MINIMUM WEIGHT t-COMPOSITION OF AN INTEGER

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Abstract. If $p \geq t$ are positive integers, a t-composition of p is an ordered t-tuple of positive integers summing p. If $T = (s_1, s_2, \ldots, s_t)$ is a t-composition of p and W is a $p-(t-1) \times t$ matrix, call $W(T) = \sum_{k=1}^{t} w_{s_k k}$ the weight of the t-composition T. We show that finding a minimum weight t-composition of p reduces to the determination of a shortest path in a certain digraph with $O(tp)$ vertices. This study was motivated by a problem arising from the automobile industry, and the presented result is useful when dealing with huge location problems.

Keywords: compositions of integers, graphs, location problems, combinatorial optimization

1. Introduction and motivation

Location problems consist of selecting among a given set P of points a subset of cardinality p, so to minimize some cost function. The p-median [7] and the p-center [8] are classical examples of location problems. These problems are NP-hard and several heuristic methods have been developed, some of them exploiting the close relationship with another NP-hard problem designated the dominating set problem [6, 9, 10]. Since practical location problems have in general a huge dimension, it is a common procedure to decompose the problem into t smaller location problems [2], and produce a solution from the outcomes of each sub-problem. Each sub-problem introduces m new problems, each consists of selecting $s = 1, 2, \ldots, m$ ($m \leq p - (t-1)$) points among a subset of a given t-partition of P. The final solution results from choosing an outcome of each sub-problem, such that the sum of the cardinalities equals p. The quality of the solution depends, on one hand, on the way the problem is splitted into t sub-problems and, on the other hand,

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on which solution, among the different cardinality solutions produced in each subproblem, should be chosen to contribute to the final set of p points. In this note we will address this second issue.

This study was motivated by a problem arising from the automobile industry called the optimal diversity management problem [1, 2, 3, 4]. For this problem there is a natural decomposition into several problems such that the optimality depends entirely on the way the second issue above is addressed. We now briefly describe this problem.

Cars are purchased with a set of *active options* (airbags, air conditioned, dvd player, etc). Cars that have a given active option are prepared with the connections necessary to have that option activated. A set of active options is called a *configuration*. For technical reasons, it is not possible to produce a large variety of different configurations. Therefore, in general, cars are assembled with more active options than those that the customers have ordered. Since the global cost of adding active options is considerably high, it is essential to choose a set of, say p, different configurations that minimizes the total cost of the unnecessary options. This problem was modelled in [1] in the following way. Consider the *inclusion relation configuration digraph* $\overrightarrow{G} = (V, A)$, where the vertices are configurations and an arc $(i, j) \in A$ means that the configuration i includes the configuration j (each active option in configuration j is also active in configuration i). Note that this digraph is acyclic (that is, has no direct cycles) and it is arc transitive (that is, if $(i, j), (j, k) \in A$, then $(i, k) \in A$). A *spanning star forest* (SSF) of $\overrightarrow{G}$ is spanning subdigraph where each component is a star. Here, we consider that a star has a center, which is a vertex with zero in-degree, and all other vertices, if any, have zero out-degree and one in-degree. An example of an inclusion relation configuration digraph $\overrightarrow{G}$ and a SSF of $\overrightarrow{G}$ are depicted in Fig. 1.

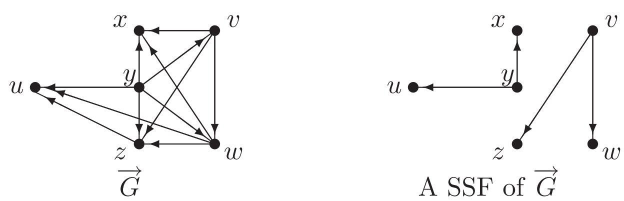

Figure 1.
An inclusion relation configuration digraph $\overrightarrow{G}$ and a SSF of $\overrightarrow{G}$.

Let c_v be the unit production cost of configuration v (i.e, the sum of the costs of all the active options in configuration v), and let n_v denote the expected number of cars with configuration v that will be sold. To each each arc (i, j) of $\overrightarrow{G}$ we assign the weight $w_{ij} = n_j(c_i - c_j)$. Notice that, for each arc (i, j), $w_{ij} > 0$ since every active option has a positive cost, configuration i strictly contains configuration j, and it is assumed that $n_i > 0$, otherwise configuration i would not be considered. The optimal diversity management problem with p configurations consists of finding a minimum weight arc sum SSF with p stars of $\overrightarrow{G}$.

In real applications, the inclusion relation configuration digraph $\overrightarrow{G}$ has several connected components $\overrightarrow{G}_1, \ldots, \overrightarrow{G}_t$. Thus, a minimum weight arc sum SSF of $\overrightarrow{G}$ with p stars consists of a minimum weight arc sum SSF with s_i stars of each $\overrightarrow{G}_i$, where s_i are such that $p = s_1 + \cdots + s_t$. This introduces a new problem which is how to choose the right s_i for each $\overrightarrow{G}_i$. We address this problem in a more general context in the next section.

2. The minimum weight of a t-composition of p

If p and t are positive integers, with $p \geq t$, a t-composition of p is an ordered t-tuple of positive integers summing p. The tuples $(2,3,1,2)$, $(3,2,2,1)$ and $(1,1,5,1)$ are examples of distinct 4-compositions of 8. We use $\mathbb{T}$ to denote the set of all t-compositions of p, that is,

$$\mathbb{T} = \{(s_1, \ldots, s_t) \in \mathbb{N}^t : \sum_{k=1}^{t} s_k = p\}.$$

For $k = 1, \ldots, t$, let $W_k = \{w_{1k}, w_{2k}, \ldots, w_{m_k k}\}$ be a collection of $m_k \leq p-(t-1)$ real numbers, and assign to every t-composition $T = (s_1, s_2, \ldots, s_t) \in \mathbb{T}$, with $s_k \leq m_k$, the weight given by

$$W(T) = \sum_{k=1}^{t} w_{s_k k},$$

referred as the weight of composition T.

We consider the problem of determining a minimum weight t-composition of p, that is, finding $(s_1^*, \ldots, s_t^*) \in \mathbb{T}$, such that

$$\sum_{k=1}^{t} w_{s_k^* k} = \min_{(s_1,\ldots,s_t)\in\mathbb{T}} \sum_{k=1}^{t} w_{s_k k}.$$

For simplicity we assume that all collections W_k have cardinality $m_k = p-(t-1)$. In this way W_k can be viewed as the k-column of a $p-(t-1) \times t$ weight matrix W. Note that there is no loss of generality since large weights can be assigned to every entry w_{ik} of W whenever $i > m_k$.

With respect to the matrix W of Table 1, the weights of the 4-compositions $(2,3,1,2)$, $(3,2,2,1)$ and $(1,1,5,1)$ are 38, 40 and 39, respectively. Later we will show that $(3,1,2,2)$ (the 4-composition of 8 highlighted in the table) is of minimum weight.

1	12	$\boxed{12}$	13	12
2	9	11	$\boxed{10}$	$\boxed{8}$
3	$\boxed{7}$	8	9	6
4	6	7	6	4
5	4	5	3	2

Table 1. A weight matrix W.

3. The algorithm

We show that identifying a minimum weight t-composition of p, with respect to matrix W, is equivalent to determine a shortest path between two vertices of a $(t+1)$-partite weighted digraph $\overrightarrow{G}_W$ constructed as follows.

- The vertex set $V(\overrightarrow{G}_W)$ is partitioned into the vertex-subsets $V_0, V_1, \ldots, V_t$, where V_0 and V_t include a single vertex labeled 0 and p, respectively. For $k = 1, 2, \ldots t-1$, the vertices of V_k are labeled $k, k+1, \ldots, k+p-t$.
- For $k = 1, 2, \ldots, p$, there is an arc $uv \in E(\overrightarrow{G}_W)$ connecting vertices $u \in V_{k-1}$ and $v \in V_k$ if and only if $u < v$. No further arcs exist.
- The weight of each arc $uv \in E(G_W)$, with $u \in V_{k-1}$ and $v \in V_k$, is equal to the entry $w_{(v-u)k}$ of matrix W.

The $0, t$-paths in $\overrightarrow{G}_W$ and the t-compositions of p are in one to one correspondence. The path $P = (0, v_1, v_2, \ldots, v_{t-1}, p)$ corresponds to the k-composition $(v_1, v_2 - v_1, \ldots, v_{t-1} - v_{t-2}, p - v_{t-1})$, and the weight of P is equal to $w_{v_1 1} + w_{v_2 - v_1 2} + \cdots + w_{v_{t-1} - v_{t-2} t-1} + w_{p - v_{t-1} t}$. Thus, a shortest $0, p$-path of $\overrightarrow{G}_W$ corresponds to a minimum weighted t-composition of p.

More formally, the algorithm for determining a minimum weight t-composition of p can be described as follows.

Algorithm (minimum weight t-composition of integer p)

Input: $t, p \in \mathbb{N}$, with $t \leq p$, and a $p-(t-1) \times t$ matrix W of weights;

1. Construct the $(t+1)$-partite digraph $\overrightarrow{G}_W$ as follows:
   $V_0 := \{0\}$;
   For $k = 1$ to $t-1$ **do** $V_k := \{k, k+1, \ldots, k+p-t\}$;
   $V_t := \{p\}$;
   $V(\overrightarrow{G}_W) := V_0 \cup V_1 \cup \cdots \cup V_t$;
   For $k = 1$ to t **do** $E_k := \{uv : u \in V_{k-1}, v \in V_k, u < v\}$;
   $E(\overrightarrow{G}_W) := E_1 \cup E_2 \cup \cdots \cup E_t$;
2. Determine a shortest path in $\overrightarrow{G}_W$, between the vertices 0 and p:
   $$0 = v_0 \to v_1 \to \cdots \to v_{t-1} \to v_t = p;$$
3. **For** $k = 1$ to t **do** $s_k^* := v_k - v_{k-1}$;
   Return $T^* = (s_1^*, \ldots, s_t^*)$.

The minimum weight is $W(T^*) = \sum_{k=1}^{t} w_{s_k^* k}$.

4. A numerical example

Let us use the algorithm of the previous section to determine a minimum weight 4-composition of 8 with respect to the weight matrix W of Table 1.

The 5-partite digraph $\overrightarrow{G}_W$ constructed according to the step 1 of the algorithm is depicted in Figure 2. The reduced size numbers are the arc weights, and the remaining numbers are the labels of the vertices.

To perform step 2 we used a specialized version of Dijkstra's algorithm [5]. Table 2 reports the values produced by that algorithm on the digraph $\overrightarrow{G}_W$ of Figure 2. Column k records, for every vertex $v \in V_k$, the weight w_v^k of a shortest $0, v$-path (of length k), together with the vertex $p(v)$ that precedes v on that path. Note that

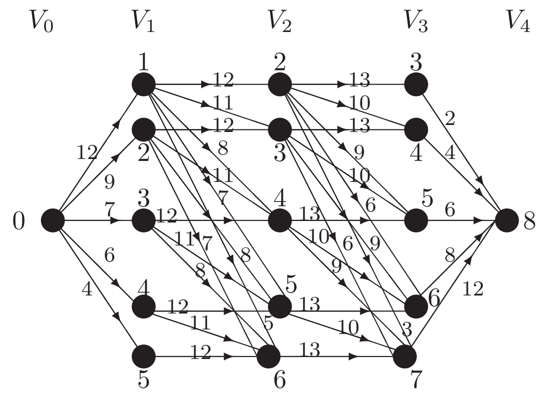

FIGURE 2.
Digraph $\overrightarrow{G}_W$ with edge weights calculated from the matrix W of Table 1

for determining w_v^k and $p(v)$ only the values of w_u^{k-1} and $p(u)$, with $u \in V_{k-1}$ and $uv \in E(\overrightarrow{G}_W)$, are needed. Hence, a shortest $0, t$-path is obtained in $O(E(\overrightarrow{G}_W))$ time.

	1		2		3		4	
vertex v	w_v	$p(v)$	w_v	$p(v)$	w_v	$p(v)$	w_v	$p(v)$
1	12	0						
2	9	0	24	1				
3	7	0	21	2	37	2		
4	6	0	19	3	34	3		
5	4	0	17	2	31	3		
6			15	3	29	4		
7					27	5		
8							37	6

TABLE 2. Determination of the shortest $0, 8$-path of the digraph of Figure 2, using an adaptation of Dijkstra's algorithm.

Dijkstra's algorithm outcomes 37 as the weight of the shortest $0, 8$-path $0 \to 3 \to 4 \to 6 \to 8$. From the above path the algorithm defines the 4-composition $T^* = (s_1^*, s_2^*, s_3^*, s_4^*) = (3, 1, 2, 2)$ of 8, which is a minimum 4-composition of 8. Note that another optimal composition can be derived from the alternative shortest $0, 8$-path $0 \to 2 \to 3 \to 5 \to 8$.

5. CONCLUSION

Motivated by the optimal diversity management problem from automobile industry, where is very common to deal with a inclusion relation configuration digraph with t components, the minimum weight t-composition of a positive integer p is introduced and it is reduced to the determination of a shortest path between the vertices 0 and p of a digraph $\overrightarrow{G}_W$ which has $(t-1)(p-(t-1))+2$ vertices and $\frac{t-2}{2}((p-(t-1))^2+(p-(t-1)))+2(p-(t-1))$ arcs. This result can help in solving location problems, such as the one described above, putting together solutions obtained from different sub-problems.

References

[1] A. Agra, D.M. Cardoso, J.O. Cerdeira, M. Miranda and E. Rocha, "The minimum weight spanning star forest model of the optimal diversity management problem", Cadernos de Matemática, Universidade de Aveiro CM07(I-08), 2007.

[2] P. Avella, M. Boccia, C. D. Martino, G. Oliviero, A. Sforza, "A decomposition approach for a very large scale optimal diversity management problem," *4OR* **3**(1), 23–37, 2005.

[3] O. Briant, "Étude théorique et numérique du problème de la gestion de la diversité," PhD thesis, Institut National Polytechnique de Grenoble, 2000.

[4] O. Briant, D. Naddef, "The optimal diversity management problem," *Operations Research* **52**(4), 515–526, 2004.

[5] E.W. Dijkstra, "A note on two problems in connexion with graphs," *Numerische Mathematik* **1**, 269-271, 1959.

[6] D. Hochbaum and D. Shmoys, "A best possible heuristic for the k-center problem," *Math Oper Res* **10**, 180-184, 1985.

[7] P. Jarvinen, J. Rajala, H. Sinervo, "A branch and bound algorithm for seeking the p-median," *Operations Research*, **20**, 173–178, 1972.

[8] O. Kariv and S.L. Hakini, "An algorithmic approach to network location problems. Part 1: The p-Centers," SIAM J. Appl. Math. **37**, 513-538, 1979.

[9] J. Martinich, "A vertex-closing approach to the p-Center problem," *Naval Res Log*, **35**, 185-201, 1988.

[10] J. Plesnik, "A heuristic for the p-Center problem in graphs," *Discrete Appl Math*, **17**, 263-268, 1987.