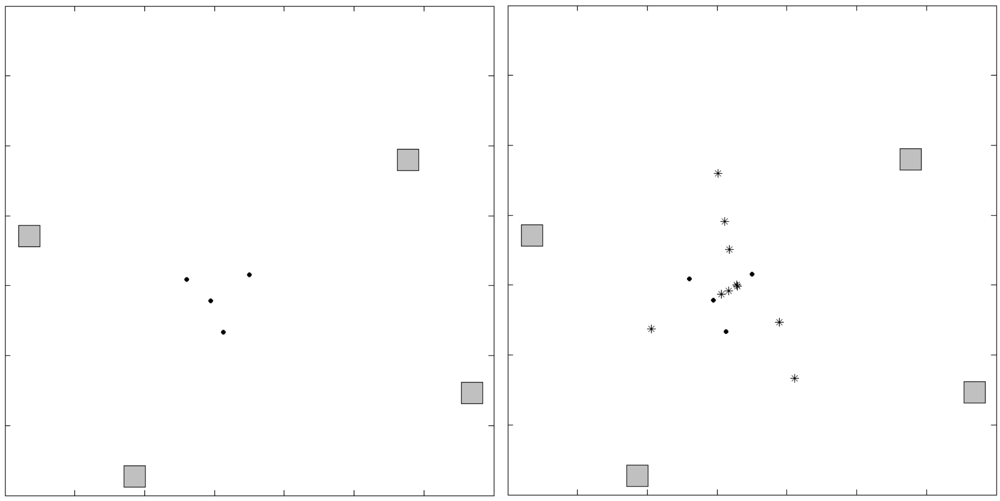

Figure 15: Left: Initial scenario. Rigth: Set of candidate optimal solutions

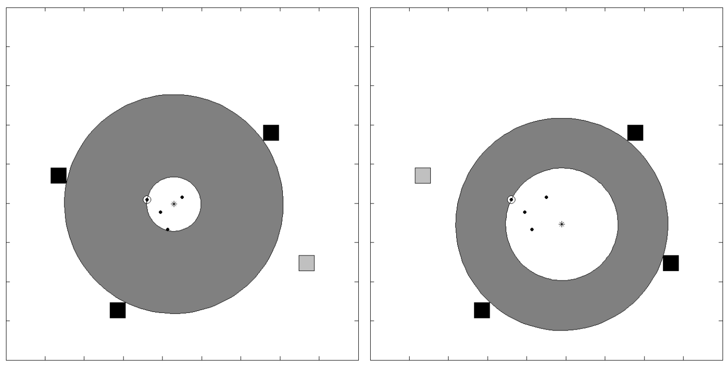

Figure 16: Candidates type 1. Area of the annulus: 183.27 and 132.32, respectively

small circle. These active elements (points and squares) lie on the boundary of the balls $B(x_0, r_+)$ and $B(x_0, r_-)$, respectively, where x_0 is represented via an star. Maximizing the objective function is equivalent to maximizing the area of the annulus defined by the boundaries of the two balls.

In Figure 17, the three candidate solutions have two active points and two squares. In Figure 18, we show five candidates with two active squares and one active element, and x_0, the location of the facility, is a breakpoint of the bisector defined by the two active squares. Observe that, although the active elements are the same for the three first pictures with this configuration, the solutions are different because the centers of the balls are different breakpoints of the same bisector. Due to the definition of breakpoint, one of the active squares in this kind of solutions has a vertex as the active element, but the adjacent edge touches tangently the ball $B(x_0, r_-)$. Hence, one can say that the two elements (the vertex and the edge) can be considered as active.

In this case, there are no candidate solutions with a configuration of type 4 or 5.

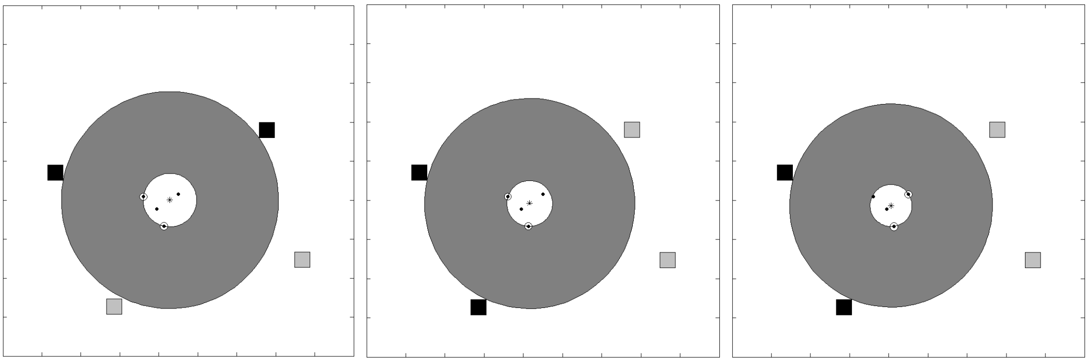

Figure 17: Candidates type 2. Area of the annulus: 182.32, 171.45 and 160.97, respectively

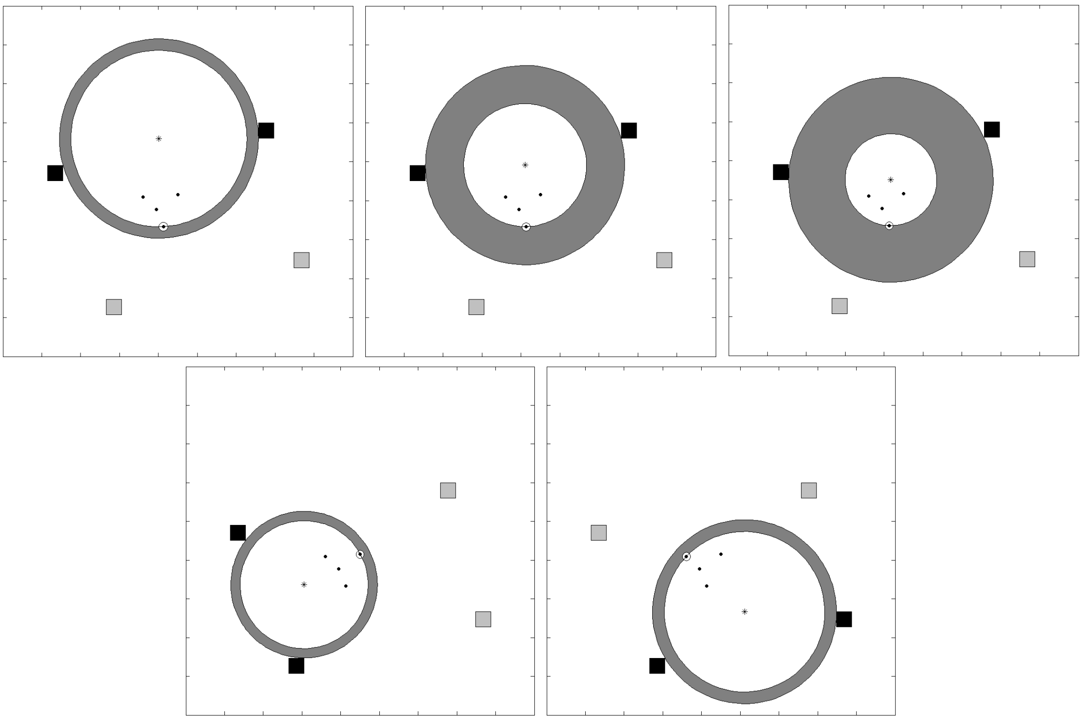

Figure 18: Candidates type 3. Area of the annulus: 35.648, 101.54, 138.16, 21.53 and 33.932, respectively

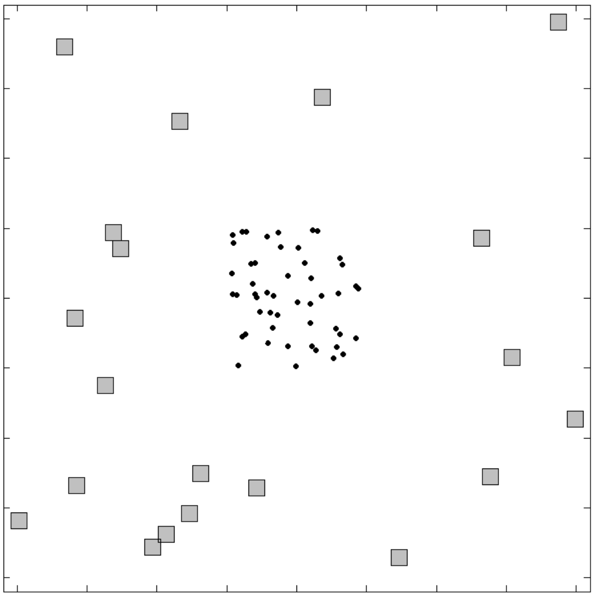

Figure 19: Initial scenario (50 points and 20 squares)

If we compare the ten areas (that is, the values of the objective function), we obtain that the first picture in Figure 16 is the optimal solution of our problem.

5.2 Other random datasets

Other larger databases have been generated to run the algorithm. In the next one, we have generated at random 50 points for the group G_+ and 20 points as the centers of the squares of G_- (the area for every square is the same), coming from two uniform distributions. Figure 19 shows a picture of the dataset.
By means of the method described in Section 4, all the candidate optimal solutions have been studied. Figure 20 shows two pictures, with different zoom levels, of all the candidate locations we have obtained, represented via stars. The stars which are far from the set of squares and points represent local optima with a negative value of the objective function. These solutions have at least two active points associated (configurations of type 2 and 5). In practice, if the dataset is spherically separable (in the sense that there exists a sphere separating the two sets of elements), the global optimum will not have a negative value of the objective function, but the formulation of our problem allows this kind of solutions as local optima.
Figure 21 shows the optimal solution for this dataset. The solution x_0, represented via an star, has two active points associated (those with a small circle around) and two active squares (in both of them, the point lying on the boundary of the ball is a vertex). The value of the objective function in this case (the area of the annulus) is 481.29.
Finally, we show a bigger random database, with 100 points generated via a uniform distribution and 50 squares (with a smaller area than in the previous cases). The initial scenario can be observed in Figure 22.

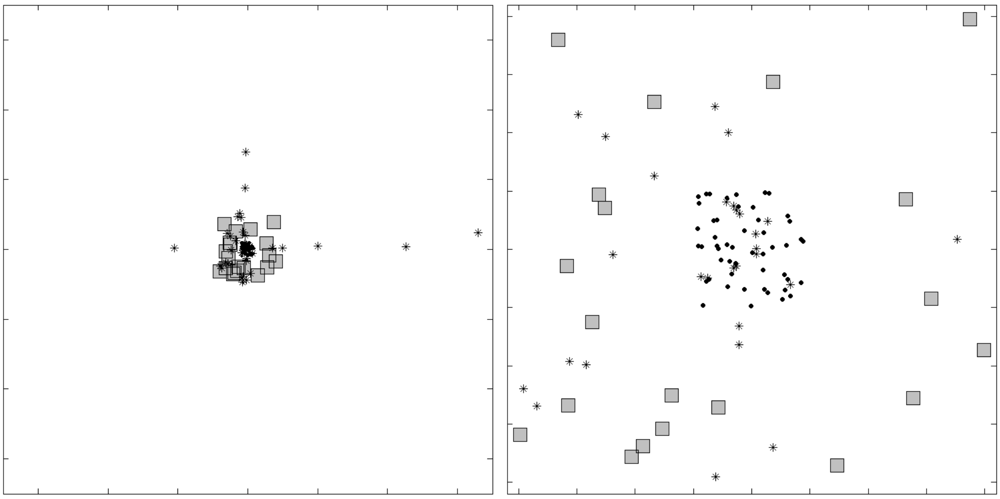

Figure 20: Candidates to optimal solution

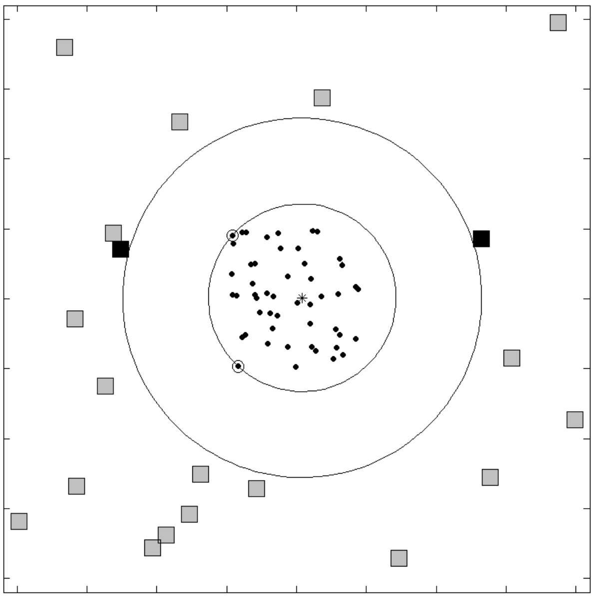

Figure 21: Optimal solution

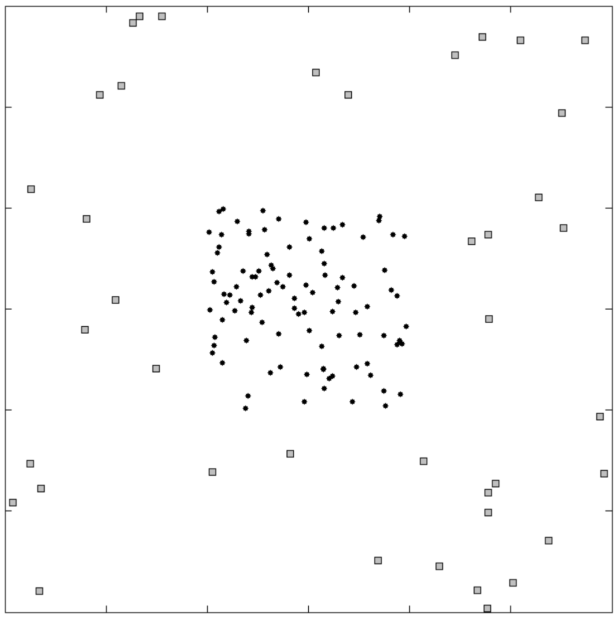

Figure 22: Initial scenario (100 points and 40 squares)

In Figure 23, one can observe the set of candidate optimal solutions (two different zoom levels). In this case, we have a lot of solutions with negative value of the objective function. However, the database is spherically separable, hence, we have some solutions with a positive value of the objective function. All these candidates with positive value are depicted in Figure 24. Finally, the optimal location of the facility is depicted in Figure 25. The two balls also appear in the picture. The solution has four active elements associated, two active points and two active polygons. The ball of radius r_- touches these two squares on one vertex of each square.

6 Conclusions and extensions

In this work, the location of a single semi-obnoxious facility in the Euclidean plane with repelling areas has been solved. The idea of maximizing a margin, as done in techniques coming from the field of Data Mining, such as Support Vector Machines, has been introduced to define the concept of solution.
The problem has been formulated via a nonlinear continuous optimization problem and necessary conditions for optimality have been deduced. These conditions state that every candidate solution must have at least four active elements (except for some especial cases), two of them belonging to the group whose associated ball is bigger and one of them belonging to the other group. Likewise, other conditions have been obtained by studying the intersection of the convex hulls of the sets of active elements and the sets of groups, respectively.
With these necessary conditions, it is proved that a finite dominating set of solutions can be built in order to obtain the optimal solution. This dominating set of solutions has been

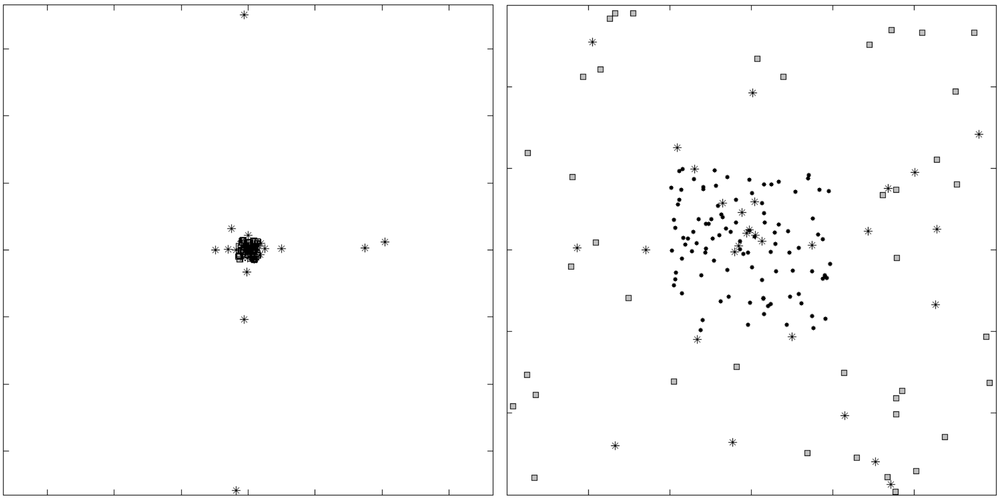

Figure 23: Candidates to be optimal solution

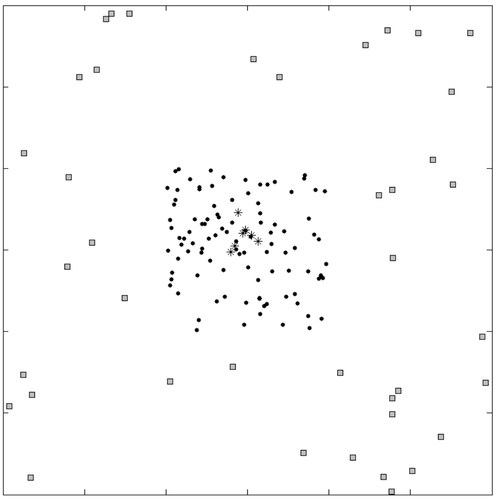

Figure 24: Candidates with positive value of the objective function

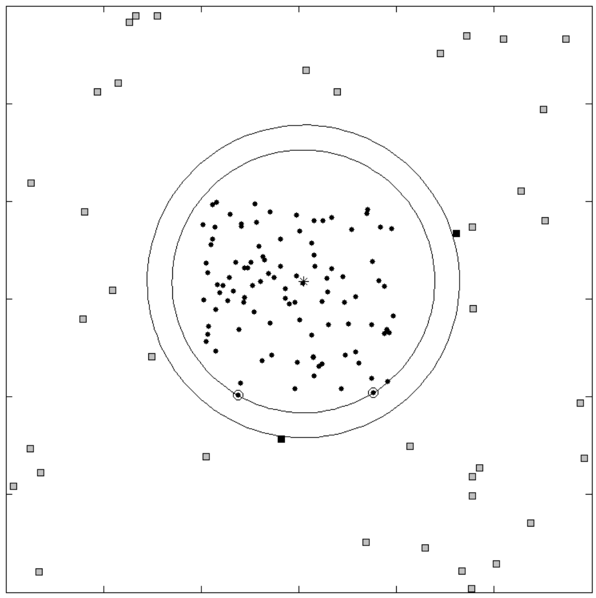

Figure 25: Optimal solution

constructed algorithmically, and this algorithm has been implemented and some numerical results have been given.
The concept of solution for this problem can be extended by considering other types of balls (such as ellipsoids, for example) and the problem can also be extended to higher dimensions.
For higher dimensions, heuristics techniques must be used to obtain a solution, and as well for large databases, if we want to decrease the CPU running time for obtaining a solution. A possibility, now under study, would be to use metaheuristics, such as VNS, [8, 9], for which a neighbourhood structure can be easily defined.

References

[1] R. Blanquero and E. Carrizosa. A D.C. Biobjective Location Model. *Journal of Global Optimization*, 23:139–154, (2002).

[2] E. Carrizosa and F. Plastria. Location of Semi-Obnoxious Facilities. *Studies in Locational Analysis*, 12:1–27, (1999).

[3] P. C. Chen, P. Hansen, B. Jaumard, and H. Tuy. Weber's Problem with Attraction and Repulsion. *Journal of Regional Science*, 32:467–486, (1992).

[4] N. Cristianini and J. Shawe-Taylor. *An Introduction to Support Vector Machines and other Kernel-based Learning Methods.* Cambridge University Press, Cambridge, (2000).

[5] J. M. Díaz-Báñez, F. Hurtado, H. Meijer, D. Rappaport, and T. Sellares. The Largest Empty Annulus Problem. *International Journal of Computational Geometry and Applications*, 13(4):317–325, (2003).

[6] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf. *Computational Geometry: Algorithms and Applications*. Springer-Verlag, Berlin, (1997).

[7] E. Erkut and S. Neuman. Analytical Models for Locating Undesirable Facilities. *European Journal of Operational Research*, 40:275–291, (1989).

[8] P. Hansen and N. Mladenovic. Variable Neighborhood Search: Principles and Applications. *European Journal of Operational Research*, 130:449–467, (2001).

[9] N. Mladenovic and P. Hansen. Variable Neighborhood Search. *Computers and Operations Research*, 24:1097–1100, (1997).

[10] S. Nickel and E. M. Dudenhoffer. Weber's Problem with Attraction and Repulsion under Polyhedral Gauges. *Journal of Global Optimization*, 11:409–432, (1997).

[11] Y. Ohsawa. Bicriteria Euclidean Location Associated with Maximin and Minimax Criteria. *Naval Research Logistics*, 47:581–592, (2000).

[12] Y. Ohsawa, F. Plastria, and K. Tamura. Euclidean Push-Pull Partial Covering Problems. *Computers and Operations Research*, 33:3566–3582, (2006).

[13] A. Okabe, B. Boots, and K. Sugihara. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. John Wiley and Sons, Chichester, (1992).

[14] M. I. Shamos and D. Hoey. Closest-Point Problems. In *Proc. of the 16th Annual IIEE Symposium on Foundations of Computer Science*, pages 151–162, (1975).

[15] H. Tuy, F. Al-Khayal, and F. Zhou. A D.C. Optimization Method for Single Facility Location Problems. *Journal of Global Optimization*, 7:209–227, (1995).

[16] V. N. Vapnik. *Statistical Learning Theory*. John Wiley and Sons, New York, (1998).