A GRADIENT-BASED APPROACH FOR COMPUTING NASH EQUILIBRIA OF LARGE SEQUENTIAL GAMES

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ABSTRACT. We propose a new gradient based scheme to approximate Nash equilibria of large sequential two-player, zero-sum games. The algorithm uses modern smoothing techniques for saddle-point problems tailored specifically for the polytopes used in the Nash equilibrium problem.

1. Introduction

Equilibrium problems have long attracted the interest of game theorists and optimizers. The Nash equilibrium problem for two-player zero-sum sequential games can be modeled using linear programming [12]. In principle, these linear programs can be solved using any state-of-the-art linear programming solver (see Section 2). However, for most interesting games, the size of the game tree and the corresponding linear program is enormous.

This problem seems to be particularly challenging from an optimization perspective. The polytope in the linear programming formulation tends to be highly degenerate. This is one of the reasons active-set methods seem to perform poorly on this class of problems [2]. Interior-point methods, despite their great iteration complexity do not fare any better; they pay an enormous per iteration price arising from the matrix factorizations required at each iteration. For the size of games we are interested in solving, it is unlikely that an interior-point method will be able to complete even a single iteration in a reasonable amount of time.

Our approach sidesteps these difficulties by solving the basic saddle-point version of the problem using a smoothing technique proposed by Nesterov [7, 8]. The smoothing step involves perturbing the saddle point problem in order to transform it into a minimization problem of a convex function with Lipschitz gradient. The latter problem in turn is amenable to an efficient gradient-based method. This method has minimal memory requirements and a cost per iteration that is linear in the size of the game tree. In Section 3 we provide a summary of Nesterov's recent results [7, 8].

At the heart of our approach is the construction of a family of prox functions for the polytopes arising in the saddle-point formulation of zero-sum two-player sequential games. These prox functions provide the perturbation used in the smoothing portion of the algorithm. Moreover, the performance of the algorithm depends entirely on finding appropriate prox functions for the constraint set of the problem. In Section 4 we present an inductive construction of *nice* prox functions for sequential games based on a particular "lifting" operation used in convex analysis. The lifting operation produces a simple recursion involving subproblems over much simpler sets.

We conclude by considering games with a particular *uniform structure*. For these games we prove that the number of iterations required for obtaining an ϵ-solution is $O(T^2/\epsilon)$ where T is sublinear in the size of the game tree.[1]

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[1] Since the solutions we compute have a duality gap of ϵ, we are actually computing ϵ-Nash equilibria in which each player's incentive to deviate from the strategy is at most ϵ. In fact, our solutions are ϵ-minimax solutions since each player's payoff guarantee does not depend on the opponent playing any particular strategy.

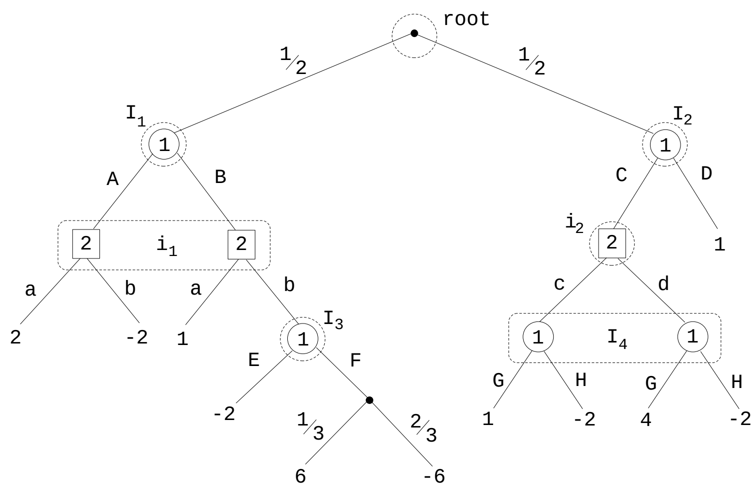

FIGURE 1. A graphical depiction of a two-player sequential game. The dotted lines enclose decision nodes in the same information set. The labels on the decision nodes indicate which player is making the decision while the labels on the edges indicate the choices. Chance nodes are unlabeled and the edges are labeled with the probability of each chance move occurring.

2. Sequential games

In this section we review sequential games. In particular, we describe the sequence form of a game and the resulting linear programming formulation [12], focusing on the realization polytope that arises from this formulation.

2.1. **Preliminaries.** A finite sequential game is represented as an *extensive form game* [9]. The *root* of the tree represents the starting point of the game while the *leaves* represent the termination points. The remaining internal nodes are the *decision nodes*; they indicate points in the game where one of the players (possibly the chance player) has to make a choice. The edges of the game tree are oriented from the root towards the leaves. Each edge leaving a particular decision node is labeled with the player's choice.

A game tree may contain decision nodes that model random events in the game. It is customary to associate these nodes with a fictitious *chance player*. The choices available to the chance player are called *chance moves*, which allow the modeling of situations in which the next branch of the game tree is reached randomly according to the probabilities dictated by the game.

The decision nodes are partitioned into *information sets* [5, 9]. Each information set is associated with exactly one player; moreover, the player has exactly the same set of choices at every decision node in an information set. We assume, without loss of generality, that the choices at distinct information sets are distinct. If u is an information set then C_u denotes the set of choices at u.

Example 1. Figure 1 illustrates a sequential game with two players. The information sets belonging to the first player are $I_1, \ldots, I_4$ and the information sets belonging to the second player are i_1 and i_2. Observe that nodes belonging to the same information set have the same set of choices and that the set of choices corresponding to distinct information sets are disjoint. For example, $C_{i_1} = \{a, b\}$ and $C_{i_2} = \{c, d\}$. □

2.2. **Sequence form.** The *sequence form* of a game leads to a concise representation of strategies. For two-player zero-sum games, this representation yields a concise linear programming formulation [3, 10, 11, 12].[2] Each node in the game tree determines a unique sequence of choices from the root to that node for each of the players. A *sequence* for a player is represented by a string consisting of the labels on the edges corresponding to the choices made by the player and in the order the choices were made. The number of sequences is thus bounded by the number of nodes in the game tree.

Let U and S be the set of all information sets and the set of all sequences belonging to one of the players. The empty sequence ε is the sequence in which the player has not yet made a choice. Let $\sigma\colon U \to S$ be the function that maps each information set $u \in U$ to the sequence of choices made by the player to reach u. We denote the image $\sigma(u)$ by σ_u. The set of all sequences S associated with the player can be expressed as

$$S := \{\varepsilon\} \cup \{\sigma_u c\colon u \in U, c \in C_u\}.$$

Note that σ is not necessarily one-to-one since many information sets may be reached by the same sequence of choices by the player. Two different information sets are *parallel* if they both belong to the same player and are preceded by the same sequence of choices by that player, *i.e.* the information sets u and $v \in U$ are parallel if $\sigma_u = \sigma_v$.

Example 2. The set of all information sets belonging to the first and second player in Figure 1 are

$$U_1 = \{I_1, I_2, I_3, I_4\} \quad \text{and} \quad U_2 = \{i_1, i_2\}.$$

The set of all sequences for the first and second player are

$$S_1 = \{\varepsilon, A, B, C, D, BE, BF, CG, CH\} \quad \text{and} \quad S_2 = \{\varepsilon, a, b, c, d\}.$$

The sequence of choices made by the first player to reach the information set I_1 is $\sigma_{I_1} = \varepsilon$ while to reach information set I_3, the sequence is $\sigma_{I_3} = B$. Note that concatenating B to the empty sequence results in the string B. The string BF corresponds to the sequence in which the first player makes the choice B followed by the choice F. This sequence does not lead to another information set for the first player but to one of two terminal nodes (leaves) determined by a chance move after the first player chooses F.

Observe that $\sigma_{I_1} = \sigma_{I_2} = \varepsilon$ and both information sets belong to the first player; thus I_1 and I_2 are parallel information sets. Finally observe that I_3 succeeds I_1 while I_4 succeeds I_2 and $\sigma_{I_3} = B \neq C = \sigma_{I_4}$. □

A *realization plan* for the player is any vector $\mathbf{x} \in \mathbb{R}^S_+$ such that

$$x(\varepsilon) = 1 \quad \text{and} \quad x(\sigma_u) = \sum_{c \in C_u} x(\sigma_u c) \quad \text{for all } u \in U. \tag{1}$$

The *complex* (or *realization polytope*) associated with a player is the set of realization plans for that player. Realization plans induce *reduced behavioral strategies* which specify a probability distribution over choices at all information sets except those for which a previous choice in the path to the root has probability zero. The complex is the convex hull of realization plans that induce deterministic reduced behavioral strategies [12]. Furthermore, the extreme points of the complex are 0-1 vectors. There are exactly $1 + |U|$ equations defining the complex for the player. This system of equations has a coefficient matrix of dimension $(1 + |U|) \times |S|$ with entries -1, 0 and 1. The vector on the right-hand side is all zeros except for the first entry which is equal to one.

Example 3. Continuing with the game shown in Figure 1, the complex corresponding to the first player is $Q_1 := \{\mathbf{x} \in \mathbb{R}^{S_1}_+ : E\mathbf{x} = \mathbf{e}\}$, where

$$E = \begin{array}{c} \\ \text{root} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{array} \begin{array}{c} \begin{array}{ccccccccc} \varepsilon & A & B & C & D & BE & BF & CG & CH \end{array} \\ \left[\begin{array}{ccccccccc} 1 & & & & & & & & \\ -1 & 1 & 1 & & & & & & \\ -1 & & & 1 & 1 & & & & \\ & & -1 & & & 1 & 1 & & \\ & & & -1 & & & & 1 & 1 \end{array}\right] \end{array}, \quad \mathbf{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} \text{root} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}.$$

[2]The sequence form representation requires that the players have *perfect recall*; this means that all the decision nodes in a player's information set are preceded by the same sequence of moves by that player. We assume that perfect recall holds in all the results that follow.

The missing entries in the coefficient matrix are zero. Similarly, the complex corresponding to the second player is $Q_2 := \{\mathbf{y} \in \mathbb{R}_+^{S_2} : F\mathbf{y} = \mathbf{f}\}$, where

$$
\begin{array}{c}
\\ \\ F = \\ \\
\end{array}
\begin{array}{c}
\\ \\ \text{root} \\ i_1 \\ i_2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \varepsilon & a & b & c & d \end{array} \\ \\
\left[\begin{array}{ccccc}
1 & & & & \\
-1 & 1 & 1 & & \\
-1 & & & 1 & 1
\end{array}\right]
\end{array},
\quad
\mathbf{f} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]
\begin{array}{c} \text{root} \\ i_1 \\ i_2 \end{array}.
$$

□

Let S_1 and S_2 be the set of all sequences for the first and second player respectively. Then the payoffs at the leaves for the first player can be represented by a matrix in $\mathbb{R}^{S_1 \times S_2}$, that is, a matrix whose rows are indexed by the sequences in S_1 and whose columns are indexed by the sequences in S_2. Each player has their own payoff matrix and in the case of zero-sum games the payoff matrices are negatives of one another.

Consider a pair of sequences $(s, t) \in S_1 \times S_2$. Suppose that the pair (s, t) leads to a leaf. Then the (s, t)-entry in the matrix is the sum, over all the leaves reachable by the pair of sequences (s, t), of the payoff at each leaf multiplied by the probability of chance moves (if any) leading to that leaf. Otherwise, if the pair of sequences (s, t) do not lead to a leaf then the (s, t)-entry in the payoff matrix is zero.

Example 4. The payoff matrix for the first player in the game shown in Figure 1 is

$$
\mathcal{A} =
\begin{array}{c}
\varepsilon \\ A \\ B \\ C \\ D \\ BE \\ BF \\ CG \\ CH
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \varepsilon & a & b & c & d \end{array} \\
\left[\begin{array}{ccccc}
 & & & & \\
 & 1 & -1 & & \\
 & 1/2 & & & \\
 & & & & \\
1/2 & & & & \\
 & & -1 & & \\
 & & -1 & & \\
 & & & 1/2 & 2 \\
 & & & -1 & -1
\end{array}\right]
\end{array}.
$$

For example, to calculate the (B, a)-entry in the matrix we look at all leaves with the sequence (B, a). Since there is a single leaf with this sequence, we multiply the payoff at that leaf by any chance moves along the path to the leaf. The payoff is 1 with a single chance move that occurs with probability 1/2; thus the (B, a)-entry is 1/2.

Observe that there are two leaves corresponding to the sequences (BF, b). Thus the (BF, b)-entry in the matrix is the sum of the payoffs at both leaves multiplied by the probability of the chance moves along the paths leading to the leaves. The calculation for the (BF, b)-entry in the matrix is

$$[\mathcal{A}]_{(BF,b)} = \frac{1}{2}\left(\frac{1}{3}\right)(6) + \frac{1}{2}\left(\frac{2}{3}\right)(-6) = -1.$$

□

3. Smoothing techniques

In this section we describe Nesterov's approach as it applies to our problem. Throughout the paper we assume that the matrix A contains the payoffs for the first player.

Let S_1 and S_2 denote the set of all sequences that can be played by the first and second player respectively. Identify $Q_1 \subseteq \mathbb{R}^{S_1}$ and $Q_2 \subseteq \mathbb{R}^{S_2}$ as the complexes associated with the players. Suppose that the first player picks a realization plan $\mathbf{x}$. The second player's best response is the solution to

$$f(\mathbf{x}) = \min_{\mathbf{y} \in Q_2} \langle A\mathbf{y}, \mathbf{x} \rangle. \tag{2}$$

The first player's goal is to maximize her payoff, that is, she wants to solve the problem $\max\{f(\mathbf{x})\colon \mathbf{x} \in Q_1\}$. Similarly, if the second player chooses a realization plan $\mathbf{y}$ then his opponent's best response is the solution to

$$\phi(\mathbf{y}) = \max_{\mathbf{x}\in Q_1} \langle A\mathbf{y}, \mathbf{x}\rangle. \tag{3}$$

Consequently, the second player's optimization problem is $\min\{\phi(\mathbf{y})\colon \mathbf{y} \in Q_2\}$. Using linear programming duality [12], it follows that

$$\max_{\mathbf{x}\in Q_1} f(\mathbf{x}) = \max_{\mathbf{x}\in Q_1} \min_{\mathbf{y}\in Q_2} \langle A\mathbf{y}, \mathbf{x}\rangle = \min_{\mathbf{y}\in Q_2} \max_{\mathbf{x}\in Q_1} \langle A\mathbf{y}, \mathbf{x}\rangle = \min_{\mathbf{y}\in Q_2} \phi(\mathbf{y}). \tag{4}$$

The problem stated in (4) is the saddle-point formulation of the Nash equilibrium problem for zero-sum two-player games.

Next fix two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ associated with $\mathbb{R}^{S_1}$ and $\mathbb{R}^{S_2}$ (these can be *any* two norms). Given a norm $\|\cdot\|_1$ for $\mathbb{R}^{S_1}$, the standard norm for the dual space $(\mathbb{R}^{S_1})^*$ is defined as

$$\|\mathbf{u}\|_1^* = \max_{\|\mathbf{x}\|_1=1} \left\{\langle \mathbf{u}, \mathbf{x}\rangle_1 \colon \mathbf{x} \in \mathbb{R}^{S_1}\right\}.$$

We will consider A as a linear map from $\mathbb{R}^{S_2}$ to $(\mathbb{R}^{S_1})^*$. Then the vector norms $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the following operator norm

$$\|A\|_{2,1} = \max\ \{\langle A\mathbf{y}, \mathbf{x}\rangle_1 \colon \|\mathbf{x}\|_1 \le 1, \|\mathbf{y}\|_2 \le 1\}.$$

We will not explicitly indicate the norm when it can be inferred from the context.

Recall that a function $d\colon \mathbb{R}^n \to \mathbb{R}$ is *strongly convex* on a convex subset $X \subseteq \mathbb{R}^n$ if there exists $\rho > 0$ such that

$$d(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \le \alpha d(\mathbf{x}) + (1-\alpha)d(\mathbf{y}) - \frac{1}{2}\rho\alpha(1-\alpha)\|\mathbf{x}-\mathbf{y}\|^2 \quad \text{for all } \alpha \in [0,1] \text{ and all } \mathbf{x}, \mathbf{y} \in X. \tag{5}$$

If d is also differentiable in X then each of the following conditions is equivalent to (5) for the same $\rho > 0$ [6, Theorem 2.1.9]

$$d(\mathbf{y}) \ge d(\mathbf{x}) + \langle \nabla d(\mathbf{x}), \mathbf{y} - \mathbf{x}\rangle + \frac{1}{2}\rho\|\mathbf{x}-\mathbf{y}\|^2 \quad \text{for all } \mathbf{x}, \mathbf{y} \in X, \tag{6}$$

and,

$$\langle \nabla d(\mathbf{x}) - \nabla d(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle \ge \rho\|\mathbf{x}-\mathbf{y}\|^2 \quad \text{for all } \mathbf{x}, \mathbf{y} \in X. \tag{7}$$

The constant ρ is the *strong convexity parameter* of $d(\mathbf{x})$.

Assume that d_1 and d_2 are two strongly convex functions over Q_1 and Q_2 respectively with strong convexity parameters ρ_1 and ρ_2, and their minimum values are zero over their respective domains. Let

$$D_1 = \max_{\mathbf{x}\in Q_1} d_1(\mathbf{x}) \quad \text{and} \quad D_2 = \max_{\mathbf{y}\in Q_2} d_2(\mathbf{y}).$$

Theorem 3.1. (Nesterov, [8]) *There exists*[3] *an algorithm that creates a sequence of points* $(\mathbf{x}^k, \mathbf{y}^k) \in Q_1 \times Q_2$ *such that*

$$0 \le \phi(\mathbf{y}^k) - f(\mathbf{x}^k) \le \frac{4\,\|A\|}{k+1}\sqrt{\frac{D_1 D_2}{\rho_1\rho_2}}. \tag{8}$$

Each iteration of the algorithm needs to perform some elementary operations, three matrix-vector multiplications by A and requires the exact solution of three subproblems of the form

$$\max_{\mathbf{x}\in Q_1} \{\langle \mathbf{g}, \mathbf{x}\rangle - d_1(\mathbf{x})\} \quad \textit{or} \quad \max_{\mathbf{y}\in Q_2} \{\langle \mathbf{g}, \mathbf{y}\rangle - d_2(\mathbf{y})\}. \tag{9}$$

The solutions to the subproblems (9) are critical to the performance of the algorithm used in Theorem 3.1 since they are solved at each iteration. Consider a function d defined over a compact convex set $Q \subseteq \mathbb{R}^n$. We say that d is a *nice* prox function for Q if it satisfies the following three conditions:

(i) d is strongly convex and continuous everywhere in Q and is differentiable in the relative interior of Q.

[3] We include a reification of this existence result in Appendix A.

(ii) The minimum value that d attains in Q is zero.
(iii) The maps $\text{smax}(d, \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ and $\text{sargmax}(d, \cdot)\colon \mathbb{R}^n \to Q$ defined as

$$\text{smax}(d, \mathbf{g}) := \max_{\mathbf{x}\in Q} \{\langle \mathbf{g}, \mathbf{x}\rangle - d(\mathbf{x})\}$$
$$\text{sargmax}(d, \mathbf{g}) := \operatorname*{argmax}_{\mathbf{x}\in Q} \{\langle \mathbf{g}, \mathbf{x}\rangle - d(\mathbf{x})\}$$

have easily computable expressions. Ideally, the solutions should be available in closed-form (in terms of $\mathbf{g}$).

Assume that d is a nice prox function with strong convexity parameter ρ and attains a maximum value of D over Q. From the convergence bound stated in Theorem 3.1 we choose to measure the *niceness* of the prox function d by the ratio ρ/D. It is important to note that ρ depends on the specific vector norm used in the analysis. Thus one needs to be careful when making statements using this measure. For example, simply scaling the vector norm can increase the ratio ρ/D. However, this scaling causes an inflation in the induced operator norm and may not result in a more accurate convergence bound. We discuss this in more detail in Section 5.

Example 5. This example is taken from Nesterov [7]. Consider the simplices $\Delta_m \subseteq \mathbb{R}^m$ and $\Delta_n \subseteq \mathbb{R}^n$. The entropy prox functions (or entropy distances) over both simplices are

$$d_1(x_1, \ldots, x_m) = \ln m + \sum_{i=1}^m x_i \ln x_i, \quad \text{and} \quad d_2(y_1, \ldots, y_n) = \ln n + \sum_{i=1}^n y_i \ln y_i.$$

The functions d_1 and d_2 are strongly convex and continuous in Δ_m and Δ_n respectively. The functions are also differentiable in the relative interiors of the simplices. The strong convexity parameters $\rho_1 = \rho_2 = 1$ are obtained using the 1-norm, i.e., $\|\mathbf{x}\| = \sum_{i=1}^m |x_i|$ and $\|\mathbf{y}\| = \sum_{i=1}^n |y_i|$. This pair of vector norms induce the following operator norm of any linear map $A\colon \mathbb{R}^n \to (\mathbb{R}^m)^*$

$$\|A\| = \max_{i,j} |A_{ij}|.$$

The functions d_1 and d_2 also attain a minimum value of zero and have the following niceness parameters

$$\frac{\rho_1}{D_1} = \frac{1}{\ln m} \quad \text{and} \quad \frac{\rho_2}{D_2} = \frac{1}{\ln n}.$$

The functions $\text{smax}(d_1, \mathbf{g})$ and $\text{sargmax}(d_1, \mathbf{g})$ have the following closed-form expressions

$$\text{smax}(d_1, \mathbf{g}) = \ln \sum_{i=1}^m e^{g_i}, \quad \text{and} \quad \text{sargmax}(d_1, \mathbf{g})_i = \frac{e^{g_i}}{\sum\limits_{i=1}^m e^{g_i}}.$$

Another prox function for simplices is the (squared) Euclidean distance from the center of the simplex, that is,

$$d_1(x_1, \ldots, x_m) = \sum_{i=1}^m \left(x_i - \frac{1}{m}\right)^2, \quad \text{and} \quad d_2(y_1, \ldots, y_n) = \sum_{i=1}^n \left(y_i - \frac{1}{n}\right)^2.$$

Both d_1 and d_2 are strongly convex, continuous and differentiable in Δ_m and Δ_n respectively. The strong convexity parameters $\rho_1 = \rho_2 = 1$ are obtained using the Euclidean norm. The induced operator norm for any linear map $A\colon \mathbb{R}^n \to (\mathbb{R}^m)^*$ is the spectral norm, that is

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}(A^T A)$ is the largest eigenvalue of $A^T A$. Note that the spectral norm is typically much larger than the induced norm for the entropy functions. The functions d_1 and d_2 attain a minimum value of zero and have the following niceness parameters

$$\frac{\rho_1}{D_1} = \frac{1}{\left(1 - \frac{1}{m}\right)} = \frac{m}{m-1} \quad \text{and} \quad \frac{\rho_2}{D_2} = \frac{1}{\left(1 - \frac{1}{n}\right)} = \frac{n}{n-1}.$$

The subproblems $\text{smax}(d_1, \mathbf{g})$ and $\text{sargmax}(d_1, \mathbf{g})$ are easily computable though they do not have simple closed-form expressions. We note that these subproblems can be solved in time linear in the dimension of the simplex. A detailed discussion on using the Euclidean prox function over simplices can be found in [1].

□

4. Prox functions for the complex

In this section we provide a general procedure to convert a family of nice prox functions for simplices into a nice prox function for the complex. We will work with the complex for a particular player and as such we will assume that the information sets, choices and sequences all correspond to that same player.

4.1. **The structure of complexes.** Let U, $\{C_u : u \in U\}$ denote the set of information sets and sets of choices for a particular player. For $i \geq 0$ define U^i as the subset of all information sets that the player can be in after making exactly i choices in the game. Thus if the player makes at most t choices in the game then U is partitioned as $U = U^0 \cup \cdots \cup U^{t-1}$.

Recall that for each information set $u \in U^i$ we use σ_u to denote the unique sequence of choices made by the player to reach u. Let S denote the set of all sequences of the player in the game. Then the partition of the information sets stated above induces the following natural partition of the set of sequences: $S = S^0 \cup \cdots \cup S^t$ where

$$S^i := \begin{cases} \{\varepsilon\} & \text{if } i = 0, \\ \{\sigma_u c \colon c \in C_u,\, u \in U^{i-1}\} & \text{if } i > 0. \end{cases}$$

The *length* of a sequence is the number of choices the player makes in that sequence. Thus the set S^i is the set of sequences of length i. We use $S^{[i]}$ to indicate the set of all sequences of length at most i, that is, $S^{[i]} := S^0 \cup S^1 \cup \cdots \cup S^i$. Similarly, $U^{[i-1]}$ indicates the union $U^0 \cup \cdots \cup U^{i-1}$. Now define $Q_i(U,S)$ as the complex for a particular player with sequences of length less than or equal to i, that is

$$Q_i(U,S) := \left\{ \mathbf{x} \in \mathbb{R}_+^{S^{[i]}} : x(\varepsilon) = 1,\, x(\sigma_u) = \sum_{c \in C_u} x(\sigma_u c),\ \text{for all } u \in U^{[i-1]} \right\}. \tag{10}$$

If a player makes at most t choices in the game then the polytope $Q_t(U,S)$ describes the set of all the player's realization plans. For all nonnegative integers $i \leq t$, the set $Q_i(U,S)$ is the projection of $Q_t(U,S) \subseteq \mathbb{R}^{S^{[t]}}$ onto the subspace $\mathbb{R}^{S^{[i]}}$. Given a subset of sequences $T \subseteq S$ and a realization plan $\mathbf{x} \in \mathbb{R}^S$, the vector $\mathbf{x}(T)$ denotes the projection of $\mathbf{x}$ onto $\mathbb{R}^T$, that is, $\mathbf{x}(T) := (x(s) \colon s \in T)$.

Define the set-valued map $\iota \colon S \to 2^U$ as follows

$$\iota(s) = \{u \in U \colon \sigma_u = s\}.$$

For each $s \in S$, the set $\iota(s)$ is the set of (parallel) information sets that the sequence s leads to. Observe that $\iota(s)$ can be empty. Indeed $\iota(s)$ is empty if and only if the sequence s does not lead to any other information set for the player. For example, if the longest sequence in the game for the player has length t then $\iota(s) = \emptyset$ for all $s \in S^t$.

For any information set $u \in U$ and any sequence $s \in S$ define $u^+ \subseteq S$ and $s^+ \subseteq S$ as

$$u^+ := \{\sigma_u c \colon c \in C_u\} \quad \text{and} \quad s^+ := \bigcup_{u \in \iota(s)} u^+.$$

Note that the set u^+ is nonempty for any information set $u \in U$. Using this notation we can rewrite (10) as

$$Q_i(U,S) := \left\{ \mathbf{x} \in \mathbb{R}_+^{S^{[i]}} : x(\varepsilon) = 1,\, x(s) = \sum_{\hat{s} \in u^+} x(\hat{s}) \text{ for all } u \in \iota(s),\, s \in S^{[i-1]},\, \iota(s) \neq \emptyset \right\},$$

or equivalently as,

$$Q_i(U,S) := \left\{ \mathbf{x} \in \mathbb{R}_+^{S^{[i]}} : \mathbf{x}(S^{[i-1]}) \in Q_{i-1}(U,S),\, x(s) = \sum_{\hat{s} \in u^+} x(\hat{s}) \text{ for all } u \in \iota(s),\, s \in S^{i-1},\, \iota(s) \neq \emptyset \right\}. \tag{11}$$

The sets s^+ can be used to partition $S^{[i]}$ as follows

$$S^{[i]} = S^{[i-1]} \cup \bigcup_{s \in S^{i-1}} s^+. \tag{12}$$

Example 6. For the game shown in Figure 1 we determined for the first player that

$$U = \{I_1, I_2, I_3, I_4\} \quad \text{and} \quad S = \{\varepsilon, A, B, C, D, BE, BF, CG, CH\}.$$

Since the longest sequence of choices for the first player is two, we have

$$U^0 = \{I_1, I_2\}, \quad U^1 = \{I_3, I_4\}, \quad U^{[0]} = \{I_1, I_2\}, \quad U^{[1]} = \{I_1, I_2, I_3, I_4\},$$
$$S^0 = \{\varepsilon\}, \quad S^1 = \{A, B, C, D\}, \quad S^2 = \{BE, BF, CG, CH\},$$
$$S^{[0]} = \{\varepsilon\}, \quad S^{[1]} = \{\varepsilon, A, B, C, D\}, \quad S^{[2]} = \{\varepsilon, A, B, C, D, BE, BF, CG, CH\}.$$

Observe that the sequences leading to the information sets I_1 and I_3 are $\sigma_{I_1} = \varepsilon$ and $\sigma_{I_3} = B$. Since the set of choices at I_1 and I_3 are $C_{I_1} = \{A, B\}$ and $C_{I_3} = \{E, F\}$, it follows that

$$I_1^+ = \{A, B\} \quad \text{and} \quad I_3^+ = \{BE, BF\}.$$

Given the sequences ε, A and B, the set of information sets that these sequences lead to are

$$\iota(\varepsilon) = \{I_1, I_2\}, \quad \iota(A) = \emptyset, \quad \iota(B) = \{I_3\}.$$

Observe that the first player does not make any more choices after making the choice A. Since the sequence of choices A does not lead to another information set belonging to the first player, $\iota(A) = \emptyset$. So,

$$\varepsilon^+ = I_1^+ \cup I_2^+ = \{A, B, C, D\}, \quad A^+ = \emptyset, \quad B^+ = \{BE, BF\}.$$

□

4.2. **Constructing prox functions for complexes.** A key ingredient in our construction is the following "lifting" step. Consider any function $\Phi \colon D \to \mathbb{R}$ where D is a closed subset of $[0,1]^n$. We define the function $\bar{\Phi} \colon [0,1] \times D \to \mathbb{R}$ as

$$\bar{\Phi}(x, \mathbf{y}) = \begin{cases} x \cdot \Phi\left(\frac{\mathbf{y}}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

If Φ is continuous and D is compact then a simple limiting argument shows that $\bar{\Phi}$ is continuous everywhere in $[0,1] \times D$. Also if $x > 0$ and $\nabla\Phi(\mathbf{y}/x)$ exists then $\nabla\bar{\Phi}(x, \mathbf{y})$ exists. In fact, in such cases

$$\begin{aligned} \nabla_x \bar{\Phi}(x, \mathbf{y}) &= \Phi\left(\tfrac{\mathbf{y}}{x}\right) - \left\langle \nabla\Phi\left(\tfrac{\mathbf{y}}{x}\right), \tfrac{\mathbf{y}}{x} \right\rangle, \\ \nabla_{\mathbf{y}} \bar{\Phi}(x, \mathbf{y}) &= \nabla\Phi\left(\tfrac{\mathbf{y}}{x}\right). \end{aligned} \tag{13}$$

Given an information set $u \in U$ let $\Delta(u^+)$ denote the simplex

$$\Delta(u^+) := \left\{ \mathbf{x} \in \mathbb{R}_+^{u^+} : \sum_{s \in u^+} x(s) = 1 \right\}.$$

Similarly, given a sequence $s \in S$ such that $\iota(s) \neq \emptyset$, let $\Pi(s)$ denote the following Cartesian product of simplices

$$\Pi(s) := \prod_{u \in \iota(s)} \Delta(u^+) \subseteq [0,1]^{s^+}.$$

Recall that the sets u^+ for $u \in U$ are pairwise disjoint; thus the Cartesian product $\Pi(s)$ is well-defined. We can use Π to express the set $Q_i(U, S)$ in (10) and (11) as

$$Q_i(U,S) = \left\{ \mathbf{x} \in \mathbb{R}_+^{S^{[i]}} : \mathbf{x}(S^{[i-1]}) \in Q_{i-1}(U,S),\ \mathbf{x}(s^+) \in x(s)\Pi(s),\ \text{for all } s \in S^{i-1}, \iota(s) \neq \emptyset \right\}. \tag{14}$$

Assume that for each information set $u \in U$ there is a given nice prox function ψ_u for the simplex $\Delta(u^+)$. For $s \in S$ with $\iota(s) \neq \emptyset$ define $\Psi_s \colon \Pi(s) \to \mathbb{R}$ as follows

$$\Psi_s\left(\mathbf{x}(s^+)\right) = \sum_{u \in \iota(s)} \psi_u\left(\mathbf{x}(u^+)\right).$$

Since $\{\psi_u : u \in U\}$ is a family of nice prox functions for the simplices $\{\Delta(u^+) : u \in U\}$ it follows that $\{\Psi_s : s \in S, \iota(s) \neq \emptyset\}$ is a family of nice prox functions for the Cartesian products $\{\Pi(s) : s \in S, \iota(s) \neq \emptyset\}$. We use ϱ_s to denote the strong convexity parameter of Ψ_s. Finally, observe that the functions $\bar{\psi}$ and $\bar{\Psi}$ are continuous everywhere in their domains.

We define the family of functions $d_i : Q_i(U,S) \to \mathbb{R}$ for $i = 1, \ldots, t$ as follows

$$d_1\left(\mathbf{x}(S^{[1]})\right) = \bar{\Psi}_\varepsilon\left(x(\varepsilon), \mathbf{x}(\varepsilon^+)\right) = \Psi_\varepsilon\left(\mathbf{x}(\varepsilon^+)\right), \tag{15}$$

and for $i \in \{2, \ldots, t\}$

$$d_i\left(\mathbf{x}(S^{[i]})\right) = d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) + \sum_{\substack{s \in S^{i-1} \\ \iota(s) \neq \emptyset}} \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right). \tag{16}$$

In the sequel we will assume that the sum over sequences $s \in S^{i-1}$ in (16) is only considered over those sequences $s \in S^{i-1}$ such that $\iota(s) \neq \emptyset$. We now state the main theorem in the paper.

Theorem 4.1. *The functions $d_i : Q_i(U,S) \to \mathbb{R}$ defined in (16) are nice prox functions for the complexes $Q_i(U,S)$ for $i = 1, \ldots, t$.*

The theorem follows from the following three lemmas.

Lemma 4.2. *The function d_i is continuous and strongly convex everywhere in $Q_i(U,S)$ and is differentiable in the relative interior of $Q_i(U,S)$.*

Lemma 4.3. *The minimum value of d_i over $Q_i(U,S)$ is zero.*

Lemma 4.4. *The problems $\operatorname{smax}\left(d_i, \mathbf{g}(S^{[i]})\right)$ and $\operatorname{sargmax}\left(d_i, \mathbf{g}(S^{[i]})\right)$ can be computed recursively as follows. For $i = 1$*

$$\operatorname{smax}\left(d_i, \mathbf{g}(S^{[1]})\right) = \operatorname{smax}\left(\Psi_\varepsilon, \mathbf{g}(\varepsilon^+)\right) + g(\varepsilon),$$

and $\mathbf{x}^*((S^{[1]})) = \operatorname{sargmax}(d, \mathbf{g}(S^{[1]}))$ is

$$x^*(\varepsilon) = 1, \quad \textit{and} \quad \mathbf{x}^*(\varepsilon^+) = \operatorname{sargmax}(\Psi_\varepsilon, \mathbf{g}(\varepsilon^*)).$$

For $i > 1$, define $\bar{\mathbf{g}} \in \mathbb{R}^{S^{[i-1]}}$ as

$$\begin{array}{ll} \bar{\mathbf{g}}\left(S^{[i-2]}\right) & = \mathbf{g}\left(S^{[i-2]}\right) \\ \bar{g}(s) & = g(s) + \operatorname{smax}\left(\Psi_s, \mathbf{g}\left(s^+\right)\right) \quad \text{for } s \in S^{i-1}, \iota(s) \neq \emptyset. \end{array}$$

Then

$$\operatorname{smax}(d_i, \mathbf{g}) = \operatorname{smax}(d_{i-1}, \bar{\mathbf{g}}),$$

and the vector $\mathbf{x}^* = \operatorname{sargmax}(d_i, \mathbf{g}) \in Q_i(U,S)$ is given by

$$\begin{array}{ll} \mathbf{x}^*\left(S^{[i-1]}\right) & = \operatorname{sargmax}(d_{i-1}, \bar{\mathbf{g}}), \\ \mathbf{x}^*\left(s^+\right) & = x^*(s) \cdot \operatorname{sargmax}\left(\Psi_s, \mathbf{g}\left(s^+\right)\right) \quad \text{for } \ s \in S^{i-1}, \iota(s) \neq \emptyset. \end{array}$$

□

Before presenting the proofs of the lemmas we will illustrate the basic objects used in Theorem 4.1 in the following example.

Example 7. Consider the family $\{\psi_u\}$ of prox functions where ψ_u is the entropy distance over $\Delta(u^+)$, that is

$$\psi_u\left(\mathbf{x}\left(u^+\right)\right) = \ln|u^+| + \sum_{s \in u^+} x(s) \ln x(s).$$

Now consider the information set I_3 in the game shown in Figure 1. We note that $\sigma_{I_3} = B$ and $C_{I_3} = \{E, F\}$; thus $I_3^+ = \{BE, BF\}$ and

$$\psi_{I_3}\left(\mathbf{x}\left(I_3^+\right)\right) = \ln 2 + x(BE)\ln x(BE) + x(BF)\ln x(BF).$$

Observe that the informations sets reachable by the empty sequence ε are I_1 and I_2, that is, $\iota(\varepsilon) = \{I_1, I_2\}$. Since $C_{I_1} = \{A, B\}$ and $C_{I_2} = \{C, D\}$ we have $I_1^+ = \{A, B\}$, $I_2^+ = \{C, D\}$ and $\varepsilon^+ = \{A, B, C, D\}$. Thus

$$\begin{aligned}\Psi_\varepsilon\left(\mathbf{x}(\varepsilon^+)\right) &= \psi_{I_1}(\mathbf{x}(I_1^+)) + \psi_{I_2}(\mathbf{x}(I_2^+)) \\ &= [\ln 2 + x(A)\ln x(A) + x(B)\ln x(B)] + [\ln 2 + x(C)\ln x(C) + x(D)\ln x(D)]\,.\end{aligned}$$

Now we can derive the prox functions $d_i\colon Q_i(U,S) \to \mathbb{R}$ for the first player. First note that $B^+ = I_3^+ = \{BE, BF\}$ and $C^+ = I_4^+ = \{CG, CH\}$ and so

$$\begin{aligned}d_1(\mathbf{x}(S^{[1]})) &= \Psi_\varepsilon\left(\mathbf{x}(\varepsilon^+)\right) \\ d_2(\mathbf{x}(S^{[2]})) &= d_1(\mathbf{x}(S^{[1]})) + \bar{\Psi}_B\left(\mathbf{x}(B^+)\right) + \bar{\Psi}_C\left(\mathbf{x}(C^+)\right) \\ &= d_1(\mathbf{x}(S^{[1]})) + \left[x(B)\ln 2 + x(B)\cdot\frac{x(BE)}{x(B)}\ln\frac{x(BE)}{x(B)} + x(B)\cdot\frac{x(BF)}{x(B)}\ln\frac{x(BF)}{x(B)}\right] \\ &\qquad + \left[x(C)\ln 2 + x(C)\cdot\frac{x(CG)}{x(C)}\ln\frac{x(CG)}{x(C)} + x(C)\cdot\frac{x(CH)}{x(C)}\ln\frac{x(CH)}{x(C)}\right] \\ &= d_1(\mathbf{x}(S^{[1]})) + \left[x(B)\ln 2 + x(BE)\ln\frac{x(BE)}{x(B)} + x(BF)\ln\frac{x(BF)}{x(B)}\right] \\ &\qquad + \left[x(C)\ln 2 + x(CG)\ln\frac{x(CG)}{x(C)} + x(CH)\ln\frac{x(CH)}{x(C)}\right].\end{aligned}$$

Now we calculate $\operatorname{smax}(d_2, \mathbf{g}(S^{[2]}))$. First, recall that $S^1 = \{A, B, C, D\}$ and that

$$A^+ = \emptyset, \quad B^+ = \{BE, BF\}, \quad C^+ = \{CG, CH\}, \quad D^+ = \emptyset.$$

Using the decomposition used in Lemma 4.4 and the closed-form solution provided in Example 5 for the entropy prox function over the simplex, we obtain

$$\begin{aligned}\operatorname{smax}\left(\Psi_B, \mathbf{g}\left(B^+\right)\right) &= \ln\left(\exp\{g(BE)\} + \exp\{g(BF)\}\right), \\ \operatorname{smax}\left(\Psi_C, \mathbf{g}\left(C^+\right)\right) &= \ln\left(\exp\{g(CG)\} + \exp\{g(CH)\}\right),\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}^*\left(B^+\right) &= (x^*(BE), x^*(BF)) = \tfrac{x^*(B)}{\exp\{g(BE)\}+\exp\{g(BF)\}}\left(\exp\{g(BE)\}, \exp\{g(BF)\}\right), \\ \mathbf{x}^*\left(C^+\right) &= (x^*(CG), x^*(CH)) = \tfrac{x^*(C)}{\exp\{g(CG)\}+\exp\{g(CH)\}}\left(\exp\{g(CG)\}, \exp\{g(CH)\}\right).\end{aligned} \tag{17}$$

Thus

$$\begin{aligned}\bar{\mathbf{g}}(S^{[1]}) = (\bar{g}(\varepsilon), \bar{g}(A), \bar{g}(B), \bar{g}(C), \bar{g}(D)) = \\ (g(\varepsilon), g(A), g(B) + \ln\left(\exp\{g(BE)\} + \exp\{g(BF)\}\right), \\ g(C) + \ln\left(\exp\{g(CG)\} + \exp\{g(CH)\}\right), g(D))\,.\end{aligned}$$

To finish we now have

$$\operatorname{smax}(d_1, \bar{\mathbf{g}}(S^{[1]})) = \ln(\exp\{\bar{g}(A)\} + \exp\{\bar{g}(B)\} + \exp\{\bar{g}(C)\} + \exp\{\bar{g}(D)\}) + g(\varepsilon),$$

and $\operatorname{sargmax}(d_1, \bar{\mathbf{g}}(S^{[1]}))$ is $x^*(\varepsilon) = 1$ and $(x^*(A), x^*(B), x^*(C), x^*(D)) = \operatorname{sargmax}(\Psi_\varepsilon, \bar{\mathbf{g}})$, that is

$$\begin{aligned}x^*(A) &= \tfrac{\exp\{\bar{g}(A)\}}{\exp\{\bar{g}(A)\}+\exp\{\bar{g}(B)\}}, \quad x^*(B) = \tfrac{\exp\{\bar{g}(B)\}}{\exp\{\bar{g}(A)\}+\exp\{\bar{g}(B)\}}, \\ x^*(C) &= \tfrac{\exp\{\bar{g}(C)\}}{\exp\{\bar{g}(C)\}+\exp\{\bar{g}(D)\}}, \quad x^*(D) = \tfrac{\exp\{\bar{g}(D)\}}{\exp\{\bar{g}(C)\}+\exp\{\bar{g}(D)\}}.\end{aligned} \tag{18}$$

It follows from Lemma 4.4 that

$$\operatorname{smax}\left(d_2(\mathbf{x}(S^{[2]})), \mathbf{g}\right) = \operatorname{smax}\left(d_1(\mathbf{x}(S^{[1]})), \bar{\mathbf{g}}\right).$$

Moreover, substituting $x^*(B)$ and $x^*(C)$ in (17) we can determine $x^*(BE)$, $x^*(BF)$, $x^*(CG)$ and $x^*(CH)$.

□

The proof of Lemma 4.2 will provide an estimate for the strong convexity parameter of d_i. This estimate will be used extensively in the next section to derive the complexity results for our approach.

Proof of Lemma 4.2. Recall that the functions $\{\bar{\Psi}_s : s \in S, \iota(s) \neq \emptyset\}$ are continuous in their domains, and that if the functions $\{\Psi_s : s \in S, \iota(s) \neq \emptyset\}$ are differentiable in the relative interiors of their domains then the functions $\bar{\Psi}$ are also differentiable in the relative interiors of their domains. It follows that for each $i = 1, \ldots, t$ the function d_i is continuous everywhere in $Q_i(U, S)$ and differentiable in the relative interior of $Q_i(U, S)$.

To prove the strong convexity of the functions d_i we proceed by induction on i. For the base case we have

$$d_1\left(\mathbf{x}(S^{[1]})\right) = \Psi_\varepsilon\left(\mathbf{x}(\varepsilon^+)\right).$$

By assumption Ψ_ε is strongly convex everywhere in $Q_1(U, S)$ with strong convexity parameter ϱ_ε. Hence d_1 is strongly convex with parameter ϱ_ϵ.

For $i > 1$ we will first prove the strong convexity of d_i for points in the relative interior of $Q_i(U, S)$ (this is where d_i is differentiable). Then using a limiting argument we will show that strong convexity holds everywhere in $Q_i(U, S)$.

Recall that

$$d_i\left(\mathbf{x}(S^{[i]})\right) = d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) + \sum_{s \in S^{i-1}} \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right). \tag{19}$$

Since we are considering points in the relative interior of $Q_i(U, S)$ it follows that $x(s) > 0$ for all $s \in S$. Thus we can introduce the following change of variable: let $\mathbf{z}(s^+) = \mathbf{x}(s^+)/x(s)$ for all $s \in S$. So for points $\mathbf{x}(S^{[i]})$ in the relative interior of $Q_i(U, S)$ we have

$$\mathbf{x}(S^{[i]}) = \left(\mathbf{x}(S^{[i-1]}), \mathbf{x}(S^i)\right) = \left(\mathbf{x}(S^{[i-1]}), \left(x(s)\mathbf{z}(s^+)\right)_{s \in S^{i-1}}\right).$$

By the inductive hypothesis, the function d_{i-1} is differentiable and strongly convex in the relative interior of $Q_{i-1}(U, S)$ with strong convexity parameter ρ_{i-1}. Using the characterization given in (7), for any pair of points $\mathbf{x}(S^{[i-1]})$, $\tilde{\mathbf{x}}(S^{[i-1]}) \in Q_{i-1}(U, S)$

$$\left\langle \nabla d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) - \nabla d_{i-1}\left(\tilde{\mathbf{x}}(S^{[i-1]})\right), \mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\rangle \geq \rho_{i-1}\left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2. \tag{20}$$

Similarly, since each Ψ_s is differentiable and strongly convex with strong convexity parameter ϱ_s, it follows from (6) that for all $\mathbf{z}(s^+)$, $\tilde{\mathbf{z}}(s^+) \in \Pi(s)$

$$\Psi_s\left(\tilde{\mathbf{z}}(s^+)\right) \geq \Psi_s\left(\mathbf{z}(s^+)\right) + \left\langle \nabla\Psi_s\left(\mathbf{z}(s^+)\right), \tilde{\mathbf{z}}(s^+) - \mathbf{z}(s^+)\right\rangle + \frac{1}{2}\varrho_s\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2. \tag{21}$$

To prove the strong convexity of d_i for points in relint $Q_i(U, S)$, we will show that there exists $\rho_i > 0$ such that for any pair of points $\mathbf{x}(S^{[i]})$ and $\tilde{\mathbf{x}}(S^{[i]}) \in Q_i(U, S)$ the following inequality holds

$$\left\langle \nabla d\left(\mathbf{x}(S^{[i]})\right) - \nabla d\left(\tilde{\mathbf{x}}(S^{[i]})\right), \mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\rangle \geq \rho_i\left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\|^2. \tag{22}$$

Using (13) and (19) together with some elementary calculations, it follows that for $\mathbf{x}(S^{[i]}) \in$ relint $Q_i(U, S)$

$$\begin{aligned}
&\nabla_{\mathbf{x}(S^{[i-2]})} d_i\left(\mathbf{x}(S^{[i]})\right) = \nabla_{\mathbf{x}(S^{[i-2]})} d_{i-1}\left(\mathbf{x}(S^{[i-2]})\right), \\
&\quad \nabla_{x(s)} d_i\left(\mathbf{x}(S^{[i]})\right) = \nabla_{x(s)} d_{i-1}\left(\mathbf{x}(S^{i-1})\right) + \Psi_s\left(\mathbf{z}(s^+)\right) - \left\langle \nabla_{\mathbf{z}(s^+)}\Psi_s\left(\mathbf{z}(s^+)\right), \mathbf{z}(s^+)\right\rangle \quad \text{for each } s \in S^{i-1}, \\
&\quad \nabla_{x(s^+)} d_i\left(\mathbf{x}(S^{[i]})\right) = \nabla_{\mathbf{z}(s^+)}\Psi_s\left(\mathbf{z}(s^+)\right) \quad \text{for each } s \in S^{i-1}.
\end{aligned}$$

Now we will bound the LHS of (22).

$$\left\langle \nabla d\left(\mathbf{x}(S^{[i]})\right) - \nabla d\left(\tilde{\mathbf{x}}(S^{[i]})\right), \mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\rangle$$

$$= \left\langle \nabla d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) - \nabla d_{i-1}\left(\tilde{\mathbf{x}}(S^{[i-1]})\right), \mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\rangle + \sum_{s\in S^{i-1}} \left[\Psi_s\left(\mathbf{z}(s^+)\right) - \Psi_s\left(\tilde{\mathbf{z}}(s^+)\right)\right]\cdot[x(s)-\tilde{x}(s)] - \sum_{s\in S^{i-1}} \left[\left\langle \nabla\Psi_s\left(\mathbf{z}(s^+)\right), \mathbf{z}(s^+)\right\rangle - \left\langle \nabla\Psi_s\left(\tilde{\mathbf{z}}(s^+)\right), \tilde{\mathbf{z}}(s^+)\right\rangle\right]\cdot[x(s)-\tilde{x}(s)] + \sum_{s\in S^{i-1}} \left\langle \nabla\Psi_s\left(\mathbf{z}(s^+)\right) - \nabla\Psi_s\left(\tilde{\mathbf{z}}(s^+)\right), \mathbf{x}(s^+) - \tilde{\mathbf{x}}(s^+)\right\rangle$$

$$= \left\langle \nabla d_{i-1}\left(\mathbf{x}(S^{[i-1]}\right) - \nabla d_{i-1}\left(\tilde{\mathbf{x}}(S^{[i-1]})\right), \mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\rangle + \sum_{s\in S^{i-1}} \left[\Psi_s\left(\mathbf{z}(s^+)\right) - \Psi_s\left(\tilde{\mathbf{z}}(s^+)\right)\right]\cdot[x(s)-\tilde{x}(s)] + \sum_{s\in S^{i-1}} \tilde{x}(s)\left\langle \nabla\Psi_s\left(\mathbf{z}(s^+)\right), \mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\rangle + \sum_{s\in S^{i-1}} x(s)\left\langle \nabla\Psi_s\left(\tilde{\mathbf{z}}(s^+)\right), \tilde{\mathbf{z}}(s^+) - \mathbf{z}(s^+)\right\rangle$$

$$= \left\langle \nabla d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) - \nabla d_{i-1}\left(\tilde{\mathbf{x}}(S^{[i-1]})\right), \mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\rangle + \sum_{s\in S^{i-1}} x(s)\cdot\left[\Psi_s\left(\mathbf{z}(s^+)\right) - \Psi_s\left(\tilde{\mathbf{z}}(s^+)\right) + \left\langle \nabla\Psi_s\left(\tilde{\mathbf{z}}(s^+)\right), \tilde{\mathbf{z}}(s^+) - \mathbf{z}(s^+)\right\rangle\right] + \sum_{s\in S^{i-1}} \tilde{x}(s)\cdot\left[\Psi_s\left(\tilde{\mathbf{z}}(s)\right) - \Psi_s\left(\mathbf{z}(s)\right) + \left\langle \nabla\Psi_s\left(\mathbf{z}(s^+)\right), \mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\rangle\right]$$

$$\geq \rho_{i-1}\left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s\in S^{i-1}} \varrho_s\left(\frac{x(s)+\tilde{x}(s)}{2}\right)\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2 \qquad \text{by (20) and (21).}$$

Let $\hat{x}(s) = (x(s)+\tilde{x}(s))/2$ for $s \in S^{i-1}$. Thus to prove (22) it is enough to determine $\rho_i > 0$ such that

$$\rho_{i-1}\left\|\mathbf{x}(S^{[i-1]} - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s\in S^{i-1}} \varrho_s\hat{x}(s)\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2 \geq \rho_i\left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\|^2. \tag{23}$$

Next, we bound the RHS of (23). Applying the triangle inequality we get

$$\left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\| \leq \left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\| + \sum_{s\in S^{i-1}} \left\|x(s)\mathbf{z}(s^+) - \tilde{x}(s)\tilde{\mathbf{z}}(s^+)\right\|. \tag{24}$$

Working on the summation above and applying the triangle inequality once more we obtain

$$\begin{aligned}
\sum_{s\in S^{i-1}} &\left\|x(s)\mathbf{z}(s^+) - \tilde{x}(s)\tilde{\mathbf{z}}(s^+)\right\| \\
&= \sum_{s\in S^{i-1}} \left\|x(s)\mathbf{z}(s^+) - \hat{x}(s)\mathbf{z}(s^+) + \hat{x}(s)\mathbf{z}(s^+) - \hat{x}(s)\tilde{\mathbf{z}}(s^+) + \hat{x}(s)\tilde{\mathbf{z}}(s^+) - \tilde{x}(s)\tilde{\mathbf{z}}(s^+)\right\| \\
&= \sum_{s\in S^{i-1}} \left\|\left(\frac{x(s)-\tilde{x}(s)}{2}\right)\mathbf{z}(s^+) + \hat{x}(s)\left[\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right] + \left(\frac{x(s)-\tilde{x}(s)}{2}\right)\tilde{\mathbf{z}}(s^+)\right\| \\
&\leq \sum_{s\in S^{i-1}} \frac{1}{2}\left|x(s)-\tilde{x}(s)\right|\cdot\left\|\mathbf{z}(s^+) + \tilde{\mathbf{z}}(s^+)\right\| + \sum_{s\in S^{i-1}} \hat{x}(s)\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|.
\end{aligned}$$

Because each $\Pi(s)$ is compact there exists a constant $M \geq 1$ such that

$$\sum_{s\in S^{i-1}} \frac{1}{2}\left|x(s)-\tilde{x}(s)\right|\cdot\left\|\mathbf{z}(s^+) + \tilde{\mathbf{z}}(s^+)\right\| \leq (M-1)\left\|\mathbf{x}(S^{i-1}) - \tilde{\mathbf{x}}(S^{i-1})\right\|.$$

Thus from (24) it follows that

$$\left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\| \leq M \left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\| + \sum_{s \in S^{i-1}} \hat{x}(s) \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|.$$

Hence to show (23) it suffices to find $\rho_i > 0$ such that

$$\rho_{i-1} \left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s \in S^{i-1}} \varrho_s \hat{x}(s) \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2 \geq \rho_i \left(M \left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\| + \sum_{s \in S^{i-1}} \hat{x}(s) \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\| \right)^2. \tag{25}$$

Since $Q_{i-1}(U, S)$ is compact there exists $\kappa_{i-1} > 0$ such that $\sum_{s \in S^{i-1}} \hat{x}(s) \leq \kappa_{i-1}$. Now we apply the Cauchy-Schwarz inequality to the RHS of (25):

$$\left(M \left\|\mathbf{x}(S^{[i-1]} - \tilde{\mathbf{x}}(S^{[i-1]})\right\| + \sum_{s \in S^{i-1}} \hat{x}(s) \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\| \right)^2 \leq \left(M^2 + \sum_{s \in S^{i-1}} \frac{\hat{x}(s)}{\kappa_{i-1}} \right) \left(\left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s \in S^{i-1}} \kappa_{i-1} \hat{x}(s) \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2 \right).$$

Since $\sum_{s \in S^{i-1}} \hat{x}(s) \leq \kappa_{i-1}$, it follows that

$$\left(M \left\|\mathbf{x}(S^{[i-1]} - \tilde{\mathbf{x}}(S^{[i-1]}\right\| + \sum_{s \in S^{i-1}} \hat{x}(s) \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\| \right)^2 \leq (M^2 + 1) \left(\left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s \in S^{i-1}} \kappa_{i-1} \hat{x}(s) \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2 \right).$$

At this point we have simplified our problem to finding $\rho_i > 0$ such that

$$[\rho_{i-1} - (M^2+1)\rho_i] \cdot \left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s \in S^{i-1}} \hat{x}(s)[\varrho_s - \kappa_{i-1}(M^2+1)\rho_i] \cdot \left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2 \geq 0. \tag{26}$$

Observe that the LHS of (26) is a weighted sum of squares. Consequently, for the LHS to be nonnegative it suffices for each one of the weights to be nonnegative. Using the fact that $0 \leq \hat{x}(s) \leq 1$ for all $s \in S^{i-1}$ it follows that (26) holds as long as

$$0 < \rho_i \leq \min_{s \in S^{i-1}} \left\{ \frac{\rho_{i-1}}{M^2+1}, \frac{\varrho_s}{\kappa_{i-1}(M^2+1)} \right\}. \tag{27}$$

So far we have shown that there exists $\rho_i > 0$ such that for all $\alpha \in [0, 1]$ and any pair of points $\mathbf{x}(S^{[i]})$, $\tilde{\mathbf{x}}(S^{[i]}) \in \text{relint } Q_i(U, S)$

$$d\left(\alpha \mathbf{x}(S^{[i]}) + (1-\alpha)\tilde{\mathbf{x}}(S^{[i]})\right) \leq \alpha d\left(\mathbf{x}(S^{[i]})\right) + (1-\alpha) d\left(\tilde{\mathbf{x}}(S^{[i]})\right) - \frac{1}{2}\rho_i \alpha(1-\alpha) \left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\|^2. \tag{28}$$

Now we will extend this inequality to all points in $Q_i(U, S)$. Consider any pair of points $\mathbf{x}(S^{[i]})$, $\tilde{\mathbf{x}}(S^{[i]}) \in Q_i(U, S)$ and some point $\mathbf{x}^\circ(S^{[i]})$ in the relative interior of $Q_i(U, S)$. Define the points

$$\mathbf{x}(S^{[i]})_\mu = (1-\mu)\mathbf{x}(S^{[i]}) + \mu \mathbf{x}^\circ(S^{[i]}) \quad \text{and} \quad \tilde{\mathbf{x}}(S^{[i]})_\mu = (1-\mu)\tilde{\mathbf{x}}(S^{[i]}) + \mu \mathbf{x}^\circ(S^{[i]}).$$

Notice that $(\mathbf{x},\mathbf{y})_\mu$ and $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_\mu \in \text{relint } Q_i(U,S)$ for all $\mu \in (0,1]$. So for all $\mu \in (0,1]$ we have the following strong convexity inequality using the value of ρ_i determined for points in relint $Q_i(U,S)$ above

$$\begin{aligned}(29)\quad & d_i\left(\alpha\mathbf{x}(S^{[i]})_\mu + (1-\alpha)\tilde{\mathbf{x}}(S^{[i]})_\mu\right) \leq \\ & \alpha d_i\left(\mathbf{x}(S^{[i]})\right)_\mu + (1-\alpha)d_i\left(\tilde{\mathbf{x}}(S^{[i]})\right)_\mu - \frac{1}{2}\rho_i\alpha(1-\alpha)\left\|\mathbf{x}(S^{[i]})_\mu - \tilde{\mathbf{x}}(S^{[i]})_\mu\right\|^2.\end{aligned}$$

Since d_i is continuous, inequality (29) will also hold in the limit, that is, as $\mu \downarrow 0$. Finally, observe that as $\mu \downarrow 0$ the points $\mathbf{x}(S^{[i]})_\mu \to \mathbf{x}(S^{[i]})$ and $\tilde{\mathbf{x}}(S^{[i]})_\mu \to \tilde{\mathbf{x}}(S^{[i]})$. Therefore inequality (28) holds for all pairs of points $\mathbf{x}(S^{[i]})$ and $\tilde{\mathbf{x}}(S^{[i]}) \in Q_i(U,S)$. □

Recall that in order for the functions d_i to be nice prox functions, their minimum value over the complex must be zero. The following proof establishes that the functions d_i indeed satisfy this property.

Proof of Lemma 4.3. The proof follows from Lemma 4.4 by setting $\mathbf{g}(S^{[i]}) := \mathbf{0}$. □

The following proof of Lemma 4.4 shows that the problems

$$(30)\quad \begin{aligned}\text{smax}\left(d_i,\mathbf{g}(S^{[i]})\right) &= \max_{\mathbf{x}\in Q_i(U,S)} \left\{\langle \mathbf{g}(S^{[i]}),\mathbf{x}(S^{[i]})\rangle - d_i\left(\mathbf{x}(S^{[i]})\right)\right\},\\ \text{sargmax}\left(d_i,\mathbf{g}(S^{[i]})\right) &= \operatorname*{argmax}_{\mathbf{x}\in Q_i(U,S)} \left\{\langle \mathbf{g}(S^{[i]}),\mathbf{x}(S^{[i]})\rangle - d_i\left(\mathbf{x}(S^{[i]})\right)\right\}\end{aligned}$$

are easily computable. In the general case, the proof sets up a backward recursion by defining $\bar{\mathbf{g}}(S^{[i-1]}) \in \mathbf{R}^{S^{[i-1]}}$ as

$$(31)\quad \begin{aligned}\bar{\mathbf{g}}\left(S^{[i-2]}\right) &= \mathbf{g}\left(S^{[i-2]}\right) && \text{if } i > 1,\\ \bar{g}(s) &= g(s) + \text{smax}\left(\Psi_s,\mathbf{g}\left(s^+\right)\right) && \text{for } s \in S^{i-1}, \iota(s) \neq \emptyset,\end{aligned}$$

and then showing that for $i > 1$ the maximization problem in (30) is equivalent to

$$(32)\quad \begin{aligned}\max\quad & \left\langle \bar{\mathbf{g}}(S^{[i-1]}),\, \mathbf{x}(S^{[i-1]})\right\rangle - d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right)\\ \text{s.t.}\quad & \mathbf{x}(S^{[i-1]}) \in Q_{i-1}(U,S),\end{aligned}$$

and

$$(33)\quad \mathbf{x}(s^+) = x(s)\cdot \text{sargmax}\left(\Psi_s,\mathbf{g}(s^+)\right) \qquad \text{for all } s \in S^{i-1}, \iota(s) \neq \emptyset.$$

Observe that for each $s \in S^{i-1}$ such that $\iota(s) \neq \emptyset$ the function Ψ_s is just the sum of nice prox functions ψ's over disjoint simplices. Thus the problems $\text{smax}\left(\Psi_s,\mathbf{g}(s^+)\right)$ and $\text{sargmax}\left(\Psi_s,\mathbf{g}(s^+)\right)$ have easily computable solutions. More precisely for any $s \in S$ such that $\iota(s) \neq \emptyset$ we have

$$\text{sargmax}\left(\Psi_s, g(s^+)\right) = \left(\text{sargmax}\left(\psi_u, g(u^+)\right) : u \in \iota(s)\right).$$

Using the fact that $x(\varepsilon) = 1$ we will show that $\mathbf{x}(\varepsilon^+) = \text{sargmax}(\Psi_\varepsilon,\mathbf{g}(\varepsilon^+))$. The rest of the realization plan $\mathbf{x}$ is uniquely determined by propagating $\mathbf{x}(\varepsilon^+)$ and working forward through the recursion. So it is enough to prove that the problem in (32) is equivalent to the original problem stated in (30) for each $i = 2,\ldots,t$.

Proof of Lemma 4.4. As before we will proceed by induction on i. For the base case we have

$$d_1\left(\mathbf{x}(S^{[1]})\right) = \bar{\Psi}_\varepsilon\left(x(\varepsilon),\mathbf{x}(\varepsilon^+)\right) = \Psi_\varepsilon\left(\mathbf{x}(\varepsilon^+)\right).$$

Since $x(\varepsilon) = 1$ it follows that the optimal solution to (30) is $x(\varepsilon) = 1$ and

$$\mathbf{x}(\varepsilon^+) = \text{sargmax}\left(\Psi_\varepsilon, g(\varepsilon^+)\right),$$

and the optimal value is

$$g(\varepsilon) + \sum_{u\in\iota(\varepsilon)} \text{smax}\left(\psi_u, g(u^+)\right) = g(\varepsilon) + \text{smax}\left(\Psi_\varepsilon, g(\varepsilon^+)\right).$$

For $i > 1$ we have

$$d_i(\mathbf{x}(S^{[i]})) = d_{i-1}(\mathbf{x}(S^{[i-1]})) + \sum_{s\in S^{i-1}} \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right), \tag{34}$$

where $\mathbf{x}(S^{[i-1]}) \in Q_{i-1}(U,S)$. For any $\mathbf{x}(S^{[i]}) \in Q_i(U,S)$ such that $x(s) = 0$ for some $s \in S^{i-1}$, $\iota(s) \neq \emptyset$ it follows that

$$\mathbf{x}(s^+) = 0 \cdot \operatorname{sargmax}\left(\Psi_s, \mathbf{g}(s^+)\right)$$

and

$$0 = \left\langle \mathbf{g}(s^+), \mathbf{x}(s^+)\right\rangle - \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right) = x(s) \cdot \operatorname{smax}\left(\Psi_s, \mathbf{g}(s^+)\right). \tag{35}$$

On the other hand if $x(s) > 0$ then $\mathbf{x}(S^{[i]}) \in Q_i(U,S)$ implies that $\mathbf{x}(s^+)/x(s) \in \Pi(s)$. Moreover,

$$\left\langle \mathbf{g}(s^+), \mathbf{x}(s^+)\right\rangle - \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right) = x(s) \cdot \left(\left\langle \mathbf{g}(s^+), \frac{\mathbf{x}(s^+)}{x(s)}\right\rangle - \Psi_s\left(\frac{\mathbf{x}(s^+)}{x(s)}\right)\right). \tag{36}$$

From (34), (35) and (36) we can deduce that (30) is equivalent to

$$\begin{aligned}\max_{\mathbf{x}(S^{[i-1]})\in Q_{i-1}(U,S)} \Bigg[& \left\langle \mathbf{g}(S^{[i-1]}), \mathbf{x}(S^{[i-1]})\right\rangle - d_{i-1}(\mathbf{x}(S^{[i-1]})) + \sum_{\substack{s\in S^{i-1}\\ x(s)=0}} x(s) \cdot \operatorname{smax}\left(\Psi_s, \mathbf{g}(s^+)\right) \\ & + \sum_{\substack{s\in S^{i-1}\\ x(s)>0}} x(s) \max_{\frac{\mathbf{x}(s^+)}{x(s)}\in\Pi(s)} \left\{\left\langle g(s^+), \frac{\mathbf{x}(s^+)}{x(s)}\right\rangle - \Psi_s\left(\frac{\mathbf{x}(s^+)}{x(s)}\right)\right\}\Bigg].\end{aligned} \tag{37}$$

For those terms with $x(s) > 0$ we can express the maximization subproblem above as

$$\max_{\frac{\mathbf{x}(s^+)}{x(s)}\in\Pi(s)} \left\{\left\langle \mathbf{g}(s^+), \frac{\mathbf{x}(s^+)}{x(s)}\right\rangle - \Psi_s\left(\frac{\mathbf{x}(s^+)}{x(s)}\right)\right\} = \operatorname{smax}\left(\Psi_s, \mathbf{g}(s^+)\right). \tag{38}$$

From the definition of smax and sargmax it follows that the maximum value of the subproblem in (38) will be attained when

$$\frac{\mathbf{x}(s^+)}{x(s)} = \operatorname{sargmax}\left(\Psi_s, \mathbf{g}(s^+)\right). \tag{39}$$

To obtain the problem shown in (32), apply (38) and (39) to (37) and use the following identity that holds by definition of $\bar{\mathbf{g}}$

$$\left\langle \mathbf{g}(S^{[i-1]}), \mathbf{x}(S^{[i-1]})\right\rangle + \sum_{s\in S^{i-1}} x(s) \cdot \operatorname{smax}(\Psi_s, \mathbf{g}(s^+)) = \left\langle \bar{\mathbf{g}}, \mathbf{x}(S^{[i-1]})\right\rangle.$$

□

Remark 4.5. The results in this section can be easily extended to "weighted" versions. More precisely, for positive weights $w_0, \ldots, w_{t-1}$ we define the functions $d_i \colon Q_i(U,S) \to \mathbb{R}$ as follows

$$d_i(\mathbf{x}(S^{[i]})) = d_{i-1}(\mathbf{x}(S^{[i-1]})) + w_{i-1} \sum_{s\in S^{i-1}} \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right). \tag{40}$$

Clearly the functions d_i remain continuous, differentiable, strongly convex and retain their minimum value of zero. Also the subproblems remain easy to solve. For our purpose the most important change is in the estimate of the strong convexity parameter. Using the notation from Lemma 4.3 the strong convexity estimate stated in (27) becomes

$$0 < \rho_i \leq \min_{s\in S^{i-1}} \left\{\frac{\rho_{i-1}}{M^2+1}, \frac{w_{i-1}\cdot \varrho_s}{\kappa_{i-1}(M^2+1)}\right\}. \tag{41}$$

All the complexity results obtained in the following section will arise from weighted prox functions, that is, prox functions of the form shown in (40). A key step in the analyses to follow is choosing weights so as to maximize the niceness parameters of the prox functions.

5. Complexity results

In this section we derive complexity results using Theorem 3.1 for the entropy and Euclidean prox functions. We introduce the family of *uniform games* and exploit the structure in these games to provide simple bounds. Next we identify the key parameters in a (uniform) sequential game that affect the overall iteration complexity of the algorithm.

5.1. **Uniform complexes and uniform games.** A player's complex $Q_t(U, S)$ is *uniform* of type $\mathcal{Q}(n, k, b, t)$ if:

i. For each information set $u \in U$, the number of choices is n, that is, $|C_u| = |u^+| = n$.
ii. For each information set $u \in U$, the number of choices that lead to another information set for that player is b.
iii. For each sequence $s \in S$ such that $\iota(s) \neq \emptyset$ the number of parallel information sets reachable by s is k, that is $|\iota(s)| = k$. Equivalently, given an information set $u \in U$ the number of information sets parallel to u (including u itself) is k, that is, $|\iota(\sigma_u)| = k$.
iv. The length of the longest sequence of choices for the player is t, that is, $S = S^0 \cup S^1 \cup \cdots \cup S^t$.

A sequential zero-sum two-player game is *uniform* if the complexes belonging to both players are uniform (but not necessarily of the same type). Note that the parameters n, k, b and t, are sufficient to determine the dimensions of the constraint matrix E defining the complex $Q_t(U, S)$. The number of rows in the matrix E is

$$1 + k + k(kb) + k(kb)^2 + \cdots + k(kb)^{t-1} = 1 + k\left(\frac{(kb)^t - 1}{kb - 1}\right),$$

Similarly, the number of columns in E is

$$1 + nk + nk(kb) + nk(kb)^2 + \cdots + nk(kb)^{t-1} = 1 + nk\left(\frac{(kb)^t - 1}{kb - 1}\right).$$

Example 8. The following coefficient matrix describes a complex that is uniform of type $\mathcal{Q}(3, 2, 1, 2)$

$$E = \begin{bmatrix}
1 & & & & & & & & & & & & & & & & & & \\
-1 & 1 & 1 & 1 & & & & & & & & & & & & & & & \\
-1 & & & & 1 & 1 & 1 & & & & & & & & & & & & \\
 & -1 & & & & & & 1 & 1 & 1 & & & & & & & & & \\
 & -1 & & & & & & & & & 1 & 1 & 1 & & & & & & \\
 & & & & -1 & & & & & & & & & 1 & 1 & 1 & & & \\
 & & & & -1 & & & & & & & & & & & & 1 & 1 & 1
\end{bmatrix}.$$

The parameter n describes the number of ones that appear in a row of E (except for the first row). The parameter k describes the number of negative ones that appear in a column of E (if any appear at all). So in this example, $n = 3$ and $k = 2$.

Describing the role of the parameter b is a little more complicated. Given a row we look for the columns with entries of one. The number of such columns with negative ones appearing below them (if any) is described by the parameter b. In this example, consider the second row; the entries with ones in this row appear in the second, third and fourth column. There is only one column with negative ones appearing below it (the second column) and so $b = 1$.

This structure is recursive and the parameter t describes the level (or depth) of the recursion. In this example, $t = 2$.

□

5.2. **The maximum value of prox functions over uniform complexes.** The maximum value attained by the prox function plays an important role in the iteration bound. However this involves maximizing a convex function over a polytope, which may not be straightforward. But we can easily compute the maximum value of the derived prox function over the complex if the basic prox function over the simplex satisfies a certain (natural) condition.

Lemma 5.1. *Let $\phi\colon \Delta_n \to \mathbb{R}$ be any function which attains its (finite) maximum value at* every *extreme point of the simplex Δ_n. For any $\mathbf{g} \in \mathbb{R}^n$ suppose that $g_j = \max\{g_i : i = 1, \ldots, n\}$. Then*

$$\mathbf{e}^j \in \operatorname*{argmax}_{\mathbf{x}\in\Delta_n} \{\phi(\mathbf{x}) + \langle \mathbf{g}, \mathbf{x}\rangle\}, \tag{42}$$

where $\mathbf{e}^j$ is the j^{th} standard basis vector.

Proof. Let $\xi = \max\{\phi(\mathbf{x})\colon \mathbf{x} \in \Delta_n\}$. Any point $\mathbf{x} \in \Delta_n$ can be expressed as

$$\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{e}^i, \quad \sum_{i=1}^n \lambda_i = 1, \text{ and } \quad \lambda_i \geq 0 \quad \text{for all } i = 1, \ldots, n.$$

To prove (42), observe that

$$\phi(\mathbf{x}) + \langle \mathbf{g}, \mathbf{x}\rangle \leq \xi + \sum_{i=1}^n g_i\lambda_i \leq \xi + g_j \sum_{i=1}^n \lambda_i = \xi + g_j = \phi(\mathbf{e}^j) + \langle \mathbf{g}, \mathbf{e}^j \rangle .$$

□

Lemma 5.1 extends naturally to the Cartesian product of simplices, that is, each function of the form $\Psi_s(\mathbf{x}(s^+)) + \langle \mathbf{g}(s^+), \mathbf{x}(s^+)\rangle$ for $s \in S$, $\iota(s) \neq \emptyset$ will attain its maximum value at some extreme point of $\Pi(s)$. Moreover, Ψ_s will attain its maximum value at every extreme point of $\Pi(s)$. We now demonstrate that a similar condition applies to the complex. The following lemma states that there exists a 0–1 (deterministic) realization plan that maximizes the prox function constructed for the complex.

Lemma 5.2. *If the prox functions $\{\Psi_s : s \in S, \iota(s) \neq \emptyset\}$ used to build d_i in Theorem 4.1 achieve their maximum value at every extreme point of the the sets $\Pi(s)$ then functions of the form $d_i\left(\mathbf{x}(S^{[i]})\right) + \left\langle \mathbf{g}(S^{[i]}), \mathbf{x}(S^{[i]})\right\rangle$ attain their maximum value at some extreme point of $Q_i(U, S)$.*

Proof. The proof of the lemma will be very similar to the proofs of Lemma 4.3 and Lemma 4.4. As before we proceed by induction on i. The case when $i = 1$ follows immediately from Lemma 5.1. For $i > 1$ we can write

$$\max_{\mathbf{x}(S^{[i]})\in Q_i(U,S)} \left\{ d_i\left(x(S^{[i]})\right) + \left\langle \mathbf{g}(S^{[i]}), \mathbf{x}(S^{[i]})\right\rangle \right\} \tag{43}$$

$$= \max_{\mathbf{x}(S^{[i-1]})\in Q_{i-1}(U,S)} \left\{ d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) + \left\langle \mathbf{g}(S^{[i-1]}), \mathbf{x}(S^{[i-1]})\right\rangle + \sum_{s\in S^{i-1}} \left[\max_{\frac{\mathbf{x}(s^+)}{x(s)}\in\Pi(s)} \left\langle \mathbf{g}(s^+), \mathbf{x}(s^+)\right\rangle + \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right) \right] \right\}$$

For each $s \in S^{i-1}$ let $\bar{\mathbf{z}}(s^+)$ be an extreme point maximizer of

$$\max_{\mathbf{z}(s^+)\in\Pi(s)} \left\{ \left\langle \mathbf{g}(s^+), \mathbf{z}(s^+)\right\rangle + \Psi_s\left(\mathbf{z}(s^+)\right) \right\}.$$

We know that such a maximizer exists by Lemma 5.1. Define $\bar{\mathbf{g}} \in \mathbb{R}^{S^{[i-1]}}$ as follows

$$\bar{g}(s) = \begin{cases} g(s) + \langle \mathbf{g}(s^+), \bar{\mathbf{z}}(s^+)\rangle + \Psi_s(\bar{\mathbf{z}}(s^+)) & \text{for } s \in S^{i-1}, \iota(s) \neq \emptyset, \\ g(s) & \text{otherwise.} \end{cases}$$

Observe that for any point $\tilde{\mathbf{x}}(S^{[i]}) \in Q_i(U, S)$

$$\begin{aligned} d_i\left(\tilde{\mathbf{x}}(S^{[i]}\right) + \left\langle \mathbf{g}(S^{[i]}), \tilde{\mathbf{x}}(S^{[i]})\right\rangle &\leq d_{i-1}\left(\tilde{\mathbf{x}}(S^{[i-1]})\right) + \left\langle \bar{\mathbf{g}}(S^{[i-1]}), \tilde{\mathbf{x}}(S^{[i-1]})\right\rangle \\ &\leq \max_{\mathbf{x}(S^{[i-1]})\in Q_{i-1}} \left\{ d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) + \left\langle \bar{\mathbf{g}}(S^{[i-1]}), \mathbf{x}(S^{[i-1]})\right\rangle \right\}. \end{aligned} \tag{44}$$

Using the inductive hypothesis we know that the following problem has an extreme point solution $\mathbf{x}^*(S^{[i-1]}) \in Q_{i-1}(U,S)$

$$\max_{\mathbf{x}(S^{[i-1]})\in Q_{i-1}} \left\{ d_{i-1}\left(\mathbf{x}(S^{[i-1]})\right) + \left\langle \bar{\mathbf{g}}(S^{[i-1]}), \mathbf{x}(S^{[i-1]})\right\rangle \right\}.$$

We now extend $\mathbf{x}^*(S^{[i-1]})$ to the point $\mathbf{x}^*(S^{[i]}) \in Q_i(U,S)$ by setting

$$\mathbf{x}^*(s^+) = \begin{cases} \mathbf{0} & \text{if } x(s) = 0, \\ \bar{\mathbf{z}}(s) & \text{if } x(s) = 1. \end{cases}$$

The result follows from the inequalities in (44) and observing that

$$d_i\left(\mathbf{x}^*(S^{[i]}\right) + \left\langle \mathbf{g}(S^{[i]}), \mathbf{x}^*(S^{[i]})\right\rangle = d_{i-1}\left(\mathbf{x}^*(S^{[i-1]})\right) + \left\langle \bar{\mathbf{g}}(S^{[i-1]}), \mathbf{x}^*(S^{[i-1]})\right\rangle \tag{45}$$

□

Remark 5.3. An alternative condition for the simplices can be substituted in the lemma above; namely, if the prox functions ψ for the simplices satisfy the property that each function over the simplex of the form $\psi(\mathbf{x}) + \langle \mathbf{g}, \mathbf{x}\rangle$ attains its maximum value at some extreme point of the simplex, then we can make the same claim for the induced prox function for the complex. However, for the two prox functions that we will be using specifically, the condition stated in the lemma is easier to verify.

The following pair of lemmas provide an expression for the maximum value of the weighted prox functions

$$d_i\left(\mathbf{x}(S^{[i]})\right) = d_{i-1}(\mathbf{x}(S^{[i-1]})) + w_{i-1}\sum_{s\in S^{i-1}} \bar{\Psi}_s\left(x(s), \mathbf{x}(s^+)\right) \quad \text{for } i = 1,\ldots,t, \tag{46}$$

over uniform complexes where $w_0,\ldots,w_{t-1}$ are positive constants.

Lemma 5.4. *For any uniform complex of type $\mathcal{Q}(n,k,b,t)$ the following inequality holds for all $1 \le i \le t$*

$$\sum_{s\in S^{i-1}} x(s) \le k^{i-1}.$$

Proof. We proceed by induction on i. The base case ($i = 1$) follows immediately since $x(\varepsilon) = 1 = k^0$. For $i > 1$ we have

$$\sum_{s\in S^i} x(s) = \sum_{\hat{s}\in S^{i-1}}\sum_{s\in\hat{s}^+} x(s) = \sum_{\substack{\hat{s}\in S^{i-1}\\ \iota(\hat{s})\ne\emptyset}}\sum_{u\in\iota(\hat{s})}\sum_{s\in u^+} x(s) = \sum_{\substack{\hat{s}\in S^{i-1}\\ \iota(\hat{s})\ne\emptyset}}\sum_{u\in\iota(\hat{s})} x(\hat{s}) = \sum_{\hat{s}\in S^{i-1}} x(\hat{s})\cdot|\iota(\hat{s})|.$$

The first and second steps above follow respectively from $S^i = \bigcup_{\hat{s}\in S^{i-1}}\hat{s}^+$ and $\hat{s}^+ = \bigcup_{u\in\iota(\hat{s})} u^+$. The third step follows because $\sigma_u = \hat{s}$ for each $u \in \iota(\hat{s})$, and thus $x(\hat{s}) = x(\sigma_u) = \sum_{c\in C_u} x(\sigma_u c) = \sum_{s\in u^+} x(s)$.

Since the complex is uniform of type $\mathcal{Q}(n,k,b,t)$ we have $\iota(\hat{s}) = k$ for all $\hat{s}\in S^{i-1}$ such that $\iota(\hat{s}) \ne \emptyset$. The result follows by applying the inductive hypothesis to the sum above.

□

Lemma 5.5. *Suppose that we are given a uniform complex of type $\mathcal{Q}(n,k,b,t)$ and a prox function $\psi\colon \Delta_n \to \mathbb{R}$ that attains its maximum ξ value at every extreme point of the simplex Δ_n. Then the maximum value of the induced prox function for the complex $Q_t(U,S)$ is*

$$D = kw_0\xi + k^2w_1\xi + k^3w_2\xi + \cdots + k^t w_{t-1}\xi. \tag{47}$$

Proof. We will prove the claim by induction on t. For $t = 1$ the claim follows from Lemma 5.1 since the variables $\mathbf{x}(S^1)$ belong to k disjoint simplices. For $t > 1$ we will use equation (45) from the proof of Lemma 5.2 with $\mathbf{g} := \mathbf{0}$, that is

$$d_t\left(\mathbf{x}^*(S^{[t]}\right) + \left\langle \mathbf{g}(S^{[t]}), \mathbf{x}^*(S^{[t]})\right\rangle = d_{t-1}\left(\mathbf{x}^*(S^{[t-1]})\right) + w_{t-1}\left\langle \bar{\mathbf{g}}(S^{[t-1]}), \mathbf{x}^*(S^{[t-1]})\right\rangle,$$

where

$$\bar{g}(s) = \begin{cases} g(s) + \langle \mathbf{g}(s^+), \bar{\mathbf{z}}(s^+)\rangle + \Psi_s(\bar{\mathbf{z}}(s^+)) &= k\xi \quad \text{for } s\in S^{t-1}, \iota(s)\ne\emptyset, \\ g(s) &= 0 \quad \text{otherwise.} \end{cases}$$

Thus if $\mathbf{x}^*(S^{[t]})$ is an extreme point maximizer of d_t then

$$\begin{aligned}
\max_{\mathbf{x}(S^{[t]})\in Q_t(U,S)} d_t\left(\mathbf{x}(S^{[t]}\right) &= d_t\left(\mathbf{x}^*(S^{[t]})\right)\\
&= d_{t-1}\left(\mathbf{x}^*(S^{[t-1]})\right) + w_{t-1}\left\langle \bar{\mathbf{g}}(S^{[t-1]}), \mathbf{x}^*(S^{[t-1]})\right\rangle\\
&= d_{t-1}\left(\mathbf{x}^*(S^{[t-1]})\right) + w_{t-1}\left\langle \bar{\mathbf{g}}(S^{t-1}), \mathbf{x}^*(S^{t-1})\right\rangle\\
&= d_{t-1}\left(\mathbf{x}^*(S^{[t-1]})\right) + w_{t-1}k\xi\left(\sum_{\substack{s\in S^{t-1}\\ \iota(s)\neq\emptyset}} x^*(s)\right). \qquad (48)
\end{aligned}$$

By Lemma 5.4 we know that

$$\sum_{s\in S^{t-1}} x^*(s) \le k^{t-1}.$$

The result follows by applying the inequality above and the inductive hypothesis to (48).

□

5.3. **Estimates of strong convexity for uniform complexes.** In the proof of Lemma 4.2 we obtained a bound on the strong convexity parameter for a prox function over the complex using an inductive construction. In a uniform complex $\mathcal{Q}(n,k,b,t)$ the polytopes $\Pi(s)$ for $s\in S$, $\iota(s)\neq\emptyset$ are identical up to labeling of the coordinates. Thus the functions $\{\Psi_s\colon s\in S, \iota(s)\neq\emptyset\}$ have the same strong convexity parameter which we denote by $\varrho(k)$.

For $i=1$

$$d_1\left(\mathbf{x}(S^{[1]})\right) = w_0\cdot\bar{\Psi}_\varepsilon\left(x(\varepsilon),\mathbf{x}(\varepsilon^+)\right) = w_0\cdot\Psi_\varepsilon\left(\mathbf{x}(\varepsilon^+)\right).$$

Thus the strong convexity parameter ρ_1 is just $w_0\varrho(k)$.

In the proof of Lemma 4.2 we defined κ_{i-1} as any constant such that

$$\kappa_{i-1} \ge \sum_{s\in S^{i-1}} x(s).$$

Lemma 5.4 shows that in a uniform game of type $\mathcal{Q}(n,k,b,t)$ that we can choose $\kappa_{i-1}=k^{i-1}$, that is,

$$k^{i-1} \ge \sum_{s\in S^{i-1}} x(s). \qquad (49)$$

And so the bound for ρ_2 obtained from (41) is

$$\rho_2 = \min_{s\in S^1}\left\{\frac{\rho_1}{M^2+1}, \frac{w_1\cdot\varrho_s}{k(M^2+1)}\right\} = \min\left\{\frac{w_0\cdot\varrho(k)}{M^2+1}, \frac{w_1\cdot\varrho(k)}{k(M^2+1)}\right\} \qquad (50)$$

In the next subsection we will shown the quantity M depends on the prox function, but is independent of i and is constant for uniform complexes. The estimate for ρ_i when $i>2$ will be similar to the case discussed above in (50).

Before presenting our results for the entropy and Euclidean prox functions we will briefly discuss the types of weights we will be using in the prox functions given in (46) for the complex. For a positive constant W consider the sequence of weights $w_0=1/k$ and $w_i=W^{i-1}$ for $i=2,\dots,t-1$. The estimate of the strong convexity parameter for the prox functions with these weights over a uniform complex of type $\mathcal{Q}(n,k,b,t)$ is

$$\rho_i = \begin{cases} \frac{\varrho(k)}{k} & \text{if } i=1,\\ \min\left\{\frac{\rho_{i-1}}{M^2+1}, \frac{W^{i-2}\varrho(k)}{k^{i-1}(M^2+1)}\right\} & \text{if } i\in\{2,\dots,t\}. \end{cases} \qquad (51)$$

If kW is strictly less than one, then an upper bound on the maximum value is

$$D = w_0k\xi + k^2\xi + k^3W\xi + \cdots + k^tW^{t-2}\xi = w_0k\xi + k^2\xi\sum_{\ell=0}^{t}(kW)^t \le \left(w_0k + \frac{k^2}{1-kW}\right)\xi. \qquad (52)$$

For the entropy prox function, the bound $\rho_{i-1}/(M^2+1)$ decreases much more rapidly than $\varrho(k)/[k^{i-1}(M^2+1)]$. By choosing an appropriate factor for the weights we can leverage the fact that the strong convexity estimate for ρ_i is at most the least of these quantities. More precisely, the factor is chosen so that the second argument of the minimum appearing in (51) matches the first argument. By setting $W = \frac{k}{M^2+1}$, the strong convexity estimate in (51) becomes

$$\rho_i = \frac{\varrho(k)}{k \cdot (M^2+1)^{i-1}} \quad \text{for } i = 1, \ldots, t. \tag{53}$$

Provided that $k^2/(M^2+1) < 1$, the maximum value of the prox functions over the uniform complex is

$$D \leq \left(1 + \frac{k^2}{1 - \frac{k}{M^2+1}}\right)\xi. \tag{54}$$

This weighting scheme results in a significant decrease in the maximum value of the prox functions over the complex and hence an improvement in the niceness parameter ρ/D.

We will be using both the entropy prox function and the Euclidean (quadratic) prox function for simplices and their natural extensions to Cartesian products of simplices. Both of these functions are twice continuously differentiable in the interior of the sets $\Pi(s)$ for $s \in S$, $\iota(s) \neq \emptyset$. The following characterization (see [6]) will be useful for determining the strong convexity parameters for the prox functions over the Cartesian products of simplices: The function Ψ_s is strongly convex on $\Pi(s)$ with strong convexity parameter $\varrho(k) > 0$ if for all $(\mathbf{z}^1, \ldots, \mathbf{z}^k) \in \Pi(s)$

$$\left\langle \mathbf{h}, \nabla^2 \Psi_s(\mathbf{z}^1, \ldots, \mathbf{z}^k)\mathbf{h} \right\rangle \geq \varrho(k)\|\mathbf{h}\|^2. \tag{55}$$

Let $\iota(s) = \{u_1, \ldots, u_k\}$. Elementary calculations show that in the interior of $\Pi(s)$ the Hessian of Ψ_s is given by the following block diagonal matrix

$$\nabla^2 \Psi_s(\mathbf{z}^1, \ldots, \mathbf{z}^k) = \begin{bmatrix} \nabla^2 \psi_{u_1}(\mathbf{z}^1) & 0 & \cdots & 0 \\ 0 & \nabla^2 \psi_{u_2}(\mathbf{z}^2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \nabla^2 \psi_{u_k}(\mathbf{z}^k) \end{bmatrix}. \tag{56}$$

5.4. **The entropy prox function.** In Example 5 we mentioned that the entropy distance

$$\psi_n(\mathbf{x}) = \ln n + \sum_{j=1}^{n} x_j \ln x_j$$

is a prox function for the simplex Δ_n where the analysis used the 1-norm, that is, $\|\mathbf{x}\| = \sum_{i=1}^n |x_i|$ (see [7] for details). Moreover, the maximum value of ψ_n is $\ln n$ for the $(n-1)$-dimensional simplex Δ_n and is attained at every extreme point.

We will also use the 1-norm in the analysis of Ψ_s over the Cartesian product $\Pi(s)$. Suppose that the sets $\Pi(s)$ arise from a uniform complex of type $\mathcal{Q}(n, k, b, t)$. Let $\mathbf{h} = (\mathbf{h}^1, \ldots, \mathbf{h}^k)$, where each vector $\mathbf{h}^\ell = (h_1^\ell, \ldots, h_n^\ell)$ for $\ell = 1, \ldots, k$.

Using the expression for the Hessian given in (56) specialized to the entropy prox function we find that

$$\left\langle \mathbf{h}, \nabla^2 \Psi_s(\mathbf{z}^1, \ldots, \mathbf{z}^k)\mathbf{h} \right\rangle = \sum_{\ell=1}^{k}\sum_{j=1}^{n} \frac{(h_j^\ell)^2}{z_j^\ell}.$$

Applying a variant of the Cauchy-Schwarz inequality yields

$$\left(\sum_{\ell=1}^{k}\sum_{j=1}^{n} |h_j^\ell|\right)^2 \leq \left(\sum_{\ell=1}^{k}\sum_{j=1}^{n} z_j^\ell\right)\left(\sum_{\ell=1}^{k}\sum_{j=1}^{n} \frac{(h_j^\ell)^2}{z_j^\ell}\right). \tag{57}$$

Since each $\mathbf{z}^\ell$ is a point in an $(n-1)$-dimensional simplex Δ_n we have that $\sum_{\ell=1}^k \sum_{j=1}^n z_j^\ell = k$.

Applying the strong convexity characterization stated in (55) it follows that the strong convexity parameter for $\Psi_s(\mathbf{z}^1, \ldots, \mathbf{z}^k)$ is $1/k$ for points in the interior of $\Pi(s)$. We can use a limiting argument similar to the one in Lemma 4.2 to extend the result to points on the boundary of $\Pi(s)$.

Next, we determine the constant M introduced in Lemma 4.2 for the entropy prox functions. Recall that the constant $M \geq 1$ is defined so that

$$\sum_{s \in S^{i-1}} \frac{1}{2}|x(s) - \tilde{x}(s)| \cdot \left\| \frac{\mathbf{x}(s^+)}{x(s)} + \frac{\tilde{\mathbf{x}}(s^+)}{x(s)} \right\| \leq (M-1) \cdot \left\| \mathbf{x}(S^{i-1}) - \tilde{\mathbf{x}}(S^{i-1}) \right\|.$$

Since $\mathbf{x}(s^+)/x(s)$ and $\tilde{\mathbf{x}}(s^+)/x(s) \in \Pi(s)$ for all $s \in S^{i-1}$ it follows when using the 1-norm that

$$\sum_{s \in S^{i-1}} \frac{1}{2}|x(s) - \tilde{x}(s)| \cdot \left\| \frac{\mathbf{x}(s^+)}{x(s)} + \frac{\tilde{\mathbf{x}}(s^+)}{x(s)} \right\| \leq k \left(\sum_{s \in S^{i-1}} |x(s) - \tilde{x}(s)| \right) = k \|\mathbf{x}(S^{i-1}) - \tilde{\mathbf{x}}(S^{i-1})\|.$$

Hence we can take $M = k + 1$ and obtain the following result for uniform complexes.

Theorem 5.6. *Consider the uniform complex $\mathcal{Q}(n, k, b, t)$ with weights $w_0 = \frac{1}{k}$ and $W = \frac{k}{(k+1)^2+1}$. Then the function d_t defined in (46) derived from the entropy prox function for the simplex has the following properties*

$$\rho_t = \frac{1}{k^2 \cdot [(k+1)^2 + 1]^{t-1}}, \quad D_t \leq \left(\frac{5k^2}{4} + 1 \right) \ln n, \text{ and } \quad \frac{D_t}{\rho_t} \leq \frac{1}{2} k^2 [(k+1)^2 + 1]^t \ln n$$

Proof. The bounds on the strong convexity parameter and the maximum value are obtained by substituting $M = k+1$, $\xi = \ln n$ and $W = \frac{k}{(k+1)^2+1}$ into (53) and (54) respectively and using $\frac{k}{(k+1)^2+1} \leq \frac{1}{5}$. For the reciprocal of the niceness parameter observe that

$$\begin{aligned}
\frac{D_t}{\rho_t} &\leq k^2 \cdot [(k+1)^2 + 1]^{t-1} \left(\frac{5k^2}{4} + 1 \right) \ln n \\
&= \frac{5}{4} k^2 \cdot [(k+1)^2 + 1]^{t-1} \left(k^2 + 2k + 2 - 2k - \frac{6}{5} \right) \ln n \\
&= \frac{5}{4} k^2 \cdot [(k+1)^2 + 1]^{t} \left(1 - \frac{2k + \frac{6}{5}}{k^2 + 2k + 2} \right) \ln n \\
&\leq \frac{5}{4} k^2 \cdot [(k+1)^2 + 1]^{t} \left(\frac{9}{25} \right) \ln n \\
&\leq \frac{1}{2} k^2 \cdot [(k+1)^2 + 1]^{t} \ln n.
\end{aligned}$$

□

Now suppose that we are given a uniform game in which the players' complexes are of types $\mathcal{Q}(n_i, k_i, b_i, t_i)$ for $i = 1$ and 2. Recall that if the vector norms $\|\cdot\|_1$ and $\|\cdot\|_2$ used in the analysis are 1-norms then the induced operator norm is the largest entry in the matrix in absolute value, that is, $\|A\|_{2,1} = \max_{i,j} |A_{ij}|$. Let G be the reciprocal of the geometric mean of the niceness parameters for the weighted entropy prox functions of both players, that is,

$$G := \frac{1}{2} k_1 k_2 \sqrt{[((k_1+1)^2+1)^{t_1} \ln n_1][((k_2+1)^2+1)^{t_2} \ln n_2]},$$

An immediate consequence of Theorem 5.6 is the following iteration bound for the entropy prox functions in this uniform game.

Corollary 5.7. *Given $\epsilon > 0$, if we run the algorithm in Theorem 3.1 for $\left\lceil \frac{4G}{\epsilon}(\max|A_{ij}|) - 1 \right\rceil$ iterations, we obtain primal and dual feasible solutions $\bar{\mathbf{x}} \in Q_1$ and $\bar{\mathbf{y}} \in Q_2$ such that*

$$\phi(\bar{\mathbf{y}}) - f(\bar{\mathbf{x}}) = \max_{\mathbf{x} \in Q_1} \langle A\bar{\mathbf{y}}, \mathbf{x} \rangle - \min_{\mathbf{y} \in Q_2} \langle A\mathbf{y}, \bar{\mathbf{x}} \rangle \leq \epsilon.$$

Setting $n = \max\{n_1, n_2\}$, $k = \max\{k_1, k_2\}$ and $t = \max\{t_1, t_2\}$ we can state a simpler (but slightly weaker) result by setting

$$G := \frac{1}{2} k^2 [(k+1)^2 + 1]^t \ln n.$$

Recall that the game tree has at least $nk(kb)^{t-1}$ nodes (including the leaves). If we take $\|A\|_{2,1} = 1$ then the constant in the convergence bound is almost independent in terms of nb^{t-1} and just slightly more than quadratic in terms of k^t. We also note that the cost of each iteration is linear in the number of sequences, and hence linear in the size of the game tree.

5.5. **The Euclidean prox function.** In Example 5 we mentioned that the function

$$\psi_n(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\right)^2$$

is a prox function for the simplex Δ_n with strong convexity parameter equal to one when using the Euclidean norm. The maximum value of ψ_n is $\left(1 - \frac{1}{n}\right)$ for Δ_n and it is easy to see that the function attains its maximum value at every extreme point of the simplex.

Once again assume that we are given a uniform complex of type $\mathcal{Q}(n, k, b, t)$. Specializing (56) for the induced quadratic prox function Ψ_s over the Cartesian product $\Pi(s)$ we find that the Hessian $\nabla^2\Psi_s(\mathbf{x}(s^+))$ is the identity matrix of dimension kn. Consequently, $\varrho(k) = 1$ for the functions $\{\Psi_s : s \in S, \iota(s) \neq \emptyset\}$ by the characterization stated in (55).

Although we could use Lemma 4.2 directly to determine the strong convexity parameter for the Euclidean prox function, we will modify the argument slightly to take advantage of some the nice properties of the Euclidean norm.

First observe that for any $\mathbf{x}(s^+) \in \Pi(s)$ such that $\iota(s) \neq \emptyset$

$$\|\mathbf{x}(s^+)\| = \sqrt{\sum_{u\in\iota(s)} \|\mathbf{x}(u^+)\|^2} \leq \sqrt{|\iota(s)|} = \sqrt{k}.$$

Now (using the notation in Lemma 4.2) we modify the proof of Lemma 4.2 after (23) as follows

$$\left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\|^2 = \left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s\in S^{i-1}} \left\|x(s)\mathbf{z}(s^+) - \tilde{x}(s)\tilde{\mathbf{z}}(s^+)\right\|^2. \tag{58}$$

Recall that we set $\hat{x}(s) = (x(s) + \tilde{x}(s))/2$. Working on the summation above by applying the triangle inequality and then Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\sum_{s\in S^{i-1}} \left\|x(s)\mathbf{z}(s^+) - \tilde{x}(s)\tilde{\mathbf{z}}(s^+)\right\|^2 \\
&\quad= \sum_{s\in S^{i-1}} \left\|\left(\frac{x(s)-\tilde{x}(s)}{2}\right)[\mathbf{z}(s^+) + \tilde{\mathbf{z}}(s^+)] + \hat{x}(s)\cdot[\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)]\right\|^2 \\
&\quad\leq \sum_{s\in S^{i-1}} \left[\sqrt{k}\cdot|x(s)-\tilde{x}(s)| + \hat{x}(s)\cdot\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|\right]^2 \\
&\quad\leq \sum_{s\in S^{i-1}} \left[(\sqrt{k})^2 + \hat{x}(s)\right]\cdot\left[|x(s)-\tilde{x}(s)|^2 + \hat{x}(s)\cdot\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2\right] \\
&\quad\leq (k+1)\sum_{s\in S^{i-1}} \left(|x(s)-\tilde{x}(s)|^2 + \hat{x}(s)\cdot\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2\right) \\
&\quad= (k+1)\left(\left\|\mathbf{x}(S^{i-1}) - \tilde{\mathbf{x}}(S^{i-1})\right\|^2 + \sum_{s\in S^{i-1}} \hat{x}(s)\cdot\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2\right) \\
&\quad\leq (k+1)\left(\left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + \sum_{s\in S^{i-1}} \hat{x}(s)\cdot\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2\right).
\end{aligned}$$

Applying this to (58) we get

$$\left\|\mathbf{x}(S^{[i]}) - \tilde{\mathbf{x}}(S^{[i]})\right\|^2 \leq (k+2)\cdot\left\|\mathbf{x}(S^{[i-1]}) - \tilde{\mathbf{x}}(S^{[i-1]})\right\|^2 + (k+1)\sum_{s\in S^{i-1}} \hat{x}(s)\cdot\left\|\mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+)\right\|^2. \tag{59}$$

Thus to determine a bound on the strong convexity parameter ρ_i so that (23) holds, it suffices to find ρ_i such that

$$[\rho_{i-1} - (k+2)\rho_i] \cdot \left\| \mathbf{x}(S^{[i-1]} - \tilde{\mathbf{x}}(S^{[i-1]}) \right\|^2 + \sum_{s \in S^{i-1}} [\varrho_s - (k+1)\rho_i] \cdot \hat{x}(s) \left\| \mathbf{z}(s^+) - \tilde{\mathbf{z}}(s^+) \right\|^2 \geq 0.$$

Note that LHS is a weighted sum of squares and so it is enough to ensure that the coefficients of the squared terms are nonnegative for the LHS to be nonnegative. Using the fact that $0 \leq \hat{x}(s) \leq 1$ for all $s \in S^{i-1}$ it follows that (59) holds as long as

$$0 < \rho_i \leq \min_{s \in S^{i-1}} \left\{ \frac{\rho_{i-1}}{k+2}, \frac{\varrho_s}{k+1} \right\} = \min \left\{ \frac{\rho_{i-1}}{k+2}, \frac{1}{k+1} \right\}. \tag{60}$$

As we did for the entropy prox functions, we introduce weights to take advantage of the form of the recursion in (60). Recall that the weighted prox functions d_i are defined as

$$d_i \left(\mathbf{x}(S^{[i]}) \right) = d_{i-1}(\mathbf{x}(S^{[i-1]})) + w_{i-1} \sum_{s \in S^{i-1}} \bar{\Psi}_s \left(x(s), \mathbf{x}(s^+) \right) \quad \text{for } i = 1, \ldots, t, \tag{61}$$

Set $w_0 = \frac{k+2}{k+1}$ and $w_i = W^{i-1}$ where $W = \frac{1}{k+2}$. Then for $i > 1$

$$0 < \rho_i \leq \min_{s \in S^{i-1}} \left\{ \frac{\rho_{i-1}}{k+2}, \frac{\varrho_s}{k+1} \right\} = \min \left\{ \frac{\rho_{i-1}}{k+2}, \frac{W^{i-1}}{k+1} \right\}. \tag{62}$$

Using these weights we obtain the following result for uniform complexes.

Theorem 5.8. *Consider the uniform complex $\mathcal{Q}(n, k, b, t)$ with weights $w_0 = \frac{k+2}{k+1}$ and $W = \frac{1}{k+2}$. Then the function d_t defined in (61) derived from the Euclidean prox function for the simplex has the following properties*

$$\rho_t = \frac{1}{(k+1)(k+2)^{t-2}}, \quad D_t \leq \frac{3}{2} k(k+1) \left(1 - \frac{1}{n} \right), \text{ and } \quad \frac{D_t}{\rho_t} \leq \frac{3}{2} k(k+1)^2 (k+2)^{t-2} \left(1 - \frac{1}{n} \right).$$

Proof. Since $\varrho(k) = 1$ it follows that $\rho_1 = w_0 = (k+1)(k+2)^{-1}$. The result for $t > 1$ follows immediately from the recursion in (62). Substituting the given weights and $\xi = \left(1 - \frac{1}{n}\right)$ into (52) yields the following bound on the maximum value of the function d_t over $Q_t(U, S)$

$$D_t \leq \left[k \left(\frac{k+2}{k+1} \right) + \frac{k^2}{1 - \frac{k}{k+2}} \right] \cdot \left(1 - \frac{1}{n} \right) \leq \left[k \left(\frac{k+2}{k+1} \right) + \frac{3}{2} k^2 \right] \cdot \left(1 - \frac{1}{n} \right) \leq \frac{3}{2} k(k+1) \left(1 - \frac{1}{n} \right).$$

Thus result for the reciprocal of the niceness parameter follows immediately.

□

Now suppose that we are given a uniform game in which the players' complexes are of types $\mathcal{Q}(n_i, k_i, b_i, t_i)$ for $i = 1$ and 2. Recall that the vector norms $\| \cdot \|_1$ and $\| \cdot \|_2$ used in the analysis are the Euclidean norms and thus the induced operator norm is the spectral norm. Let G be the reciprocal of the geometric mean of the niceness parameters for the weighted Euclidean prox functions of both players, that is,

$$G := \frac{3}{2}(k_1+1)(k_2+1) \sqrt{ \left[k_1 (k_1+2)^{t_1 - 2} \left(1 - \frac{1}{n_1} \right) \right] \left[k_2 (k_2+2)^{t_2 - 2} \left(1 - \frac{1}{n_2} \right) \right] },$$

An immediate consequence of Theorem 5.8 is the following iteration bound for the Euclidean prox functions in this uniform game.

Corollary 5.9. *Given $\epsilon > 0$, if we run the algorithm in Theorem 3.1 for $\lceil \frac{4G}{\epsilon} \lambda_{\max}^{1/2}(A^T A) - 1 \rceil$ iterations, we obtain primal and dual feasible solutions $\bar{\mathbf{x}} \in Q_1$ and $\bar{\mathbf{y}} \in Q_2$ such that*

$$\phi(\bar{\mathbf{y}}) - f(\bar{\mathbf{x}}) = \max_{\mathbf{x} \in Q_1} \langle A\bar{\mathbf{y}}, \mathbf{x} \rangle - \min_{\mathbf{y} \in Q_2} \langle A\mathbf{y}, \bar{\mathbf{x}} \rangle \leq \epsilon.$$

Setting $n = \max\{n_1, n_2\}$, $k = \max\{k_1, k_2\}$ and $t = \max\{t_1, t_2\}$ we can state a simpler (but slightly weaker) result by setting

$$G := \frac{3}{2}k(k+1)^2(k+2)^{t-2}\left(1 - \frac{1}{n}\right).$$

Recall that the game tree has at least $nk(kb)^{t-1}$ nodes (including the leaves). Ignoring the factor due to the operator norm, we note that the constant in the convergence bound is almost independent in terms of nb^{t-1} and slightly more than linear in terms of k^t. As we mentioned in Example 5 the subproblems can be solved in linear time and so the work per iteration for the algorithm in Theorem 3.1 is linear in the size of the game tree.

Remark 5.10. The niceness parameter of the Euclidean prox function is much better than the niceness parameter of the entropy prox function. However, we still cannot conclude that the Euclidean prox function will converge faster than the entropy prox function. Recall that the niceness parameter is only one of the main factors appearing in the iteration bounds given in Corollary 5.7 and Corollary 5.9. A second key factor is the induced matrix norm.

In general the spectral norm is much larger than the operator norm $\|A\| = \max_{i,j} |A_{i,j}|$. We can bound the spectral norm as follows

$$\lambda_{\max}^{1/2}(A^T A) \leq \sqrt{\|A\|_1 \cdot \|A\|_\infty}.$$

This bound is the square root of the product of the largest column sum with the largest row sum where the sums are taken over the absolute values of the entries. If the payoffs are at most one in absolute value, then the bound above for the spectral norm can be linear in the size of the game tree. In fact, we can construct examples for which this bound is tight.

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Appendix A. Nesterov's algorithm

For completeness, we include a description of one of Nesterov's algorithms in [8] specialized for problem (4). Assume d_1 and d_2 are given nice prox functions for Q_1 and Q_2 respectively. For $\mu_1, \mu_2 > 0$ consider

$$f_{\mu_2}(\mathbf{x}) = \min_{\mathbf{y} \in Q_2} \left\{\langle A\mathbf{y}, \mathbf{x}\rangle + \mu_2 d_2(\mathbf{y})\right\} \quad \text{and} \quad \phi_{\mu_1}(\mathbf{y}) = \max_{\mathbf{x} \in Q_1} \left\{\langle A\mathbf{y}, \mathbf{x}\rangle - \mu_1 d_1(\mathbf{x})\right\}.$$

The algorithm below generates iterates $(\mathbf{x}^k, \mathbf{y}^k, \mu_1^k, \mu_2^k)$ with μ_1^k, μ_2^k decreasing to zero and such that the following *excessive gap condition* is satisfied at each iteration:

$$f_{\mu_2^k}(\mathbf{x}^k) \geq \phi_{\mu_1^k}(\mathbf{y}^k). \tag{63}$$

It is intuitively clear that for μ_1, μ_2 small $f_{\mu_2} \approx f$ and $\phi_{\mu_1} \approx \phi$. Since $f(\mathbf{x}) \le \phi(\mathbf{y})$ for $\mathbf{x} \in Q_1,\ \mathbf{y} \in Q_2$, it is also intuitively clear that the excessive gap condition (63) ensures that $f(\mathbf{x}^k) \approx \phi(\mathbf{y}^k)$ for μ_1^k, μ_2^k small.

The detailed algorithm is as follows.

Input: Nice prox functions d_1, d_2 for $Q_1 \subseteq \mathbb{R}^{S_1}, Q_2 \subseteq \mathbb{R}^{S_2}$ respectively, $A \in \mathbb{R}^{S_1 \times S_2}$, and a positive integer K

Output: $\mathbf{x}^* \in Q_1, \mathbf{y}^* \in Q_2$ with $0 \le \max\limits_{\mathbf{x} \in Q_1} \langle A\mathbf{y}^*, \mathbf{x}\rangle - \min\limits_{\mathbf{y} \in Q_2} \langle A\mathbf{y}, \mathbf{x}^*\rangle \le \frac{4\,\|A\|}{K+1}\sqrt{\frac{D_1 D_2}{\rho_1 \rho_2}}$

(1) $\mu_1^0 = \mu_2^0 = \frac{\|A\|}{\sqrt{\rho_1 \rho_2}}$
(2) $\hat{\mathbf{y}} = \mathrm{sargmax}\,(d_2, \mathbf{0})$
(3) $\mathbf{x}^0 = \mathrm{sargmax}\left(d_1, \frac{1}{\mu_1^0} A\hat{\mathbf{y}}\right)$
(4) $\mathbf{y}^0 = \mathrm{sargmax}\left(d_2, \nabla d_2\,(\hat{\mathbf{x}}) + \frac{1}{\mu_2^0} A^{\mathrm{T}}\mathbf{x}^0\right)$
(5) `For` $k = 0, 1, \ldots, K$`:`
  (a) $\tau = \frac{2}{k+3}$
  (b) `If` k `is even:  /* Shrink` μ_2 `*/`
    (i) $\breve{\mathbf{y}} = \mathrm{sargmax}\left(d_2, -\frac{1}{\mu_2^k} A^{\mathrm{T}}\mathbf{x}^k\right)$
    (ii) $\hat{\mathbf{y}} = (1-\tau)\mathbf{y}^k + \tau\breve{\mathbf{y}}$
    (iii) $\hat{\mathbf{x}} = \mathrm{sargmax}\left(d_1, \frac{1}{\mu_1^k} A\hat{\mathbf{y}}\right)$
    (iv) $\tilde{\mathbf{y}} = \mathrm{sargmax}\left(d_2, \nabla d_2\,(\breve{\mathbf{y}}) + \frac{\tau}{(1-\tau)\mu_2^k} A^{\mathrm{T}}\hat{\mathbf{x}}\right)$
    (v) $\mathbf{x}^{k+1} = (1-\tau)\mathbf{x}^k + \tau\hat{\mathbf{x}}$
    (vi) $\mathbf{y}^{k+1} = (1-\tau)\mathbf{y}^k + \tau\tilde{\mathbf{y}}$
    (vii) $\mu_2^{k+1} = (1-\tau)\mu_2^k$
  (c) `If` k `is odd:  /* Shrink` μ_1 `*/`
    (i) $\breve{\mathbf{x}} = \mathrm{sargmax}\left(d_1, \frac{1}{\mu_1^k} A\mathbf{y}^k\right)$
    (ii) $\hat{\mathbf{x}} = (1-\tau)\mathbf{x}^k + \tau\breve{\mathbf{x}}$
    (iii) $\hat{\mathbf{y}} = \mathrm{sargmax}\left(d_2, -\frac{1}{\mu_2^k} A^{\mathrm{T}}\hat{\mathbf{x}}\right)$
    (iv) $\tilde{\mathbf{x}} = \mathrm{sargmax}\left(d_1, \nabla d_1\,(\breve{\mathbf{x}}) - \frac{\tau}{(1-\tau)\mu_1^k} A\hat{\mathbf{y}}\right)$
    (v) $\mathbf{y}^{k+1} = (1-\tau)\mathbf{y}^k + \tau\hat{\mathbf{y}}$
    (vi) $\mathbf{x}^{k+1} = (1-\tau)\mathbf{x}^k + \tau\tilde{\mathbf{x}}$
    (vii) $\mu_1^{k+1} = (1-\tau)\mu_1^k$