Some Relations Between Facets of Low- and High-Dimensional Group Problems

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Abstract

In this paper, we introduce an operation that creates families of facet-defining inequalities for high-dimensional infinite group problems using facet-defining inequalities of lower-dimensional group problems. We call this family *sequential-merge inequalities* because they are produced by applying two group cuts one after the other and because the resultant inequality depends on the order of the operation. The sequential-merge inequalities can be used to generate inequalities whose continuous variables coefficients are stronger than those of the constituent low-dimensional cuts. We also show that they can be used to derive the three-gradient facet-defining inequality introduced by Dey and Richard. We then introduce a general scheme for generating valid inequalities for lower-dimensional group problems using valid inequalities of higher-dimensional group problems. We present conditions under which this construction generates facet-defining inequalities when applied to sequential-merge inequalities. We show that this procedure generates a subfamily of the two-step MIR inequalities of Dash and Günlük.

1 Introduction

Branch-and-cut algorithms are the cornerstone of solution approaches for Integer Programming; see Marchand et al. [2002] and Johnson et al. [2000]. In the last decade, a vast amount of research has been directed towards generating strong general-purpose cutting planes that are easy to separate. One approach to generate strong cutting planes is to use the constraints of the problems one at a time. This approach has proven to be successful in many cases. However it seems that stronger cutting planes could be obtained if information obtained from multiple constraints of the problem were considered simultaneously.

In a series of papers, Gomory [1969], Gomory and Johnson [1972a,b], Gomory et al. [2003], Gomory and Johnson [2003] and Johnson [1974] showed how group relaxations could be used to generate cutting planes for general mixed integer programs. Although the group-theoretic approach can be applied to problems with m constraints (m-dimensional group problems), most research has considered only one-dimensional group relaxations; see Gomory and Johnson [1972a,b], Gomory et al. [2003], Gomory and Johnson [2003], Aráoz et al. [2003], Miller et al. [2006], Richard et al. [2006] and Dash and Günlük [2006a] for descriptions and derivations of large families of facet-defining inequalities for one-dimensional group problems. In particular, the Gomory mixed Integer cut (GMIC) has been empirically proven to be one of the most useful cutting plane for solving general mixed integer programs; see Bixby et al. [2000] and Bixby and Rothberg [2007].

Recent numerical studies using general one-dimensional group cuts by Fischetti and Saturni [2007] and by Dash and Günlük [2006b] suggest that the effectiveness of group cuts may be improved by using information from multiple tableau rows simultaneously and by improving the coefficients of continuous variables. Gomory and Johnson [2003] also observed that the advantage of using high-dimensional group cuts is not only that they use information from multiple constraints simultaneously, but they also represent continuous variables better. Therefore, discovering families of facet-defining inequalities for high-dimensional group problems is an important area of research towards deriving stronger cutting planes for general MIPs. There are, however, only a few papers that focus on the systematic study of group problems with multiple constraints. Johnson [1974] presents general theoretical results for group relaxations of Mixed Integer Programs with multiple constraints. Recently, Dey and Richard [2006] introduced tools to study two-dimensional infinite group problems and introduced two families of facet-defining inequalities for two-dimensional group relaxations. In particular, we note that only very few families of facet-defining inequalities are known for two-dimensional group problems and none are known for higher-dimensional group problems.

In this paper, we present a procedure that we call *sequential-merge*. This procedure generates facets of higher-dimensional group problems from facets of lower-dimensional group problems. To the best of our knowledge, this is the first known family of facet-defining inequalities for general high-dimensional infinite group problems. We also show in this paper how facets of high-dimensional group problems can be used to generate facets of low-dimensional group problems.

In Section 2 we present fundamental results and concepts about the group approach. We also describe a relationship between valid inequalities of group problems and certain types of lifting functions. In Section 3, we describe the sequential-merge procedure. We show that this procedure shares some relationship with the two-step MIR procedure of Dash and Günlük [2006a] and can be used to explain the family of three-gradient facets obtained by Dey and Richard [2006]. In Section 4 we prove that, under mild conditions, the sequential-merge procedure generates facets for high-dimensional infinite group problems. In Section 5 we analyze the types of inequalities that can be generated for low-dimensional group problems using high-dimensional sequential-merge group cuts. Surprisingly, even though the sequential-merge procedure generates facet-defining inequalities for high-dimensional group problems under very general conditions, the conditions under which it generates facets for low-dimensional group problems are more restrictive. We conclude in Section 6 with directions of future research.

An extended abstract of some of the results in Sections 2-4 specific to the case of two-dimensional group problems will be presented at IPCO XII meet to be held in Ithaca, June 2007.

2 Group Approach and Lifting

In this section, we present fundamental results about group problems and describe the notion of valid and facet-defining inequalities for these problems. These results were introduced and proven by Gomory and Johnson [1972a, 2003] and Johnson [1974]. Although some of these results were originally presented in the context of the one-dimensional group problem, most of the proofs are independent of the dimension of the group. We therefore present them in the more general setting. We then present some relations between valid inequalities of group problems and certain lifting functions that extend results from Richard et al. [2006]. These relations will allow us to give an intuitive interpretation of the sequential-merge operation as a two-stage cut generation procedure in Section 3.

First, we denote by I^m the group of real m-dimensional vectors where the group operation is addition modulo 1 componentwise, i.e., $I^m = \{(x_1, x_2 ... x_m) \mid 0 \leq x_i < 1 \; \forall 1 \leq i \leq m\}$. We refer to the vector $(0, 0, ..., 0) \in I^m$ as o. Because it is clear from context, the symbol $+$ is used to denote both the addition in $\mathbb{R}^m$ and I^m. Next we give a formal definition of the group problem.

Definition 1 (Gomory and Johnson, 1972a, Johnson, 1974) *For $r \in I^m$ with $r \neq o$, the group problem $PI(r, m)$ is the set of functions $t : I^m \to \mathbb{R}$ such that*

1. *t has a finite support, i.e., $t(u) > 0$ for a finite subset of I^m.*
2. *$t(u)$ is a non-negative integer for all $u \in I^m$,*
3. *$\sum_{u \in I^m} u t(u) = r$.* □

Next we define the concept of a valid inequality for the group problem.

Definition 2 (Gomory and Johnson, 1972a, Johnson, 1974) *A function $\phi : I^m \rightarrow \mathbb{R}_+$ is said to define a valid inequality for PI(r,m) if $\phi(o) = 0$, $\phi(r) = 1$ and $\sum_{u \in I^m} \phi(u)t(u) \geq 1$, $\forall t \in PI(r, m)$.* □

In the remainder of this paper, we will use the terms valid function and valid inequality interchangeably. It can be verified that given the simplex tableau $\sum_{i=1}^{n} a_i x_i = b$ of an integer program P with m rows, the inequality $\sum_{i=1}^{n} \phi(\mathbb{P}(a_i))x_i \geq 1$ is valid for P if ϕ is valid for $PI(r, m)$, $\mathbb{P}(a_i) = (a_{1i}(mod 1), a_{2i}(mod 1), ...a_{mi}(mod 1))$ and $\mathbb{P}(b) = r$; see Gomory and Johnson [2003]. We next describe necessary conditions for valid inequalities ϕ to be strong.

Definition 3 (Gomory and Johnson, 1972a) *A valid inequality ϕ for $PI(r, m)$ is said to be subadditive if $\phi(u) + \phi(v) \geq \phi(u + v)$, $\forall u, v \in I^m$.* □

Gomory and Johnson [1972a] prove that all valid functions of $PI(r, m)$ that are not subadditive are dominated by valid subadditive functions of $PI(r, m)$. Therefore it is sufficient to study the valid subadditive functions of $PI(r, m)$. Next we introduce a definition to characterize strong inequalities.

Definition 4 (Gomory and Johnson, 1972a) *A valid inequality ϕ is minimal for $PI(r, m)$ if there does not exist a valid function ϕ^* for $PI(r, m)$ different from ϕ such that $\phi^*(u) \leq \phi(u)$ $\forall u \in I^m$.* □

We next present a series of result characterizing minimal functions.

Theorem 5 (Gomory and Johnson, 1972a) *If ϕ is a valid function for $PI(r, m)$ and $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$ then ϕ is minimal.* □

Theorem 6 (Gomory and Johnson, 1972a) *A valid function ϕ is minimal for $PI(r, m)$ iff ϕ is subadditive and $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$.* □

Minimal inequalities for $PI(r, m)$ are strong because they are not dominated by any single valid inequality. However, there is a stronger class of valid inequalities that Gomory and Johnson refer to as facet-defining inequalities. We present the definition of these inequalities next.

Definition 7 (Facet, Gomory and Johnson, 2003) *Let $P(\phi) = \{t \in PI(r, m) | \sum_{u \in I^m,\ t(u)>0} \phi(u)t(u) = 1\}$. We say that an inequality ϕ is facet-defining for $PI(r, m)$ if there does not exist a valid function ϕ^* such that $P(\phi^*) \supsetneq P(\phi)$.* □

Gomory and Johnson [2003] proved that all facet-defining inequalities are minimal inequalities. To prove that a function is facet-defining, Gomory and Johnson [2003] introduced a tool that they refer to as the Facet Theorem. We describe this result in Theorem 9 and introduce necessary definitions next.

Definition 8 (Equality Set, Gomory and Johnson, 2003) *For each point $u \in I^2$, we define $g(u)$ to be the variable corresponding to the point u. We define the set of equalities of ϕ to be the system of equations $g(u) + g(v) = g(u + v)$ for all $u, v \in I^2$ such that $\phi(u) + \phi(v) = \phi(u + v)$. We denote this set as $E(\phi)$.* □

Theorem 9 (Facet Theorem, Gomory and Johnson, 2003) *If ϕ is minimal and subadditive, and if ϕ is the unique solution of $E(\phi)$ then ϕ is a facet.* □

Two related results will also be used to prove that functions are facets. These are presented next.

Proposition 10 (Dey, 2007) *Let ϕ be a valid, subadditive and minimal function for $PI(r, m)$. If ϕ is not facet-defining then there exists a valid subadditive and minimal function ϕ' such that $E(\phi') \supsetneq E(\phi)$.* □

Proposition 11 (Dey, 2007) *Let ϕ be a valid, subadditive and minimal function for $PI(r, m)$. If there exists a valid subadditive and minimal function ϕ' such that $E(\phi') \supsetneq E(\phi)$, then ϕ is not facet-defining.* □

The following result of Aczél [1966] is helpful in proving that $E(\phi)$ has an unique solution.

Proposition 12 (Aczél, 1966) *Let K be the closed interval $[0, \epsilon] \subset \mathbb{R}$ for $\epsilon > 0$. If $g : K \rightarrow \mathbb{R}$ is such that $g(x) + g(y) = g(x + y)$ $\forall x, y \in K$ and $g(x) \geq 0$ for arbitrarily small $x \in K$, then $g(x) = cx$ $\forall x \in K$, where $c \in \mathbb{R}_+$.* □

The Facet Theorem is the only tool that has been used to date to prove that valid functions are facet-defining inequalities. Therefore, all known continuous facets of $PI(r, 1)$ satisfy the following property.

Definition 13 *Let ϕ be a valid continuous function for $PI(r, k)$ where $k \in \mathbb{Z}_+$ and $r \in I^k$. We say that the solution of $E(\phi)$ is unique up to scaling if for any other continuous function $\phi' : I^k \to \mathbb{R}_+$, $E(\phi') \supseteq E(\phi)$ implies that $\phi' = c\phi$ for $c \in \mathbb{R}_+$.* □

We observe that all facets for infinite group problems known to date are also piecewise linear. A function ϕ is defined to be piecewise linear if I^m can be divided into polytopes such that the function ϕ is linear over each polytope; see Gomory and Johnson [2003] and Dey and Richard [2006]. Further, Gomory and Johnson [2003] conjectured that all facets of infinite group problems are piecewise linear. Therefore, when introducing tools to prove that inequalities are facet-defining, it is usual to assume that the inequality under study is piecewise linear. Next we present in Theorem 15 a result regarding the continuity of valid functions of $PI(r, m)$ that is used in the proof of the Sequential-Merge Theorem of Section 4. Theorem 15 is proven using the following preliminary result.

Theorem 14 (Dey et al., 2006) *If a valid function ϕ for $PI(r, m)$ satisfies the following conditions*

1. $\phi(x) + \phi(y) \geq \phi(x + y)\ \forall x, y \in I^m$,
2. $lim_{h\downarrow 0} \frac{\phi(hd)}{h}$ *exists for all* $d \in \mathbb{R}^m$,

then ϕ is continuous. □

Theorem 15 *Let ϕ be a minimal piecewise linear and continuous function for $PI(r, m)$. If ψ is a valid function for $PI(r, m)$ such that $E(\phi) \subseteq E(\psi)$ then ψ is continuous.*

Proof: Since ϕ is minimal $\phi(x) + \phi(r - x) = \phi(r)\ \forall x \in I^m$. By assumption $E(\phi) \subseteq E(\psi)$, and therefore ψ satisfies these equalities. Since ψ is also valid, it follows from a Theorem 5 that ψ is minimal. We conclude from Theorem 6 that ψ is subadditive.

Let $d \in \mathbb{R}^m$ be any direction and denote the line segment between the origin and the point $d\epsilon$ as $[0, \epsilon]$. We choose ϵ sufficiently small so that ϕ is linear along the line segment $[0, \epsilon]$. This can be done since ϕ is assumed to be piecewise linear. Then $\phi(x) + \phi(y) = \phi(x + y)$ for all $x, y \in [0, \frac{\epsilon}{2}]$. Since ψ satisfies these equalities and $\psi(x) \geq 0\ \forall x \in I^m$, it follows from Proposition 12 that $\psi(x) = cx$ for $x \in [0, \epsilon]$. We conclude that ψ satisfies both conditions of Theorem 14 and so ψ is continuous. □

Generating strong inequalities for group problems is often difficult. Richard et al. [2006] showed that lifting can be used to derive valid and facet-defining inequalities for one-dimensional group problems. The family of facet-defining inequalities we present here is also easier to derive using lifting functions. In the remainder of this section, given any $x \in I^m$, we denote $\tilde{x}$ as the element of $\mathbb{R}^m$ with the same numerical value as x.

Definition 16 (Lifting-Space Representation) *Given a valid inequality ϕ for $PI(r, m)$, we define the lifting-space representation of ϕ as $[\phi]_r : \mathbb{R}^m \to \mathbb{R}$ where*

$$[\phi]_r(x) = \sum_{i=1}^{m} x_i - \sum_{i=1}^{m} \tilde{r}_i \phi(\mathbb{P}(x)).$$

□

To illustrate the idea that motivates this definition, we discuss the case where $m = 1$. Consider a row of the simplex tableau $\sum_{i=1}^{n} a_i x_i = a_0$ of an integer program, where $a_i \in \mathbb{R}\ \forall i \in \{1, ..., n\}$, the fractional part r of a_0 is nonzero, and x_i are nonnegative integer variables. If ϕ is a valid function for $PI(r, 1)$ we have that $\sum_{i=1}^{n} \phi(a_i)x_i \geq 1$ is a valid cut for the original IP. Multiplying this cut with $\tilde{r}$ and then subtracting it from the original row we obtain $\sum_{i=1}^{n} [\phi]_r(a_i)x_i \leq [\phi]_r(r)$. One well-known example of the relation between the group-space and the lifting-space representation of an inequality is that of the Gomory Mixed Integer Cut (GMIC) and the Mixed Integer Rounding (MIR) inequality. It can be easily verified that the form in which MIR is presented is $[GMIC]_r$. Thus, intuitively, the construction of the lifting-space representation given in Definition 16 is a generalization of the relation that GMIC shares with MIR to other group cuts of one and higher dimensions.

Propositions 17 and 19 are generalizations of results from Richard et al. [2006].

Proposition 17 *If ϕ is valid function for $PI(r,m)$,*

1. *$[\phi]_r(x+e_i) = [\phi]_r(x)+1$, where e_i is the i^{th} unit vector of $\mathbb{R}^m$. We say that $[\phi]_r$ is pseudo-symmetric.*
2. *$[\phi]_r$ is superadditive iff ϕ is subadditive.*

Proof:

1. $[\phi]_r(x+e_i) = \sum_{i=1}^m x_i + 1 - \sum_{i=1}^m \tilde{r}_i \phi(\mathbb{P}(x)) = [\phi]_r(x)+1$.
2. Assume first that ϕ is subadditive. For any $x, y \in \mathbb{R}^m$, we have

$$\begin{aligned}
[\phi]_r(x)+[\phi]_r(y) &= \sum_{i=1}^m x_i - \sum_{i=1}^m \tilde{r}_i \phi(\mathbb{P}(x)) + \sum_{i=1}^m y_i - \sum_{i=1}^m \tilde{r}_i \phi(\mathbb{P}(y)) \\
&\leq \sum_{i=1}^m (x_i+y_i) - \sum_{i=1}^m \tilde{r}_i \phi(\mathbb{P}(x+y)) \\
&= [\phi]_r(x+y).
\end{aligned}$$

Now assume that $[\phi]_r$ is superadditive over $\mathbb{R}^m$. For any $x, y \in I^m$ we have

$$\begin{aligned}
\phi(x)+\phi(y) &= \frac{\sum_{i=1}^m \tilde{x}_i - [\phi]_r(\tilde{x})}{\sum_{i=1}^m \tilde{r}_i} + \frac{\sum_{i=1}^m \tilde{y}_i - [\phi]_r(\tilde{y})}{\sum_{i=1}^m \tilde{r}_i} \\
&= \frac{\sum_{i=1}^m (\tilde{x}_i+\tilde{y}_i) - [\phi]_r(\tilde{x}) - [\phi]_r(\tilde{y})}{\sum_{i=1}^m \tilde{r}_i} \\
&\geq \frac{\sum_{i=1}^m (\tilde{x}_i+\tilde{y}_i) - [\phi]_r(\tilde{x}+\tilde{y})}{\sum_{i=1}^m \tilde{r}_i} \\
&= \phi(\mathbb{P}(\tilde{x}+\tilde{y})) \\
&= \phi(x+y).
\end{aligned}$$

□

Motivated by Definition 16, we define next the inverse operation.

Definition 18 (Group-Space Representation) *Given a superadditive function $\psi : \mathbb{R}^m \to \mathbb{R}$ that is pseudo-symmetric, we define the group-space representation of ψ as $[\psi]_r^{-1} : I^m \to \mathbb{R}$ where $[\psi]_r^{-1}(x) = \frac{\sum_{i=1}^m \tilde{x}_i - \psi(x)}{\sum_{i=1}^m \tilde{r}_i}$.*
□

In Figure 1, an aggregation facet of the two-dimensional group problem is shown in its group- and lifting-space representation. The inequality was proven to be facet-defining in Dey and Richard [2006].

Proposition 19 *A valid group-space function $g : I^m \to \mathbb{R}$ is minimal iff $[g]_r$ is superadditive and $[g]_r(x) + [g]_r(r-x) = 0$.*

Proof: We know from Theorem 6 that g is minimal, iff g is subadditive and $g(x)+g(r-x) = 1$. Also,

$$\begin{aligned}
& \frac{\sum_{i=1}^m x_i - [g]_r(x)}{\sum_{i=1}^m r_i} + \frac{\sum_{i=1}^m (r_i - x_i) - [g]_r(r-x)}{\sum_{i=1}^m r_i} = 1 \\
\iff \quad & \frac{\sum_{i=1}^m r_i - [g]_r(x) - [g]_r(r-x)}{\sum_{i=1}^m r_i} = 1 \\
\iff \quad & [g]_r(x) + [g]_r(r-x) = 0.
\end{aligned}$$

The result then follows from Proposition 17. □

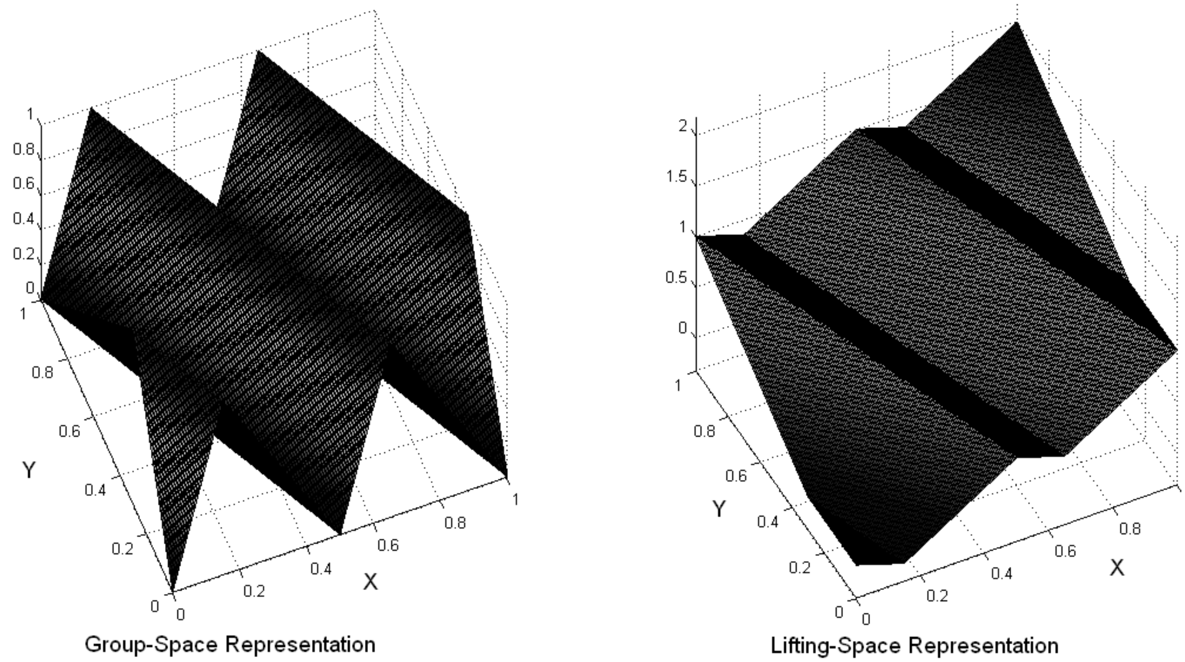

Figure 1: Group-space and lifting-space representations of an aggregation facet of PI(r,2).

3 Sequential-Merge Inequalities for High-Dimensional Group Problems

In this section, we introduce an operation that produces valid inequalities for $PI(r, m{+}1)$ from valid inequalities of $PI(r', 1)$ and $PI(r'', m)$. To simplify the notation, we denote x̃ by x since the symbol is clear from the context.

Definition 20 (Sequential-merge inequality) *Assume that g and h are valid functions for $PI(r_1, 1)$ and $PI(r_2, m)$ respectively. We define the sequential-merge of g and h as the function $g\Diamond h : I^{m+1} \to \mathbb{R}_+$ where*

$$g\Diamond h(x_1, x_2) = [[g]_{r_1}(x_1 + [h]_{r_2}(x_2))]_r^{-1}(x_1, x_2) \tag{1}$$

and $r = (r_1, r_2)$. In this construction, we refer to g as the outer function and to h as the inner function. □

Figure 2 gives an example of facet-defining inequality for $PI((0.4, 0.3), 2)$ that is obtained from the sequential-merge of a GMIC [Gomory and Johnson, 1972a] for $PI(0.4, 1)$ with a two-step MIR [Dash and Günlük, 2006a] for $PI(0.3, 1)$.

We observe that there is an intuitive interpretation to the construction presented in Definition 20. Given $m+1$ rows of a simplex tableau, we first generate a cutting plane in the lifting-space of the last m rows. This cutting plane is added to the first row of the tableau to generate a combined inequality. Finally, a one-dimensional cutting plane is generated from the combined inequality. Proposition 22 states that the group-space representation of inequalities generated using this procedure is valid for $PI(r, m+1)$ when the outer function is nondecreasing in the lifting-space.

Before we prove this result, we present a formula for the sequential-merge inequality in terms of the group representation of its inner and outer functions. For $x \in I^m$, we use the notation $X = \sum_{i=1}^m x_i$ for $x \in I^m$. Note that the summation defining X is performed in $\mathbb{R}$, i.e., we add the numeric value of each component of x to obtain X.

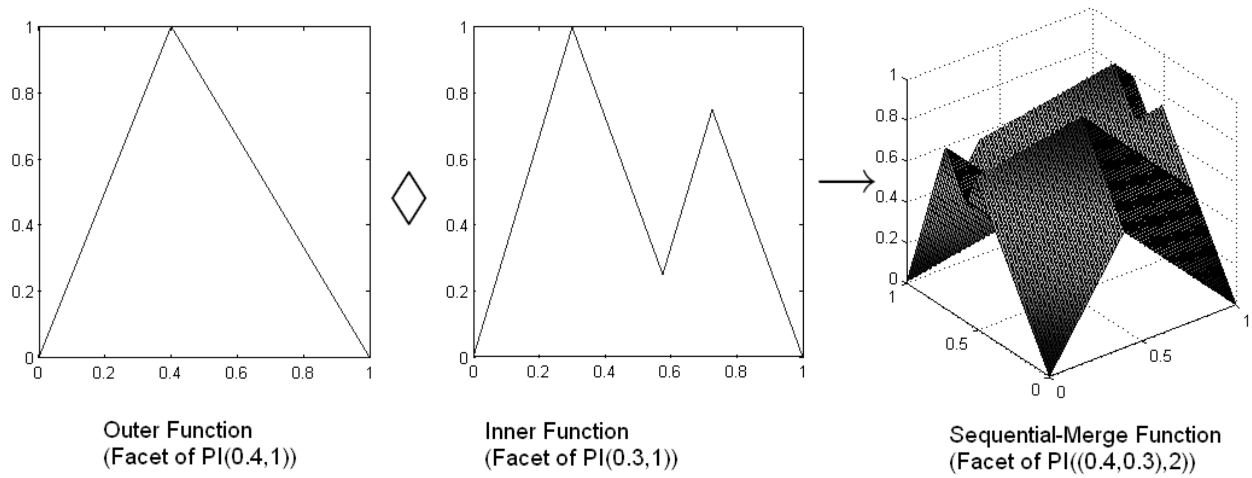

Figure 2: Example of sequential-merge operation.

Proposition 21 *Let g and h be valid inequalities for $PI(r_1,1)$ and $PI(r_2,m)$ respectively. Then* $g\Diamond h(x_1,x_2) = \frac{R_2 h(x_2)+r_1 g(\mathbb{P}(x_1+X_2-R_2h(x_2)))}{r_1+R_2}$.

Proof: By Definition 16, $[h]_{r_2}(x_2) = X_2 - R_2h(x_2)$. Therefore $[g]_{r_1}(x_1+[h]_{r_2}(x_2)) = x_1 + X_2 - R_2h(x_2) - r_1 g(\mathbb{P}(x_1+X_2-R_2h(x_2)))$. Let $x_1+X_2-R_2h(x_2) = p+q$, where $p = \mathbb{P}(x_1+X_2-R_2h(x_2))$ and $q \in \mathbb{Z}$. Then $[g]_{r_1}(x_1+[h]_{r_2}(x_2)) = p+q-r_1g(p)$. Finally using Definition 18, $[[g]_{r_1}(x_1+[h]_{r_2}(x_2))]_r^{-1} = \frac{x_1+X_2-p-q+r_1g(p)}{r_1+R_2}$ $= \frac{R_2h(x_2)+r_1g(\mathbb{P}(x_1+X_2-R_2h(x_2)))}{r_1+R_2}$. □

Using Proposition 21 it is easy to verify that the sequential-merge operator $\Diamond$ is non-commutative in the case $m = 1$. ($h\Diamond g$ is not defined for $m > 1$). In the next proposition, we prove that sequential-merge inequalities are valid for the $m+1$-dimensional group problems.

Proposition 22 *If g and h are valid subadditive functions for $PI(r_1,1)$ and $PI(r_2,m)$ respectively, and $[g]_{r_1}$ is nondecreasing, then $g\Diamond h$ is a valid subadditive function for $PI(r,m+1)$ where $r \equiv (r_1,r_2)$.*

Proof: To prove that $g\Diamond h$ is valid, it suffices to show that (1) $g\Diamond h(x_1,x_2) \geq 0 \ \forall (x_1,x_2) \in I^{m+1}$, (2) $g\Diamond h(r_1,r_2) = 1$ and (3) $g\Diamond h$ is subadditive.

1. Since g and h are valid inequalities, they are nonnegative by definition. Therefore it follows from Proposition 21 that $g\Diamond h(x_1,x_2) \geq 0$, $\forall (x_1,x_2) \in I^{m+1}$.

2. Using substitution and the fact that $g(r_1) = h(r_2) = 1$, we obtain that $g\Diamond h(r_1,r_2) = 1$.

3. Because of Proposition 17, it is sufficient to verify that $[g]_{r_1}(x_1+[h]_{r_2}(x_2))$ is superadditive. Since g and h are subadditive, it follows again from Proposition 17 that $[g]_{r_1}$ and $[h]_{r_2}$ are superadditive. For any (x_1,x_2) and (y_1,y_2) we obtain $[g]_{r_1}(x_1+[h]_{r_2}(x_2)) + [g]_{r_1}(y_1+[h]_{r_2}(y_2)) \leq [g]_{r_1}(x_1+[h]_{r_2}(x_2)+y_1+[h]_{r_2}(y_2)) \leq [g]_{r_1}(x_1+y_1+[h]_{r_2}(x_2+y_2))$ where the last inequality holds because $[g]_{r_1}$ is nondecreasing and $[h]_{r_2}$ is superadditive.

□

Figure 3 illustrates all the different types of valid two-dimensional inequalities that can be obtained using GMIC, a two-step MIR and a three-slope facet of the one-dimensional group problem as inner and outer functions in the sequential-merge construction. These inequalities are valid for $PI(r,2)$ since the three building functions used have nondecreasing lifting-space representations. We next show that the inequalities obtained in this way are strong.

Proposition 23 *If g and h are valid, minimal functions for $PI(r_1,1)$ and $PI(r_2,m)$, and $[g]_{r_1}$ is nondecreasing, then $g\Diamond h$ is a minimal function for $PI(r,m+1)$, where $r \equiv (r_1,r_2)$.*

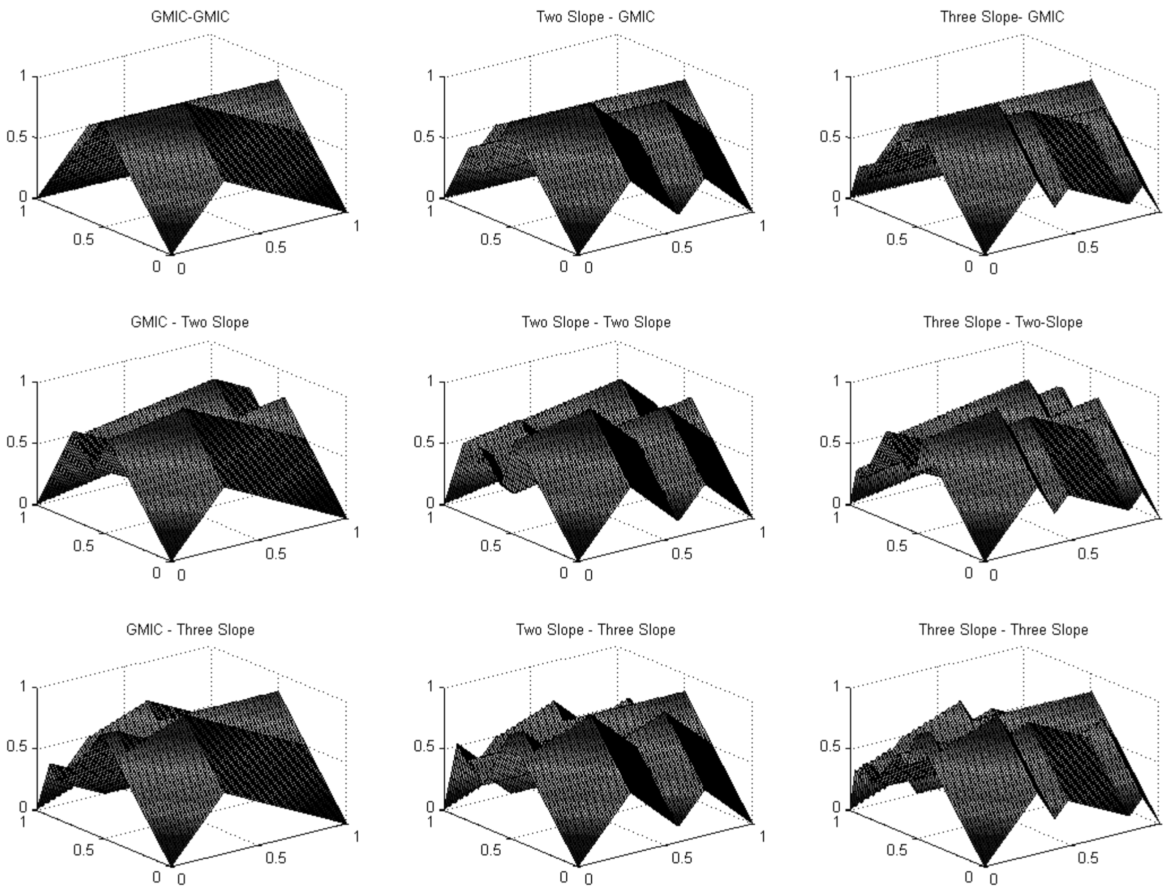

Figure 3: Examples of sequential-merge inequalities for $PI(r,2)$.(From Dey and Richard, 2007)

Proof: It follows from Proposition 22, that $[g\Diamond h]_{(r_1,r_2)}$ is superadditive. It therefore follows from Proposition 19 that we only need to show that $[g\Diamond h]_r(x_1,x_2)+[g\Diamond h]_r(r_1-x_1,r_2-x_2)=0$. We have

$$\begin{aligned} g\Diamond h(x_1,x_2)+g\Diamond h(r_1-x_1,r_2-x_2) &= [g]_{r_1}(x_1+[h]_{r_2}(x_2))+[g]_{r_1}(r_1-x_1+[h]_{r_2}(r_2-x_2)) \\ &= [g]_{r_1}(x_1+[h]_{r_2}(x_2))+[g]_{r_1}(r_1-x_1-[h]_{r_2}(x_2)) \\ &= [g]_{r_1}(x_1+[h]_{r_2}(x_2))-[g]_{r_1}(x_1+[h]_{r_2}(x_2)) \\ &= 0 \end{aligned}$$

where the second equality holds because h is minimal and the third equality holds because g is minimal. $\square$

We next give two examples of known valid inequalities for group problems that can be obtained using the sequential-merge procedure.

Proposition 24 *Consider $\kappa(x)=[[\xi]_r(x+[\xi]_r(x))]^{-1}_{(r,r)}(x,x)$, where ξ is the GMIC, i.e., $\kappa(x)$ is the sequential-merge inequality obtained using the same constraint twice and using GMIC as both the inner and outer function. Then $\kappa(x)$ is a two-step MIR function of Dash and Günlük [2006a].*

Proof: For a right-hand-side of b, the two-step MIR of Dash and Günlük [2006a] is represented as

$$g^{b,\alpha}(v) = \begin{cases} \frac{v(1-\rho\tau)-k(v)(\alpha-\rho)}{\rho\tau(1-b)} & \text{if } v - k(v)\alpha < \rho \\ \frac{k(v)+1-\tau v}{\tau(1-b)} & \text{if } v - k(v)\alpha \geq \rho \end{cases} \tag{2}$$

where $\rho = b - \alpha\lfloor b/\alpha \rfloor$, $\tau = \lceil b/\alpha \rceil$, $k(v) = \min\{\lceil v/\alpha \rceil, \tau\} - 1$. For $\alpha = \frac{b}{1+b}$, $\tau = \lceil 1 + b \rceil = 2$, $\rho = \frac{b^2}{1+b}$ and

$$k(v) = \begin{cases} 0 & v < \alpha \\ 1 & v \geq \alpha, \end{cases} \tag{3}$$

the function is

$$g^{b,\frac{b}{1+b}}(v) = \begin{cases} \frac{1+b-2b^2}{2b^2(1-b)}v & v < \frac{b^2}{1+b} \\ \frac{1-2v}{2(1-b)} & \frac{b^2}{1+b} \leq v < \frac{b}{1+b} \\ \frac{v(1+b-2b^2)-b(1-b)}{2b^2(1-b)} & \frac{b}{1+b} \leq v < b \\ \frac{1-v}{1-b} & b \leq v < 1. \end{cases} \tag{4}$$

Using Proposition 21, $k(x) = \frac{r_2\xi(x)+r_1\xi(\mathbb{P}(2x-r_2\xi(x)))}{r_1+r_2}$. It is easy to verify that this function is the automorphism of $g^{b,\frac{b}{1+b}}$ with $r = 1 - b$. $\square$

We observe that sequential-merge procedure shares some relations with the two-step MIR procedure of Dash and Günlük [2006a]. An important difference however is that the sequential-merge procedure uses, in general, different rows of a simplex tableau. Also the two-step MIR procedure only uses MIR inequalities as constituent functions. We will define, compare and discuss further the properties of the sequential-merge procedure for one-dimensional group problems in Section 5.

We describe in the next proposition another family of facets for the two-dimensional group problem that can be obtained using the sequential-merge procedure. The proof involves simple algebraic manipulations and is omitted.

Proposition 25 *Consider $\rho(x, y) = [[\xi]_{r_1}(x + [\xi]_{r_2}(y))]^{-1}_{(r_1,r_2)}(x, y)$, where ξ is the GMIC, i.e. $\rho(x, y)$ is the sequential-merge inequality obtained using GMIC as both the inner and outer function. This inequality is the function $\psi \circ \varphi$ where ψ is three-gradient facet-defining inequality for $PI((r_1, r_2), 2)$ of Dey and Richard [2006] and φ is the automorphism of I^2, $\varphi(x, y) = (1 - y, x)$.* $\square$

4 Facet-defining Sequential-Merge Inequalities for $PI(r, m+1)$

In this section, we derive conditions under which sequential-merge inequalities are facet-defining for the $m+1$-dimensional group problem $PI(r, m+1)$. We begin by studying some geometric properties of $g\Diamond h$.

Definition 26 *We define the set of points $\{(x, y) \mid x = (-Y + R_2h(y))(mod1)\}$ as the support of the function $g\Diamond h$. We denote the support of $g\Diamond h$ by $\mathcal{S}(g\Diamond h)$.* $\square$

In Figure 4 we illustrate the support of a function $g\Diamond h$ for the case where $m = 1$ and the inner function is the three-slope facet defining inequality of Gomory and Johnson [2003] with a right-hand-side of 0.2. The support of $g\Diamond h$ is important because it contains all the equalities that h satisfies. In particular, we show in the next proposition that for every equality that h satisfies there exists an equality that $g\Diamond h$ satisfies that involves points of its support.

Proposition 27 *Let g and h be valid subadditive inequalities for $PI(r_1, 1)$ and $PI(r_2, m)$ respectively and let $[g]_{r_1}$ be nondecreasing. If $v_1, v_2 \in I^m$ are such that $h(v_1) + h(v_2) = h(v_1 + v_2)$ and $(u_1, v_1), (u_2, v_2) \in \mathcal{S}(g\Diamond h)$ then*

1. $(u_1 + u_2, v_1 + v_2) \in \mathcal{S}(g\Diamond h)$.
2. $g\Diamond h(u_1, v_1) + g\Diamond h(u_2, v_2) = g\Diamond h(u_1 + u_2, v_1 + v_2)$

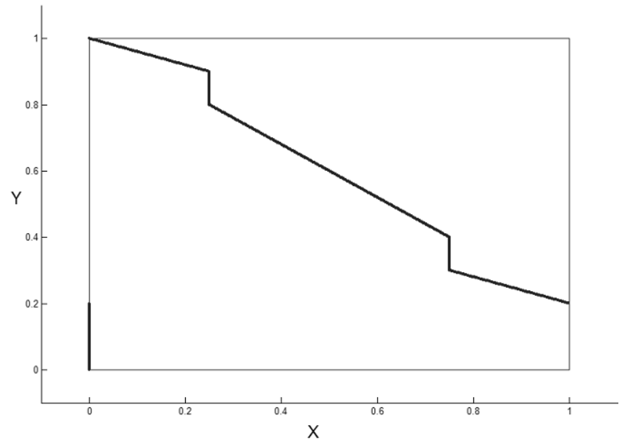

Figure 4: $\mathcal{S}(g\Diamond h)$ when h is the three-slope facet of Gomory and Johnson [2003] (From Dey and Richard, 2007)

Proof: Assume that $h(v_1) + h(v_2) = h(v_1 + v_2)$. Let $(u_1, v_1), (u_2, v_2) \in \mathcal{S}(g\Diamond h)$. From Definition 26 we know that $u_1 = (-V_1 + R_2h(v_1))(mod1)$ and $u_2 = (-V_2 + R_2h(v_2))(mod1)$.

1. $u_1 + u_2 = (-V_1 + R_2h(v_1))(mod1) + (-V_2 + R_2h(v_2))(mod1) = [-(V_1 + V_2) + R_2(h(v_1) + h(v_2))](mod1)$ $= [-(V_1 + V_2) + R_2h(v_1 + v_2)](mod1)$ where the last equality holds because $h(v_1) + h(v_2) = h(v_1 + v_2)$. Therefore, $(u_1 + u_2, v_1 + v_2) \in \mathcal{S}(g\Diamond h)$.

2. For any point $(u, v) \in \mathcal{S}(g\Diamond h)$ we have $\mathbb{P}(u+V-R_2h(v)) = 0$. Therefore $g\Diamond h(u, v) = \frac{R_2h(v)+r_1g(\mathbb{P}(u+V-R_2h(v)))}{r_1+R_2}$ $= \frac{R_2}{r_1+R_2}h(v)$. Now $g\Diamond h(u_1, v_1) + g\Diamond h(u_2, v_2) = \frac{R_2}{r_1+R_2}h(v_1) + \frac{R_2}{r_1+R_2}h(v_2) = \frac{R_2}{r_1+R_2}h(v_1 + v_2) = g\Diamond h(u_1 + u_2, v_1 + v_2)$.

□

Intuitively, because the function $g\Diamond h$ has the equalities of h on its support, $E(g\Diamond h)$ will have an unique solution on its support up to scaling whenever $E(h)$ has a unique solution up to scaling. We next give a formal proof of this observation that will be used in the proof of the Sequential-Merge Theorem 32.

Proposition 28 *Let g and h be continuous, piecewise linear valid inequalities for $PI(r_1, 1)$ and $PI(r_2, m)$ respectively. Assume that $[g]_{r_1}$ be nondecreasing, and that $E(h)$ has an unique solution up to scaling. If ψ is a valid function for $PI(r, m+1)$ such that $E(\psi) \supseteq E(g\Diamond h)$, then $\psi(u_1, u_2) = cg\Diamond h(u_1, u_2) = \frac{cR_2}{r_1+R_2}h(u_2)$ $\forall (u_1, u_2) \in \mathcal{S}(g\Diamond h)$ where c is a nonnegative real number.*

Proof: Note first that Proposition 15 implies that ψ is continuous. Let $\eta : I^m \to I^{m+1}$ be the function, $\eta(u) = ((-U + R_2h(u))(mod1), u)$. Since h is continuous, η is a continuous function. Define now the function $h' : I^m \to \mathbb{R}_+$ as $h'(u_2) = \frac{r_1+R_2}{R_2}\psi(u_1, u_2)$ where $(u_1, u_2) \in \mathcal{S}(g\Diamond h)$, i.e., $h'(u) = \frac{r_1+R_2}{R_2}\psi \circ \eta(u)$. We conclude h' is a continuous function since h and η are continuous functions.

Now let $h(y_1) + h(y_2) = h(y_1 + y_2)$ be any equality that h satisfies. Then using Proposition 27 we obtain that $g\Diamond h(x_1, y_1) + g\Diamond h(x_2, y_2) = g\Diamond h(x_1 + x_2, y_1 + y_2)$ where $(x_1, y_1), (x_2, y_2), (x_1 + x_2, y_1 + y_2) \in \mathcal{S}(g\Diamond h)$. Since $E(\psi) \supseteq E(g\Diamond h)$, we obtain that $\psi(x_1, y_1) + \psi(x_2, y_2) = \psi(x_1 + x_2, y_1 + y_2)$. Finally by the construction of h' we obtain $h'(y_1) + h'(y_2) = h'(y_1 + y_2)$. Thus $E(h') \supseteq E(h)$. However $E(h)$ has an unique solution up to scaling. Therefore $h'(u) = ch(u)$, or equivalently, $\psi(u_1, u_2) = \frac{cR_2}{r_1+R_2}h(u_2) = cg\Diamond h(u_1, u_2)$ $\forall (u_1, u_2) \in \mathcal{S}(g\Diamond h)$. Moveover since ψ is a nonnegative function, c is nonnegative. □

Although Proposition 28 establishes that $E(g\Diamond h)$ has an unique solution up to scaling on its support, it falls short of proving that $E(g\Diamond h)$ has an unique solution over I^{m+1}. Therefore, we identify in Propositions 29 and 31 some equalities that $g\Diamond h$ satisfies that help in extending the result of Proposition 28 to I^{m+1}.

Proposition 29 *If g and h are valid inequalities for $PI(r_1, 1)$ and $PI(r_2, m)$ respectively and if $[g]_{r_1}$ is nondecreasing, then $g\Diamond h(x_1, y_1) + g\Diamond h(\delta, 0) = g\Diamond h(x_1 + \delta, y_1)$ $\forall \delta \in I^1$ and $\forall (x_1, y_1) \in \mathcal{S}(g\Diamond h)$.*

Proof: Because $(x_1, y_1) \in \mathcal{S}(g\Diamond h)$, we know from Proposition 27 that $\mathbb{P}(x_1 + Y_1 - R_2 h(y_1)) = 0$. Therefore it can be verified that $g\Diamond h(x_1, y_1) = \frac{R_2 h(y_1)}{r_1+R_2}$ and $g\Diamond h(x_1+\delta, y_1) = \frac{R_2 h(y_1)+r_1 g(\delta)}{r_1+R_2}$. Since $g\Diamond h(\delta, 0) = \frac{r_1 g(\delta)}{r_1+R_2}$ we obtain that $g\Diamond h(x_1, y_1) + g\Diamond h(\delta, 0) = g\Diamond h(x_1+\delta, y_1)$. $\square$

Next we present a simple result characterizing valid functions that are nondecreasing in their lifting-space representation.

Proposition 30 *Let f be a continuous, subadditive valid inequality for $PI(r,k)$. If the function $[f]_r$ is nondecreasing then $f(x) = \frac{X}{R}$ for $\{x \in I^k \mid 0 \le x_i \le r_i \ \forall i \in \{1, ..., m\}\}$.*

Proof: Since f is valid, we have that $f(r) = 1$. Therefore, $[f]_r(r) = R - Rf(r) = 0$. Now consider $x \in \mathbb{R}^m$ such that $0 \le x_i \le r_i \ \forall i \in \{1, ..., m\}$. Let $y \in \mathbb{R}^m$ be such that $y_i = r_i - x_i \ \forall i \in \{1, ..., m\}$. Since f is subadditive, $[f]_r$ is superadditive. Then we have $[f]_r(x) + [f]_r(y) \le [f]_r(x+y) = [f]_r(r) = 0$. Finally, since $[f]_r$ is nondecreasing we have that $0 = [f]_r(0) \le [f]_r(x)$ and $0 = [f]_r(0) \le [f]_r(y)$. Therefore $[f]_r(x) = 0$ or $f(x) = \frac{X}{R}$. $\square$

Proposition 31 *Assume that g and h are valid subadditive inequalities for $PI(r,1)$ and $PI(r,m)$ respectively. Assume also that $[g]_{r_1}$ and $[h]_{r_2}$ are nondecreasing functions. Then $g\Diamond h(x_1, x_2) = \frac{x_1+X_2}{r_1+R_2} \ \forall (x_1, x_2) \in \{(x_1, x_2) \mid 0 \le x_1 \le r_1, 0 \le (x_2)_i \le (r_2)_i \forall 1 \le i \le m\}$. Furthermore $g\Diamond h(u_1, v_1) + g\Diamond h(u_2, v_2) = g\Diamond h(u_1 + u_2, v_1 + v_2)$ whenever $u_1, u_2, u_1+u_2 \le r_1$ and $v_1, v_2, v_1+v_2 \le r_2$.*

Proof: Since g is a valid subadditive function and $[g]_{r_1}$ is nondecreasing, we have from Proposition 30 that $g(x_1) = \frac{x_1}{r_1} \ \forall 0 \le x_1 \le r_1$. Similarly $h(x_2) = \frac{X_2}{R_2}$. The result then follows from Proposition 21. $\square$

Theorem 32 (Sequential-Merge Theorem) *Assume that g and h are continuous, piecewise linear, facet-defining inequalities of $PI(r_1, 1)$ and $PI(r_2, m)$ respectively. Assume also that $E(g)$ and $E(h)$ are unique up to scaling and that $[g]_{r_1}$ and $[h]_{r_2}$ are nondecreasing. Then $g\Diamond h$ is a facet-defining inequality for $PI((r_1, r_2), m+1)$.*

Proof: Assume by contradiction that $g\Diamond h$ is not facet-defining. Therefore we conclude from Theorem 9 that there exists a valid inequality $\psi \neq g\Diamond h$ such that $E(\psi) \supseteq E(g\Diamond h)$. To obtain a contradiction, we now prove that $\psi(u) = g\Diamond h(u) \ \forall u \in I^{m+1}$. The proof is in two steps. First we prove that $g\Diamond h(u) = \psi(u) \ \forall u \in \mathcal{S}(g\Diamond h)$. Then we prove that $g\Diamond h(u) = \psi(u) \ \forall u \in I^{m+1}$. This will provide the required contradiction.

We know from Proposition 31 that $g\Diamond h(x_1, x_2) = \frac{x_1+X_2}{r_1+R_2} \ \forall 0 \le x_1 \le r_1$ and $0 \le (x_2)_i \le (r_2)_i \ \forall 1 \le i \le m$. Because ψ satisfies all the equalities of $g\Diamond h$, we have that

$$\psi(x_1, 0) + \psi(0, x_2) = \psi(x_1, x_2) \quad \forall 0 \le x_1 \le r_1, 0 \le (x_2)_i \le (r_2)_i \quad \forall 1 \le i \le m. \tag{5}$$

In particular, because $g\Diamond h(u_1, 0) + g\Diamond h(u_2, 0) = g\Diamond h(u_1 + u_2, 0) \ \forall (u_1, u_2)$ in the line segment between $(0,0)$ and $(\frac{r_1}{2}, 0)$, ψ must satisfy these equalities. Therefore by Proposition 12,

$$\psi(kr_1, 0) = k\psi(r_1, 0) \quad \forall k \in [0,1]. \tag{6}$$

Similarly it can be shown that

$$\psi(0, kr_2) = k\psi(0, r_2) \quad \forall k \in [0,1]. \tag{7}$$

Further, ψ must be minimal as $g\Diamond h$ is, and therefore

$$\psi(r_1, 0) + \psi(0, r_2) = 1. \tag{8}$$

Let $\overline{1}$ be the vector $(1, 1, ..., 1)$. Because h is a piecewise linear continuous function, there exists $0 < \Delta < 1$ such that $h(\overline{1} - \delta r_2) = \delta\gamma \ \forall 0 \le \delta \le \Delta$ for some $\gamma \ge 0$. Select $\delta > 0$ such that $\delta < \min\{\Delta, 1, \frac{r_1}{R_2 + R_2\gamma}\}$. Now $g\Diamond h(0, r_2) = \frac{R_2}{r_1+R_2}$, $g\Diamond h(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2) = \frac{R_2}{r_1+R_2} h(\overline{1} - \delta r_2) = \frac{R_2\gamma\delta}{r_1+R_2}$. Since $r_2 - \delta r_2 < r_2$, we have from Proposition 30 that $h(r_2 - \delta r_2) = \frac{R_2 - \delta R_2}{R_2}$. Also $\delta\gamma R_2 + \delta R_2 < r_1$, implies that $g(\gamma R_2\delta + \delta R_2) = \frac{\gamma R_2\delta + \delta R_2}{r_1}$. Using these two relations, we obtain that $g\Diamond h(\delta R_2 + R_2\gamma\delta, r_2 - \delta r_2) = \frac{R_2 + R_2\gamma\delta}{r_1+R_2}$. Thus, $g\Diamond h(0, r_2) + g\Diamond h(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2) = g\Diamond h(\delta R_2 + R_2\gamma\delta, r_2 - \delta r_2)$. Therefore ψ satisfies the same equation, i.e.,

$$\psi(0, r_2) + \psi(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2) = \psi(\delta R_2 + R_2\gamma\delta, r_2 - \delta r_2). \tag{9}$$

Since $\delta < \frac{r_1}{R_2+R_2\gamma}$ we have that $\delta R_2 + R_2\gamma\delta < r_1$. Also $r_2 - \delta r_2 < r_2$. Therefore using (5), (6), (7) we obtain

$$\psi(\delta R_2 + R_2\gamma\delta, r_2 - \delta r_2) = \frac{\delta R_2 + R_2\gamma\delta}{r_1}\psi(r_1, 0) + \frac{R_2 - \delta R_2}{R_2}\psi(0, r_2). \tag{10}$$

Using (9) and (10) we obtain

$$\psi(0, r_2) + \psi(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2) = \frac{\delta R_2 + R_2\gamma\delta}{r_1}\psi(r_1, 0) + \frac{R_2 - \delta R_2}{R_2}\psi(0, r_2). \tag{11}$$

From Proposition 28, we obtain that $\psi(u) = cg\diamondsuit h(u)\ \forall u \in \mathcal{S}(g\diamondsuit h)$. In particular, we have

$$c = \frac{\psi(0, r_2)}{g\diamondsuit h(0, r_2)} \tag{12}$$

since $(0, r_2)$ belongs to $\mathcal{S}(g\diamondsuit h)$ and $g\diamondsuit h(0, r_2) \neq 0$. It is easy to verify that $(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2) \in \mathcal{S}(g\diamondsuit h)$ and $(0, r_2) \in \mathcal{S}(g\diamondsuit h)$. Therefore by using Proposition 28 again, we obtain

$$\begin{aligned}\psi(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2) &= cg\diamondsuit h(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2)\\ &= \frac{g\diamondsuit h(\delta R_2 + R_2\gamma\delta, \overline{1} - \delta r_2)}{g\diamondsuit h(0, r_2)}\psi(0, r_2) = \gamma\delta\psi(0, r_2).\end{aligned} \tag{13}$$

Substituting (13) in (11) we obtain

$$r_1\psi(0, r_2) = R_2\psi(r_1, 0). \tag{14}$$

Thus using (8) and (14), we conclude that $\psi(r_1, 0) = \frac{r_1}{r_1+R_2} = g\diamondsuit h(r_1, 0)$ and $\psi(0, r_2) = \frac{R_2}{r_1+R_2} = g\diamondsuit h(0, r_2)$. Along with (12), this implies that $c = 1$ and $\psi(u) = g\diamondsuit h(u)\ \forall u \in \mathcal{S}(g\diamondsuit h)$.

Next note that since ψ satisfies all the equalities of $g\diamondsuit h$, ψ satisfies the equalities satisfied by $g\diamondsuit h$ along the x_1-axis. It is easy to verify that the equalities of $g\diamondsuit h$ along the x_1-axis are the same as those of g since $g\diamondsuit h(x_1, 0) = \frac{r_1}{r_1+R_2}g(x_1)$. Since $E(g)$ is unique up to scaling $\psi(u) = c'g\diamondsuit h(u)$ for all u along the x_1-axis. Also $\psi(r_1, 0) = g\diamondsuit h(r_1, 0)$. Therefore $\psi(u) = g\diamondsuit h(u)$ for all u along the x_1-axis.

Finally consider any point $(u, v) \in I^{m+1}$. There exists δ such that $(u_1, v) + (\delta, 0) = (u, v)$, where $(u_1, v) \in \mathcal{S}(g\diamondsuit h)$. Since ψ satisfies all the inequalities of $g\diamondsuit h$, it follows from Proposition 29 that $\psi(u_1, v) + \psi(\delta, 0) = \psi(u, v)$. But $\psi(u_1, v) = g\diamondsuit h(u_1, v)$ since (u_1, v) belongs to $\mathcal{S}(g\diamondsuit h)$ and $\psi(\delta, 0) = g\diamondsuit h(\delta, 0)$ as $(\delta, 0)$ belongs to the x_1-axis. Therefore, $\psi(u, v) = g\diamondsuit h(u, v)$. □

We conclude from Theorem 32 that all the functions illustrated in Figure 3 are facet-defining for the two-dimensional group problem.

Theorem 32 presents some sufficient conditions under which the sequential-merge operation generates facet-defining inequalities for high-dimensional group problems from facet-defining inequalities of lower-dimensional group problems. An interesting question is to determine which of these conditions are necessary for the sequential-merge operator to generate facet-defining inequalities. In Theorem 32, we assumed the technical conditions that g and h are facet-defining and $E(g)$ and $E(h)$ are unique up to scaling. It can be verified that when g is not facet-defining then $g\diamondsuit h$ is not facet-defining, thus showing that the condition that g is facet-defining is necessary for $g\diamondsuit h$ to be facet-defining. The condition that $E(g)$ is unique up to scaling is therefore not very restrictive as all known facet-defining inequalities for $PI(r, 1)$ satisfy this condition; see Gomory and Johnson [2003]. On the other hand, the conditions that h is facet-defining and $E(h)$ is unique up to scaling in Theorem 32 seem more restrictive. We conjecture that there may exist functions $g\diamondsuit h$ that are facet-defining but do not satisfy these conditions.

We now extend the above results to the mixed integer case. To this end, we use a result from Johnson [1974] which states that the coefficient of a continuous variable μ_ϕ in a minimal group cut ϕ can be found as $\mu_\phi(u) = lim_{h\to 0^+}\frac{\phi(\mathbb{P}(hu))}{h}$ where $u \in \mathbb{R}^{m+1}$ is the column vector of coefficients of the continuous variable.

The following proposition describes how the sequential-merge facets obtained for $PI(r, m+1)$ can be generalized to $m+1$-dimensional mixed integer group cuts.

Proposition 33 *Let $x \in \mathbb{R}^1$, $y \in \mathbb{R}^m$ and let $c_g^+ = lim_{\epsilon \to 0^+} \frac{g(\epsilon)}{\epsilon} = \frac{1}{r_1}$, $c_g^- = lim_{\epsilon \to 0^+} \frac{g(1-\epsilon)}{\epsilon}$, $c_h(y) = lim_{\epsilon \to 0^+} \frac{h(\epsilon y)}{\epsilon}$. The coefficients of the continuous variables for $g\Diamond h$ are given by*

$$\mu_{g\Diamond h}(x,y) = \begin{cases} \frac{R_2 c_h(y) + r_1 c_g^+(x+Y-R_2c_h(y))}{r_1+R_2} & \text{if } (x+Y-R_2c_h(y)) \geq 0 \\ \frac{R_2 c_h(y) - r_1 c_g^-(x+Y-R_2c_h(y))}{r_1+R_2} & \text{if } (x+Y-R_2c_h(y)) \leq 0. \end{cases} \tag{15}$$

Proof: The proof for the two cases are similar. Therefore, we give the proof only for the first case. From Johnson [1974] we obtain

$$\mu_{g\Diamond h}(u) = lim_{\epsilon \to 0^+} \frac{g\Diamond h(\mathbb{P}(\epsilon u))}{\epsilon}. \tag{16}$$

Since h is piecewise linear, $h(\epsilon y) = \epsilon c_h(y)$ for sufficiently small ϵ's. Therefore $\mathbb{P}(\epsilon x + \epsilon Y - R_2 h(\epsilon y)) = \mathbb{P}(\epsilon x + \epsilon Y - \epsilon R_2 c_h(y))$. Again, for sufficiently small ϵ's, we have that $g(\mathbb{P}(\epsilon x + \epsilon Y - \epsilon R_2 c_h(y))) = \epsilon c_g^+(x + Y - R_2 c_h(y))$ since $x + Y - R_2 c_h(y) \geq 0$. Therefore we obtain

$$\begin{aligned} \frac{g\Diamond h(\mathbb{P}(\epsilon(x,y)))}{\epsilon} &= \frac{R_2 h(\epsilon y) + r_1 g(\mathbb{P}(\epsilon x + \epsilon Y - R_2 h(\epsilon y)))}{(r_1+R_2)\epsilon} \\ &= \frac{R_2 \epsilon c_h(y) + \epsilon r_1 c_g^+(x+Y-R_2c_h(y))}{(r_1+R_2)\epsilon} \\ &= \frac{R_2 c_h(y) + r_1 c_g^+(x+Y-R_2c_h(y))}{(r_1+R_2)}. \end{aligned}$$

This completes the proof. □

Next we illustrate on an example how the sequential-merge procedure can be applied to mixed integer programs.

Example 34 *Consider the following mixed integer set*

$$\frac{1}{3}x - \frac{1}{3}y \leq 1 \tag{17}$$

$$\frac{1}{3}x + \frac{2}{3}y \leq \frac{3}{2} \tag{18}$$

$$x, y \in \mathbb{Z}_+. \tag{19}$$

whose feasible region is represented in Figure 5. We introduce continuous non-negative slack variables s_1 and s_2 and perform simplex iterations to obtain the tableau

$$x + 2s_1 + s_2 = \frac{7}{2} \tag{20}$$

$$y - s_1 + s_2 = \frac{1}{2}. \tag{21}$$

Using Proposition 33 and using GMIC as both the inner and outer functions, we obtain the sequential-merge cut $s_1 + 2s_2 \geq 1$. This cut is illustrated in Figure 5 and is equivalent to $x + y \leq 3$. It can easily be verified that this inequality is facet-defining for the convex hull of solutions to (17), (18) and (19).

It is also easily shown that the two GMICs generated from the individual rows of the tableau are $4s_1+2s_2 \geq 1$ and $2s_1 + 2s_2 \geq 1$. They are equivalent to $x \leq 3$ and $2x + y \leq 6$ respectively. Interestingly, it can be shown that these inequalities are not facet-defining for the convex hull of solutions to (17), (18) and (19). This illustrates that the cuts generated from individual rows can be weaker than those generated by the sequential-merge procedure.

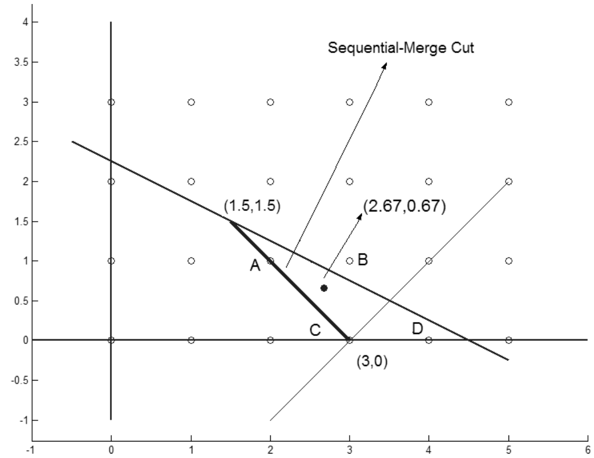

Figure 5: The Sequential-Merge Inequality.

In this example an even stronger result can be shown as it can be proven that there exists no split disjunction that can be used to derive the cut $x_1 + x_2 \le 3$. To prove that no such disjunction exists, we show that the point $(\frac{8}{3}, \frac{2}{3})$ belongs to the rank-1 split closure of (17), (18) and (19). This provides the desired proof as $(\frac{8}{3}, \frac{2}{3})$ is cut off by $x_1 + x_2 \le 3$. Consider the integer points $A \equiv (2,1)$, $B \equiv (3,1)$, $C \equiv (3,0)$, $D \equiv (4,0)$.

Consider now any split disjunction

$$P_1 \equiv \{(x,y) \in \mathbb{R}^2 \mid \pi_1 x + \pi_2 y \le \pi_0\} \tag{22}$$

and

$$P_2 \equiv \{(x,y) \in \mathbb{R}^2 \mid \pi_1 x + \pi_2 y \ge \pi_0 + 1\} \tag{23}$$

where $\pi_0, \pi_1, \pi_2 \in \mathbb{Z}$. Denote the LP relaxation of (17), (18) and (19) as LPR. Denote also the intersection of P_1 with LPR as V_1 and the intersection of P_2 with LPR as V_2. A point p is not cut off by the disjunctive cut if it is a convex combinations of points in V_1 and V_2. There are eight cases to be considered:

1. *$A, B, C, D \in P_1$: It can be easily verified that the point $(\frac{8}{3}, \frac{2}{3}) \in LPR$. Also $(\frac{8}{3}, \frac{2}{3}) \in P_1$ since $(\frac{8}{3}, \frac{2}{3}) = \frac{1}{3}(A + B + C)$. Therefore $(\frac{8}{3}, \frac{2}{3}) \in V_1$ and the disjunction does not cut off the point $(\frac{8}{3}, \frac{2}{3})$.*

2. *$A, B, C \in P_1$, $D \in P_2$: The proof is identical to that of case 1.*

3. *$A, B, D \in P_1$, $C \in P_2$: It can be easily verified that the point $(\frac{8}{3}, \frac{2}{3}) \in LPR$. Also $(\frac{8}{3}, \frac{2}{3}) \in P_1$ since $(\frac{8}{3}, \frac{2}{3}) = \frac{2}{3}A + \frac{1}{3}D$. Therefore $(\frac{8}{3}, \frac{2}{3}) \in V_1$ and the disjunction does not cut off the point $(\frac{8}{3}, \frac{2}{3})$.*

4. *$A, C, D \in P_1$, $B \in P_2$: The proof is identical to that of case 3.*

5. *$B, C, D \in P_1$, $A \in P_2$: It can be verified that $(\frac{10}{3}, \frac{1}{3}) \in LPR$. Also $(\frac{10}{3}, \frac{1}{3}) \in P_1$ since $(\frac{10}{3}, \frac{1}{3}) = \frac{1}{3}(B + C + D)$. Therefore $(\frac{10}{3}, \frac{1}{3}) \in V_1$. Also $A \in LPR$ and $A \in P_2$, i.e., $A \in V_2$. Finally, note that $(\frac{8}{3}, \frac{2}{3}) = \frac{1}{2}(\frac{10}{3}, \frac{1}{3}) + \frac{1}{2}A$. Therefore $(\frac{8}{3}, \frac{2}{3})$ is a convex combination of points in V_1 and V_2 and the disjunction does not cut off the point $(\frac{8}{3}, \frac{2}{3})$.*

6. *$A, B \in P_1$, $C, D \in P_2$: It can be verified that the point $(\frac{5}{2}, 1) \in LPR$. Also $(\frac{5}{2}, 1) = \frac{1}{2}(A + B)$. Therefore, $(\frac{5}{2}, 1) \in V_1$. Also $C \in V_2$ since $C \in LPR$. Finally, note that $(\frac{8}{3}, \frac{2}{3}) = \frac{2}{3}(\frac{5}{2}, 1) + \frac{1}{3}C$. Therefore $(\frac{8}{3}, \frac{2}{3})$ is a convex combination of points in V_1 and V_2 and the disjunction does not cut off the point $(\frac{8}{3}, \frac{2}{3})$.*

7. *$A, C \in P_1$, $B, D \in P_2$: It can be verified that $(\frac{9}{4}, \frac{3}{4}) \in LPR$. Also $(\frac{9}{4}, \frac{3}{4}) \in P_1$ since $(\frac{9}{4}, \frac{3}{4}) = \frac{3}{4}A + \frac{1}{4}C$. Therefore, $(\frac{9}{4}, \frac{3}{4}) \in V_1$. It can be verified that the point $(\frac{7}{2}, \frac{1}{2}) \in LPR$. Also $(\frac{7}{2}, \frac{1}{2}) \in P_2$ since $(\frac{7}{2}, \frac{1}{2}) = \frac{1}{2}(B + D)$. Finally, note that $(\frac{8}{3}, \frac{2}{3}) = \frac{2}{3}(\frac{9}{4}, \frac{3}{4}) + \frac{1}{3}(\frac{7}{2}, \frac{1}{2})$. Therefore $(\frac{8}{3}, \frac{2}{3})$ is a convex combination of points in V_1 and V_2 and the disjunction does not cut off the point $(\frac{8}{3}, \frac{2}{3})$.*

8. $A, D \in P_1$, $B, C \in P_2$: *This case is not possible.*

$\square$

It can be seen from Proposition 33 that sequential-merge inequalities have very diverse continuous variable coefficients. To understand the strength of the continuous coefficients in sequential-merge inequalities we consider the following example.

Example 35 *Consider a continuous variable with coefficient $(u_1, u_2) \in \mathbb{R}^2$ with $u_1 > 0$, $u_2 < 0$, $u_1 + u_2 + r_2 c_h^- u_2 = 0$. The coefficient of this continuous variable in $g \Diamond h$ is $\frac{r_2}{r_1+r_2}(-u_2 c_h^-)$. If the group cut h was used to generate a cut from the second constraint alone, the coefficient of the continuous variable would be $-u_2 c_h^- > \frac{r_2}{r_1+r_2}(-u_2 c_h^-)$. Similarly, if the group cut g was derived using the first constraint alone, the coefficient of the continuous variable would be $\frac{1}{r_1} u_1$. Since $u_1 + u_2 + r_2 c_h^- u_2 = 0$ the coefficient of the continuous variable using $g \Diamond h$, $\frac{r_2}{r_1+r_2}(-u_2 c_h^-) = \frac{u_1+u_2}{r_1+r_2} < \frac{1}{r_1} u_1$ as $u_2 < 0$. Therefore in this case the continuous coefficients generated using the two different cuts g and h individually will be strictly weaker than those generated using $g \Diamond h$.* $\square$

We conclude from Example 35 that if both the inner and outer functions used in the sequential-merge procedure are GMICs then the coefficient generated for the continuous variable is stronger than the coefficient generated using each of the individual group cuts for a continuous variable with coefficient (u_1, u_2) where $u_1 > 0$, $u_2 < 0$, $u_1 + u_2 + r_2 c_h^- u_2 = 0$ (i.e., the coefficients of the sequential-merge inequalities are not dominated by those of the GMICs). This result is significant because it can be proven that GMIC generates the strongest possible coefficients for continuous variables among all facets of one-dimensional group problems. Note also that although the above discussion was based on the specific case where $u_1 > 0$, $u_2 < 0$ and $u_1 + u_2 + r_2 c_h^- u_2 = 0$, there exists a larger range of continuous variables coefficients for which the sequential-merge procedure yields inequalities whose coefficients are not dominated by the continuous coefficients of the one-dimensional group cuts derived from the individual rows.

5 Projected Sequential-Merge Inequality

In this section, we study methods to construct sequential-merge inequalities for low-dimensional group problems from sequential-merge inequalities for higher-dimensional group problems. Further, we derive conditions for the resulting inequalities to be strong.

Consider first a valid subadditive function $\phi : I^{m+1} \rightarrow \mathbb{R}_+$ for a $m+1$-dimensional group problem. One intuitive procedure to generate a cut for the m-dimensional group problem from ϕ is to first duplicate one row of the m-dimensional system of constraints, scale the two identical rows with integer multipliers and then generate a $m+1$-dimensional cut for the expanded system. Algebraically, this construction corresponds to defining the m-dimensional function

$$f(x_1, x_2, ..., x_m) = \phi(x_1, x_2, ..., n_1 x_m, n_2 x_m). \tag{24}$$

The procedure presented in (24) is one of many possible ways of deriving inequalities of m-dimensional group problems using inequalities of $m+1$-dimensional group problems. We leave the investigation of different constructions for future research. Moreover, while the procedure presented in (24) can be applied to any $m+1$-dimensional inequality, we study it only with respect to inequalities that are constructed by iteratively applying sequential-merge operators to one-dimensional facet-defining inequalities, i.e., we analyze $m+1$-dimensional facet-defining inequalities ϕ of the form $(g_1 \Diamond (g_2 \Diamond ... (g_m \Diamond g_{m+1})))$ where each g_i is facet-defining for $PI(r_i, 1)$ and $[g_i]_{r_i}$ is nondecreasing. Based on the procedure presented in (24), we are interested in deriving conditions under which m-dimensional functions of the form $(g_1 \Diamond (g_2 \Diamond ... (g_m \Diamond g_{m+1})))(x_1, x_2, .., n_1 x_m, n_2 x_m)$ are facet-defining. Observe first that if we prove that the function $(g_m \Diamond g_{m+1})(n_1 x_m, n_2 x_m)$ is a facet-defining inequality for the one-dimensional group problem and is non-decreasing in its lifting-space representation, it will follow from the sequential-merge theorem that the 'projected' m-dimensional function $f(x_1, x_2, ..., x_m)$ is facet-defining. Therefore, we begin this section by deriving conditions for which one-dimensional infinite group cuts projected from two-dimensional sequential-merge inequalities are facet-defining. This result is presented

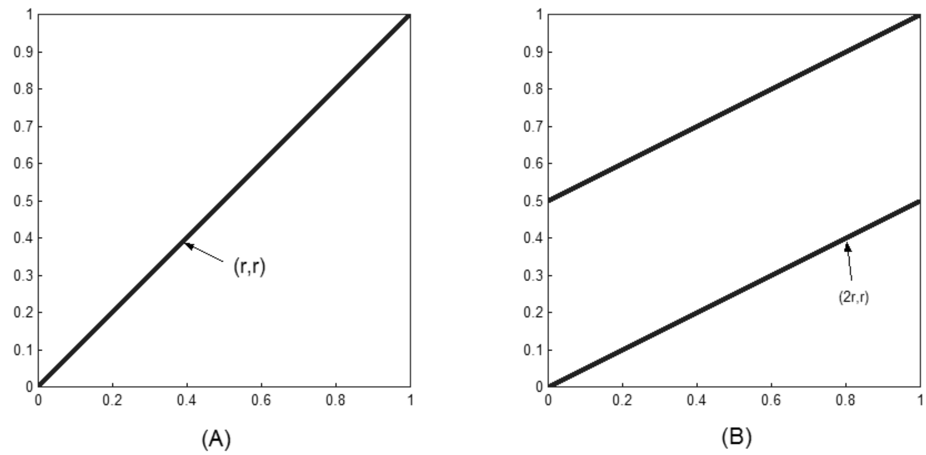

Figure 6: Generating one-dimensional sequential-merge inequality.

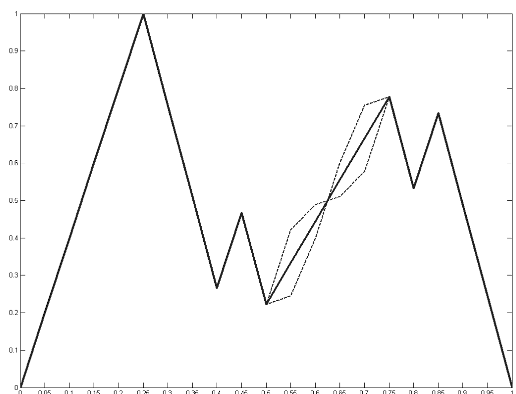

Figure 7: The function $[[\xi]_r(x + [\xi]_{2r}(2x))]^{-1}_{(r,2r)}$ where ξ is the GMIC, is not extreme for $PI(r, 1)$.

in Lemma 40 and is then used to prove Theorem 41, a more general result that characterizes the projection of $m+1$-dimensional sequential-merge inequalities.

Geometrically, when $m = 1$, and $n_1 = n_2 = 1$, the procedure in (24) corresponds to creating a one-dimensional cut by computing the value of ϕ along the 'diagonal' of I^2. Figure 6(A) illustrates this case, while Figure 6(B) shows the case when $n_1 = 2$. The construction of two-step MIR inequalities using sequential-merge inequalities, that was presented in Proposition 24 is an example of this idea.

It was proven in Section 4 that under very general conditions, the sequential-merge procedure $\Diamond$ creates facet-defining inequalities for high-dimensional group problems from facet-defining inequalities for low-dimensional group problems. Surprisingly, the one-dimensional projection does not always produce facet-defining inequalities for $PI(r, 1)$ even when the constituent functions are facet-defining for the one-dimensional group problem. In fact, the result does not even necessarily hold when the constituent inequalities are GMICs. Figure 7 shows an example where $n_2 = 2$ and both the inner and outer functions are GMICs. It can be verified that this function (denoted as ϕ) is not facet-defining by showing that it is not extreme, i.e., $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$.

Figure 8 shows another example of this observation where the outer function is GMIC, the inner function is a three-slope facet-defining inequality, and $n_1 = n_2 = 1$.

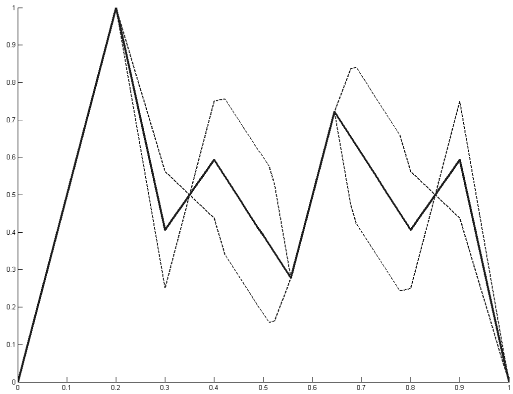

Figure 8: The function $[[\xi]_r(x + [h]_r(x))]^{-1}_{(r,r)}$ where h is a three-slope facet and ξ is the GMIC, is not extreme for $PI(r, 1)$.

Interestingly, when the inner function is GMIC and $n_2 = 1$, the sequential-merge procedure generates facet-defining inequalities for $PI(r, 1)$. We introduce some notation to represent this subfamily of one-dimensional projected sequential-merge inequalities.

Definition 36 *Let $n \in \mathbb{Z}_+$ be such that $n \geq 1$, $r < 1/n$. Given a valid inequality g for $PI(nr, 1)$ that is such that $[g]_{nr}$ is a nondecreasing function, we define the function $g\Diamond^1_n h : I^1 \to \mathbb{R}_+$ as $g\Diamond^1_n h(x) = [[g]_{nr}(nx + [h]_r(x))]^{-1}_{(nr,r)}(nx, x)$.*

In the remainder of this section ξ is used to denote the GMIC. In order to prove Lemma 40, we begin by studying in Propositions 38 and 39 some properties of the family of inequalities obtained using $\Diamond^1_n$. Note first that, using Proposition 21, it is easy to verify that

$$g\Diamond^1_n \xi(x) = \frac{1}{n+1}(\xi(x) + ng(\mathbb{P}((n+1)x - r\xi(x))). \tag{25}$$

Next we identify some intervals of I^1 over which $g\Diamond^1_n\xi$ satisfies some equalities. Define $x_k = \frac{k-(k-1)r}{(n+1)(1-r)+r}$ and $y_k = \frac{(1-r)(k+nr)+r}{(n+1)(1-r)+r}$ for $k \in \{0, ..., n\}$. Also define $x_{n+1} = 1$. It can be easily verified that $x_k < y_k$ for all $k \in \{0, ..., n\}$. The following proposition establishes some properties of the intervals $[x_k, y_k]$ and the function $g\Diamond^1_n\xi$.

Proposition 37 *Let g be a continuous valid subadditive function for $PI(nr, 1)$ and $r < \frac{1}{n}$, then*

1. $x_k \geq r \ \forall k \in \{1, ..., n\}$.
2. $x_k < y_k \leq 1 \ \forall k \in \{1, ..., n\}$.
3. $k \leq (n+1)x - r\xi(x) \leq k + nr \ \forall x \in [x_k, y_k] \ \forall k \in \{1, ..., n\}$.
4. $(x_{k_1} + x_{k_2})(mod 1) = x_{k_1+k_2-(n+1)} \ \forall k_1, k_2 \in \{1, ..., n\}$ *such that* $k_1 + k_2 \geq n + 1$.
5. $g\Diamond^1_n\xi(x) = \frac{x}{r} - \frac{k}{(n+1)r} \ \forall x \in [x_k, y_k] \ \forall k \in \{1, ..., n\}$.
6. $g\Diamond^1_n\xi(x_{k_1}) + g\Diamond^1_n\xi(x_{k_2}) = g\Diamond^1_n\xi(x_{k_1+k_2-(n+1)}) \ \forall k_1, k_2 \in \{1, ..., n\}$ *such that* $k_1 + k_2 \geq n + 1$.
7. $[g\Diamond^1_n\xi]_r$ *is nondecreasing.*

Proof:

1. First observe that $x_{k+1} > x_k$ since $x_{k+1} - x_k = \frac{1-r}{(n+1)(1-r)+r}$. Therefore to show that the result holds, it suffices to verify that $x_1 > r$. Now $x_1 - r = \frac{1-nr+nr^2+r}{(n+1)(1-r)+r} > 0$ since $r < \frac{1}{n}$.

2. Note that $x_k < y_k < y_{k+1}$. Therefore to show that the result holds, it suffices to verify that $y_n < 1$. Now $y_n = \frac{(1-r)(n+nr)+r}{(1-r)(1+n)+r} < 1$ since $n + nr < n + 1$.

3. Since $x \geq r$ from part (1), we have that $\xi(x) = \frac{1-x}{1-r}$. Therefore $(n+1)x - r\xi(x)$ is a nondecreasing function of x. Hence it suffices to verify the value of the function $(n+1)x - r\xi(x)$ at x_k and y_k to prove the result. When $x = x_k$, we obtain

$$(n+1)x_k - r\xi(x_k) = (n+1)\frac{k-(k-1)r}{(n+1)(1-r)+r} - \frac{r(n+1-k)}{(n+1)(1-r)+r} = k. \tag{26}$$

   Similarly, when $x = y_k$, we obtain

$$\begin{aligned}(n+1)y_k - r\xi(y_k) &= (n+1)\frac{((1-r)(n+kr)+r)}{(n+1)(1-r)+r} - \frac{r(n+1-k-nr)}{(n+1)(1-r)+r} \\ &= k + nr. \end{aligned} \tag{27}$$

4. Since $k_1 + k_2 \geq n+1$, $(k_1+k_2)(1-r) + 2r \geq (n+1)(1-r) + r$. Therefore

$$\begin{aligned}(x_{k_1} + x_{k_2})(mod 1) &= \frac{k_1-(k_1-1)r}{(n+1)(1-r)+r} + \frac{k_2-(k_2-1)r}{(n+1)(1-r)+r} - 1 \\ &= \frac{k_1+k_2-(k_1+k_2-2)r}{(n+1)(1-r)+r} - 1 \\ &= \frac{k_1+k_2-(k_1+k_2-2)r-(n+1)(1-r)-r}{(n+1)(1-r)+r} \\ &= \frac{k_1+k_2-(n+1)-(k_1+k_2-(n+1)-1)r}{(n+1)(1-r)+r} \\ &= x_{k_1+k_2-(n+1)}. \end{aligned} \tag{28}$$

5. From (3) we know that, $k \leq (n+1)x - r\xi(x) \leq k + nr$ $\forall x \in [x_k, y_k]$. Therefore

$$\begin{aligned} g\Diamond_n^1 h(x) &= \frac{1}{n+1}(\xi(x) + ng(\mathbb{P}((n+1)x - r\xi(x))) \\ &= \frac{1-x}{(1-r)(1+n)} + \frac{n}{1+n}g((n+1)x - r\frac{1-x}{1-r} - k) \\ &= \frac{1-x}{(1-r)(1+n)} + \frac{1}{(1+n)r}((n+1)x - r\frac{1-x}{1-r} - k). \end{aligned} \tag{29}$$

   Note that the last equality follows from the fact that $[g]_{nr}$ is a nondecreasing function and therefore using Proposition 30 we obtain that $g(u) = \frac{u}{nr}$ $\forall 0 \leq u \leq nr$. Simplifying (29) further, we obtain

$$g\Diamond_n^1 h(x) = \frac{x}{r} - \frac{k}{(n+1)r}.$$

6. Using (5) we obtain

$$\begin{aligned} g\Diamond_n^1\xi(x_{k_1}) + g\Diamond_n^1\xi(x_{k_2}) &= \frac{x_{k_1}}{r} - \frac{k_1}{(n+1)r} + \frac{x_{k_2}}{r} - \frac{k_2}{(n+1)r} \\ &= \frac{x_{k_1}+x_{k_2}}{r} + \frac{k_1+k_2}{(n+1)r} \\ &= \frac{x_{k_1}+x_{k_2}-1}{r} + \frac{k_1+k_2-(n+1)}{(n+1)r} \\ &= g\Diamond_n^1\xi(x_{k_1+k_2-(n+1)}). \end{aligned} \tag{30}$$

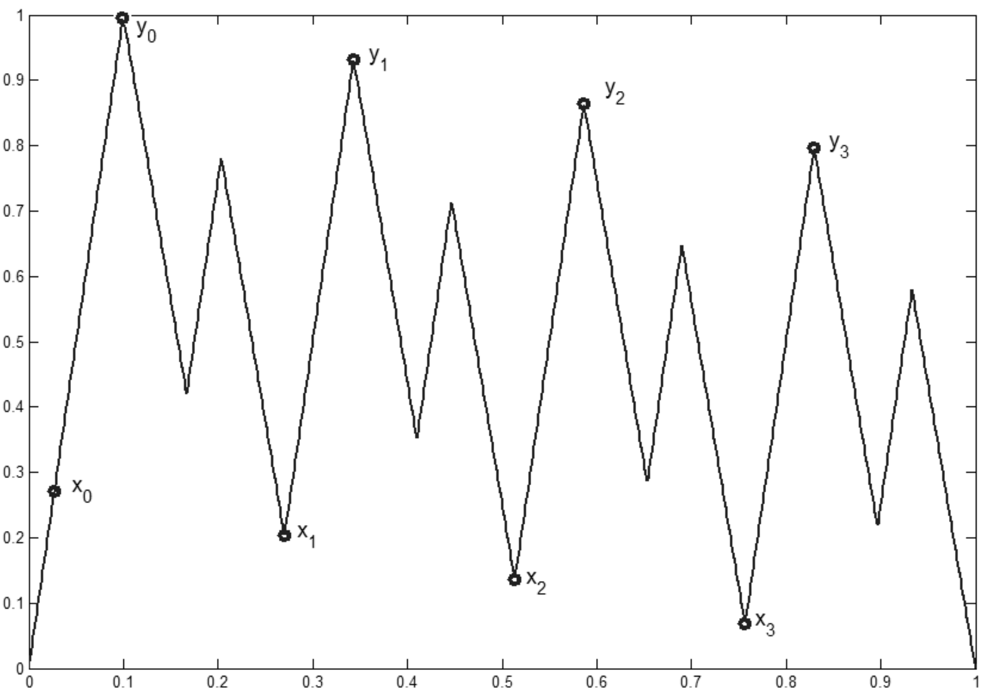

Figure 9: The intervals $[x_k, y_k]$.

7. First observe that Proposition 30 shows that $g(u) = \frac{u}{nr}$ $\forall 0 \le u \le nr$. Therefore $g\Diamond_n^1\xi(x) = \frac{x}{r}$ for $x \in [0,r]$, i.e., $[g\Diamond_n^1\xi]_r(x) = 0$ for $x \in [0,r]$. Assume now by contradiction that $[g\Diamond_n^1\xi]_r$ is not nondecreasing. Since g is continuous, $[g\Diamond_n^1\xi]_r$ is continuous. Therefore there exists two points a and $a+\epsilon$, with $\epsilon < r$ for which $[g\Diamond_n^1\xi]_r(a) > [g\Diamond_n^1\xi]_r(a+\epsilon)$. However, this implies that $[g\Diamond_n^1\xi]_r(a) + [g\Diamond_n^1\xi]_r(\epsilon) > [g\Diamond_n^1\xi]_r(a+\epsilon)$ which contradicts the fact that $g\Diamond_n^1\xi$ is subadditive.

□

Figure 9 illustrates the intervals $[x_k, y_k]$ of the function $g\Diamond_n^1\xi$ where $n = 3$ and g is a two-step MIR inequality. From this figure, we can observe some properties proven in Proposition 37 that are common to all projected sequential-merge one-dimensional inequalities. First the value of the function $g\Diamond_n^1\xi$ is $\frac{x}{r}$ in the interval $[0,r]$. Also, the value of the function $g\Diamond_n^1\xi$ is fixed in the intervals $[x_k, y_k]$ $\forall k \in \{1, ..., n\}$ and is independent of g. Therefore these intervals provide a large number of equalities that help in proving that $g\Diamond_n^1\xi$ is facet-defining for $PI(r,1)$. In contrast, the value of the function in the intervals $[y_k, x_{k+1}]$ depends on the function g. Intuitively, the shape of the function $g\Diamond_n^1\xi$ in the interval $[y_k, x_{k+1}]$ is similar to the shape of g in the interval $[r,1)$. We next describe additional equalities that are found in sequential-merge inequalities.

Proposition 38 *Let g be a continuous, valid subadditive inequality for $PI(rn,1)$. The equality $g\Diamond_n^1\xi(a) + g\Diamond_n^1\xi(x_n) = g\Diamond_n^1\xi(a+x_n)$ holds for all $a \in [x_n, 1)$.*

Proof: Using part (4) of Proposition 37 we have that $x_n + x_n = x_{n-1}$. Therefore $x_{n-1} \le x_n + a \le x_n$. Also it is clear that the numerical value of $x_n + a$ is $x_n + a - 1$. There are two cases:

1. $n > 1$. First it follows from (26) that $(n+1)x_n - r\xi(x_n) = n$ which implies that $g\Diamond_n^1\xi(x_n) = \frac{\xi(x_n)}{n+1}$. Then because $r \le x_1 \le a + x_n \le x_n$ we conclude that

$$\xi(a) + \xi(x_n) = \xi(a + x_n) \tag{31}$$

since $\xi(a) + \xi(x_n) = \frac{1-a}{1-r} + \frac{1-x_n}{1-r} = \frac{1-(a+x_n-1)}{1-r} = \xi(a+x_n)$. Now $g\Diamond_n^1\xi(a+x_n) = \frac{1}{n+1}[\xi(a+x_n) + ng((n+1)(a+x_n) - r\xi(a+x_n))] = \frac{1}{n+1}[\xi(a) + \xi(x_n) + ng((n+1)a - r\xi(a) + n)] = g\Diamond_n^1\xi(a) + g\Diamond_n^1\xi(x_n)$ where the second equality follows from (26) and (31).

2. $n = 1$. There are two subcases:

   (a) $x_1 + a \geq r$. In this case, the proof is exactly the same as that presented when $n > 1$.

   (b) $x_1 + a \leq r$. Then it is easy to verify that $x_1 \leq a \leq y_1$ since $x_1 + y_1 = r$. Using part (5) of Proposition 37 we obtain that $g\Diamond_1^1\xi(a) = \frac{a}{r} - \frac{1}{2r}$ and $g\Diamond_1^1\xi(x_1) = \frac{x_1}{r} - \frac{1}{2r}$. Thus, $g\Diamond_1^1\xi(a) + g\Diamond_1^1\xi(x_1) = \frac{a}{r} - \frac{1}{2r} + \frac{x_1}{r} - \frac{1}{2r} = \frac{x_1+a}{r} - \frac{1}{r} = \frac{x_1+a-1}{r} = g\Diamond_1^1\xi(x_1 + a)$.

□

The following proposition shows that every equality that the function g satisfies, yields an equality for the function $g\Diamond_n^1\xi$. This result is used in the proof of Lemma 40 to prove that the projected sequential-merge functions are facet-defining.

Proposition 39 *For $x \in I^1$ define $\overline{x} = x\frac{1-r}{(n+1)(1-r)+r}$. Then*

1. $\mathbb{P}((1-\overline{x}) + (1-\overline{y})) = (1 - \overline{x} - \overline{y})\ \forall x, y \in I^2$.
2. *For* $x \in I^1$, $1 - \overline{x} \geq x_n$.
3. *If* $g(1-x) + g(1-y) = g(\mathbb{P}(2-x-y))$ *then* $g\Diamond_n^1\xi(1-\overline{x}) + g\Diamond_n^1\xi(1-\overline{y}) = g\Diamond_n^1\xi(1-\overline{x}-\overline{y})$.

Proof:

1. $\mathbb{P}((1-\overline{x})+(1-\overline{y})) = \mathbb{P}(1-x\frac{1-r}{(n+1)(1-r)+r} + 1 - y\frac{1-r}{(n+1)(1-r)+r}) = \mathbb{P}(2-(x+y)\frac{1-r}{(n+1)(1-r)+r})$. Now observe that since $n+1 \geq 2$, $\frac{(x+y)(1-r)}{(n+1)(1-r)+r} < 1$. Therefore, $\mathbb{P}(2-(x+y)\frac{1-r}{(n+1)(1-r)+r}) = (1-(x+y)\frac{1-r}{(n+1)(1-r)+r})$ $= 1 - \overline{x} - \overline{y}$.

2. $1 - \overline{x} = 1 - x\frac{1-r}{(n+1)(1-r)+r} = \frac{(n+1-x)(1-r)+r}{(n+1)(1-r)+r} \geq \frac{n(1-r)+r}{(n+1)(1-r)+r} = x_n$ where the last inequality follows from the fact that the numerical value of x is less than or equal to 1.

3. Since $1 - \overline{x} \geq x_n \geq r$ we have that $\xi(1-\overline{x}) = \frac{\overline{x}}{1-r} = \frac{1}{1-r}\left(x\frac{1-r}{(n+1)(1-r)+r}\right) = \frac{x}{(n+1)(1-r)+r}$. Next we compute

$$\begin{aligned} g\Diamond_n^1\xi(1-\overline{x}) &= \frac{1}{n+1}[\xi(1-\overline{x}) + ng(\mathbb{P}((n+1)(1-\overline{x}) - r\xi(1-\overline{x})))] \\ &= \frac{1}{n+1}\left[\frac{x}{(n+1)(1-r)+r} + ng(\mathbb{P}((n+1)-x))\right] \\ &= \frac{1}{n+1}\left[\frac{x}{(n+1)(1-r)+r} + ng(1-x)\right]. \end{aligned} \tag{32}$$

Therefore we obtain,

$$\begin{aligned} &g\Diamond_n^1\xi(1-\overline{x}) + g\Diamond_n^1\xi(1-\overline{y}) \\ &= \frac{1}{n+1}\left[\frac{x+y}{(n+1)(1-r)+r} + ng(1-x) + ng(1-y)\right] \\ &= \frac{1}{n+1}\left[\frac{x+y}{(n+1)(1-r)+r} + ng(\mathbb{P}(2-x-y))\right]. \end{aligned} \tag{33}$$

We now consider the following two cases:

   (a) $x + y \leq 1$. In this case, $\mathbb{P}(2-x-y) = (1-x-y)$. Then $\frac{1}{n+1}[\frac{x+y}{(n+1)(1-r)+r} + ng(\mathbb{P}(2-x-y))]$ $= \frac{1}{n+1}\left[\frac{x+y}{(n+1)(1-r)+r} + ng(1-x-y)\right] = g\Diamond_n^1\xi(1-\overline{x+y}) = g\Diamond_n^1\xi(1-\overline{x}-\overline{y})$.

   (b) $1 < x+y \leq 2$. In this case, $\mathbb{P}(2-x-y) = (2-x-y)$. First note that $g\Diamond_n^1\xi(x_n) = \frac{1}{(n+1)((n+1)(1-r)+r)}$. Therefore we obtain from (33) that $\frac{1}{n+1}\left[\frac{x+y}{(n+1)(1-r)+r} + ng(\mathbb{P}(2-x-y))\right] = \frac{1}{n+1}[\frac{x+y-1}{(n+1)(1-r)+r} + ng(1-(x+y-1))] + \frac{1}{((n+1)(1-r)+r)(n+1)} = g\Diamond_n^1\xi(1-\overline{x+y-1}) + g\Diamond_n^1\xi(x_n)$. Since $0 \leq x+y-1 \leq 1$, $1-\overline{x+y-1} \geq x_n$. Finally, using Proposition 38 we obtain that $g\Diamond_n^1\xi(1-\overline{x+y-1}) + g\Diamond_n^1\xi(x_n) = g\Diamond_n^1\xi(x_n+1-\overline{x+y-1}) = g\Diamond_n^1\xi(1-\frac{(x+y)(1-r)}{(n+1)(1-r)+r} + \frac{1-r}{(n+1)(1-r)+r} + \frac{n(1-r)+r}{(n+1)(1-r)+r}) = g\Diamond_n^1\xi(1-\overline{x}-\overline{y})$.

□

Next we present the lemma that proves that the operator $\diamondsuit_n^1$ generates facet-defining inequalities for $PI(r,1)$ when the constituent inequality g is facet-defining for $PI(nr,1)$.

Lemma 40 *Let g be a facet-defining inequality for $PI(nr,1)$ for which $E(g)$ is unique up to scaling and $[g]_{nr}$ is nondecreasing, then $g\diamondsuit_n^1\xi$ is facet-defining for $PI(r,1)$ and $E(g\diamondsuit_n^1\xi)$ is unique upto scaling.*

Proof: First note that it can easily be verified from Proposition 23 that $g\diamondsuit_n^1\xi$ is minimal and subadditive. Assume now by contradiction that $g\diamondsuit_n^1\xi$ is not facet-defining. Then there exists a valid subadditive function $\psi \neq g\diamondsuit_n^1\xi$ such that $E(\psi) \supseteq E(g\diamondsuit_n^1\xi)$. By using a variant of Proposition 15 we can verify that ψ is a continuous function.

To obtain a contradiction, we will prove that $\psi = g\diamondsuit_n^1\xi$. This proof will be made in three steps. First, we will use the fact that $E(\psi) \supseteq E(g\diamondsuit_n^1\xi)$ to derive some properties of ψ. In particular, we will show that the value of ψ is same as that of $g\diamondsuit_n^1\xi$ on the intervals $[x_k, y_k]$. Second using this property, a new function $g' : I^1 \to \mathbb{R}_+$ will be constructed from ψ. It will be shown that $E(g') \supseteq E(g)$. This result will in turn be used to prove that $g' = g$ since g is unique up to scaling. Third we will use the fact that $g' = g$ to prove that $\psi = g\diamondsuit_n^1\xi$. This proof also establishes that $E(g\diamondsuit_n^1\xi)$ is unique upto scaling.

First we establish a few properties of ψ. Because $g\diamondsuit_n^1\xi(x) = \frac{x}{r}\ \forall x \in [0,r]$ it is easy to verify using Proposition 12 that $\psi(x) = \frac{x}{r}\ \forall x \in [0,r]$.

Next observe that $g\diamondsuit_n^1\xi(x) = \frac{x}{r} - \frac{k}{(n+1)r}\ \forall x \in [x_k, y_k]$ where $1 \leq k \leq n$. Since $\psi(x) = \frac{x}{r}\ \forall x \in [0,r]$ it is easy to prove using Proposition 12 that $\psi(x) = \frac{x}{r} + c_k\ \forall x \in [x_k, y_k]$ where $1 \leq k \leq n$. We also have from Proposition 38 that $g\diamondsuit_n^1\xi(x_{k_1}) + g\diamondsuit_n^1\xi(x_{k_2}) = g\diamondsuit_n^1\xi(x_{k_1+k_2-(n+1)})$ whenever $k_1 + k_2 \geq n+1$. Because ψ satisfies these equalities, we have that

$$\psi(x_1) + \psi(x_n) = \psi(x_0) \text{ and } \psi(x_2) + \psi(x_n) = \psi(x_1).$$

It follows that $\psi(x_2) = \psi(x_0) - 2\psi(x_n)$. Similarly, it can be verified that

$$\psi(x_j) = \psi(x_0) - j\psi(x_n). \tag{34}$$

Therefore $\psi(x_n) = \psi(x_0) - n\psi(x_n)$, or $\psi(x_n) = \frac{\psi(x_0)}{n+1} = \frac{g\diamondsuit_n^1\xi(x_0)}{n+1}$ where the last equality holds since $x_0 \leq r$. It is easy to verify using part (5) of Proposition 37 that $\frac{g\diamondsuit_n^1\xi(x_0)}{n+1} = g\diamondsuit_n^1\xi(x_n)$. Therefore, $\psi(x_n) = g\diamondsuit_n^1\xi(x_n)$. Finally using (34), we obtain that $\psi(x_j) = \psi(x_0) - j\psi(x_n) = g\diamondsuit_n^1\xi(x_j)$. Thus $c_k = \frac{k}{n+1}\ \forall 1 \leq k \leq n$ and

$$\psi(x) = \frac{x}{r} - \frac{k}{(n+1)r} \quad \forall x \in [x_k, y_k]. \tag{35}$$

Next we construct from ψ a function $g' : I^1 \to \mathbb{R}$ and show that $E(g') \supseteq E(g)$.

The function is defined as

$$g'(1-x) = \frac{(n+1)}{n}\psi(1-\overline{x}) - \frac{x}{n((n+1)(1-r)+r)}. \tag{36}$$

Note first that g' is a continuous function over I^1 since ψ is a continuous function over I^1. Now consider the following

1. $g'(nr) = g'(1-(1-nr)) = \frac{(n+1)}{n}\psi(1-\overline{(1-nr)}) - \frac{1-nr}{n((n+1)(1-r)+r)}$. It can be verified that $1-\overline{(1-nr)} = y_n$. Therefore using (35) we obtain $\psi(1-\overline{(1-nr)}) = \psi(y_n) = \frac{1+(1+n)(n)(1-r)}{(1+n)((1+n)(1-r)+r)}$. Thus, $g'(nr) = \frac{(1+n)n(1-r)+1}{n((n+1)(1-r)+r)} - \frac{1-nr}{n((n+1)(1-r)+r)} = 1$.

2. We next show that $E(g') \supseteq E(g)$. Let $g(1-x) + g(1-y) = g(1-x+y)$ be any equality satisfied by g. We show that g' also satisfy this equality. Now $g'(1-x) + g'(1-y) = \frac{(n+1)}{n}\psi(1-\overline{x}) + \frac{(n+1)}{n}\psi(1-\overline{y}) - \frac{x}{n((n+1)(1-r)+r)} - \frac{y}{n((n+1)(1-r)+r)} = \frac{(n+1)}{n}\psi(1-(\overline{x}+\overline{y})) - \frac{x+y}{n((n+1)(1-r)+r)}$ where the last equality follows from Proposition 39. There are two cases:

(a) $x + y \leq 1$. By definition $g'(1-(x+y)) = \frac{(n+1)}{n}\psi(1-(\overline{x}+\overline{y})) - \frac{x+y}{n((n+1)(1-r)+r)}$, which completes the proof.

(b) $x + y > 1$. By Proposition 38, $g\Diamond_n^1\xi(1-\overline{(x+y-1)}) + g\Diamond_n^1\xi(x_n) = g\Diamond_n^1\xi(1-\overline{x+y})$. Since ψ also satisfies these equalities,

$$\frac{(n+1)}{n}\psi(1-\overline{(x+y-1)}) + \frac{(n+1)}{n}\psi(x_n) = \frac{(n+1)}{n}\psi(1-\overline{x+y}).$$

Using (35) we obtain that $\frac{(n+1)}{n}\psi(x_n) = \frac{1}{n((n+1)(1-r)+r)}$. This completes the proof since

$$\begin{aligned}
&\frac{(n+1)}{n}\psi(1-(\overline{x}+\overline{y})) - \frac{x+y}{n((n+1)(1-r)+r)} \\
&= \frac{(n+1)}{n}\psi(1-\overline{(x+y-1)}) - \frac{x+y-1}{n((n+1)(1-r)+r)} \\
&= g'(1-(x+y-1)).
\end{aligned} \tag{37}$$

Now since g is unique up to scaling and g' is a continuous function with $g'(nr) = g(nr) = 1$ we obtain that $g' = g$. Then (36) and (32) imply that $\psi = g\Diamond_n^1\xi$, which completes the proof that $g\Diamond_n^1\xi$ is facet-defining.

Finally note that since the assumption $E(\psi) \supseteq E(g\Diamond_n^1\xi)$ led to the proof that $\psi = g\Diamond_n^1\xi$, we conclude that $E(g\Diamond_n^1\xi)$ has an unique solution up to scaling. □

In Figure 10(B), we present an example of a projected sequential-merge facet-defining inequality for the one-dimensional group problem. The function is obtained by using Gomory and Johnson [2003] three-slope facet (presented in Figure 10(A)) as the outer function and by choosing $n = 1$. The one-dimensional functions that are obtained using sequential-merge are very different from most known facets of one-dimensional group problems. Most known facets of the one-dimensional group problems have linear segments whose slopes either equal to those at the origin or are aligned in such a way that their linear extrapolation passes through the origin; see Gomory and Johnson [2003], Dash and Günlük [2006a] and Miller et al. [2006]. This observation is illustrated in Figure 10(A) where the middle segment of the function has a linear extrapolation (represented by the dotted line) that passes through the origin. We observe however that there are intervals in I^1 for $g\Diamond_n^1\xi$ where the function has a slope which is neither equal to those of the origin nor has an extrapolation that passes through the origin. The line segment $[p, q]$ in Figure 10(B) provides an example. Thus the family of functions $g\Diamond_n^1\xi$ does not only form a new and large family of facets for $PI(r, 1)$, but also has geometric attributes that are not seen in currently known facets of $PI(r, 1)$.

To conclude this section, we present the main result of this section that uses Lemma 40 to give conditions under which projected sequential-merge inequality are facet-defining when $m > 1$ in (24).

Theorem 41 (Projected Sequential-Merge Theorem) *Let $\phi : I^{m+1} \to \mathbb{R}_+$ be a facet-defining inequality of $PI((r_1, ..., r_{m+1}), m+1)$ that is of the form $g_1\Diamond g_2\Diamond g_3...\Diamond\xi$ where g_i's are piecewise linear, continuous, facet-defining inequalities of $PI(r_i, 1)$ such that $[g_i]$ is nondecreasing and $E(g_i)$ is unique up to scaling. Then $\phi' : I^m \to \mathbb{R}_+$ defined as $g_1\Diamond g_2\Diamond g_3...\Diamond g_m\Diamond_n^1\xi$ is facet-defining for $PI((r_1, ..., r_m/n), m)$.*

Proof: The function $g_m\Diamond_n^1\xi$ is facet-defining for $PI(r_m/n, 1)$ and $E(g_m\Diamond_n^1\xi)$ is unique upto scaling from Lemma 40. Moreover, using Proposition 37 we conclude that $[g_m\Diamond_n^1\xi]$ is nondecreasing. Therefore, using Theorem 32, we obtain that ϕ' is facet-defining for $PI((r_1, ..., r_m/n), m)$. □

6 Conclusion

In this paper we presented a general sequential-merge procedure to produce facet-defining inequalities for high-dimensional group problems using facet-defining inequalities of lower-dimensional group problems. We proved that, under very general conditions, sequential-merge inequalities are facet-defining. These inequalities illustrate that it is possible to strengthen the coefficient of continuous variables of group cuts by considering several constraints simultaneously. In particular, it is possible to obtain inequalities that are not dominated by

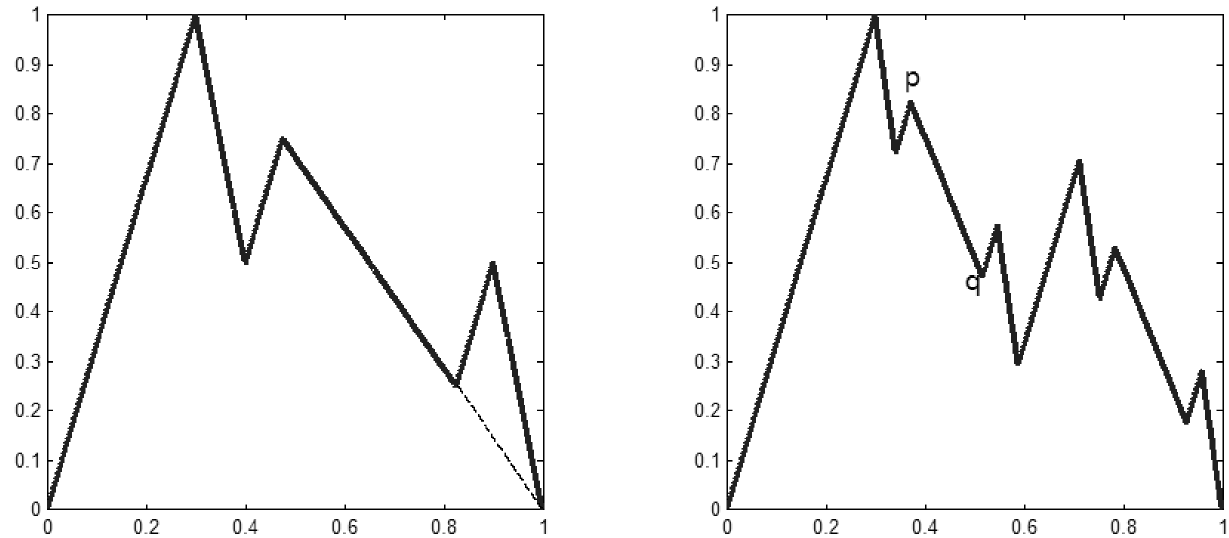

Figure 10: The function g and Sequential-Merge facet $g\Diamond_n^1\xi$ for one-dimensional group problem.

group cuts generated from individual constraints. Sequential-merge inequalities are also interesting because they show strong relations between facet-defining inequalities of one-dimensional and high-dimensional group problems.

We also showed that the sequential-merge approach can be used to generate inequalities for the m-dimensional group problem using inequalities of the $m+1$-dimensional group problem. In particular, we presented conditions under which facets of m-dimensional group problems can be extracted from facets of $m+1$-dimensional group problem. Two interesting observations can be made with respect to these inequalities. The first is that the conditions that guarantee that projected sequential-merge inequalities are facet-defining are restrictive. This is surprising since the conditions for generating sequential-merge inequalities for high-dimensional group problem from lower-dimensional group problem were found to be general. The second observation is that one-dimensional projected sequential-merge inequalities display patterns not seen in currently known facets of one-dimensional group problems.

A few important theoretical and practical questions arise from this paper. The first is that, to date, all the known facet-defining inequalities for high-dimensional group problems can be derived from facet-defining inequalities of one-dimensional group problems. An interesting challenge is to discover group cuts for two- and higher-dimensional infinite group problem that are difficult to derive from one-dimensional group cuts.

Finally, because sequential-merge inequalities are related to the two-step MIR inequalities of Dash and Günlük [2006a] and because these cuts have been shown to be successful in practice (see Dash et al. [2006]), we expect sequential-merge inequalities to also be successful in practice. An important direction of future research is to determine the computational potential of these cuts.

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