The Value of Information in the Newsvendor Problem

Gokhan Metan* Aurélie Thiele†

May 2007

Abstract

In this work, we investigate the value of information when the decision-maker knows whether a perishable product will be in high, moderate or low demand before placing his order. We derive optimality conditions for the probability of the baseline scenario under symmetric distributions and analyze the impact of the cost parameters on simulation experiments. Our results provide managers with deeper insights into the information that will help them reach better decisions.

1 Introduction

We consider the problem of managing the inventory of a perishable product subject to random demand, in the presence of both holding and backorder penalties; the goal is to minimize the total expected cost. This setting, known as the "newsvendor problem", has been investigated in the literature from two perspectives: (i) under the assumption that the demand distribution is known (see Porteus [3] for a review), and more recently, (ii) using distribution-free techniques (see Gallego and Moon [1], Bertsimas and Thiele [4]). In this short communication, we take a middle ground and analyze the impact on the optimal strategy of gaining advance information on the random variable.

*Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA 18015, gom204@lehigh.edu. Work supported in part by NSF grant DMI-0540143.

†P.C. Rossin Assistant Professor, Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA 18015, aurelie.thiele@lehigh.edu. Work supported in part by NSF grant DMI-0540143. Corresponding author.

Specifically, we assume that the manager learns before he places his order whether the product will be in high, moderate (baseline) or low demand. Demand is symmetric and the two extreme scenarios are defined so that they have the same probability; hence, our focus is on understanding what the decision-maker needs to know about the the middle scenario in order to take the most effective action. In other words, *for which likelihood of the baseline scenario is the expected cost minimized?* The approach we propose here differs from Perakis and Roels [2] in several major aspects, as they use a minimax regret criterion and assume no distributional information.

Notations. Throughout this note, we denote by interval 1 the region of moderate demand (i.e., the baseline demand region, which includes the mean), and interval 2, 3 the region of low and high demand, respectively. With $\overline{d}_0$ the mean of the demand, we define $\widehat{d}_0$ such that region 1 is the interval $[\overline{d}_0 - \widehat{d}_0, \overline{d}_0 + \widehat{d}_0]$. We also use the following notation:

- c : unit ordering cost,
- h : unit holding cost,
- b : unit backorder cost,
- X : the random demand,
- F : the cumulative distribution of the demand before information is revealed,
- f : the probability density function of the demand before information is revealed,
- π_j : probability that the random demand falls within interval j, $j = 1, 2, 3$,
- S_j : basestock level for demand values that fall in interval j, $j = 1, 2, 3$.

Initial inventory is zero. Finally, we assume that the unit backorder cost exceeds the unit ordering cost, as is common in practice as well as in the literature.

2 Optimal strategy

The structure of the newsvendor's strategy requires us to consider two related problems: (i) how much to order in each region? (ii) how to define each region? While (i) can be addressed following an approach similar to the classical newsboy setting, the novelty of the analysis lies in (ii). The total expected cost is formulated as:

$$\begin{aligned}
E[C(S_1,S_2,S_3)] &= c[\pi_1 S_1+\pi_2 S_2+\pi_2 S_3]+\pi_2\left(h\int_{\bar{d_0}+\hat{d_0}}^{S_2}(S_2-X)\frac{f(x)}{\bar{F}(\bar{d_0}+\hat{d_0})}dx+b\int_{S_2}^{\infty}(X-S_2)\frac{f(x)}{\bar{F}(\bar{d_0}+\hat{d_0})}dx\right)\\
&\quad+\pi_1\left(h\int_{\bar{d_0}-\hat{d_0}}^{S_1}(S_1-X)\frac{f(x)}{F(\bar{d_0}+\hat{d_0})-F(\bar{d_0}-\hat{d_0})}dx+b\int_{S_1}^{\bar{d_0}+\hat{d_0}}(X-S_1)\frac{f(x)}{F(\bar{d_0}+\hat{d_0})-F(\bar{d_0}-\hat{d_0})}dx\right)\\
&\quad+\pi_2\left(h\int_0^{S_3}(S_3-X)\frac{f(x)}{F(\bar{d_0}-\hat{d_0})}dx+b\int_{S_3}^{\bar{d_0}-\hat{d_0}}(X-S_3)\frac{f(x)}{F(\bar{d_0}-\hat{d_0})}dx\right)\\
&= c[\pi_1 S_1+\pi_2 S_2+\pi_2 S_3]+h\left(\int_{\bar{d_0}+\hat{d_0}}^{S_2}F(x)dx-F(\bar{d_0}+\hat{d_0})\big(S_2-(\bar{d_0}+\hat{d_0})\big)\right)+b\left(\int_{S_2}^{\infty}\bar{F}(x)dx\right)\\
&\quad+h\left(\int_{\bar{d_0}-\hat{d_0}}^{S_1}F(x)dx-F(\bar{d_0}-\hat{d_0})\big(S_1-(\bar{d_0}-\hat{d_0})\big)\right)+b\left(F(\bar{d_0}+\hat{d_0})(\bar{d_0}+\hat{d_0}-S_1)-\int_{S_1}^{\bar{d_0}+\hat{d_0}}F(x)dx\right)\\
&\quad+h\left(\int_0^{S_3}F(x)dx\right)+b\left(F(\bar{d_0}-\hat{d_0})(\bar{d_0}-\hat{d_0}-S_3)-\int_{S_3}^{\bar{d_0}-\hat{d_0}}F(x)dx\right),
\end{aligned} \tag{1}$$

where we have used that $\pi_1 = F(\bar{d_0}+\hat{d_0}) - F(\bar{d_0}-\hat{d_0})$ and $\pi_2 = \pi_3 = \bar{F}(\bar{d_0}+\hat{d_0}) = F(\bar{d_0}-\hat{d_0})$ by definition of the demand subregions. Theorem 2.1 provides the optimal basestock levels.

Theorem 2.1 *The basestock levels that minimize Equation (1) are given by:*

$$\begin{aligned}
S_1^* &= F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right),\\
S_2^* &= F^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right),\\
S_3^* &= F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right).
\end{aligned} \tag{2}$$

Proof: We use the convexity of the objective function and set the partial derivatives to zero (recall that $\pi_1 + 2\,\pi_2 = 1$):

$$
\begin{aligned}
\text{With respect to } S_1\text{:} \quad c\pi_1 + (b+h)F(S_1) - hF(\bar{d}_0 - \hat{d}_0) - bF(\bar{d}_0 + \hat{d}_0) &= 0, \\
\text{With respect to } S_2\text{:} \quad c\pi_2 + (b+h)F(S_2) - hF(\bar{d}_0 + \hat{d}_0) - b &= 0, \\
\text{With respect to } S_3\text{:} \quad c\pi_2 + (h+b)F(S_3) - bF(\bar{d}_0 - \hat{d}_0) &= 0.
\end{aligned} \tag{3}
$$

□

Remark: When $\pi_1 \to 1$, $S_1^* \to F^{-1}\left(\frac{b-c}{b+h}\right)$, which is exactly the classical newsvendor solution.

Injecting S_1^*, S_2^* and S_3^* into Equation (1) yields the optimal expected cost as a function of π_1 only (see Equation (4)), and Theorem 2.2 provides the optimal value of the probability π_1 for the decision-maker to take most advantage of the advance information.

$$
\begin{aligned}
E[C^*(\pi_1)] = \; & c\left[\pi_1 F^{-1}\left(\frac{1}{2} + \frac{b-h-2c}{2(b+h)}\pi_1\right) + \frac{1-\pi_1}{2}F^{-1}\left(1 - \frac{(1-\pi_1)(h+c)}{2(b+h)}\right) + \frac{1-\pi_1}{2}F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)\right] \\
& + h\left[\int_{F^{-1}\left(\frac{1+\pi_1}{2}\right)}^{F^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)} F(x)dx - \left(\frac{1+\pi_1}{2}\right)\left(F^{-1}\left(1 - \frac{(1-\pi_1)(h+c)}{2(b+h)}\right) - F^{-1}\left(\frac{1+\pi_1}{2}\right)\right)\right] \\
& + b\left[\int_{F^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)}^{\infty} \bar{F}(x)dx\right] \\
& + h\left[\int_{F^{-1}\left(\frac{1-\pi_1}{2}\right)}^{F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)} F(x)dx - \left(\frac{1-\pi_1}{2}\right)\left(F^{-1}\left(\frac{1}{2} + \frac{b-h-2c}{2(b+h)}\pi_1\right) - F^{-1}\left(\frac{1-\pi_1}{2}\right)\right)\right] \\
& + b\left[\left(\frac{1+\pi_1}{2}\right)\left(F^{-1}\left(\frac{1+\pi_1}{2}\right) - F^{-1}\left(\frac{1}{2} + \frac{b-h-2c}{2(b+h)}\pi_1\right)\right) - \int_{F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)}^{F^{-1}\left(\frac{1+\pi_1}{2}\right)} F(x)dx\right] \\
& + h\left[\int_0^{F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)} F(x)dx\right] \\
& + b\left[\left(\frac{1-\pi_1}{2}\right)\left(F^{-1}\left(\frac{1-\pi_1}{2}\right) - F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)\right) - \int_{F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)}^{F^{-1}\left(\frac{1-\pi_1}{2}\right)} F(x)dx\right]
\end{aligned} \tag{4}
$$

Theorem 2.2 *The optimal π_1^* quantity satisfies the following equation:*

$$
\begin{aligned}
& \left[c + \frac{h-b}{2}\right]F^{-1}\left(\frac{1}{2} + \frac{b-h-2c}{2(b+h)}\pi_1\right) - \left[\frac{c+h}{2}\right]F^{-1}\left(1 - \frac{(1-\pi_1)(h+c)}{2(b+h)}\right) \\
& + \left[\frac{b-c}{2}\right]F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right) + \left[\frac{h+b}{2}\right]\left(F^{-1}\left(\frac{1+\pi_1}{2}\right) - F^{-1}\left(\frac{1-\pi_1}{2}\right)\right) = 0
\end{aligned} \tag{5}
$$

Proof: The proof follows from studying the derivative of the cost function with respect to π_1 and is provided in appendix. □.

Remarks:

- When the demand is Normally distributed, Equation (5) yields:

$$\left[c+\frac{h-b}{2}\right]\Phi^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)-\left[\frac{c+h}{2}\right]\Phi^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)$$
$$+\left[\frac{b-c}{2}\right]\Phi^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)+\left[\frac{h+b}{2}\right]\left(\Phi^{-1}\left(\frac{1+\pi_1}{2}\right)-\Phi^{-1}\left(\frac{1-\pi_1}{2}\right)\right) \quad = \quad 0 \qquad (6)$$

  As a result, the optimal π_1^* is independent of the parameters of the distribution, i.e., the mean and the standard deviation.

- When the demand is Uniformly distributed, it is easy to check that the optimal solution to Equation (5) is $\pi_1^* = 1/3$, i.e., it is optimal to have three regions of equal probability.

Figure 1 shows the behavior of the cost function (i.e., Equation (4)) as a function of π_1 for two parameter settings: $c = 1$, $h = 5$, $b = 10$ and $c = 10$, $h = 1$, $b = 10.2$. (The value of 10.2 was chosen for b to be close to c but not equal, in line with our assumptions.) We observe that poorly selected values of π_1 affect system performance significantly. Due to the convexity of the cost function, we have a unique π_1^* value that minimizes the objective function; here, the minimum is achieved at $\pi_1^* = 0.395$ and $\pi_1^* = 0.485$ for the first and second parameter settings, respectively. When $\pi_1 \to 1$, the expected cost reaches its maximum. Such a result was expected because this case corresponds to the classical newsvendor problem with no advance information.

To further investigate the impact of the cost parameters on the optimal probability π_1^*, we solve Equation (5) for various parameter settings using Matlab. Figures 2 through 4 show the optimal π_1 values for different problem instances. In Figure 2, we observe that π_1^* takes values within the range of $[0.39, 0.50]$. In addition, the ordering cost seems to be the most dominant parameter – compared to the holding cost – that influences the value of π_1. Increasing the value of c for $c > 3$ results in an increase of the optimal π_1^* value, which can also be interpreted as an expansion of region 1.

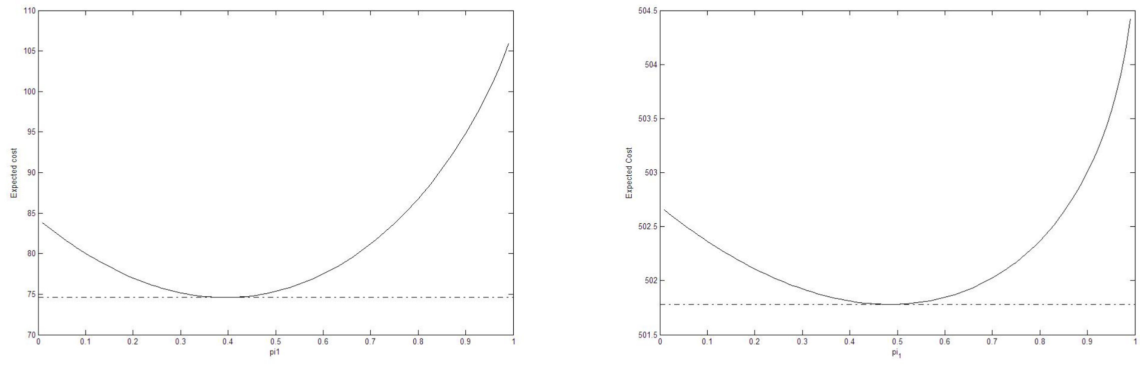

Figure 1: Expected cost with respect to π_1 for demand distribution Normal(50, 10^2): (i) $c = 1$, $h = 5$, $b = 10$ is given on the left; (ii) $c = 10$, $h = 1$, $b = 10.2$ is given on the right.

This increase continues until $c = b$, in which case every possible π_1 value satisfies Equation (5).

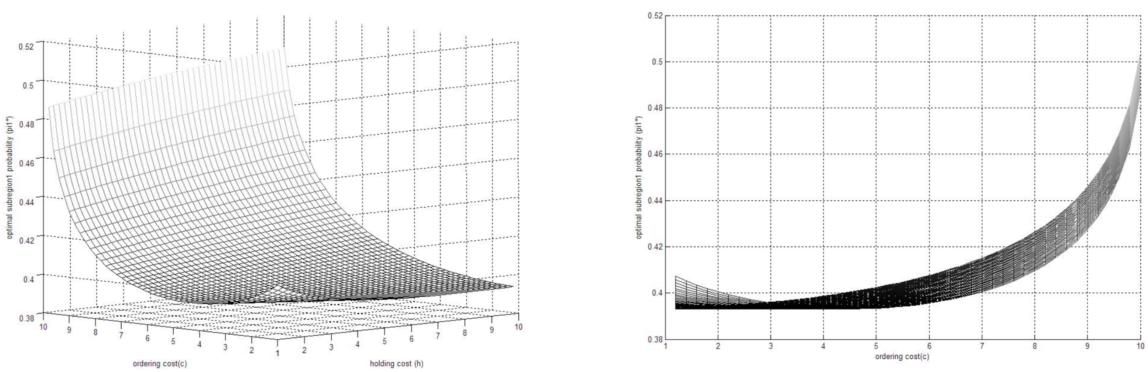

Figure 2: Optimal π_1^* with respect to ordering (c) and holding (h) costs for $b = 10.2$ and under the demand distribution Normal(50, 10^2). (Right panel shows all optimal π_1^* as a function of c.)

Next, we set the value of the ordering cost parameter to 1 and explore the connection between the optimal π_1^* and the cost parameters b and h within the range of $[1, 10]$ for each. Figure 3 depicts these results as a three-dimensional surface. We observe that the parameter b has a significant impact on the optimal π_1^* value. As in the previous case, the holding cost appears to only have a minor effect on determining the optimal probability for Region 1. Decreasing the value of b results in an increase in π_1^*, i.e., an expansion of Region 1. We already observe this same behavior of increasing π_1^* value when we increase the value of c up to the value of b (see Figure 2). Finally, we

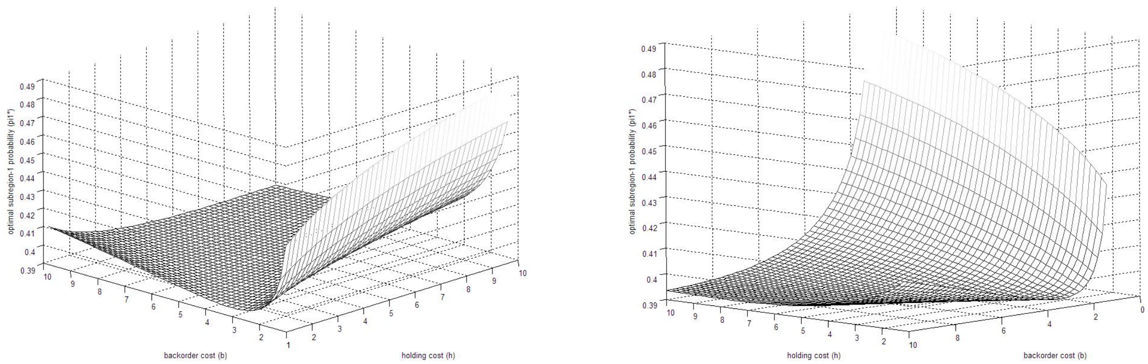

Figure 3: Optimal π_1^* with respect to holding (h) and backordering (b) costs for $c = 1$ and under the demand distribution Normal(50, 10^2). (Left panel shows 3D graph with backorder cost on the left horizontal axis and holding on the right. On the right panel order of costs is reversed.)

solve Equation 5 for various c and b parameter combinations, and obtain Figure 4. In particular, we consider a range of $[1, 10]$ for the values of both parameters under the constraint that $c < b$. The graph on the right panel of Figure 4 is the projection of the graph on the left on the $c + b = 10$ plane. We note again that the value of π_1^* increases as ordering and backordering costs converge (i.e., $c \to b$).

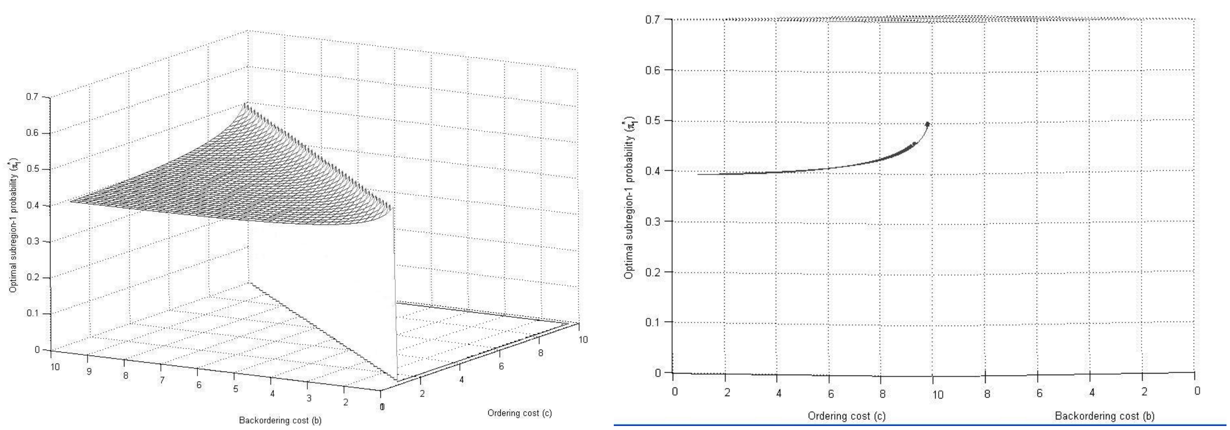

Figure 4: Optimal π_1^* with respect to ordering (c) and backordering (b) costs for $h = 5$ and under the demand distribution Normal(50, 10^2)

References

[1] Gallego, G., I. Moon. 1993. The distribution-free newsboy problem: review and extensions. *J. Oper. Res. Soc.* **44** 825-834.

[2] Perakis, G., G. Roels. The price of information: Inventory management with limited information about demand. Extended Abstracts of 2005 MSOM Student Paper Competition. *Manufacturing and Service Operations Management* **8**, 102-104.

[3] Porteus, E. 2002. *Foundations of Stochastic Inventory Theory.* Stanford Business Books, Palo Alto, CA.

[4] Bertsimas, D., A. Thiele. 2005. A data-driven approach to newsvendor problems. Technical report, Massachusetts Institute of Technology, Sloan School of Management, Cambridge, MA.

A Proof of Theorem 2.2

The result follows from:

$$\begin{aligned}
\frac{dE[C^*(\pi_1)]}{d\pi_1} = {} & c\Bigg[F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)+\pi_1\frac{dF^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)}{d\pi_1}-\frac{1}{2}F^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right) \\
& +\frac{1-\pi_1}{2}\frac{dF^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)}{d\pi_1}-\frac{1}{2}F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)+\frac{1-\pi_1}{2}\frac{dF^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)}{d\pi_1}\Bigg] \\
& +h\Bigg[\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)\frac{dF^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)}{d\pi_1}-\frac{1+\pi_1}{2}\frac{dF^{-1}\left(\frac{1+\pi_1}{2}\right)}{d\pi_1} \\
& -\frac{1}{2}\left(F^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)-F^{-1}\left(\frac{1+\pi_1}{2}\right)\right)-\frac{1+\pi_1}{2}\Bigg(\frac{dF^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)}{d\pi_1} \\
& -\frac{dF^{-1}\left(\frac{1+\pi_1}{2}\right)}{d\pi_1}\Bigg)\Bigg]-b\Bigg[\left(1-\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)\right)\frac{dF^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)}{d\pi_1}\Bigg] \\
& +h\Bigg[\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)\frac{dF^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)}{d\pi_1}-\frac{1-\pi_1}{2}\frac{dF^{-1}\left(\frac{1-\pi_1}{2}\right)}{d\pi_1}
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\left(F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)-F^{-1}\left(\frac{1-\pi_1}{2}\right)\right)-\frac{1-\pi_1}{2}\left(\frac{dF^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)}{d\pi_1}\right. \\
& \left.\left.-\frac{dF^{-1}\left(\frac{1-\pi_1}{2}\right)}{d\pi_1}\right)\right]+b\left[\frac{1}{2}\left(F^{-1}\left(\frac{1+\pi_1}{2}\right)-F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)\right)\right. \\
& +\left(\frac{1+\pi_1}{2}\right)\left(\frac{dF^{-1}\left(\frac{1+\pi_1}{2}\right)}{d\pi_1}-\frac{dF^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)}{d\pi_1}\right)-\left(\frac{1+\pi_1}{2}\right)\frac{dF^{-1}\left(\frac{1+\pi_1}{2}\right)}{d\pi_1} \\
& \left.+\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)\frac{dF^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)}{d\pi_1}\right]+h\left[\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)\frac{dF^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)}{d\pi_1}\right] \\
& +b\left[\left(-\frac{1}{2}\right)\left(F^{-1}\left(\frac{1-\pi_1}{2}\right)-F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)\right)+\left(\frac{1-\pi_1}{2}\right)\left(\frac{dF^{-1}\left(\frac{1-\pi_1}{2}\right)}{d\pi_1}\right.\right. \\
& \left.\left.-\frac{dF^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)}{d\pi_1}\right)-\left(\frac{1-\pi_1}{2}\right)\frac{dF^{-1}\left(\frac{1-\pi_1}{2}\right)}{d\pi_1}+\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)\frac{dF^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)}{d\pi_1}\right] \\
= \quad & \frac{dF^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)}{d\pi_1}(0)+\frac{dF^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)}{d\pi_1}(0)+\frac{dF^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)}{d\pi_1}(0) \\
& +\left[\frac{dF^{-1}\left(\frac{1+\pi_1}{2}\right)}{d\pi_1}+\frac{dF^{-1}\left(\frac{1-\pi_1}{2}\right)}{d\pi_1}\right](0)+\left[c+\frac{h-b}{2}\right]F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right) \\
& -\left[\frac{c+h}{2}\right]F^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right)+\left[\frac{b-c}{2}\right]F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right) \\
& +\left[\frac{h+b}{2}\right]\left(F^{-1}\left(\frac{1+\pi_1}{2}\right)-F^{-1}\left(\frac{1-\pi_1}{2}\right)\right) \\
= \quad & \left[c+\frac{h-b}{2}\right]F^{-1}\left(\frac{1}{2}+\frac{b-h-2c}{2(b+h)}\pi_1\right)-\left[\frac{c+h}{2}\right]F^{-1}\left(1-\frac{(1-\pi_1)(h+c)}{2(b+h)}\right) \\
& +\left[\frac{b-c}{2}\right]F^{-1}\left(\frac{(1-\pi_1)(b-c)}{2(b+h)}\right)+\left[\frac{h+b}{2}\right]\left(F^{-1}\left(\frac{1+\pi_1}{2}\right)-F^{-1}\left(\frac{1-\pi_1}{2}\right)\right).
\end{aligned}
$$