SMALL CHVÁTAL RANK

TRISTRAM BOGART AND REKHA R. THOMAS

Abstract. We introduce a new measure of complexity of integer hulls of rational polyhedra called the small Chvátal rank (SCR). The SCR of an integer matrix A is the number of rounds of a Hilbert basis procedure needed to generate all normals of a sufficient set of inequalities to cut out the integer hulls of all polyhedra $\{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ as $\mathbf{b}$ varies. The SCR of A is bounded above by the Chvátal rank of A and is hence finite. We exhibit examples where SCR is much smaller than Chvátal rank. When the number of columns of A is at least three, we show that SCR can be arbitrarily high proving that, in general, SCR is not a function of dimension alone. For polytopes in the unit cube we provide a lower bound for SCR that is comparable to the known lower bounds for Chvátal rank in that situation. We use the notion of supernormality to completely characterize matrices for which SCR equals zero.

1. Introduction

The study of integer hulls of rational polyhedra is a fundamental area of research in integer programming and discrete geometry. For a matrix $A \in \mathbb{Z}^{m\times n}$ and vector $\mathbf{b} \in \mathbb{Z}^m$, consider the rational polyhedron $Q_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ and its integer hull $Q_{\mathbf{b}}^I :=$ convex hull$(Q_{\mathbf{b}} \cap \mathbb{Z}^n)$. An algorithm for computing $Q_{\mathbf{b}}^I$ from the inequality description of $Q_{\mathbf{b}}$ is given by the *Chvátal-Gomory* procedure [8, §23]. This method involves iteratively adding rounds of *cutting planes* to $Q_{\mathbf{b}}$ until $Q_{\mathbf{b}}^I$ is obtained. The *Chvátal rank* of $A\mathbf{x} \leq \mathbf{b}$ is the minimum number of rounds of cuts needed in the Chvátal-Gomory procedure to obtain $Q_{\mathbf{b}}^I$, and the *Chvátal rank* of A is the maximum over the Chvátal ranks of $A\mathbf{x} \leq \mathbf{b}$ as $\mathbf{b}$ varies in $\mathbb{Z}^m$. The Chvátal-Gomory procedure and the Chvátal ranks defined above are all finite [8, §23].

In this paper we fix a matrix $A \in \mathbb{Z}^{m\times n}$ of rank n and look at the problem of finding normals of a sufficient set of inequalities to cut out all integer hulls $Q_{\mathbf{b}}^I$ as $\mathbf{b}$ varies in $\mathbb{Z}^m$. By this we mean finding a matrix M such that for each $Q_{\mathbf{b}}^I$ there exists an integer vector $\mathbf{d}$ such that $Q_{\mathbf{b}}^I = \{\mathbf{x} \in \mathbb{R}^n : M\mathbf{x} \leq \mathbf{d}\}$. Theorem 17.4 in [8] proves the existence of such an M. The method is to first prove that if Δ is the maximum absolute value of a minor of A, then every $Q_{\mathbf{b}}^I$ can be described by inequalities whose normal vectors have entries of absolute value at most $n^{2n}\Delta^n$. Then M is taken to be the matrix whose rows are all of the non-zero integer vectors in the cone generated by the

Date: June 22, 2007.
Both authors were partially supported by NSF grant DMS-0401047.

rows of A whose entries have absolute value at most $n^{2n}\Delta^n$. For example, the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has $\Delta = 3$ and $n^{2n}\Delta^n = 144$ and its rows positively span $\mathbb{R}^2$. Therefore, the rows of M would be all the non-zero integer vectors in the box

$$\{(x, y) \in \mathbb{R}^2 \,:\, -144 \leq x, y \leq 144\}.$$

However, it suffices to use

$$M = \begin{pmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & -1 \\ -1 & -2 \\ -1 & -1 \end{pmatrix}$$

and every row of M is actually a facet normal in some $Q^I_{\mathbf{b}}$. In fact, if $A \in \mathbb{Z}^{m\times 2}$ has rank two, we will show that there is always an M with at most $m\Delta$ rows (Corollary 4.9). Thus a sufficient M may be much smaller than the one constructed in [8, Theorem 17.4].

As in this example, we are interested in matrices M that are as economical as possible. We introduce a modified version of the Chvátal-Gomory procedure called *iterated basis normalization* (IBN) that in principle constructs M. The *small Chvátal rank* (SCR) of A is the least number of rounds of IBN necessary to generate an M that works for all $Q^I_{\mathbf{b}}$ as $\mathbf{b}$ varies. For a fixed $\mathbf{b}$, the *small Chvátal rank* of $A\mathbf{x} \leq \mathbf{b}$ is the least number of iterations of IBN needed to obtain an M that describes that particular $Q^I_{\mathbf{b}}$. It can be shown that the SCR of A (respectively $A\mathbf{x} \leq \mathbf{b}$) is at most its Chvátal rank and is hence finite. In contrast, IBN may not terminate when $n \geq 3$. These definitions and preliminaries are described in Section 2.

In Section 3 we show that when $n = 2$, the SCR of A is at most one while it is known that Chvátal rank can be arbitrarily large. Matrices with SCR one and Chvatal rank arbitrarily large are shown to exist for all $n \geq 2$. It is also shown that for a certain standard linear relaxation of the *coclique polytope* of the complete graph K_n, SCR is either one or two depending on the parity of n while Chvátal rank is known to be $\mathcal{O}(\log n)$. These examples support our use of the adjective "small".

In contrast, we show in Section 5 that in dimension greater than or equal to three, SCR may grow exponentially in the bit size of the matrix A, asymptotically just as fast as Chvátal rank. Thus for $n > 2$, SCR is not a function of m and n alone. Using a construction by Alon and Vũ of 0/1 matrices with large determinants, we also show that SCR can be large even when $Q_{\mathbf{b}}$ is contained in the unit cube. The Chvátal rank of polytopes in the unit cube has been well studied. Our methods provide alternate ways to prove lower bounds on Chvátal rank.

In Section 4 we establish the relationship between SCR of A and *supernormality* of the vector configuration, $\mathcal{A}$, consisting of the rows of A [6]. The Chvátal rank of A is zero if and only if $\mathcal{A}$ is unimodular. We prove the analogous result that the SCR of A is zero if and only if $\mathcal{A}$ is supernormal. In this case, A also serves as M which implies that for every $\mathbf{b}$, $Q_{\mathbf{b}}^I$ has the form $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{d}\}$ for some integer $\mathbf{d}$. In this sense, supernormality is a generalization of unimodularity. We produce a family of vector configurations in growing dimension that are supernormal but not unimodular, answering a question in [6].

The *integrality gap* of a vector $\mathbf{c} \in \mathbb{Z}^n$ with respect to $Q_{\mathbf{b}}$ is

$$\max\{\mathbf{cx} : \mathbf{x} \in Q_{\mathbf{b}}\} - \max\{\mathbf{cx} : \mathbf{x} \in Q_{\mathbf{b}}^I\}.$$

A standard technique in estimating the Chvátal rank of $A\mathbf{x} \leq \mathbf{b}$ is to bound the integrality gap of a vector $\mathbf{c}$ on $Q_{\mathbf{b}}$ and to use that to bound the number of iterations of the Chvátal-Gomory procedure on $Q_{\mathbf{b}}$. For the small Chvátal rank of A we only care about the normals of the inequalities describing $Q_{\mathbf{b}}^I$ and not about their right-hand-sides. Therefore, integrality gap does not appear to be a useful tool in the study of SCR. The problem here is to understand how deep in the IBN/Chvátal-Gomory procedure the last facet normal of an integer hull $Q_{\mathbf{b}}^I$ will be generated. Integrality gap and the tools needed for SCR are complementary approaches to understanding the difference between $Q_{\mathbf{b}}^I$ and $Q_{\mathbf{b}}$.

2. Main Definitions

Fix a matrix $A \in \mathbb{Z}^{m \times n}$ of rank n, so $m \geq n$. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be the vector configuration in $\mathbb{Z}^n$ consisting of the rows of A. For each $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{Z}^m$, define the polyhedron

$$Q_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$$

and its *integer hull* (using conv for *convex hull*)

$$Q_{\mathbf{b}}^I := \operatorname{conv}\{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} \leq \mathbf{b}\}.$$

Since the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are used as normal vectors of inequalities that cut out polyhedra, it is reasonable to assume they are *primitive* (i.e, the greatest common divisor of the entries in each vector is one).

Assumption 2.1. Assume that each row of A is primitive.

If the rows of A are not primitive, divide each row by its greatest common divisor. Given a system $A\mathbf{x} \leq \mathbf{b}$, this division may cause a component of $\mathbf{b}$ to become fractional; in this case simply round any such component down to its floor. The polytope $Q_{\mathbf{b}}$ will not be preserved by this operation, but $Q^I_{\mathbf{b}}$ will be, which is enough for our purposes.

The Chvátal-Gomory procedure [4], [8, §23] for computing $Q^I_{\mathbf{b}}$ works as follows. For each minimal face F of $Q_{\mathbf{b}}$, set

$$\mathcal{A}_F := \{\mathbf{a}_i : \mathbf{a}_i \cdot \mathbf{x} = b_i \ \ \forall\ \mathbf{x} \in F\}.$$

If $\mathbf{h}$ is an integer vector in $\mathrm{cone}(\mathcal{A}_F)$, the cone generated by $\mathcal{A}_F$, and $\mathbf{f} \in F$, then the inequality $\mathbf{h}\cdot\mathbf{x} \leq \mathbf{h}\cdot\mathbf{f}$ is valid on $Q_{\mathbf{b}}$, so the inequality $\mathbf{h}\cdot\mathbf{x} \leq \lfloor \mathbf{h}\cdot\mathbf{f} \rfloor$ is valid on $Q^I_{\mathbf{b}}$. Recall that a **Hilbert basis** of a rational polyhedral cone K is a set of integer vectors $\mathbf{h}_1, \ldots, \mathbf{h}_t$ in K such that every integer vector $\mathbf{z} \in K$ can be written as a non-negative integer combination of $\mathbf{h}_1, \ldots, \mathbf{h}_t$. The new polyhedron $Q^{(1)}_{\mathbf{b}}$ defined by the inequalities $\mathbf{h}\cdot\mathbf{x} \leq \lfloor \mathbf{h}\cdot\mathbf{f} \rfloor$ for every minimal face F of $Q_{\mathbf{b}}$ and every vector $\mathbf{h}$ in the minimal Hilbert basis of $\mathrm{cone}(\mathcal{A}_F)$ satisfies

$$Q^I_{\mathbf{b}} \subseteq Q^{(1)}_{\mathbf{b}} \subseteq Q_{\mathbf{b}}.$$

Furthermore, every non-integral vertex of $Q_{\mathbf{b}}$ is cut off by this procedure (that is, lies outside $Q^{(1)}_{\mathbf{b}}$). For $i \geq 2$, inductively define $Q^{(i)}_{\mathbf{b}} := (Q^{(i-1)}_{\mathbf{b}})^{(1)}$. The **Chvátal rank** of $A\mathbf{x} \leq \mathbf{b}$ is the smallest number t such that $Q^{(t)}_{\mathbf{b}} = Q^I_{\mathbf{b}}$. The **Chvátal rank** of A is the maximum over all $\mathbf{b} \in \mathbb{Z}^m$ of the Chvátal ranks of $A\mathbf{x} \leq \mathbf{b}$. The procedure and the ranks are finite [8, Chapter 23].

To study just the normals of inequalities needed for the integer hulls $Q^I_{\mathbf{b}}$ for all $\mathbf{b}$, we modify the Chvátal-Gomory procedure as follows. An n-subset $\tau \subseteq [m] := \{1, 2, \ldots, m\}$ is called a **basis** if the submatrix A_τ consisting of the rows of A indexed by τ is non-singular. Let $\mathcal{A}_\tau$ be the set of rows of A_τ. Thus $\mathcal{A}_\tau$ is a basis of $\mathbb{R}^n$. We call $\mathrm{cone}(\mathcal{A}_\tau)$ a **basis cone**.

Algorithm 2.2. Iterated Basis Normalization (IBN)
Input: $A \in \mathbb{Z}^{m\times n}$ satisfying the assumptions above. Let $\mathcal{A}$ be the set of rows of A.

1. Set $\mathcal{A}^{(0)} := \mathcal{A}$.
2. For $k \geq 1$, let $\mathcal{A}^{(k)}$ be the union of all the (unique) minimal Hilbert bases of all basis cones in $\mathcal{A}^{(k-1)}$.
3. If $\mathcal{A}^{(k)} = \mathcal{A}^{(k-1)}$, then stop.

Remark 2.3. Since each vector in $\mathcal{A}$ is primitive, $\mathcal{A} \subseteq \mathcal{A}^{(1)}$. Every vector created during IBN is also primitive and so Assumption 2.1 implies that

$$\mathcal{A} \subseteq \mathcal{A}^{(1)} \subseteq \mathcal{A}^{(2)} \subseteq \ldots.$$

Lemma 2.4. *Suppose σ is any subset of $[m]$ such that $\mathcal{A}_\sigma$ linearly spans $\mathbb{R}^n$. Then the union of the minimal Hilbert bases of the basis cones* $\mathrm{cone}(\mathcal{A}_\tau)$*, as τ varies over the bases contained in σ, is a Hilbert basis for* $\mathrm{cone}(\mathcal{A}_\sigma)$*.*

Proof: Every integer point in $\mathrm{cone}(\mathcal{A}_\sigma)$ lies in $\mathrm{cone}(\mathcal{A}_\tau)$ for some basis $\tau \subseteq \sigma$ and hence can be written as a non-negative integer combination of the Hilbert basis elements of $\mathrm{cone}(\mathcal{A}_\tau)$. □

Proposition 2.5. *For any* $\mathbf{b}$*, a sufficient set of normals needed by the Chvátal-Gomory procedure for obtaining* $Q^I_{\mathbf{b}}$ *from* $Q_{\mathbf{b}}$ *is generated by IBN.*

Proof: If F is a minimal face of some intermediate polyhedron $Q^{(i)}_{\mathbf{b}} = \{\mathbf{x} : U\mathbf{x} \leq \mathbf{u}\}$ in the Chvátal-Gomory procedure, then $\mathcal{U}_F$ linearly spans $\mathbb{R}^n$ and hence its index set σ satisfies the hypothesis of Lemma 2.4. By induction, a Hilbert basis of cone($\mathcal{U}_F$) is produced within $(i+1)$ iterations of IBN. □

Let $A^{(k)}$ denote a matrix whose rows are the elements of the set $\mathcal{A}^{(k)}$ produced in the kth iteration of IBN.

Definition 2.6.

(1) The **small Chvátal rank (SCR)** of the system of inequalities $A\mathbf{x} \leq \mathbf{b}$ defining $Q_{\mathbf{b}}$ is the smallest number k such that there is an integer vector $\mathbf{b}'$ satisfying

$$Q^I_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : A^{(k)}\mathbf{x} \leq \mathbf{b}'\}.$$

(2) The SCR of a matrix A is the supremum of the SCRs of all systems of the form $A\mathbf{x} \leq \mathbf{b}$ as $\mathbf{b}$ varies in $\mathbb{Z}^m$.

Proposition 2.7. *For any* $\mathbf{b} \in \mathbb{Z}^m$*, the SCR of* $A\mathbf{x} \leq \mathbf{b}$ *is at most the Chvátal rank of the same system, and the SCR of* $A \in \mathbb{Z}^{m\times n}$ *is at most the Chvátal rank of* A*. In particular, SCR is always finite.*

Proof: By Lemma 2.4 and Proposition 2.5, if the Chvátal rank of $A\mathbf{x} \leq \mathbf{b}$ is t, then within t iterations, IBN generates the normals of all inequalities needed in the Chvátal-Gomory procedure on $A\mathbf{x} \leq \mathbf{b}$. □

We now note that if $Q^I_{\mathbf{b}} = \emptyset$ there is always a $\mathbf{b}' \in \mathbb{Z}^m$ such that $Q^I_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}'\}$ regardless of the SCR of A. This allows us to say that the SCR of A is the smallest positive integer k such that for each $\mathbf{b} \in \mathbb{Z}^m$ with $Q^I_{\mathbf{b}} \neq \emptyset$, there is an integral $\mathbf{b}'$ such that $Q^I_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : A^{(k)}\mathbf{x} \leq \mathbf{b}'\}$.

(1) Suppose $Q_{\mathbf{b}} = \emptyset$. Then $Q^I_{\mathbf{b}} = \emptyset$ and $A\mathbf{x} \leq \mathbf{b}$ is infeasible and we can choose $\mathbf{b}' = \mathbf{b}$ to get $Q^I_{\mathbf{b}} = \emptyset = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$.

(2) Suppose $Q_{\mathbf{b}} \neq \emptyset$ but $Q^I_{\mathbf{b}} = \emptyset$. Then the recession cone of $Q_{\mathbf{b}}$, which is $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{0}\}$ is not full-dimensional. Hence the inequalities $A\mathbf{x} \leq \mathbf{0}$ imply an equality which means that there is a positive dependency among a subset of the rows of A. Without loss of generality, for some k with $1 \leq k < m$, there exists $\lambda_1, \ldots, \lambda_k > 0$ and rational such that

$$\lambda_1\mathbf{a}_1 + \cdots + \lambda_k\mathbf{a}_k = -\mathbf{a}_{k+1}.$$

This implies that both $\mathbf{a}_{k+1}\mathbf{x} \leq b_{k+1}$ and $-\mathbf{a}_{k+1}\mathbf{x} \leq \lambda_1 b_1 + \cdots + \lambda_k b_k$ are valid for $Q_{\mathbf{b}}$. Rewriting, both $\mathbf{a}_{k+1}\mathbf{x} \leq b_{k+1}$ and $\mathbf{a}_{k+1}\mathbf{x} \geq -(\lambda_1 b_1 + \cdots + \lambda_k b_k)$ are valid for $Q_{\mathbf{b}}$. We need to show that we can change the right-hand-side $\mathbf{b}$ of $A\mathbf{x} \leq \mathbf{b}$ to some $\mathbf{b}'$ such that $A\mathbf{x} \leq \mathbf{b}'$ will be infeasible and hence will serve as a representation for $Q^I_{\mathbf{b}}$. To get this, replace $b_1, \ldots, b_k$ by numbers that are sufficiently negative to get a new right-hand-side vector $\mathbf{b}'$ and the implied inequality

$$-\mathbf{a}_{k+1}\mathbf{x} = (\lambda_1 \mathbf{a}_1 + \cdots + \lambda_k \mathbf{a}_k)\mathbf{x} \leq -M$$

from the inequality system $A\mathbf{x} \leq \mathbf{b}'$ where M is a large positive number. In other words, we get the inequality $\mathbf{a}_{k+1}\mathbf{x} \geq M$. The choices involved allow M to be made larger than b_{k+1}, making $A\mathbf{x} \leq \mathbf{b}'$ infeasible and hence a representation for $Q_{\mathbf{b}}^I$.

If IBN terminates after $k+1$ iterations (since $\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)}$) then k is an upper bound on the SCR of A and hence also on the SCR of $A\mathbf{x} \leq \mathbf{b}$ for all $\mathbf{b}$. So a natural first question to ask is whether IBN always terminates.

Lemma 2.8. *When $n = 2$, $\mathcal{A}^{(2)} = \mathcal{A}^{(1)}$.*

Proof: Pick $\mathbf{r}, \mathbf{s} \in \mathcal{A}^{(1)} \subset \mathbb{R}^2$ such that cone$(\mathbf{r}, \mathbf{s})$ is a basis cone. Let $\mathbf{t}_1 := \mathbf{r}, \mathbf{t}_2, \ldots, \mathbf{t}_{k-1}, \mathbf{t}_k := \mathbf{s}$ be the elements of $\mathcal{A}^{(1)}$ in cone$(\mathbf{r}, \mathbf{s})$ in cyclic order from $\mathbf{r}$ to $\mathbf{s}$. Then for each $i \in \{1, \ldots, k-1\}$, cone$(\mathbf{t}_i, \mathbf{t}_{i+1})$ is unimodular. (This is an artifact of $\mathbb{R}^2$. See [7, Corollary 3.11] for a proof.) Hence a Hilbert basis of cone$(\mathbf{r}, \mathbf{s})$ is contained in $\{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$. Thus $A^{(2)} = \mathcal{A}^{(1)}$. □

When $n > 2$, IBN need not terminate. An example appears in [6]; we independently discovered another as follows.

Example 2.9. Take

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 1 & 1 \\ 2 & 5 & 5 \\ 1 & 4 & 3 \end{pmatrix}.$$

For each positive integer j, set

$$\mathbf{u}_j := (j, 2j+2, 2j+1) \text{ and } \mathbf{v}_j := (j, 2j+1, 2j).$$

Note that $\mathbf{u}_1 = (1, 4, 3)$ is a row of A. To show that IBN does not terminate on $\mathcal{A}$, one can check the following two assertions. We omit the details.

(1) For each $j \geq 1$, the unique minimal Hilbert basis of the cone generated by $(0, 3, 1)$, $(1, 1, 1)$, and $\mathbf{u}_j$ includes $\mathbf{v}_j$.
(2) For each $j \geq 1$, the unique minimal Hilbert basis of the cone generated by $(0, 3, 1)$, $(2, 5, 5)$, and $\mathbf{v}_j$ includes $\mathbf{u}_{j+1}$.

For a fixed $\mathbf{b}$, with $Q_{\mathbf{b}}$ full-dimensional, one can in principle compute SCR of $A\mathbf{x} \leq \mathbf{b}$ by first computing $Q_{\mathbf{b}}^I$ using the Chvátal-Gomory procedure and thus knowing the facet normals of $Q_{\mathbf{b}}^I$. However, just as we do not know a systematic way to compute the Chvátal rank of a matrix A, we also do not know an algorithm to compute the SCR of A. Knowing a superset of the normals of inequalities needed to cut out all $Q_{\mathbf{b}}^I$ does not help in computing SCR since there may be vectors in this superset that are never going to be generated by IBN. There are several methods in the literature for computing such supersets such as the method in [8, Theorem 17.4] or via *atomic fibers*

[1]. Regardless, we will see that in many instances one can calculate or bound SCR.

3. Contrasting small Chvátal rank with Chvátal rank

In this section we justify our use of the adjective "small" by showing that SCR can be very small even for families of matrices whose Chvátal rank tend to infinity. Note that a lower bound on the SCR or Chvátal rank of a system $A\mathbf{x} \leq \mathbf{b}$ is also a lower bound on the corresponding rank of A. On the other hand, an upper bound on either rank of A is an upper bound on the corresponding rank of $A\mathbf{x} \leq \mathbf{b}$ for any $\mathbf{b}$.

Proposition 3.1. *If $A \in \mathbb{Z}^{m\times 2}$, then the SCR of A is at most one.*

Proof: This follows immediately from Lemma 2.8. □

Proposition 3.2. *There are systems $A\mathbf{x} \leq \mathbf{b}$ with $A \in \mathbb{Z}^{3\times 2}$ whose SCRs are one but whose Chvátal ranks are arbitrarily large.*

Proof: Consider the family of inequality systems $A_j\mathbf{x} \leq \mathbf{b}_j$, $j = 1, 2, \ldots$ where

$$A_j = \begin{pmatrix} -1 & 0 \\ 1 & 2j \\ 1 & -2j \end{pmatrix} \text{ and } \mathbf{b}_j = (0, 2j, 0)^t.$$

The polyhedron $Q_{\mathbf{b}_j}$ determined by the jth system is a triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$ and $(j, 1/2)$, and its integer hull is the line segment with endpoints $(0,0)$ and $(0,1)$. It is noted in [8, §23.3] that the Chvátal rank of $A_j\mathbf{x} \leq \mathbf{b}_j$ is at least j. In contrast, Proposition 3.1 proves that the SCR of any matrix A with only two columns is at most one. For these systems, SCR is one. □

This family can be modified to produce families in higher dimensions.

Theorem 3.3. *For any $n \geq 2$ and $m \geq n+1$, there are systems $A\mathbf{x} \leq \mathbf{b}$ with $A \in \mathbb{Z}^{m\times n}$ whose SCRs are one but whose Chvátal ranks are arbitrarily large.*

Proof: We prove this by induction using the systems $A_j\mathbf{x} \leq \mathbf{b}_j$ in Proposition 3.2 as base cases. It suffices to show that given a system $A\mathbf{x} \leq \mathbf{b}$, we can extend it in either of the following two ways without changing its Chvátal rank or SCR.

(1) Adjoin one new inequality $\mathbf{a}'\mathbf{y} \leq b'$ to the system, thus increasing m by one while fixing n.
(2) Adjoin one new row and one new column to A and one new entry to $\mathbf{b}$, thus increasing m and n by one each.

For the first construction, take $\mathbf{a}'\mathbf{y} \leq b'$ to be any inequality that is satisfied on an open set containing $Q_{\mathbf{b}}$. Such an inequality does not affect $Q_{\mathbf{b}}$, $Q_{\mathbf{b}}^I$, or the running of the Chvátal procedure, so if we had started with

$A_j\mathbf{x} \le \mathbf{b}_j$, the Chvátal rank of the new system stays unchanged. If A' is the matrix formed by adjoining the new vector $\mathbf{a}'$ to A, then since $\mathcal{A}'$ contains $\mathcal{A}$, it follows from the definition of IBN that $\mathcal{A}'^{(i)}$ contains $\mathcal{A}^{(i)}$ for all i. In particular this holds for $i = 1$, so $\mathcal{A}'^{(1)}$ contains all facet normals of the (unchanged) integer hull $Q^I_{\mathbf{b}}$, leaving SCR also unchanged.

For the second construction, set

$$A^* = \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathbb{Z}^{(m+1)\times(n+1)}$$

and form $\mathbf{b}^*$ by adjoining any integer z to the end of $\mathbf{b}$. The polyhedron $Q_{\mathbf{b}^*}$ is just the product of $Q_{\mathbf{b}}$ with the ray $(-\infty, z]$ orthogonal to the hyperplane containing $Q_{\mathbf{b}}$. The same will hold after each iteration of the Chvátal procedure and for the integer hulls, so the Chvátal rank is left unchanged. Furthermore, the new vector $(\mathbf{0}, 1)$ in $\mathcal{A}^*$ cannot contribute to any Hilbert basis constructed during IBN. This is because any linear combination of the vectors of $\mathcal{A}^*$ with multipliers in $[0, 1)$ will have a fractional last coordinate if the multiplier of $(\mathbf{0}, 1)$ is non-zero. Thus no Hilbert basis element created during IBN involves $(\mathbf{0}, 1)$ and IBN proceeds for $\mathcal{A}^*$ exactly as it did for $\mathcal{A}$, so the SCR is left unchanged by (2). □

Our next example also has SCR much less than Chvátal rank and comes from combinatorial optimization. Given a graph $G = (V(G), E(G))$ on n vertices, the *coclique polytope* of G, denoted by $P_{\mathrm{cocl}}(G)$, is the convex hull in $\mathbb{R}^n$ of all incidence vectors of cocliques (i.e., independent sets of vertices) of G. In general, an inequality description of $P_{\mathrm{cocl}}(G)$ is unknown [8, §23.5]. Let $Q(G)$ be the polytope in $\mathbb{R}^n$ given by the inequalities $x_i \ge 0$ for all $i \in V(G)$ and $x_i + x_j \le 1$ for every edge $\{i, j\} \in E(G)$. Then $P_{\mathrm{cocl}}(G) = Q(G)^I$.

When G is the complete graph K_n, its coclique polytope is just the simplex $\mathrm{conv}\{0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$. This polytope is defined by the inequalities $x_i \ge 0$ for all $i \in [n]$ and $\sum_{i=1}^n x_i \le 1$. However, $Q(K_n)$ is far larger, and the Chvátal rank of the system defining $Q(K_n)$ is $\mathcal{O}(\log(n))$ [8, Example 23.2]. In contrast, the SCR of the system defining $Q(K_n)$ is small.

Theorem 3.4. *The small Chvátal rank of the system defining $Q(K_n)$ is one if n is odd and two if n is even.*

Proof: The matrix A used to define $Q(K_n)$ consists of the rows $\mathbf{e}_i + \mathbf{e}_j$ for all $1 \le i < j \le n$ and $-\mathbf{e}_i$ for all $i \in [n]$. The SCR of this system is the smallest k such that the all-ones vector is an element of $\mathcal{A}^{(k)}$.

If n is odd, form an $n \times n$ non-singular submatrix of A, using the $\mathbf{e}_i + \mathbf{e}_j$ rows, that is the incidence matrix of an n-cycle in G. The all-ones vector is half the sum of the rows of this submatrix and is the only nontrivial integer vector in the open parallelepiped they span, so it is in the Hilbert basis of the cone spanned by these rows. In particular, the all-ones vector is an element of $\mathcal{A}^{(1)}$, and thus the small Chvátal rank is exactly one.

If n is even, the above construction will not work since the all-ones vector is in the semigroup spanned by the rows of the incidence matrix of an n-cycle and hence is not in the Hilbert basis of the cone spanned by these rows. Instead, for each $1 \leq i \leq n$, consider the $n \times n$ non-singular submatrix of A consisting of the row $-\mathbf{e}_i$ and the incidence matrix of an $(n-1)$-cycle that does not pass through vertex i. The Hilbert basis of the cone spanned by the rows of this submatrix contains the vector with all ones except for a zero in the ith position; weight the rows of the cycle by 1/2 and $-\mathbf{e}_i$ by zero. Call this vector $\mathbf{v}_i$. Hence $\mathbf{v}_i \in \mathcal{A}^{(1)}$. The sub-configuration of $\mathcal{A}^{(1)}$ consisting of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ has the all-ones vector in its Hilbert basis, so the all-ones vector appears in $\mathcal{A}^{(2)}$.

It remains to show that when n is even, the all-ones vector does not appear in $\mathcal{A}^{(1)}$. Suppose on the contrary that it does appear. Then there is a basis $\mathcal{U} := \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset \mathcal{A}$ and scalars $0 \leq c_1, \ldots, c_n < 1$ such that

$$c_1\mathbf{u}_1 + \ldots c_n\mathbf{u}_n = (1, \ldots, 1).$$

But since no component of any vector in $\mathcal{A}$ is more than one and every c_i is strictly less than one, for every i there must be at least two vectors in $\mathcal{U}$ whose ith component is one. In other words, for each vertex, $\mathcal{U}$ contains the incidence vectors of at least two edges covering that vertex. This implies that there are at least n edges contributing to $\mathcal{U}$ and since $|\mathcal{U}| = n$, every $\mathbf{u} \in \mathcal{U}$ comes from an edge and these edges cover every vertex exactly twice. Thus these edges form a set of disjoint cycles covering all the vertices. If there is only one cycle, then since n is even, as we observed earlier, the all-ones vector is not in the Hilbert basis of cone($\mathcal{U}$). If there is more than one cycle, then again the all-ones vector is not in the Hilbert basis of cone($\mathcal{U}$) for the following reason. For each cycle C (now of size less than n), the all-ones vector with support in the vertices of C is in the cone spanned by the $\mathbf{u}$'s corresponding to the edges in C. The big all-ones vector with support $[n]$ is the sum of all these all-ones vectors with smaller support since the cycles are disjoint, and hence is not in the Hilbert basis of cone($\mathcal{U}$). □

4. Supernormality and small Chvátal rank

The Chvátal rank of a matrix A is zero if and only if A is unimodular. Characterizations of higher Chvátal rank are not known. Our main goal in this section is to characterize matrices for which small Chvátal rank is zero.

Definition 4.1. A vector configuration $\mathcal{A}$ in $\mathbb{Z}^n$ is **unimodular** if for every subset $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{A}'$ is a Hilbert basis for cone($\mathcal{A}'$).

Definition 4.2. [8, Theorem 22.5] A system of linear inequalities $A\mathbf{x} \leq \mathbf{b}$ is **totally dual integral (TDI)** if for every face F of $Q_{\mathbf{b}} = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$,

$$\mathcal{A}_F := \{\mathbf{a}_i \in \mathcal{A} : \mathbf{a}_i\mathbf{x} = b_i \ \forall \ \mathbf{x} \in F\}$$

is a Hilbert basis.

Theorem 4.3. [8] *Let* $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ *be a configuration such that the matrix* A *whose rows are* $\mathbf{a}_1, \ldots, \mathbf{a}_m$ *has rank* n. *Then the following are equivalent.*

1. $\mathcal{A}$ *is unimodular.*
2. *Every basis in* $\mathcal{A}$ *is a basis of* $\mathbb{Z}^n$ *as a lattice.*
3. *Every (regular) triangulation of* $\mathcal{A}$ *is unimodular.*
4. *For all* $\mathbf{b} \in \mathbb{Z}^m$, *the inequality system* $A\mathbf{x} \leq \mathbf{b}$ *is TDI.*
5. *For all* $\mathbf{b} \in \mathbb{Z}^m$, *the polyhedron* $Q_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ *is integral.*
6. *The Chvátal rank of* A *is zero.*

For (1) $\Leftrightarrow$ (6) see [8, Section 23.4]. The rest are either standard or straightforward to derive.

Theorem 4.5 will provide a complete analogue to Theorem 4.3 when SCR replaces Chvátal rank. A vector configuration $\mathcal{A}$ in $\mathbb{Z}^n$ is **normal** if every integer point in $\text{cone}(\mathcal{A})$ is a non-negative integer combination of $\mathcal{A}$. Normality means exactly that $\mathcal{A}$ is a Hilbert basis for $\text{cone}(\mathcal{A})$, which has immediate relevance to the Chvátal-Gomory procedure. Hoşten, Maclagan, and Sturmfels generalize normality to *supernormality*, which we use as the analogue of unimodularity.

Definition 4.4. [6] A configuration $\mathcal{A}$ is **supernormal** if for every subset $\mathcal{A}'$ of $\mathcal{A}$, $\mathcal{A} \cap \text{cone}(\mathcal{A}')$ is a Hilbert basis of $\text{cone}(\mathcal{A}')$.

Following [6], we say that a system $A\mathbf{x} \leq \mathbf{b}$ is **tight** if for each $i = 1, \ldots, m$, the hyperplane $\mathbf{a}_i\mathbf{x} = b_i$ contains an integer point in $Q_{\mathbf{b}}$ and hence supports $Q_{\mathbf{b}}^I$. When the inequality system is clear, it is typical to simply say that the polyhedron $Q_{\mathbf{b}} = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ is tight. If $Q_{\mathbf{b}}^I$ is nonempty, define

$$\beta_i := \max\{\mathbf{a}_i\mathbf{x} : \mathbf{x} \in Q_{\mathbf{b}} \cap \mathbb{Z}^n\} \text{ for } i = 1, \ldots, m$$

and $\beta := (\beta_i) \in \mathbb{Z}^m$. Then $Q_{\mathbf{b}} \supseteq Q_\beta \supseteq Q_{\mathbf{b}}^I$ and Q_β is tight.

Theorem 4.5. *Let* $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ *be a configuration of primitive vectors such that the matrix* A *whose rows are* $\mathbf{a}_1, \ldots, \mathbf{a}_m$ *has rank* n. *Then the following are equivalent.*

1. $\mathcal{A}$ *is supernormal.*
2. *Every basis* $\mathcal{A}'$ *in* $\mathcal{A}$ *has the property that* $\mathcal{A} \cap \text{cone}(\mathcal{A}')$ *is a Hilbert basis of* $\text{cone}(\mathcal{A}')$, *or equivalently,* $\mathcal{A} = \mathcal{A}^{(1)}$.
3. *Every (regular) triangulation of* $\mathcal{A}$ *that uses all the vectors is unimodular.*
4. *For all* $\mathbf{b} \in \mathbb{Z}^m$, *the inequality system* $A\mathbf{x} \leq \mathbf{b}$ *is TDI whenever* $Q_{\mathbf{b}}$ *is tight.*
5. *For all* $\mathbf{b} \in \mathbb{Z}^m$, *the polyhedron* $Q_{\mathbf{b}}$ *is integral whenever* $Q_{\mathbf{b}}$ *is tight.*
6. *The SCR of* A *is zero.*

The equivalence of (1), (3), and (4) is shown in [6, Proposition 3.1 and Theorem 3.6]. Our contribution is the remaining set of equivalences.

Proof: [(1) $\Rightarrow$ (2)]: This is immediate from the definition of supernormality.

[(2) $\Rightarrow$ (3)]: Let T be a triangulation of $\mathcal{A}$ using all of the vectors and σ be a maximal simplex of T. Then the sub-configuration $\mathcal{A}_\sigma$ is a basis of $\mathcal{A}$ and by (2), $\mathcal{A}$ contains the minimal Hilbert basis of cone($\mathcal{A}_\sigma$). But since every vector in $\mathcal{A}$ is used in the triangulation T, none can lie inside or on the boundary of cone($\mathcal{A}_\sigma$) except those in $\mathcal{A}_\sigma$ itself. Thus $\mathcal{A}_\sigma$ is the Hilbert basis of its own cone. This implies that $\mathcal{A}_\sigma$ is a lattice basis, so σ is a unimodular simplex. Since σ was arbitrary, T is a unimodular triangulation.

[(5) $\Rightarrow$ (6)]: Suppose every polyhedron of the form $Q_{\mathbf{b}}$ with $\mathbf{b} \in \mathbb{Z}^m$ that is tight is also integral. Then in particular, for any $\mathbf{b} \in \mathbb{Z}^m$ with $Q_{\mathbf{b}}^I \neq \emptyset$, Q_β is integral since it is tight. But $Q_{\mathbf{b}}^I \subseteq Q_\beta \subseteq Q_{\mathbf{b}}$ which implies that $Q_\beta = Q_{\mathbf{b}}^I$ and hence the SCR of A is zero.

[(6) $\Rightarrow$ (5)]: Suppose the SCR of A is zero and $\mathbf{b} \in \mathbb{Z}^m$ such that $Q_{\mathbf{b}}$ is tight. Then no new facet normals are needed for $Q_{\mathbf{b}}^I$, so $Q_{\mathbf{b}}^I = Q_\beta \subseteq Q_{\mathbf{b}}$. Moreover, $Q_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ and $Q_\beta = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \beta\}$. This implies that for each $i = 1, \ldots, m$, $\mathbf{a}_i\mathbf{x} = b_i$ and $\mathbf{a}_i\mathbf{x} = \beta_i$ both support $Q_{\mathbf{b}}^I$. Since these are parallel hyperplanes with the same normal, it must be that $\beta = \mathbf{b}$. Since Q_β is integral, we therefore have $Q_{\mathbf{b}}$ integral.

[(4) $\Rightarrow$ (5)]: This follows from [8, Corollary 22.1c], which says that for a $\mathbf{b} \in \mathbb{Z}^m$, if $A\mathbf{x} \leq \mathbf{b}$ is TDI, then $Q_{\mathbf{b}}$ is integral.

[(5) $\Rightarrow$ (3)]: The proof is by contraposition. Suppose there exists a non-unimodular (regular) triangulation T of $\mathcal{A}$ that uses all the vectors in $\mathcal{A}$. Let $\mathcal{A}'$ be a basis in $\mathcal{A}$ whose elements form a non-unimodular facet in T. Then, no element of $\mathcal{A}\backslash\mathcal{A}'$ lies in cone($\mathcal{A}'$). Let A' be the non-singular square matrix whose rows are the elements of $\mathcal{A}'$. Then the columns of A' do not form a basis for $\mathbb{Z}^n$ and so there exists a $\mathbf{b}' \in \mathbb{Z}^n$ such that $A'\mathbf{x} = \mathbf{b}'$ has no integer solution but does have the unique rational solution $\mathbf{v}$. Let $\mathcal{A}'' := \mathcal{A}\backslash\mathcal{A}'$ and assume $A = \begin{pmatrix} A' \\ A'' \end{pmatrix}$. Form the polyhedron $Q_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : A'\mathbf{x} \leq \mathbf{b}',\ A''\mathbf{x} \leq \mathbf{b}''\}$ where $\mathbf{b} = \begin{pmatrix} \mathbf{b}' \\ \mathbf{b}'' \end{pmatrix}$ and $\mathbf{b}''$ is an integer vector chosen so that $\mathbf{v}$ is a (simple) vertex of $Q_{\mathbf{b}}$ and all hyperplanes in $A''\mathbf{x} = \mathbf{b}''$ are arbitrarily far from $\mathbf{v}$. To achieve this choose $\mathbf{b}'' = \lceil A''\mathbf{v} \rceil + \mathbf{w}$ where $\mathbf{w}$ is an integer vector with large positive entries. Check that $\mathbf{v} \in Q_{\mathbf{b}}$ and hence $\{\mathbf{x} \in Q_{\mathbf{b}} : A'\mathbf{x} = \mathbf{b}'\}$ is a non-empty face of $Q_{\mathbf{b}}$. But since $\mathbf{v}$ is the unique point on this face, $\mathbf{v}$ is a simple vertex of $Q_{\mathbf{b}}$. Now replace the ith entry of $\mathbf{b}''$ with $\max\{\mathbf{a}_i\mathbf{x} : \mathbf{x} \in Q_{\mathbf{b}} \cap \mathbb{Z}^n\}$ for each i indexing a row of A''. Name the new right-hand-side of A'' by $\mathbf{b}''$ again and the new polyhedron $Q_{\mathbf{b}}$ again. Note that this still preserves $\mathbf{v}$ as a simple vertex of $Q_{\mathbf{b}}$ since $\mathbf{w}$ is large and no $\mathbf{a} \in \mathcal{A}''$ lies in cone($\mathcal{A}'$) and therefore, no hyperplane in $A''\mathbf{x} = \mathbf{b}''$ gets pushed past $\mathbf{v}$. If we can show that every facet of $Q_{\mathbf{b}}$ coming from an inequality in $A'\mathbf{x} = \mathbf{b}'$ also contains an integer point, then $Q_{\mathbf{b}}$ would be a tight polyhedron which is not integral, proving the result.

Consider the facet $F := \{\mathbf{x} \in Q_{\mathbf{b}} : \mathbf{a}_i\mathbf{x} = b'_i\}$ where $\mathbf{a}_i\mathbf{x} = b'_i$ is one of the equations in $A'\mathbf{x} = \mathbf{b}'$. Then F is the intersection of the $(n-1)$-dimensional cone

$$K := \{\mathbf{x} \in \mathbb{R}^n \,:\, A'\mathbf{x} \leq \mathbf{b}', \mathbf{a}_i\mathbf{x} = b'_i\}$$

with vertex $\mathbf{v}$ and affine span $H := \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{a}_i\mathbf{x} = b'_i\}$ with the half-spaces coming from $A''\mathbf{x} \leq \mathbf{b}''$. Since $\mathbf{a}_i$ is primitive and b'_i is integer, H contains integer points. Again since $\mathbf{w}$ can be made arbitrarily large, there exists a large $r > 0$ such that $K \cap \{\mathbf{x} \in \mathbb{R}^n \,:\, ||\mathbf{x} - \mathbf{v}||_2 \leq r\}$ is contained in F. Therefore, since H contains integer points, we can choose $\mathbf{w}$ large enough for K, and hence F, to also contain integer points. □

Example 4.6. A trivial example shows that if we drop Assumption 2.1, then supernormality is not necessary for the SCR of A to be zero. Take

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then for each $\mathbf{b} \in \mathbb{Z}^2$, $Q_\mathbf{b} = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \leq \frac{b_1}{2},\ x_2 \leq \frac{b_2}{2}\}$. Hence $Q_\mathbf{b}$ is tight if and only if both b_1 and b_2 are even, in which case it has the unique integer vertex $(\frac{b_1}{2}, \frac{b_2}{2})$. Therefore all tight $Q_\mathbf{b}$'s are integral but A is not supernormal. It is easy to see in this example that the SCR of A is zero.

Remark 4.7. If the dimension n is fixed, then it is possible to determine whether $\mathcal{A}$ is supernormal in polynomial time. The number of basis cones is at most $\binom{m}{n}$, so it suffices to check in polynomial time whether $\mathcal{A} \cap \text{cone}(\mathcal{A}')$ is normal for each basis $\mathcal{A}'$ in $\mathcal{A}$. Barvinok and Woods [3, Theorem 7.1] show that in fixed dimension, a rational generating function for the Hilbert basis of each cone can be computed in polynomial time. We then subtract the polynomial $\sum_{\mathbf{a} \in \text{cone}(\mathcal{A}') \cap \mathcal{A}} \mathbf{x}^\mathbf{a}$ from this rational function, square the difference, and evaluate at $\mathbf{x} = (1, \ldots, 1)$. This can also be done in polynomial time [3, Theorem 2.6] and the result is zero if and only if $\mathcal{A} \cap \text{cone}(\mathcal{A}')$ is normal.

We now consider the case where $n = 2$.

Proposition 4.8. *If $A \in \mathbb{Z}^{m \times 2}$, then $\mathcal{A}^{(1)}$ is supernormal but not necessarily unimodular. Further, the vectors in $\mathcal{A}^{(1)}$ are exactly the facet normals of the full-dimensional integer polyhedra $Q^I_\mathbf{b}$ as $\mathbf{b}$ varies in $\mathbb{Z}^m$.*

Proof: The first statement follows from Lemma 2.8. Every $Q^I_\mathbf{b}$ can be cut out by inequalities whose normals are in $\mathcal{A}^{(1)}$ which in particular implies that all facet normals of all full-dimensional $Q^I_\mathbf{b}$ occur in $\mathcal{A}^{(1)}$.

To finish the proof we need to argue that each $\mathbf{h} \in \mathcal{A}^{(1)}$ is a facet normal of some $Q^I_\mathbf{b}$. Suppose $\mathbf{h}$ is in the Hilbert basis of the basis cone $\text{cone}(\mathbf{a}_i, \mathbf{a}_j)$ where $\mathbf{a}_i$ and $\mathbf{a}_j$ are consecutive vectors in the plane. For a vector $\mathbf{p} \in \mathbb{Z}^2$, define $\mathbf{p}^\perp := (-p_2, p_1)$. Set $b_i = 0$ and $b_j = \mathbf{a}_j \cdot \mathbf{h}^\perp$ and all other b_k, $k \neq i, j$ such that the inequalities $\mathbf{a}_k\mathbf{x} \leq b_k$ are very far from the intersection $\mathbf{v}$ of $\mathbf{a}_i\mathbf{x} = 0$ and $\mathbf{a}_j\mathbf{x} = -a_{j1}h_2 + a_{j2}h_1$. We will prove that the line segment with endpoints $(0, 0)$ and $\mathbf{h}^\perp$ is an edge of this $Q^I_\mathbf{b}$ and hence that $\mathbf{h}$ is a facet normal of $Q^I_\mathbf{b}$.

Note that by construction, both $(0,0)$ and $\mathbf{h}^\perp$ are on the boundary of $Q^I_{\mathbf{b}}$. Suppose the line segment joining them is not an edge of $Q^I_{\mathbf{b}}$. Then there is a sequence of consecutive lattice points $\mathbf{q}_0 := (0,0), \mathbf{q}_1, \ldots, \mathbf{q}_{t-1}, \mathbf{q}_t := \mathbf{h}^\perp$, with $t \geq 2$, on the boundary of $Q^I_{\mathbf{b}}$ and in the triangle with vertices $(0,0)$, $\mathbf{h}^\perp$ and $\mathbf{v}$. Let $\mathbf{p}_j := \mathbf{q}_j - \mathbf{q}_{j-1}$. Then $\mathbf{h}^\perp = (\mathbf{q}_1 - \mathbf{q}_0) + (\mathbf{q}_2 - \mathbf{q}_1) + \cdots + (\mathbf{q}_t - \mathbf{q}_{t-1}) = \mathbf{p}_1 + \cdots + \mathbf{p}_t$. This implies that $\mathbf{h} := \mathbf{p}_1^\perp + \cdots + \mathbf{p}_t^\perp$. By construction, each $\mathbf{p}_j^\perp$ lies in the cone spanned by $\mathbf{a}_i$ and $\mathbf{a}_j$. Since $\mathbf{p}_j^\perp \in \mathbb{Z}^2$ and $t \geq 2$, we have shown that $\mathbf{h}$ is not in the minimal Hilbert basis of $\text{cone}(\mathbf{a}_i, \mathbf{a}_j)$ which is a contradiction. Therefore, the line segment with endpoints $(0,0)$ and $\mathbf{h}^\perp$ is an edge of $Q^I_{\mathbf{b}}$. □

In contrast, when $n = 3$ not every vector in $\mathcal{A}^{(1)}$ need be a facet normal of a $Q^I_{\mathbf{b}}$. See Example 5.6.

Corollary 4.9. *For $A \in \mathbb{Z}^{m \times 2}$, with Δ the maximum absolute value of a (2×2)-minor, there exists a matrix M with at most $m\Delta$ rows such that for each $\mathbf{b} \in \mathbb{Z}^m$, there exists $\mathbf{d} \in \mathbb{Z}^m$ such that $Q^I_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n : M\mathbf{x} \leq \mathbf{d}\}$.*

Proof: The vector configuration $\mathcal{A}$, when drawn in the plane, creates a fan with m maximal cones if the fan is complete and $m - 1$ cones if the fan is not complete. The number of Hilbert basis elements of any of these maximal cones is at most $\Delta + 1$. The union of these Hilbert bases is $\mathcal{A}^{(1)}$ and its cardinality is at most $m\Delta$. □

We end this section with a family of examples that satisfy the equivalent conditions of Theorem 4.5 but not those of Theorem 4.3. Hoşten, Maclagan, and Sturmfels exhibit such a configuration in four dimensions and observe in [6]:

> It would be interesting to identify infinite families of configurations in higher dimensions which are supernormal but not unimodular. Such families might arise from graph theory or combinatorial optimization.

Proposition 4.10. *There exists an infinite family of configurations of increasing dimension which are supernormal but not unimodular.*

Proof: Let k be any positive integer and let $\mathcal{A}$ be the rows of the matrix A of size $(2k+1) \times (2k+1)$:

$$A = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

That is, A is the edge-vertex incidence matrix of an odd cycle. The determinant of A is two, so there is exactly one Hilbert basis element of $\text{cone}(\mathcal{A})$

that does not generate an extreme ray. This element is the all-ones vector $\mathbf{1}$, so $\mathcal{A}^{(1)} = \mathcal{A} \cup \{\mathbf{1}\}$.

We claim that all maximal minors of $A^{(1)}$ except for $\det(A)$ are ± 1. This implies that $\mathcal{A}^{(1)}$ equals $\mathcal{A}^{(2)}$, and hence by Theorem 4.5, $\mathcal{A}^{(1)}$ is supernormal. But since $\mathcal{A}$ is not unimodular, neither is $\mathcal{A}^{(1)}$, proving the proposition.

To prove the claim, by symmetry it suffices to check a single minor of $A^{(1)}$ different from $\det(A)$, for instance the one obtained by removing the last row of A from $A^{(1)}$. By cofactor expansion on the last column, this minor equals $-\det(D_1) + \det(D_2)$ where D_1 and D_2 are the $2k \times 2k$ matrices

$$D_1 = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

and

$$D_2 = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The last row of D_1 is the sum of its 1st, 3rd, ..., and $(2k-1)$st rows, hence $\det(D_1) = 0$. Further, D_2 is upper triangular with 1's on the main diagonal, so $\det(D_2) = 1$. □

5. Lower Bounds on SCR

In contrast to the results of Section 3, we now exhibit systems of inequalities (and hence matrices) that establish lower bounds on SCR. Recall (Proposition 2.7) that for either an inequality system or the corresponding matrix, the Chvátal rank is an upper bound for the SCR.

Theorem 5.1. *For $m = n = 3$, the small Chvátal rank of $A\mathbf{x} \le \mathbf{b}$ (and hence A) can be arbitrarily large and can grow exponentially in the size of the input.*

By adjoining inequalities that do not affect Chvátal rank or SCR, we immediately obtain the following.

Corollary 5.2. *The conclusion of Theorem 5.1 holds for any $m, n \in \mathbb{Z}$ satisfying $m \ge n \ge 3$.*

When n is fixed, Chvátal rank is known to grow no faster than exponentially in the size of the input [8, §23.3], so Theorem 5.1 shows that SCR is asymptotically as large as Chvátal rank in the worst case.

To prove Theorem 5.1, we consider the matrices

$$A_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & j & 2j-1 \end{pmatrix}$$

for $j \geq 2$ an integer. We show that the SCR of A_j is $j-1$ which is exponential in the bit size of A_j. To do this, we explicitly describe the configuration $\mathcal{A}_j^{(k)}$ for all k and prove that $\mathcal{A}_j^{(j-1)} = \mathcal{A}_j^{(j)}$, so the SCR of A_j is at most $j-1$. Then we prove that the vector $(1, j, j)$ is a facet normal of the integer hull

$$Q^I_{(0,0,j-1)^t} = \mathrm{conv}(\{\mathbf{x} \in \mathbb{Z}^3 \,:\, A_j\mathbf{x} \leq (0,0,j-1)^t\})$$

and is contained in $\mathcal{A}_j^{(j-1)}$ but not $\mathcal{A}_j^{(j-2)}$. Thus the SCR of A_j is exactly $j-1$.

For $1 \leq k \leq j-1$, define an integral polygon in $\mathbb{R}^2$ by

$$R_j^k := \mathrm{conv}\{(0,0), (k+1, k+1), (j, 2j-1-k), (j, 2j-1)\} \subseteq \mathbb{R}^2_{\geq 0}.$$

For $k = j-1$, the second and third points coincide; in all other cases the four points are distinct and in convex position. An inequality description of R_j^k is

$$R_j^k = \{(x,y) \in \mathbb{R}^2 :\ x \leq y,\ 2x \leq y+k+1,\ x \leq j,\ y \leq (\frac{2j-1}{j})x\}.$$

See Figure 1 for an illustration.

Lemma 5.3. [6, Proposition 5.1] *Let R be an integral polygon in $\mathbb{R}^2$. The configuration in $\mathbb{R}^3$ of all vectors $(1, a, b)$ such that (a, b) is an integer point in R is supernormal.*

Lemma 5.4. *For $1 \leq k \leq j-1$, $\mathcal{A}_j^{(k)}$ consists of the vector $(0, 1, 0)$ along with all vectors of the form $(1, a, b)$ where (a, b) is an integer point in R_j^k.*

Proof: Induct on k. For $k = 1$, we have

$$R_j^1 = \{(x,y) \in \mathbb{R}^2 :\ x \leq y,\ 2x \leq y+2,\ x \leq j,\ y \leq (\frac{2j-1}{j})x\}.$$

Combining the second and fourth of these inequalities gives

$$2x - 2 \leq y \leq 2x - \frac{x}{j}.$$

From the third inequality and the fact that $R_j^1 \subseteq \mathbb{R}^2_{\geq 0}$ we see that $0 \leq \frac{x}{j} \leq 1$. Thus any integer point $(a, b) \in R_j^1$ must have $b = 2a-2$, $b = 2a-1$ or $b = 2a$. This implies that $R_j^1 \cap \mathbb{Z}^2$ is contained in

$$\{\,(i, 2i)\,\} \cup \{\,(i, 2i-1)\,\} \cup \{\,(i, 2i-2)\,\} \text{ where } 0 \leq i \leq j,\ i \text{ integer}.$$

By going through each of the three sets in the union, check that

$$R_j^1 = \{(0,0)\} \cup \{(i, 2i-1) \,:\, 1 \leq i \leq j\} \cup \{(i, 2i-2) \,:\, 2 \leq i \leq j\}.$$

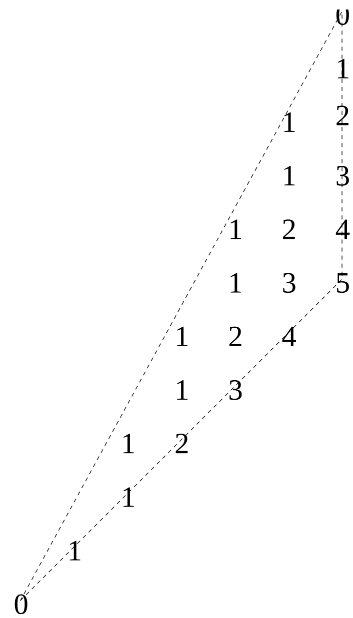

FIGURE 1. The polygon R_j^{j-1} (with $j = 6$) used to prove Theorem 5.1. Each integer point (a, b) in the polygon is labeled with the smallest number k such that $(1, a, b)$ appears in $\mathcal{A}_6^{(k)}$.

Observe that

$$(1, i, 2i-1) = (\frac{2j-2i}{2j-1}, \frac{j-i}{2j-1}, \frac{2i-1}{2j-1})A_j$$

for $1 \leq i \leq j$ and that

$$(1, i, 2i-2) = (\frac{2j-2i+1}{2j-1}, \frac{2j-i}{2j-1}, \frac{2i-2}{2j-1})A_j$$

for $2 \leq i \leq j$, so all the vectors in $\{1\} \times R_j^1 \cap \mathbb{Z}^2$ are in the fundamental parallelepiped of $\mathcal{A}_j$. Since all the first coordinates are one, no element of $\{1\} \times R_j^1 \cap \mathbb{Z}^2$ is a sum of others. Also, no two elements of $\{1\} \times R_j^1 \cap \mathbb{Z}^2$ differ by a multiple of $(0, 1, 0)$. These facts imply that $\{1\} \times R_j^1 \cap \mathbb{Z}^2$ is in the Hilbert basis of cone$(\mathcal{A}_j)$, and hence in $\mathcal{A}_j^{(1)}$. Thus $\mathcal{A}_j^{(1)}$ contains $(0, 1, 0)$ and $\{1\} \times R_j^1 \cap \mathbb{Z}^2$.

On the other hand, if $h = c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(1, j, 2j-1)$ is an integer point in the fundamental parallelepiped of $\mathcal{A}_j$ (so $0 \leq c_1, c_2, c_3 < 1$), then $c_3 = \frac{p}{2j-1}$ for some integer $1 \leq p \leq 2j-2$ and c_1 and c_2 are uniquely determined by c_3, so there is just one such h in the fundamental parallelepiped with a fixed third coordinate: the vector with that third coordinate already shown to be in the Hilbert basis.

For the induction step, first assume that $\mathcal{A}_j^{(k-1)}$ contains $\{1\} \times R_j^{k-1} \cap \mathbb{Z}^2$ for some $k \geq 2$. The difference between R_j^{k-1} and R_j^k is that the inequality $2x \leq y + k$ is replaced by $2x \leq y + k + 1$. Using this, we see that our task is to show that the set of new vectors in $\mathcal{A}_j^{(k)}$ includes the following.

$$\{(1, k+i, k+2i-1) : 1 \leq i \leq j-k\} \tag{1}$$

For each $1 \leq i \leq j-k$, the three vectors $(0,1,0)$, $(1, k+i-1, k+2i-2)$, and $(1, k+i, k+2i)$ appear in $\mathcal{A}_j^{(k-1)}$ by the induction hypothesis. The basis cone they span has normalized volume two and $(1, k+i, k+2i-1)$ (half the sum of the three vectors) is the unique integer point in the interior of the fundamental parallelepiped. Thus $(1, k+i, k+2i-1)$ appears in the Hilbert basis of the cone, and hence in $\mathcal{A}_j^{(k)}$, so $\mathcal{A}_j^{(k)}$ contains $\{1\} \times R_j^1 \cap \mathbb{Z}^2$.

We now must show that $\mathcal{A}_j^{(k)}$ does not contain any other vectors. Again, assume this is true for $k-1$. By Lemma 5.3, the previous paragraph, and the induction hypothesis, $\mathcal{A}_j^{(k-1)} \setminus \{(0,1,0)\} = R_j^k \cap \mathbb{Z}^2$ is supernormal. Thus the only bases of $\mathcal{A}_j^{(k-1)}$ that might contribute new vectors to $\mathcal{A}_j^{(k)}$ are those that include $(0,1,0)$. Any new vector arising this way must be of the form $(1,a,b)$ where $(a-1,b)$ is strictly in the interior of R_j^{k-1} and (a,b) is outside R_j^{k-1}. From the inequality description of R_j^{k-1}, this vector must indeed be of the form (1); see Figure 1. Thus no other vectors can occur in $\mathcal{A}_j^{(k)}$. □

Lemma 5.5. *The configuration* $\mathcal{A}_j^{(j-1)}$ *is supernormal.*

Proof: Applying the same argument we used to complete the proof of the previous lemma, we conclude that any vector $\mathbf{v}$ in $\mathcal{A}_j^{(j)} \setminus \mathcal{A}_j^{(j-1)}$ would have to be of the form $(1,a,b)$ where $(a-1,b)$ is a integer point in the interior of R_j^{j-1} and (a,b) is outside R_j^{j-1}. However, the inequality description of R_j^{j-1} shows it to be a triangle whose right boundary consists only of segments of the line $y = x$ and of the line $x \leq j$, so no such (a,b) exists; see Figure 1. Thus $\mathcal{A}_j^{(j)} = \mathcal{A}_j^{(j-1)}$. □

Proof of Theorem 5.1: By Lemma 5.5 the SCR of $\mathcal{A}_j$ is at most $j-1$. By Lemma 5.4, the vector $(1,j,j)^t$ appears in $\mathcal{A}_j^{(j-1)}$ but not in $\mathcal{A}_j^{(j-2)}$. So it suffices to show there is a vector $\mathbf{b}$ such that $(1,j,j)^t$ is a facet normal of the integer hull of $\{\mathbf{x} \in \mathbb{R}^3 : A_j\mathbf{x} \leq \mathbf{b}\}$.

Choose $\mathbf{b} = (0,0,j-1)^t$. We will prove that the inequality

$$(1,j,j)\,\mathbf{x} \leq \mathbf{0} \tag{2}$$

defines a facet of the integer hull

$$P_j := \{\mathbf{x} \in \mathbb{R}^3 : A_j\mathbf{x} \leq (0,0,j-1)^t\}^I.$$

Let $\mathbf{y} = (y_1, y_2, y_3)$ be any integer point in $Q_{(0,0,j-1)^t}$. We first show that $\mathbf{y}$ satisfies (2). If $y_3 \leq 0$, then since we already know $y_1, y_2 \leq 0$, immediately $\mathbf{y}$ satisfies (2). If $y_3 = 1$, then there are four cases to consider:

(1) $y_1 = y_2 = 0$: Then $(0,0,1) \in Q_{(0,0,j-1)^t}$ and from the third inequality in $A_j\mathbf{x} \leq (0,0,j-1)^t$ we would have $j \leq 0$ which is not possible since $j \geq 2$ by assumption. So this case does not occur.
(2) $y_1, y_2 < 0$: In this case $\mathbf{y}$ satisfies (2).
(3) $y_1 = 0, y_2 \leq -1$: Again, $\mathbf{y}$ satisfies (2).
(4) $y_2 = 0$: In this case, to satisfy the last inequality in $A_j\mathbf{x} \leq (0,0,j-1)^t$, $y_1 \leq -j$ and then $\mathbf{y}$ satisfies (2).

Finally, suppose $y_3 \geq 2$. Rewrite $x_1 + jx_2 + (2j-1)x_3 \leq j-1$ as

$$x_1 + jx_2 \leq (j-1) - x_3(2j-1). \tag{3}$$

Then

$$\begin{aligned} (1,j,j)\,\mathbf{y} &= y_1 + jy_2 + jy_3 \\ &\leq (j-1) - y_3(2j-1) + jy_3 \\ &= j + y_3 - jy_3 - 1 \\ &= j + y_3(1-j) - 1 \\ &\leq j + 2(1-j) - 1 \\ &= 1 - j \\ &< 0 \end{aligned}$$

where the first inequality follows from (3), the second from $y_3 \geq 2$, and the last from $j \geq 2$. Thus the inequality (2) is valid on all integer points of $Q_{(0,0,j-1)^t}$ and hence is a valid inequality of P_j.

To finish the proof we need to argue that (2) is a facet inequality of P_j. This follows from the observation that the three affinely independent integer points $(0,-1,1)^t$, $(0,0,0)^t$, and $(-j,0,1)^t$ in P_j satisfy (2) with equality. □

We now use the above family to show that not every vector in $\mathcal{A}^{(k)}$ for $k \leq \mathrm{SCR}(A)$ is needed to cut out the integer hulls $Q_{\mathbf{b}}^I$ as $\mathbf{b}$ varies. In fact, we show that there may be superfluous vectors even in $\mathcal{A}^{(1)}$.

Example 5.6. Consider the matrix A_4 in the above family. By Lemma 5.4, the vector $(1,3,4)$ is an element of $\mathcal{A}_4^{(1)}$. Since A_4 is a non-singular square matrix, the lattice spanned by its columns has finite index in $\mathbb{Z}^3$. Specifically, this index is seven, the determinant of A_4. Thus there are only seven distinct $Q_{\mathbf{b}}$'s up to lattice translation, obtained by choosing one $\mathbf{b}$ from each equivalence class of $\mathbb{Z}^3$ modulo the lattice. Using the Chvátal-Gomory procedure one can compute the seven polyhedra $Q_{\mathbf{b}}^I$ explicitly and verify that the vector $(1,3,4)$ is not a facet normal of any of them. □

Our last lower bound for SCR comes from polytopes in the unit cube. A *0/1 polytope* in $\mathbb{R}^n$ is the convex hull of any subset of $\{0,1\}^n$; that is, an integral polytope contained in the n-dimensional unit cube C_n. Many problems in combinatorial optimization can be phrased in terms of 0/1 polytopes. Accordingly, their Chvátal rank has been specifically studied, and bounds quite different from those that apply to the general case have been found.

Proposition 5.7. [5]

(1) *The Chvátal rank of any polytope contained in C_n is at most $n^2(1+\log n)$.*
(2) *There exist polytopes contained in C_n whose Chvátal rank is at least $(1+\epsilon)n$.*

Since SCR is always bounded above by Chvátal rank, the upper bound in Proposition 5.7 applies to SCR as well. We will derive a lower bound for SCR that is of the same order as that for Chvátal rank. Since our methods are quite different from those in [5], they provide a new way to prove a weaker version of its lower bound.

Theorem 5.8. *There are systems $A\mathbf{x} \leq \mathbf{b}$ that define polytopes contained in C_n whose small Chvátal ranks are at least $n/2 - o(n)$.*

Lemma 5.9. *If $\mathbf{v}$ is in a minimal Hilbert basis of the cone spanned by n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then $\|\mathbf{v}\|_\infty \leq n \cdot \max_{1\leq i\leq n} \|\mathbf{v}_i\|_\infty$.*

Proof: This is immediate because $\mathbf{v}$ is in the fundamental parallelepiped spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$; that is, it is a non-negative combination of these n vectors where all coefficients are less than one. □

Proof of Theorem 5.8: Given any 0/1 polytope Q, we can find a relaxation P contained in C_n and whose facet normals are 0/1/-1 vectors. To do this, set $\mathbf{e}_I = \sum_{i\in I} \mathbf{e}_i$ for each $I \subseteq \{1, \ldots, n\}$ where $\mathbf{e}_i$ is the ith standard unit vector in $\mathbb{R}^n$, and note that the inequality

$$\sum_{i\in I} x_i - \sum_{i\notin I} x_i \ \leq \ |I| - 1$$

is violated by $\mathbf{e}_I$ but satisfied by every other vertex of C_n. Thus we can define P by taking such an inequality for every 0/1 vector not in Q, along with the inequalities $0 \leq x_i \leq 1$ that define C_n itself. The normals of all of these inequalities are 0/1/-1 vectors. Let $\mathcal{A}$ be the configuration of all of these normals.

Using a construction by Alon and Vu [2] of 0/1 matrices with large determinant, Ziegler [9, Corollary 26] constructs an n-dimensional 0/1 polytope Q with a (relatively prime integer) facet normal $\mathbf{v}$ whose ∞-norm is at least $\frac{(n-1)^{(n-1)/2}}{2^{2n+o(n)}}$. Let P be the polytope we constructed above whose integer hull is Q, and let k be the SCR of the system $A\mathbf{x} \leq \mathbf{b}$ defining P. By definition, $\mathbf{v} \in \mathcal{A}^{(k)}$. Since $\mathcal{A}$ consists entirely of 0/1/-1 vectors, we get by inductively applying Lemma 5.9 that

$$n^k > \frac{(n-1)^{(n-1)/2}}{2^{2n+o(n)}}.$$

Taking the logarithm of both sides, we see that

$$\begin{aligned} k\log n &> \binom{n-1}{2}\log(n-1) - (2n+o(n))\log 2 \\ &= \frac{n}{2}\log(n-1) - \frac{1}{2}\log(n-1) - 2n\log 2 - o(n) \\ &= \frac{n}{2}\log n - o(n\log n) \\ &= (\frac{n}{2} - o(n))\log n \end{aligned}$$

so $k > n/2 - o(n)$, as claimed. □

Acknowledgments. We thank Sasha Barvinok, Ravi Kannan and Les Trotter for helpful inputs to this paper.

References

[1] W. W. Adams, S. Hoşten, P. Loustaunau, and L.J. Miller. SAGBI and SAGBI-Gröbner bases over principal ideal domains. *J. Symbolic Computation*, 27(1):31–47, 1999.

[2] N. Alon and V. Vu. Anti-Hadamard matrices, coin weighing, threshold gates and indecomposable hypergraphs. *J. Combinatorial Theory, Series A*, 79(1):133–160, 1997.

[3] A. Barvinok and K. Woods. Short rational generating functions for lattice point problems. *J. Amer. Math. Soc*, 16:957–979, 2003.

[4] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4:305–337, 1973.

[5] F. Eisenbrand and A.S. Schulz. Bounds on the Chvátal rank of polytopes in the 0/1 cube. *Combinatorica*, 23(2):245–261, 2003.

[6] S. Hoşten, D. Maclagan, and B. Sturmfels. Supernormal vector configurations. *J. Algebraic Combinatorics*, 19(3):297–313, 2004.

[7] D. Maclagan and R.R. Thomas. The toric Hilbert scheme of a rank two lattice is smooth and irreducible. *Journal of Combinatorial Theory, Series A*, 104:29–48, 2003.

[8] A. Schrijver. *Theory of Linear and Integer Programming.* Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1986.

[9] G. M. Ziegler. Lectures on 0/1-polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 1–41. Birkhäuser, Basel, 2000.

Department of Mathematics, University of Washington, Seattle, WA 98195-4350

E-mail address: [bogart,thomas]@math.washington.edu