# An LMI description for the cone of Lorentz-positive maps II

Roland Hildebrand *

October 6, 2008

**Abstract**

Let $L_n$ be the $n$-dimensional second order cone. A linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called positive if the image of $L_m$ under this map is contained in $L_n$. For any pair $(n,m)$ of dimensions, the set of positive maps forms a convex cone. We construct a linear matrix inequality of size $(n-1)(m-1)$ that describes this cone.

**Keywords:** second order cone, semidefinite cone
**AMS Subject Classification:** 15A48, 15A69, 90C22

## 1 Introduction

Let $K, K'$ be regular convex cones (closed convex cones, containing no lines, with non-empty interior[1]), residing in finite-dimensional real vector spaces $V, V'$. A linear map from $V$ to $V'$ that takes $K$ to $K'$ is called *$K$-to-$K'$ positive* or just *positive*, if it is clear which cones are meant. For a detailed introduction into cones of positive maps we refer to [9],[10], for applications to [3] and references cited therein.

In this contribution we consider positivity of maps with respect to two Lorentz cones. A description of the corresponding positive cones by a *nonlinear* matrix inequality, which is based on the $\mathcal{S}$-lemma, is known for decades [4]. However, a description of these cones by a *linear* matrix inequality (LMI) was long elusive. In an earlier paper [3] we derived an LMI which describes the cone of Lorentz-positive maps. However, this LMI was of exponential size. In this contribution we derive a polynomially-sized LMI. This reduction in size is achieved by exploiting the main idea in [2, Section 3]. The extreme rays and other properties of these cones were studied in [5].

The remainder of the paper is structured as follows. In the next section we introduce the necessary definitions and recall some basic facts which we need later on. In Section 3 we describe a polynomially-sized semidefinite relaxation of the cone of Lorentz-positive maps (Theorem 3.1). In Section 4 we recall the concept of Clifford algebras and consider some of their representations. In the last section we prove that the semidefinite relaxation of the Lorentz-positive cone is in fact exact (Theorem 5.6).

## 2 Notations and definitions

Throughout the paper $m, n, r$ are positive natural numbers. For convenience we also introduce the numbers $\mu = m-2$, $\nu = n-2$, $\rho = r-2$.

The components of real vectors $x \in \mathbb{R}^r$ will carry indices running from 0 to $r-1$. Accordingly, the canonical basis vectors of $\mathbb{R}^r$ will be denoted by $e_0, \dots, e_{r-1}$. The Euclidean norm will be denoted by $||\cdot||$.

We will represent linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ by real $n \times m$ matrices. The space of these matrices will be denoted by $\mathbb{R}^{n\times m}$. Further we will identify the tensor product space $\mathbb{R}^n \otimes \mathbb{R}^m$ with the space $\mathbb{R}^{n\times m}$, in such a way that for any $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ the tensor product $y \otimes x$ is identified with the real $n \times m$ matrix $yx^T$. The symbol $\otimes$ will also denote the Kronecker product of matrices. The space $\mathbb{R}^n \otimes \mathbb{R}^m$ is equipped

*LJK, Université Grenoble 1/CNRS, 51 rue des Mathématiques, 38041 Grenoble Cedex 9, France (roland.hildebrand@imag.fr).

[1] In the literature, the term *regular cone* might also be used in other contexts. Sometimes the cones we call *regular* here are called *proper*, but *proper cone* might also have different meanings. We will stick to the notation used in the conic programming literature.

with a scalar product. Namely, for matrices $A, B \in \mathbb{R}^n \otimes \mathbb{R}^m$ we have $\langle A, B\rangle = tr\,(AB^T)$, where $tr$ denotes the trace. This scalar product is the same as the one induced on $\mathbb{R}^n \otimes \mathbb{R}^m$ by the Euclidean scalar products on $\mathbb{R}^n$, $\mathbb{R}^m$.

The symbol $\oplus$ will denote the direct sum of vector spaces. For matrices $A, B$, $A \oplus B$ is a block-diagonal matrix with the two diagonal blocks $A, B$. The identity map on a vector space $V$ will be denoted by $\mathrm{id}_V$.

**Definition 2.1.** Let $V$ be a real vector space equipped with a scalar product $\langle \cdot, \cdot \rangle$ and let $K \subset V$ be a convex cone. Then the *dual* cone $K^*$ of $K$ is defined as the set of elements $y \in V$ such that $\langle x, y\rangle \geq 0$ for all $x \in K$. If $K = K^*$, then $K$ is called *self-dual*.

**Definition 2.2.** Let $V$ be a real vector space. The *convex hull* of a set $S \subset V$ is the set of all convex combinations of elements of $S$, or otherwise spoken, the smallest convex set containing $S$.

We now pass to the definition of the Lorentz cones and the cones of positive maps.

**Definition 2.3.** The cone $L_r \subset \mathbb{R}^r$ defined by

$$L_r = \left\{ (x_0, x_1, \dots, x_{r-1})^T \in \mathbb{R}^r \;\middle|\; x_0 \geq \sqrt{x_1^2 + \cdots + x_{r-1}^2} \right\}$$

is called the $r$-dimensional *second order cone* or *Lorentz cone*.

The Lorentz cone is a regular self-dual convex cone.

**Definition 2.4.** Let $M : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. We call this map *Lorentz-positive* or just *positive* if $M[L_m] \subset L_n$.

The set of Lorentz-positive maps (for fixed dimensions $m, n$) forms a regular convex cone. We denote this cone by $P_{n,m}$.

**Lemma 2.5.** *A linear map $M : \mathbb{R}^m \to \mathbb{R}^n$ is in $P_{n,m}$ if and only if for all $x \in L_m$, $y \in L_n$ we have $\langle M, yx^T\rangle \geq 0$.*

*Proof.* By self-duality of $L_n$ we have the following chain of equivalences.

$$M \in P_{n,m} \;\Leftrightarrow\; Mx \in L_n \;\forall\, x \in L_m \;\Leftrightarrow\; y^T M x = \langle M, yx^T\rangle \geq 0 \;\forall\, x \in L_m, y \in L_n. \qquad \square$$

**Corollary 2.6.** *[8] The dual cone to $P_{n,m}$ is given by the convex hull of all matrices of the form $yx^T$, where $x \in L_m$, $y \in L_n$.*

We will denote this dual cone by $\mathrm{Sep}_{n,m}$.

The following consequence of Lemma 2.5 is evident.

**Corollary 2.7.** *Let $X \subset L_m$, $Y \subset L_n$ be sets such that $L_m$ is the closure of the convex hull of the set $X_0 = \{\alpha x \,|\, \alpha \geq 0,\; x \in X\}$, and $L_n$ is the closure of the convex hull of the set $Y_0 = \{\alpha y \,|\, \alpha \geq 0,\; y \in Y\}$. Then $M \in P_{n,m}$ if and only if for all $x \in X$, $y \in Y$ we have $\langle M, yx^T\rangle \geq 0$.* $\square$

The aim of this paper is to derive a description of the cone $P_{n,m}$ as feasible set of an LMI which has polynomial size in $n, m$. Note that for $r \leq 2$ the Lorentz cone $L_r$ is polyhedral with $r$ generating rays. Hence for $\min(n, m) \leq 2$ the cone $P_{n,m}$ is a direct product of $\min(n, m)$ Lorentz cones of dimension $\max(n, m)$. In this case an LMI description of polynomial size is readily available. Throughout the paper we will hence assume $\min(n, m) \geq 3$.

Let $\mathcal{S}(n)$ be the space of real symmetric $n \times n$ matrices and $S_+(n)$ the cone of positive semidefinite (PSD) matrices in $\mathcal{S}(n)$; $\mathcal{H}(n)$ the space of complex hermitian $n \times n$ matrices and $H_+(n)$ the cone of PSD matrices in $\mathcal{H}(n)$. Denote by $\mathcal{M}(n)$ the space of real $n \times n$ matrices, and by $\mathcal{A}(n)$ the space of real skew-symmetric $n \times n$ matrices. The spaces $\mathcal{S}(n)$, $\mathcal{H}(n)$, $\mathcal{M}(n)$, and $\mathcal{A}(n)$ are equipped with the scalar product $\langle A, B\rangle = tr\,(AB^*)$. Here $B^*$ denotes the complex conjugate transpose of the matrix $B$. The cones $S_+(n)$ and $H_+(n)$ are self-dual. The ordering with respect to these cones will be denoted by $\succeq$. Let further $U(n)$ be the group of unitary $n \times n$ matrices, $O(n)$ the group of orthogonal $n \times n$ matrices, and $I_n$ the $n \times n$ identity matrix.

**Definition 2.8.** Let $V$ be a real vector space and let $G$ be a finite subgroup of the general linear group of $V$. The *group average* of an element $v \in V$ is defined as $\frac{1}{|G|} \sum_{g \in G} g(v)$.

**Lemma 2.9.** *[2, Lemma 2.11] Let $V$ be a real vector space equipped with a scalar product and let $G$ be a finite subgroup of the isometry group of $V$. Then the operation of group averaging is the orthogonal projection onto the linear subspace $L = \{v \in V \mid g(v) = v \ \forall\, g \in G\} \subset V$.*

The following result is standard in convex geometry.

**Proposition 2.10.** *Let $V$ be a real vector space equipped with a scalar product, let $L \subset V$ be a linear subspace and let $K \subset V$ be a closed convex cone. Then the dual cone to $K \cap L$ equals the closure of the orthogonal projection of $K^*$ on $L$*[2]*.*

*Proof.* Let $L^\perp$ denote the orthogonal complement of $L$.

Let $x$ be an element of the projection of $K^*$ on $L$. Then there exists $y \in L^\perp$ such that $x + y \in K^*$. Let now $z \in K \cap L$ be arbitrary. Then we have $\langle x, z\rangle = \langle x + y, z\rangle \geq 0$. Hence $x \in (K \cap L)^*$. Since $(K \cap L)^*$ is closed, the closure of the projection of $K^*$ on $L$ is also contained in $(K \cap L)^*$.

Now suppose that $x \in (K \cap L)^*$, but its distance to the closure of the projection of $K^*$ on $L$ is strictly positive. Then there exists an element $y \in L$ such that $\langle x, y\rangle < 0$, and $\langle y, z\rangle \geq 0$ for all $z$ in the projection of $K^*$ on $L$. It follows that $\langle y, z\rangle \geq 0$ also for all $z$ in $K^*$, hence $y$ is in $K$ and therefore in $K \cap L$. This leads to a contradiction and completes the proof. □

## 3 Semidefinite relaxation of $P_{n,m}$

For $r \geq 3$, define a linear map $\mathcal{W}_r : \mathbb{R}^r \to \mathcal{S}(r-1)$ by

$$\mathcal{W}_r : \quad \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \mapsto \begin{pmatrix} x_0 + x_1 & x_2 & \cdots & x_{r-1} \\ x_2 & x_0 - x_1 & & 0 \\ \vdots & & \ddots & \\ x_{r-1} & 0 & & x_0 - x_1 \end{pmatrix}. \tag{1}$$

**Theorem 3.1.** *Assume above notations. Let $M$ be a real $n \times m$ matrix satisfying the LMI*

$$\exists\, X \in \mathcal{A}(n-1) \otimes \mathcal{A}(m-1) : \qquad (\mathcal{W}_n \otimes \mathcal{W}_m)(M) + X \succeq 0. \tag{2}$$

*Then $M \in P_{n,m}$.*

Note that $\mathcal{W}_n \otimes \mathcal{W}_m$ is a linear map from $\mathbb{R}^n \otimes \mathbb{R}^m$ to $\mathcal{S}(n-1) \otimes \mathcal{S}(m-1) \subset \mathcal{S}((n-1)(m-1))$.

*Proof.* Assume the conditions of the theorem. By Corollary 2.7 the map $M$ is in $P_{n,m}$ if and only if $y^T M x \geq 0$ for all $x = (x_0, \dots, x_{m-1})^T \in \mathbb{R}^m$, $y = (y_0, \dots, y_{n-1})^T \in \mathbb{R}^n$ satisfying

$$x_0 + x_1 = 1, \quad x_0^2 = \sum_{l=1}^{m-1} x_l^2; \qquad y_0 + y_1 = 1, \quad y_0^2 = \sum_{l=1}^{n-1} y_l^2. \tag{3}$$

The subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ defined by (3) correspond to parabolic sections of the boundaries of $L_m$ and $L_n$.

Define $\tilde{x} = (x_2, \dots, x_{m-1})^T \in \mathbb{R}^{m-2}$, $\tilde{y} = (y_2, \dots, y_{n-1})^T \in \mathbb{R}^{n-2}$. Then (3) is equivalent to $x_0 = \frac{1+||\tilde{x}||^2}{2}$, $x_1 = \frac{1-||\tilde{x}||^2}{2}$ and $y_0 = \frac{1+||\tilde{y}||^2}{2}$, $y_1 = \frac{1-||\tilde{y}||^2}{2}$.

Moreover, for any matrix $X \in \mathcal{A}(n-1) \otimes \mathcal{A}(m-1)$ we get

$$\left[\begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix}\right]^T [(\mathcal{W}_n \otimes \mathcal{W}_m)(M) + X] \left[\begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix}\right] = 4y^T M x = 4\langle M, yx^T\rangle. \tag{4}$$

This can be seen as follows. Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ be arbitrary. Then we have

$$\begin{aligned} &\left[\begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix}\right]^T [(\mathcal{W}_n \otimes \mathcal{W}_m)(a \otimes b) + X] \left[\begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix}^T \mathcal{W}_n(a) \begin{pmatrix} 1 \\ \tilde{y} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix}^T \mathcal{W}_m(b) \begin{pmatrix} 1 \\ \tilde{x} \end{pmatrix} = 2\langle a, y\rangle \cdot 2\langle b, x\rangle = 4y^T(a \otimes b)x. \end{aligned}$$

[2]We would like to thank an anonymous referee who pointed out the necessity of including *closure* in the formulation of the proposition. The original version [2, Proposition 2.9] is false as stated. However, as the cones $K$ and the projections of $K^*$ used in the proofs are always closed, the results of [2] are not affected.

Equation (4) now follows by linear extension.

Thus, if $M$ satisfies LMI (2), then by (4) $y^T M x \geq 0$ for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ satisfying (3) and hence $M \in P_{n,m}$. □

The matrices $M$ satisfying (2) form a convex cone. The preceding theorem states that this cone is contained in $P_{n,m}$. Thus (2) is a semidefinite relaxation of the positive cone $P_{n,m}$.

# 4 Clifford algebras and their representations

For an introduction in the theory of Clifford algebras see [6],[11]. Our use of the Clifford algebra is essentially restricted to the use in the next section of the semidefinite description of the cone $\mathrm{Sep}_{n,m}$ provided in [3, Theorem 4.11]. In the sequel we assume $r \geq 3$.

**Definition 4.1.** Let $V_{Cl}$ be an $(r-1)$-dimensional real vector space equipped with a Euclidean norm $||\cdot||$. The *Clifford algebra* $Cl_{r-1}(\mathbb{R})$ is the universal associative real algebra with 1 which contains and is generated by $V_{Cl}$ subject to the condition $v^2 = ||v||^2$ for all $v \in V_{Cl}$.

Let $\{f_1, \dots, f_{r-1}\}$ be an orthonormal basis of $V_{Cl}$. Then the elements of this basis anti-commute and square to 1. Hence a basis of the whole Clifford algebra $Cl_{r-1}(\mathbb{R})$ is given by the ensemble of ordered products $f_{k_1} f_{k_2} \cdots f_{k_l}$ with $1 \leq k_1 < k_2 < \cdots < k_l \leq r-1$, $l = 0, \dots, r-1$. This includes the empty product, which by definition equals 1. The dimension of $Cl_{r-1}(\mathbb{R})$ is hence $2^{r-1}$.

Let now $Y$ be the $r$-dimensional subspace of $Cl_{r-1}(\mathbb{R})$ spanned by $1, f_1, \dots, f_{r-1}$. Define an isomorphism $\mathcal{Y} : \mathbb{R}^r \to Y$ by $\mathcal{Y}(e_0) = 1$, $\mathcal{Y}(e_l) = f_l$, $l = 1, \dots, r-1$.

We now proceed to the matrix representations of $Cl_{r-1}(\mathbb{R})$. We start by defining the matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Further we define multi-indexed matrices with indices of length $\rho = r-2$ by $\sigma_{k_1 k_2 \dots k_\rho} = \sigma_{k_1} \otimes \sigma_{k_2} \otimes \cdots \otimes \sigma_{k_\rho}$, where $k_j \in \{0,1,2,3\}$ for each $j$. Hence $\sigma_{k_1 k_2 \dots k_\rho}$ will be of size $2^\rho \times 2^\rho$.

Let us define a real matrix representation $\mathcal{R}^{Cl}$ of the Clifford algebra $Cl_{r-1}(\mathbb{R})$ by such multi-indexed matrices. The representation maps the unit element and the generators $f_1, \dots, f_{r-1}$ of the algebra as

$$1 \mapsto \sigma_0^{\otimes \rho}; \qquad f_k \mapsto \sigma_0^{\otimes(\rho-k)} \otimes \sigma_1 \otimes \sigma_2^{\otimes(k-1)}, \quad \text{if } k < r-1; \qquad f_{r-1} \mapsto \sigma_2^{\otimes \rho}.$$

Here $\sigma_j^{\otimes k}$ denotes the $k$-fold Kronecker product $\sigma_j \otimes \sigma_j \otimes \cdots \otimes \sigma_j$. It is not hard to verify that $\mathcal{R}^{Cl}(f_k)$ and $\mathcal{R}^{Cl}(f_l)$ anti-commute for $k \neq l$ and any $\mathcal{R}^{Cl}(f_k)$ squares to the identity matrix. Hence $\mathcal{R}^{Cl}$ is indeed a representation of $Cl_{r-1}(\mathbb{R})$. Note that $\sigma_0, \sigma_1, \sigma_2$ are symmetric, and so are Kronecker products of these matrices. We obtain the following result.

**Lemma 4.2.** *The representation $\mathcal{R}^{Cl}$ maps the space $Y \subset Cl_{r-1}(\mathbb{R})$ to a subspace of $\mathcal{S}(2^\rho)$.* □

It follows that the composition $\mathcal{R}^{Cl} \circ \mathcal{Y}$ is a linear map from $\mathbb{R}^r$ to $\mathcal{S}(2^\rho)$. Let us denote this composition by $\mathcal{R}_r$ and its image by $\mathcal{L}_r$.

There exist also complex representations of $Cl_{r-1}(\mathbb{R})$. Let us briefly restate their definition from [3, Section 4].

Define the matrices $\sigma'_k = \sigma_k$, $k = 0,1,2$, $\sigma'_3 = i\sigma_3$ as well as multi-indexed matrices $\sigma'_{k_1 \dots k_{\rho'}} = \sigma'_{k_1} \otimes \cdots \otimes \sigma'_{k_{\rho'}}$ with index length $\rho' = [(r-1)/2]$. Let us now define a representation $\mathcal{C}^{Cl}$ of the Clifford algebra $Cl_{r-1}(\mathbb{R})$ by such multi-indexed matrices. The representation maps the unit element and the generators $f_1, \dots, f_{r-1}$ of the algebra as

$$1 \mapsto {\sigma'_0}^{\otimes \rho}; \qquad f_k \mapsto {\sigma'_0}^{\otimes(\rho'-(k+1)/2)} \otimes \sigma'_1 \otimes {\sigma'_3}^{\otimes(k-1)/2}, \quad \text{if } k \text{ odd}, k < r-1;$$

$$f_k \mapsto {\sigma'}_0^{\otimes(\rho'-k/2)} \otimes {\sigma'}_2 \otimes {\sigma'_3}^{\otimes(k/2-1)}, \quad \text{if } k \text{ even}; \qquad f_{r-1} \mapsto {\sigma'_3}^{\otimes \rho}, \quad \text{if } r \text{ even}.$$

For even $r$ we define also another representation ${\mathcal{C}'}^{Cl}$, which differs from $\mathcal{C}^{Cl}$ by a sign change in the representation of the last generator, ${\mathcal{C}'}^{Cl} : f_{r-1} \mapsto -\sigma'_{3\dots3}$. Note that all matrices $\sigma'_k$ as well as their Kronecker products are complex hermitian. We have the following analog of Lemma 4.2.

**Lemma 4.3.** *The representations $\mathcal{C}^{Cl}$, $\mathcal{C}'^{Cl}$ map the space $Y \subset Cl_{r-1}(\mathbb{R})$ to a subspace of $\mathcal{H}(2^{\rho'})$.* □

It follows that the compositions $\mathcal{C}^{Cl} \circ \mathcal{Y}$, $\mathcal{C}'^{Cl} \circ \mathcal{Y}$ are linear maps from $\mathbb{R}^r$ to $\mathcal{H}(2^{\rho'})$. Let us denote these compositions by $\mathcal{C}_r$ and $\mathcal{C}'_r$, respectively.

**Theorem 4.4.** *Let $r \geq 3$ be an odd number. The representation $\mathcal{R}^{Cl}$ of $Cl_{r-1}(\mathbb{R})$ is the direct sum of $2^{(r-3)/2}$ copies of the irreducible representation $\mathcal{C}^{Cl}$. There exists a unitary matrix $U \in U(2^\rho)$ such that $\mathcal{R}^{Cl}(x) = U(I_{2^{(r-3)/2}} \otimes \mathcal{C}^{Cl}(x))U^*$ for all $x \in Cl_{r-1}(\mathbb{R})$.*

*Let $r \geq 4$ be an even number. The representation $\mathcal{R}^{Cl}$ of $Cl_{r-1}(\mathbb{R})$ is the direct sum of $2^{(r-4)/2}$ copies of the irreducible representation $\mathcal{C}^{Cl}$ and of $2^{(r-4)/2}$ copies of the irreducible representation $\mathcal{C}'^{Cl}$. There exists a unitary matrix $U \in U(2^\rho)$ such that $\mathcal{R}^{Cl}(x) = U(I_{2^{(r-4)/2}} \otimes (\mathcal{C}^{Cl}(x) \oplus \mathcal{C}'^{Cl}(x)))U^*$ for all $x \in Cl_{r-1}(\mathbb{R})$.*

We will not make use of this theorem until the last section. The proof is provided in the appendix.

### 4.1 Properties of the representation $\mathcal{R}^{Cl}$

In this subsection we restate the results of [2, Subsection 3.1].

Let $\rho \geq 1$ be an integer and put $r = \rho + 2$. A basis of the space $\mathcal{M}(2^\rho)$ is given by the set of all multi-indexed matrices $\sigma_{k_1 \dots k_\rho}$ whose index sequence has length $\rho$. Note that $\mathcal{M}(2^\rho) = \mathcal{S}(2^\rho) \oplus \mathcal{A}(2^\rho)$. A basis of $\mathcal{S}(2^\rho)$ is comprised of those multi-indexed matrices whose index sequence contains an even number of 3's. A basis of $\mathcal{A}(2^\rho)$ is comprised of those multi-indexed matrices whose index set contains an odd number of 3's.

The image $\mathcal{L}_r$ of $\mathcal{R}_r$ is a subspace of $\mathcal{S}(2^\rho)$ and hence of $\mathcal{M}(2^\rho)$. We now introduce a subgroup $G_r$ of the automorphism group of $\mathcal{M}(2^\rho)$ that leaves $\mathcal{L}_r$ elementwise invariant. For $\rho = 1$ $G_r$ will consist only of the identity automorphism. For $\rho \geq 2$ we define this group as follows.

Consider the automorphism $j$ of the space $\mathcal{M}(4)$ defined by $j : A \mapsto -\sigma_{31} A \sigma_{31}$. It is not hard to prove that this automorphism has eigenvalues $\pm 1$ and the eigenspace corresponding to the eigenvalue 1 is exactly the space spanned by the matrices $\sigma_{00}, \sigma_{01}, \sigma_{12}, \sigma_{22}, \sigma_{13}, \sigma_{23}, \sigma_{30}, \sigma_{31}$. The eigenspace of the eigenvalue $-1$ is spanned by the remaining eight double-indexed matrices.

Define automorphisms $j_k = \mathrm{id}_{\mathcal{M}(2^{\rho-2-k})} \otimes j \otimes \mathrm{id}_{\mathcal{M}(2^k)} : \mathcal{M}(2^\rho) \to \mathcal{M}(2^\rho)$ for $k = 0, \dots, \rho - 2$. The mappings $j_k$ are the conjugation by the matrix $M_k = I_{2^{\rho-2-k}} \otimes \sigma_{31} \otimes I_{2^k}$ (i.e. acting as $j_k : A \mapsto M_k A M_k^T$). Hence the $j_k$ act separately on the symmetric and the skew-symmetric part of a matrix $A \in \mathcal{M}(2^\rho)$, and can be considered as automorphisms of the spaces $\mathcal{S}(2^\rho)$ and $\mathcal{A}(2^\rho)$ when restricting the domain of definition accordingly. Moreover, the $j_k$ can be considered as automorphisms of the cone $S_+(2^\rho)$.

Consider the set of sequences $(l_1, \dots, l_\rho)$ of elements of the set $\{0, 1, 2, 3\}$. Let $\zeta$ be the subset of those sequences fulfilling $j(\sigma_{l_k l_{k+1}}) = \sigma_{l_k l_{k+1}}$ for all $k = 1, \dots, \rho - 1$. Let $\mathcal{L}'_r \subset \mathcal{S}(2^\rho)$ be the linear subspace spanned by those symmetric multi-indexed matrices $\sigma_{l_1 l_2 \dots l_\rho}$ whose multi-index satisfies $(l_1, \dots, l_\rho) \in \zeta$. Denote the orthogonal complement of $\mathcal{L}'_r$ in $\mathcal{S}(2^\rho)$ by $\mathcal{L}'^{\perp}_r$.

Let us now restate some results from [2].

**Lemma 4.5.** *[2, Lemma 3.5] The automorphism group $G_r$ generated by the automorphisms $j_k$, $k = 0, \dots, \rho - 2$, is finite. Any element of $G_r$ has eigenvalues $\pm 1$. A matrix $A \in \mathcal{S}(2^\rho)$ is invariant under the action of $G_r$ if and only if $A \in \mathcal{L}'_r$.*

**Lemma 4.6.** *[2, Lemma 3.6] $\mathcal{L}_r \subset \mathcal{L}'_r$.*

**Corollary 4.7.** *The space $\mathcal{L}_r$ is elementwise invariant under the action of the group $G_r$.* □

Denote the orthogonal complement of $\mathcal{L}_r$ in $\mathcal{L}'_r$ by $\mathcal{L}^{\perp}_r$. Let us now define a linear map $\pi_r : \mathcal{M}(2^\rho) \to \mathcal{M}(\rho + 1)$ elementwise as follows. For any matrix $A \in \mathcal{M}(2^\rho)$ the element $B_{kl}$ of the matrix $B = \pi_r(A)$ is given by the element $A_{2^{k-1}, 2^{l-1}}$ of $A$. Thus $\pi_r$ maps a given matrix of size $2^\rho \times 2^\rho$ to a principal submatrix of size $(\rho+1) \times (\rho+1)$. It follows that $\pi_r[\mathcal{S}(2^\rho)] = \mathcal{S}(\rho+1)$, $\pi_r[\mathcal{A}(2^\rho)] = \mathcal{A}(\rho+1)$, $\pi_r[S_+(2^\rho)] = S_+(\rho+1)$.

**Lemma 4.8.** *[2, Lemma 3.7] $\pi_r[\mathcal{L}^{\perp}_r] = 0$.*

## 5 Reduced LMI description of $P_{n,m}$

We will need the following result [3, Theorem 4.11].

**Theorem 5.1.** *Let* $\min(n,m) \geq 3$ *be odd. Then a matrix* $M$ *is in* $\mathrm{Sep}_{n,m}$ *if and only if* $(\mathcal{C}_n \otimes \mathcal{C}_m)(M) \succeq 0$.

*Let* $n \geq m \geq 4$*,* $m$ *even. Then a matrix* $M$ *is in* $\mathrm{Sep}_{n,m}$ *if and only if* $(\mathcal{C}_n \otimes \mathcal{C}_m)(M) \succeq 0$ *and* $(\mathcal{C}_n \otimes \mathcal{C}'_m)(M) \succeq 0$.

*Let* $m > n \geq 4$*,* $n$ *even. Then a matrix* $M$ *is in* $\mathrm{Sep}_{n,m}$ *if and only if* $(\mathcal{C}_n \otimes \mathcal{C}_m)(M) \succeq 0$ *and* $(\mathcal{C}'_n \otimes \mathcal{C}_m)(M) \succeq 0$.

**Corollary 5.2.** *Let* $m,n \geq 4$ *be even. Then a matrix* $M$ *is in* $\mathrm{Sep}_{n,m}$ *if and only if all four matrices* $(\mathcal{C}_n \otimes \mathcal{C}_m)(M)$*,* $(\mathcal{C}'_n \otimes \mathcal{C}_m)(M)$*,* $(\mathcal{C}_n \otimes \mathcal{C}'_m)(M)$*,* $(\mathcal{C}'_n \otimes \mathcal{C}'_m)(M)$ *are positive semidefinite.*

*Proof.* Assume the notations of the corollary. If the four matrices are PSD, then $M \in \mathrm{Sep}_{n,m}$ by the preceding theorem.

Suppose $m \leq n$. Let $\omega : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map that changes the sign of the last coordinate. Note that $\omega$ is an automorphism of the Lorentz cone $L_n$. Moreover, we have the relation $\mathcal{C}_n \circ \omega = \mathcal{C}'_n$. If now $M \in \mathrm{Sep}_{n,m}$, then also $M' = (\omega \otimes \mathrm{id}_{\mathbb{R}^m})(M) \in \mathrm{Sep}_{n,m}$. Hence by the preceding theorem both the matrices $(\mathcal{C}_n \otimes \mathcal{C}_m)(M)$, $(\mathcal{C}_n \otimes \mathcal{C}'_m)(M)$, and the matrices $(\mathcal{C}_n \otimes \mathcal{C}_m)(M') = (\mathcal{C}'_n \otimes \mathcal{C}_m)(M)$, $(\mathcal{C}_n \otimes \mathcal{C}'_m)(M') = (\mathcal{C}'_n \otimes \mathcal{C}'_m)(M)$ are PSD.

The case $n < m$ is proven similarly. □

Theorem 5.1 (in the case $m,n$ even Corollary 5.2) and Theorem 4.4 yield the following result.

**Corollary 5.3.** *Let* $\min(n,m) \geq 3$*. Then a matrix* $M$ *is in* $\mathrm{Sep}_{n,m}$ *if and only if* $(\mathcal{R}_n \otimes \mathcal{R}_m)(M) \succeq 0$. □

Let $\mu = m-2$, $\nu = n-2$. The image of $\mathbb{R}^{n\times m}$ under the map $\mathcal{R}_n \otimes \mathcal{R}_m$ is the subspace $\mathcal{L}_n \otimes \mathcal{L}_m \subset \mathcal{S}(2^\nu) \otimes \mathcal{S}(2^\mu) \subset \mathcal{S}(2^{\nu+\mu})$. Denote the orthogonal complement of $\mathcal{L}_n \otimes \mathcal{L}_m$ in $\mathcal{S}(2^{\nu+\mu})$ by $\mathcal{L}^\perp$. Note that $\mathcal{L}^\perp$ consists of matrices with zero trace. Hence the orthogonal projection of the cone $S_+(2^{\nu+\mu})$ on $\mathcal{L}_n \otimes \mathcal{L}_m$ is closed. Note also that the map $\mathcal{R}_n \otimes \mathcal{R}_m$ is a multiple of an isometry. By Corollary 2.6 and Proposition 2.10 we then get the following result.

**Corollary 5.4.** *Let* $\min(n,m) \geq 3$*. Then a matrix* $M$ *is in* $P_{n,m}$ *if and only if there exists* $X \in \mathcal{L}^\perp$ *such that* $(\mathcal{R}_n \otimes \mathcal{R}_m)(M) + X \succeq 0$. □

Let us summarize the results of Subsection 4.1. We have decomposed the space $\mathcal{S}(2^\rho)$ into a direct sum $\mathcal{L}_r \oplus \mathcal{L}_r^\perp \oplus {\mathcal{L}'_r}^\perp$ of mutually orthogonal subspaces and introduced a finite group $G_r$ acting on $\mathcal{S}(2^\rho)$ as well as on $\mathcal{A}(2^\rho)$. Here $r = \rho + 2$. The induced automorphisms of $\mathcal{S}(2^\rho)$ are as well automorphisms of the cone $S_+(2^\rho)$. Here $\mathcal{L}_r$ is the image of the map $\mathcal{R}_r$. The subspace $\mathcal{L}_r \oplus \mathcal{L}_r^\perp$ is the set of fixed points with respect to the action of $G_r$. Further we defined a projection $\pi_r$ of matrices in $\mathcal{M}(2^\rho)$ to principal submatrices of size $(\rho+1)\times(\rho+1)$.

In the sequel we will consider the action of the product group $G_n \times G_m$ on the tensor product space $\mathcal{M}(2^\nu) \otimes \mathcal{M}(2^\mu) = \mathcal{M}(2^{\nu+\mu})$. For any $g_n \in G_n$, $g_m \in G_m$ this action is naturally defined by $G_n \times G_m \ni (g_n, g_m) : A \otimes B \mapsto g_n(A) \otimes g_m(B)$ on tensor products and extended to the whole space $\mathcal{M}(2^{\nu+\mu})$ by linearity. The product group $G_n \times G_m$ acts separately on the subspaces $\mathcal{S}(2^\nu) \otimes \mathcal{S}(2^\mu)$ and $\mathcal{A}(2^\nu) \otimes \mathcal{A}(2^\mu)$ and hence on $\mathcal{S}(2^{\nu+\mu})$. Clearly the group elements induce automorphisms of the cone $S_+(2^{\nu+\mu})$. We will also consider the tensor product map $\pi_n \otimes \pi_m : \mathcal{M}(2^\nu) \otimes \mathcal{M}(2^\mu) \to \mathcal{M}(n-1) \otimes \mathcal{M}(m-1)$. This map assigns to any matrix in $\mathcal{M}(2^{\nu+\mu})$ a principal submatrix of size $(\mu+1)(\nu+1) \times (\mu+1)(\nu+1)$.

**Theorem 5.5.** *Let* $\min(n,m) \geq 3$ *and* $M \in P_{n,m}$*. Then there exists* $X \in \mathcal{A}(n-1) \otimes \mathcal{A}(m-1)$ *such that* $(\mathcal{W}_n \otimes \mathcal{W}_m)(M) + X \succeq 0$.

*Proof.* Assume the conditions of the theorem. By Corollary 5.4 there exists $X \in \mathcal{L}^\perp$ such that $(\mathcal{R}_n \otimes \mathcal{R}_m)(M) + X \succeq 0$. By virtue of the decomposition $\mathcal{S}(2^{\nu+\mu}) = \mathcal{S}(2^\nu) \otimes \mathcal{S}(2^\mu) \oplus \mathcal{A}(2^\nu) \otimes \mathcal{A}(2^\mu)$ we can decompose the matrix $X$ as $X^S + X^A$, where $X^S \in \mathcal{S}(2^\nu) \otimes \mathcal{S}(2^\mu)$, and $X^A \in \mathcal{A}(2^\nu) \otimes \mathcal{A}(2^\mu)$.

Let us take the group average of above LMI with respect to the action of the group $G_n \times G_m$. The matrix $(\mathcal{R}_n \otimes \mathcal{R}_m)(M)$ is an element of the subspace $\mathcal{L}_n \otimes \mathcal{L}_m$ and hence invariant under the action of the group by Corollary 4.7. Therefore this matrix is equal to its group average. Denote the group averages of the matrices $X^S, X^A$ by $X_0^S, X_0^A$. Since the group $G_n \times G_m$ acts separately on the subspaces $\mathcal{S}(2^\nu) \otimes \mathcal{S}(2^\mu)$ and $\mathcal{A}(2^\nu) \otimes \mathcal{A}(2^\mu)$, we have that $X_0^A$ is in $\mathcal{A}(2^\nu) \otimes \mathcal{A}(2^\mu)$. By Lemmas 4.5 and 2.9 $X_0^S$ is the orthogonal projection of $X^S$ on the space $(\mathcal{L}_n \oplus \mathcal{L}_n^\perp) \otimes (\mathcal{L}_m \oplus \mathcal{L}_m^\perp)$. But $X^S$ is orthogonal to the subspace $\mathcal{L}_n \otimes \mathcal{L}_m$. It follows that $X_0^S \in \mathcal{L}_n \otimes \mathcal{L}_m^\perp \oplus \mathcal{L}_n^\perp \otimes \mathcal{L}_m^\perp \oplus \mathcal{L}_n^\perp \otimes \mathcal{L}_m$.

Let us apply the map $\pi_n \otimes \pi_m$ to the averaged matrix inequality $(\mathcal{R}_n \otimes \mathcal{R}_m)(M) + X_0^S + X_0^A \succeq 0$. Since principal submatrices of positive semidefinite matrices are positive semidefinite, we obtain the smaller LMI $((\pi_n \circ \mathcal{R}_n) \otimes (\pi_m \circ \mathcal{R}_m))(M) + (\pi_n \otimes \pi_m)(X_0^S) + (\pi_n \otimes \pi_m)(X_0^A) \succeq 0$. But $\pi_n \circ \mathcal{R}_n = \mathcal{W}_n$, $\pi_m \circ \mathcal{R}_m = \mathcal{W}_m$,

$(\pi_n \otimes \pi_m)(X_0^S) = 0$ by Lemma 4.8, and $(\pi_n \otimes \pi_m)(X_0^A) \in \mathcal{A}(n-1) \otimes \mathcal{A}(m-1)$. This completes the proof. □

The preceding theorem states that LMI (2) is a necessary condition for a matrix $M$ to be a positive map. Theorem 3.1 states that it is a sufficient condition. We obtain the following result.

**Theorem 5.6.** *Let* $\min(n, m) \geq 3$. *A linear map* $M : \mathbb{R}^m \to \mathbb{R}^n$ *is in the cone of positive maps* $P_{n,m}$ *if and only if there exists* $X \in \mathcal{A}(n-1) \otimes \mathcal{A}(m-1)$ *such that* $(\mathcal{W}_n \otimes \mathcal{W}_m)(M) + X \succeq 0$, *where the maps* $\mathcal{W}_n$, $\mathcal{W}_m$ *are defined by (1).* □

LMI (2) describing the cone $P_{n,m}$ has size $(m-1)(n-1)$ and has thus a complexity which is polynomial in $n$ and $m$.

# References


[1] Karin Gatermann and Pablo A. Parrilo. Symmetry groups, semidefinite programs, and sums of squares. *Journal of Pure and Appl. Algebra*, 192(1–3):95–128, 2004.

[2] Roland Hildebrand. Exactness of sums of squares relaxations involving 3x3 matrices and Lorentz cones. *Linear Algebra and its Applications*, 426(3):815–840, 2007.

[3] Roland Hildebrand. An LMI description for the cone of Lorentz-positive maps. *Linear and Multilinear Algebra*, 55(6):551–573, 2007.

[4] Mark Grigorievich Krein and Yu. L. Shmuljan. On the plus-operators in a space with an indefinite metric. *Mat. Issled.*, 1(1):131–161, 1966.

[5] Raphael Loewy and Hans Schneider. Positive operators on the $n$-dimensional ice cream cone. *J. Math. Anal. Appl.*, 49:375–392, 1975.

[6] Ian Porteous. *Topological geometry.* van Nostrand, New York, 1969.

[7] J.-P. Serre. *Linear representations of finite groups.* Springer, Berlin, 1977.

[8] Bit Shun Tam. Some results of polyhedral cones and simplicial cones. *Linear and Multilinear Algebra*, 4(4):281–284, 1977.

[9] Bit Shun Tam. On the structure of the cone of positive operators. *Linear Algebra and its Applications*, 167:65–85, April 1992.

[10] Bit Shun Tam. *Extreme positive operators on convex cones*, pages 515–558. World Scientific, River Edge, New Jersey, 1995.

[11] Wikipedia. Clifford algebras. http://en.wikipedia.org/wiki/Clifford_algebra, 2005.


# A Proof of Theorem 4.4

## A.1 Decompositions of algebra representations

In this subsection we restate and generalize some results from [1].

Consider a compact subgroup $G \subset U(n)$ of unitary matrices. The $n$-dimensional representation of $G$ as a subgroup of $U(n)$ decomposes into irreducible representations $R_1, \dots, R_l$ of dimensions $d_1, \dots, d_l$ and with multiplicities $m_1, \dots, m_l$. We have the following result.

**Lemma A.1.** *Assume above notations. Then there exists a unitary matrix* $U_0 \in U(n)$ *with the following property. For any complex* $n \times n$ *matrix* $A$ *such that* $AU = UA$ *for all matrices* $U \in G$ *the matrix* $A_0 = U_0 A U_0^*$ *has a block-diagonal structure. Each irreducible representation* $R_k$ *gives rise to* $d_k$ *identical blocks of size* $m_k$.

The main idea of the proof is from [1].

*Proof.* Assume the conditions of the lemma. Since the elements in $G$ are represented by unitary matrices, the invariant subspaces corresponding to the different representations are mutually orthogonal. We can hence assume without restriction of generality that the matrices $U \in G$ are block-diagonal, with $m_k$ identical blocks of size $d_k$ corresponding to each irreducible representation $R_k$. Denote these blocks by $B_1(U), \dots, B_s(U)$, where $s = \sum_k m_k$.

Divide the matrix $A$ into $s \times s$ blocks $A_{k_1 k_2}$, $k_1, k_2 = 1, \dots, s$, according to the block structure introduced above. The condition $AU = UA$ then yields $A_{k_1 k_2} B_{k_2}(U) = B_{k_1}(U) A_{k_1 k_2}$ for all $U \in G$ and for every pair $k_1, k_2$ of indices. By the Schur representation lemma we then have that $A_{k_1 k_2} = 0$ if the irreducible representations corresponding to the indices $k_1, k_2$ are different, and $A_{k_1 k_2} = \alpha I$ for some complex scalar $\alpha$, if these representations are the same. Here $I$ stands for the identity matrix of appropriate size.

Thus the principal submatrix of $A$ composed of blocks corresponding to a particular representation $R_k$ is a Kronecker product $A_k \otimes I_{d_k}$, where $A_k$ is an $m_k \times m_k$ matrix composed of the individual scalars $\alpha$. An obvious permutation of rows and columns now yields the desired block-diagonal form of $A$. □

**Corollary A.2.** *Let* $\mathbf{A}$ *be an associative algebra over* $\mathbb{R}$ *(or* $\mathbb{C}$*) represented as an* $\mathbb{R}$*-subspace (or* $\mathbb{C}$*-subspace)* $L$ *of complex* $n \times n$ *matrices. Let further* $G \subset U(n)$ *be a compact group of unitary matrices, with a decomposition in irreducible representations as above, such that* $AU = UA$ *for all* $A \in L$ *and* $U \in G$*. Then there exists a unitary matrix* $U_0$ *with the following property. The matrices in the subspace* $L_0 = U_0 L U_0^*$ *have a block-diagonal structure. Each irreducible representation* $R_k$ *gives rise to* $d_k$ *identical blocks of size* $m_k$*.* □

Note that $L_0$ as well as each of its diagonal blocks is also a representation of $\mathbf{A}$. Thus the decomposition of the unitary symmetry group $G$ into irreducible representations induces a decomposition of the representation $L$ of the algebra $\mathbf{A}$ into smaller representations.

## A.2 Decomposition of the representation $\mathcal{R}^{Cl}$

The representations $\mathcal{R}^{Cl}$ and $\mathcal{C}^{Cl}$ of $Cl_2(\mathbb{R})$ are equal by definition. Therefore Theorem 4.4 is trivially valid for $r = 3$.

Let hence $r \geq 4$ and $\rho = r - 2$. Assume the notations of Subsection 4.1. Now we shall investigate the representation $\mathcal{R}^{Cl}$ of $Cl_{r-1}(\mathbb{R})$. The element $j_k \in G_r$ is by definition the conjugation with the orthogonal matrix $M_k$. Since conjugation with an orthogonal matrix commutes with matrix multiplication, we have $M_k \mathcal{R}^{Cl}(x) M_k^T = \mathcal{R}^{Cl}(x)$ for all $x \in Cl_{r-1}(\mathbb{R})$. Let $G'_r \subset O(2^\rho)$ be the group generated by the matrices $M_k$, $k = 0, \dots, \rho - 2$.

The matrices $M_k$ obey the commutation relations

$$\begin{aligned} M_k^2 &= -I_{2^\rho}, \\ M_k M_{k+1} &= -M_{k+1} M_k, \\ M_k M_l &= M_l M_k, \qquad |k - l| \geq 2. \end{aligned} \tag{5}$$

It is not hard to see that $G'_r$ is of order $2^\rho$, namely consisting of all signed ordered products $\pm M_{k_1} \cdot \dots \cdot M_{k_l}$, where $0 \leq k_1 < \dots < k_l \leq \rho - 2$. The empty product is interpreted as the identity matrix $I_{2^\rho}$ and denoted by 1. The size of the matrices in $G'_r$ is also $2^\rho$. Hence, when considered as matrix representation of itself, $G'_r$ must be reducible by the dimensionality theorem [7]. Denote this representation by $\mathcal{G}$. This induces a decomposition of the representation $\mathcal{R}^{Cl}$ as described in Corollary A.2.

Let us compute how the representation $\mathcal{G}$ decomposes into irreducible representations. The commutator of $G'_r$ is given by $\{1\}$ for $r = 4$ and by $\{\pm 1\}$ for $r \geq 5$. It is well-known that the number of irreducible representations of dimension 1 is equal to the order of the quotient group with respect to the commutator [7]. Hence $G'_r$ has 4 irreducible representations of dimension 1 for $r = 4$ and $2^{\rho-1}$ irreducible representations of dimension 1 for $r \geq 5$. Let us treat these cases separately.

For $r = 4$ the group $G'_r$ is isomorphic to $\mathbb{Z}_4$, and the representation $\mathcal{G}$ decomposes into 2 copies of the irreducible one-dimensional representation with character $i$ and 2 copies of the irreducible one-dimensional representation with character $-i$. Hence the representation $\mathcal{R}^{Cl}$ of $Cl_3(\mathbb{R})$ can be brought to a block-diagonal form with 2 equal-sized blocks. The reader will have no difficulty to show that these blocks are actually the representations $\mathcal{C}^{Cl}$, $\mathcal{C}'^{Cl}$.

To compute the decomposition into irreducible representations for $r \geq 5$, we first count the number of conjugacy classes of $G'_r$. It is clear from the commutation relations (5) that for an element $g \in G'_r$ there can be maximally 2 elements in the conjugacy class of $g$, namely $\pm g$. Those elements which form a conjugacy class on their own are exactly those which commute with all elements in the group, i.e. the elements of the centre.

**Lemma A.3.** *Let $r \geq 5$. If $r$ is even, then the centre of the group $G'_r$ is formed by the 4 elements $\pm 1, \pm \Pi_{k=0}^{(\rho-2)/2} M_{2k}$. If $r$ is odd, then the centre is formed by the two elements $\pm 1$.*

*Proof.* Let $M = \Pi_{k=0}^{\rho-2} M_k^{c_k}$ be an element of $G'_r$, where the $c_k$ equal 0 or 1. We put $c_k = 0$ for $k < 0$ and $k > \rho - 2$ for convenience. Let now $0 \leq k \leq \rho - 2$ and consider the commutator of $M$ and $M_k$. Since $M_k$ anti-commutes with the generators $M_{k-1}$ and $M_{k+1}$ and commutes with all others, we have $MM_k = (-1)^{c_{k-1}+c_{k+1}} M_k M$. Hence $M$ lies in the centre if and only if $c_{k-1} = c_{k+1}$ for all $0 \leq k \leq \rho - 2$. Recall that here $c_{-1} = c_{\rho-1} = 0$. If $r$ is odd, then this condition implies $c_k = 0$ for both even and odd indices $k$, and $M = 1$. If $r$ is even, then $c_k = 0$ for odd indices $k$, but it can assume both values 0 and 1 for even indices $k$. Hence $M = 1$ or $M = \Pi_{k=0}^{(\rho-2)/2} M_{2k}$. Now note that $-M$ lies in the centre if and only if $M$ lies in the centre. This completes the proof. □

**Corollary A.4.** *Let $r \geq 5$. For even $r$ the group $G'_r$ has $2^{\rho-1} + 2$ conjugacy classes. If $r$ is odd, then it has $2^{\rho-1} + 1$ conjugacy classes.* □

**Corollary A.5.** *Let $r \geq 5$. For even $r$ the group $G'_r$ has $2^{\rho-1}$ irreducible representations of dimension 1 and 2 irreducible representations of dimension $2^{(\rho-2)/2}$. For odd $r$ $G'_r$ has $2^{\rho-1}$ irreducible representations of dimension 1 and 1 irreducible representation of dimension $2^{(\rho-1)/2}$.*

*Proof.* The number of irreducible representations of a finite group equals the number of conjugacy classes and the sum of the squared dimensions of these representations equals the group order (dimensionality theorem) [7].

For even $r$ we have $2^{\rho-1}$ representations of dimension 1 and two other irreducible representations, say of dimensions $d_1, d_2$. Since the group order is $2^\rho$, we have $d_1^2 + d_2^2 = 2^{\rho-1}$. Now the dimension of an irreducible representation has to divide the group order [7], that is $d_1, d_2$ are powers of 2. But then $d_1 = d_2 = 2^{(\rho-2)/2}$ is the only solution of the equation $d_1^2 + d_2^2 = 2^{\rho-1}$.

For odd $r$ we have $2^{\rho-1}$ irreducible representations of dimension 1 and 1 additional irreducible representation, which has to have dimension $2^{(\rho-1)/2}$ to satisfy the dimensionality theorem. □

**Lemma A.6.** *Let $r \geq 5$. Then the orthogonal representation $\mathcal{G}$ of $G'_r$ does not contain representations of dimension 1.*

*Proof.* The reader will have no difficulty to check that the $8 \times 8$ matrices $\sigma_{031}$ and $\sigma_{310}$ have no common eigenvector. It follows that any two adjacent generators $M_k, M_{k+1}$ of $G'_r$ cannot have a common eigenvector. Were $\mathcal{G}$ to contain a representation of dimension 1, then the corresponding invariant subspace would be generated by a common eigenvector of the matrices in $G'_r$, in particular the generators $M_0, \dots, M_{\rho-2}$. This leads to a contradiction. □

**Corollary A.7.** *Let $r \geq 5$. For $r$ odd, the representation $\mathcal{G}$ of the group $G'_r$ can be decomposed into $2^{(\rho+1)/2}$ copies of an irreducible representation of dimension $2^{(\rho-1)/2}$. For $r$ even, $\mathcal{G}$ can be decomposed into $2 \cdot 2^{\rho/2}$ copies of two irreducible representations of dimension $2^{(\rho-2)/2}$, $2^{\rho/2}$ copies of each.*

*Proof.* The representation $\mathcal{G}$ has dimension $2^\rho$. If $r$ is odd, then by Corollary A.5 and Lemma A.6 there is only one irreducible representation, of dimension $2^{(\rho-1)/2}$, that can enter in $\mathcal{G}$. Hence the latter decomposes into $2^{(\rho+1)/2}$ identical copies of this irreducible representation.

For even $r$, there exist two different irreducible representations of dimension $2^{(\rho-2)/2}$ that can enter in $\mathcal{G}$. Hence there are overall $2 \cdot 2^{\rho/2}$ copies of these representations. Consider the product $M = \Pi_{k=0}^{(\rho-2)/2} M_{2k}$. By Lemma A.3 this matrix commutes with all $M_k$. Hence its eigenspaces are invariant subspaces of the representation $\mathcal{G}$. On each eigenspace there act a certain number of irreducible representations of $G'_r$ that enter in the reducible representation $\mathcal{G}$. Note also that irreducible representations that act on eigenspaces with different eigenvalues must be different, because the character of $M$ is different in these irreducible representations. For $\rho/2$ even we have $M^2 = \Pi_{k=0}^{(\rho-2)/2} M_{2k}^2 = (-1)^{\rho/2} I_{2^\rho} = I_{2^\rho}$, for $\rho/2$ odd we have $M^2 = -I_{2^\rho}$. Hence the eigenvalues of $M$ equal $\pm 1$ for even $\rho/2$ and $\pm i$ for odd $\rho/2$. It follows from the definition of the $M_k$ that $M = \sigma_{31}^{\otimes \rho/2}$ and hence all diagonal elements of $M$ are zero. But then the trace is also zero and the eigenspaces of $M$ corresponding to the two different eigenvalues must have the same dimension, and are in particular non-trivial. But since there are only two different representations, each eigenspace corresponds to irreducible representations of only one kind. Finally, since the dimensions of the two different irreducible representations are the same, and the dimensions of the eigenspaces are also the same, each of the irreducible representations enters the same number of times into $\mathcal{G}$. The proof is complete. $\square$

By Corollary A.2 we get the following result.

**Corollary A.8.** *Let $r \geq 3$ be an odd number. The representation $\mathcal{R}^{Cl}$ of $Cl_{r-1}(\mathbb{R})$ is the direct sum of $2^{(r-3)/2}$ copies of a representation $\tilde{\mathcal{C}}^{Cl}$ of dimension $2^{(r-1)/2}$. There exists a unitary matrix $U \in U(2^\rho)$ such that $\mathcal{R}^{Cl}(x) = U(I_{2^{(r-3)/2}} \otimes \tilde{\mathcal{C}}^{Cl}(x))U^*$ for all $x \in Cl_{r-1}(\mathbb{R})$.*

*Let $r \geq 4$ be an even number. The representation $\mathcal{R}^{Cl}$ of $Cl_{r-1}(\mathbb{R})$ is the direct sum of $2^{(r-4)/2}$ copies of a representation $\tilde{\mathcal{C}}^{Cl}$ and of $2^{(r-4)/2}$ copies of a representation $\tilde{\mathcal{C}'}^{Cl}$, both of dimension $2^{(r-2)/2}$. There exists a unitary matrix $U \in U(2^\rho)$ such that $\mathcal{R}^{Cl}(x) = U(I_{2^{(r-4)/2}} \otimes (\tilde{\mathcal{C}}^{Cl}(x) \oplus \tilde{\mathcal{C}'}^{Cl}(x)))U^*$ for all $x \in Cl_{r-1}(\mathbb{R})$.* $\square$

However, for odd $r$ there exists only one representation of $Cl_{r-1}(\mathbb{R})$ of dimension $2^{(r-1)/2}$, namely the spinor representation, and this representation is irreducible [11]. Hence both $\tilde{\mathcal{C}}^{Cl}$ and $\mathcal{C}^{Cl}$ must be equivalent to this spinor representation and in particular to each other.

For even $r$ there exist two representations of dimension $2^{(r-2)/2}$, namely the half-spinor or Weyl representations, and they are also irreducible. Now note that $\mathcal{C}^{Cl}$ and ${\mathcal{C}'}^{Cl}$ are not equivalent. This can be seen if one considers the images of the product $\varepsilon = f_1 f_2 \dots f_{r-1}$ (the pseudoscalar). Namely, we have $\mathcal{C}^{Cl}(\varepsilon) = (-i)^{\rho'} I_{2^{\rho'}}$, ${\mathcal{C}'}^{Cl}(\varepsilon) = -(-i)^{\rho'} I_{2^{\rho'}}$, but for equivalent representations the images must have the same trace. Hence $\mathcal{C}^{Cl}$, ${\mathcal{C}'}^{Cl}$ are equivalent to the two half-spinor representations. Likewise, the representations $\tilde{\mathcal{C}}^{Cl}$ and $\tilde{\mathcal{C}'}^{Cl}$ are not equivalent. Namely, if they were equivalent, say to $\mathcal{C}^{Cl}$, then the image of the pseudoscalar under the representation $\mathcal{R}^{Cl}$ must be proportional to the identity matrix, which is obviously not the case. Thus $\tilde{\mathcal{C}}^{Cl}$, $\tilde{\mathcal{C}'}^{Cl}$ are also equivalent to the two half-spinor representations.

Let again $r$ be odd. It rests to show that the representations $\tilde{\mathcal{C}}^{Cl}$ and $\mathcal{C}^{Cl}$ are not only equivalent, but actually unitarily equivalent. This can be seen as follows. Let $S$ be a complex $2^{\rho'} \times 2^{\rho'}$ matrix realizing the equivalence, i.e. $\tilde{\mathcal{C}}^{Cl}(x) = S\mathcal{C}^{Cl}(x)S^{-1}$ for all $x \in Cl_{r-1}(\mathbb{R})$. For all generators $f_k$ of $Cl_{r-1}(\mathbb{R})$ the images $\mathcal{C}^{Cl}(f_k)$ and $\mathcal{R}^{Cl}(f_k)$ are hermitian by definition. Hence $\tilde{\mathcal{C}}^{Cl}(f_k)$ is also hermitian by the preceding corollary. It follows by complex conjugate transposition that $\tilde{\mathcal{C}}^{Cl}(f_k) = S^{-*}\mathcal{C}^{Cl}(f_k)S^*$ for all $k = 1, \dots, r-1$. This yields $S^*S\mathcal{C}^{Cl}(f_k) = \mathcal{C}^{Cl}(f_k)S^*S$ and the matrix $S^*S$ commutes with all matrices in the image $\mathcal{C}^{Cl}[Cl_{r-1}(\mathbb{R})]$. But the representation $\mathcal{C}^{Cl}$ is irreducible, hence $S^*S$ must be proportional to the identity matrix. It follows that $S$ can be chosen to be unitary.

For even $r$ the argumentation follows the same lines of reasoning. This completes the proof of Theorem 4.4.