Generating All Efficient Extreme Points in Multiple Objective Linear Programming Problem and Its Application *

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Abstract

In this paper, simple linear programming procedure is proposed for generating all efficient extreme points and all efficient extreme rays of a multiple objective linear programming problem (VP). As an application we solve the linear multiplicative programming associated with the problem (VP).

Key words. Multiple objective linear programming, linear multiplicative programming, efficient extreme point, efficient solution.

1 Introduction

A multiple objective linear programming problem may be written as

$$\text{MIN}\{Cx \ : \ x \in M\}, \tag{VP}$$

where C is a $p \times n$ matrix with $p \geq 2$ whose rows $c^1, \cdots, c^p$ are the coefficients of p linear criterion functions $\langle c^i, x\rangle$, $i = 1, 2, ..., p$, and $M \subset \mathbb{R}^n$ is a

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nonempty polyhedral convex set. Problem (VP) is one of the most popular models that are used as aids in decision making with multiple criteria. This problem arises from various applications in engineering, industry, economics, network planning, production planning etc.(see, e.g., [7, 20, 21, 25]). For instance, for the perfect economical production plan, one wants to simultaneously minimize the cost and maximize the quality. This example illustrates a natural feature of this problem, namely, that typically the different objectives contradict each other.

Various solution concepts for problem (VP) have been proposed. The concept of an efficient solution is commonly used. In particular, a point $x^1 \in M$ is said to be an *efficient solution* for problem (VP) if there exists no $x \in M$ such that $Cx^1 \geq Cx$ and $Cx^1 \neq Cx$. Let M_E denote the set of all efficient solutions of problem (VP).

Many algorithms have been proposed to generate either all of the efficient set M_E, or a representative portion thereof, without any input from decision maker; see, e.g., [1, 2, 10, 11, 15, 24] and references therein. For a survey of these and related results see [4].

It is well known that M_E consists of a union of faces of M. While M_E is also always a connected set, generally, it is a complicated nonconvex subset of the boundary of M [16, 20]. Let M_{ex} denote the set of all extreme points of M. The set of all efficient extreme points $M_E \cap M_{ex}$ is a finite, discrete set and is smaller than all of M_E. Therefore, it ought to be more computationally practical to generate the set $M_E \cap M_{ex}$ and to present it to the decision maker without overwhelming him or her than M_E [4].

In this paper, basing on Kuhn-Tucker Theorem we express optimality conditions for efficient solutions for multiple objective linear programming problem (VP). Combining this fact and simplex pivot technique [8, 9, 13] we construct a quite easy algorithm for generating the set of all efficient extreme points and all efficient extreme rays in problem (VP). As an application we solve the linear multiplicative programming associated with the problem (VP).

2 Efficiency

Throughout the paper we use the following notations: For two vectors $y^1 = (y^1_1, ..., y^1_p)$, $y^2 = (y^2_1, ..., y^2_p) \in \mathbb{R}^p$, $y^1 \geq y^2$ denotes $y^1_j \geq y^2_j$ for $j = 1, ..., p$ and $y^1 \gg y^2$ denotes $y^1_j > y^2_j$ for $j = 1, ..., p$. As usual, $\mathbb{R}^p_+$ denotes the nonnegative orthant of $\mathbb{R}^p$.

A key result in all the sequel is the following characterization of efficiency which can be found in many places (see. e.g., Theorem 2.5, Chapter 4 [5] and Theorem 2.1.5 [23])

Theorem 2.1. *A feasible solution $x^1 \in M$ is an efficient solution to (VP) if and only if there is a positive vector $\lambda \in R^p$ (i.e. $\lambda \gg 0$) such that x^1 is an optimal solution of the linear programming problem*

$$\min\{\langle \lambda, Cx\rangle \ : \ x \in M\}. \tag{LP1}$$

It is well known that a set $\Gamma \subset \mathbb{R}^n$ is a face of M if and only if Γ equals the optimal solution set to the problem

$$\min\{\langle \alpha, x\rangle \ : \ x \in M\},$$

for some $\alpha \in \mathbb{R}^n \setminus \{0\}$. The following result is directly deduced from this fact and Theorem 2.1.

Proposition 2.2. *Let x^1 be a relative interior point of a face $\Gamma \subseteq M$. If x^1 is an efficient solution then every point of Γ is efficient solution (i.e., $\Gamma \subset M_E$).*

Let M be a polyhedral convex set in $\mathbb{R}^n$ which is defined by a system of linear inequalities

$$g_i(x) = \langle a^i, x\rangle - b_i \leq 0, \ \ i = 1, \cdots, \bar{m}, \tag{1}$$

where $a^1, \cdots, a^{\bar{m}}$ are vectors from $\mathbb{R}^n$ and $b_1, \cdots, b_{\bar{m}}$ are real numbers. Denoted by $I(x^1)$ the set of all *active indices* at $x^1 \in M$, i.e.

$$I(x^1) = \{i \in \{1, \cdots, \bar{m}\} \ : \ g_i(x^1) = 0\}. \tag{2}$$

Below is an optimality condition that will be important in helping to develop our algorithm.

Proposition 2.3 *A point $x^1 \in M$ is an efficient solution of (VP) if and only if the following linear system is consistent (has a solution)*

$$\begin{cases} \sum_{j=1}^{p} \lambda_j c^j + \sum_{i \in I(x^1)} \mu_i a^i = 0, \\ \lambda_j > 0, \forall i = 1, \cdots, p, \\ \mu_i \geq 0, \forall i \in I(x^1). \end{cases} \tag{3}$$

Proof. By Theorem 2.1, a point $x^1 \in M$ is an efficient solution of (VP) if and only if there is a positive vector $\lambda = (\lambda_1, \cdots, \lambda_p) \in I\!\!R^p$ such that x^1 is an optimal solution of the linear programming problem

$$\min\{\langle \lambda, Cx\rangle \;:\; x \in M\}. \qquad (LP1)$$

The problem $(LP1)$ can be rewritten as follows

$$\min\{f(x) \;:\; x \in M\}, \qquad (LP2)$$

where $f(x) = \langle \lambda, Cx\rangle = \langle C^T\lambda, x\rangle = \langle \sum_{j=1}^p \lambda_j c^j, x\rangle$, the real numbers $\lambda_j > 0$ for all $j = 1, \cdots, p$, the vectors $c^1, \cdots, c^p$ are rows of the matrix C and C^T stands for the transposition of the matrix C.

Since the problem $(LP2)$ is a linear programming problem, any $x^1 \in M$ is regular. By Kuhn-Tucker Theorem (see, Chapter 7 [17]), a point $x^1 \in M$ is an optimal solution of $(LP2)$ (from Theorem 2.1, x^1 is an efficient solution of (VP)) if and only if there exists a nonnegative vector $\mu = (\mu_1, \cdots, \mu_{\bar{m}}) \in I\!\!R^{\bar{m}}$ such that

$$\nabla f(x^1) + \sum_{i=1}^{\bar{m}} \mu_i \nabla g_i(x^1) = 0 \qquad (4)$$

$$\mu_i g_i(x^1) = 0, \; i = 1, \cdots, \bar{m}. \qquad (5)$$

From (5) and (2), we have

$$\mu_i = 0 \;\; \forall\, i \notin I(x^1). \qquad (6)$$

Since $f(x) = \langle \sum_{j=1}^p \lambda_j c^j, x\rangle$ and $g_i(x) = \langle a^i, x\rangle - b_i$ for all $i = 1, \cdots, \bar{m}$, it follows that $\nabla f(x^1) = \sum_{j=1}^p \lambda_j c^j$ and $\nabla g_i(x^1) = a^i$ for all $i = 1, \cdots, \bar{m}$. Combine this fact and (6), we can rewrite (4) as follows

$$\sum_{j=1}^p \lambda_j c^j + \sum_{i \in I(x^1)} \mu_i a^i = 0.$$

The proof is straight forward.■

Proposition 2.3 provides a condition for a point $x^1 \in M$ to be an efficient solution for multiple objective linear programming problem (VP). This condition is analogous to the efficient condition which presented in [15] in terms of normal cones. In that paper, it presented an algorithm for generating the set of all efficient solution M_E for problem (VP) with the strict assumption:

(*) M is nonempty polyhedral convex set in $\mathbb{R}^n$ of dimension n and the system (1) has no redundant inequalities, that is, M cannot be defined by a smaller number of inequalities of (1).

In the next section, combining this optimal condition and pivot technique of the simplex procedure we will introduce an algorithm for generating all efficient extreme points $M_E \cap M_{ex}$ and all efficient extreme rays in problem (VP). Notice that this algorithm does not require the assumption (*).

3 Determination of Efficient Extreme Points and Efficient Extreme Rays

We consider the multiple objective linear programming problem with assumption henceforth that polyhedral convex set M is the solution set of the following linear system

$$Ax = h, \; x \geq 0, \tag{7}$$

where A is a $m \times n$ matrix of rank m and h is a vector of $\mathbb{R}^m$. The problem (VP) becomes

$$\text{MIN}\{Cx \; : \; Ax = h, \; x \geq 0\}.$$

It is well known that the efficient solution set M_E is pathwise connected (Theorem 2.2, Chapter 6 [16]). Hence, to generate the set of all efficient extreme points and all efficient extreme rays in problem (VP) we just need to present the procedure that generates all the efficient vertices adjacent to a given efficient vertex $x^0 \in M$ and all the efficient extreme rays emanating from x^0.

For simplicity of exposition, assume that x^0 is nondegenerate efficient vertex. In the case of degeneracy, one can use the right hand side perturbation method of Charnes (see, e.g., Chapter 10 [8]; Chapter 6 [13]) to reduce the nondegeneracy.

Firstly, let us recall the way to determine the set of all edges emanating from the vertex x^0 for the reader's convenience (see, e.g., Chapter 3 [9]). Denote by $A_1, A_2, \cdots, A_n$ the column vectors of matrix A. Let $J_0 = \{j \in \{1, 2, \cdots, n\} \; : \; x_j^0 > 0\}$. Then we have $|J_0| = m$ and the set of m linearly independent vectors $\{A_j \; : \; j \in J_0\}$ are grouped together to form $m \times m$ basic matrix B. The variables $x_j, \; j \in J_0$ are said to be basic variables and the vectors $A_j,\ j \in J_0$ are said to be basic vectors. By the theory of linear programming, there are exact $n-m$ edges of M emanating from x^0. In particular, for each $k \notin J_0$ there is an edge emanating from x^0 whose direction $z^0(k) = (z_1^0(k), \cdots, z_n^0(k)) \in \mathbb{R}^n$ is defined by

$$z_j^0(k) = \begin{cases} -z_{jk} & \text{if } j \in J_0 \\ 0 & \text{if } j \notin J_0, j \neq k \\ 1 & \text{if } j = k, \end{cases} \tag{8}$$

where the real numbers $z_{jk},\ j \in J_0$ satisfy the system of linear equations

$$A_k = \sum_{j \in J_0} z_{jk} A_j.$$

Without loss of generality and for convenience, we assume that $J_0 = \{1, ..., m\}$ and the basis $B = (A_1, ..., A_m)$ is a $m \times m$ unit matrix. Then we have $A_k = (z_{1k}, ..., z_{mk})^T$ where $(z_{1k}, ..., z_{mk})^T$ stands for the transpose of the vector $(z_{1k}, ..., z_{mk})$. The data associated to vertex x^0 can be performed in the following Tableau 1.

Tableau 1

A_{J_0}	x_{J_0}	A_1	A_2	...	A_m	A_{m+1}	A_{m+2}	...	A_n
A_1	x_1	1	0	...	0	$z_{1\ m+1}$	$z_{1\ m+2}$	...	$z_{1\ n}$
A_2	x_2	0	1	...	0	$z_{2\ m+1}$	$z_{2\ m+2}$	...	$z_{2\ n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	x_m	0	0	...	1	$z_{m\ m+1}$	$z_{m\ m+2}$	...	$z_{m\ n}$

Denote by $\Gamma(k),\ k \in \{1, 2, ..., n\} \setminus J_0$, the edge emanating from x^0 whose direction vector $z^0(k)$ determined by (8). If nonbasic vector $A_k \leq 0$, i.e. $z_{jk} \leq 0$ for all $j \in J_0$, then $\Gamma(k)$ is an unbounded edge of M emanating from x^0. Otherwise, we obtain the new vertex $x^1 = (x_1^1, \cdots, x_n^1) \in M$ adjacent to the extreme point x^0, where

$$x_j^1 = \begin{cases} x_j^0 - \theta_0 z_{jk} & \text{if } j \in J(x^0) \setminus \{r\} \\ \theta_0 & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

and

$$\theta_0 = \min\{\frac{x_j^0}{z_{jk}} \ :\ z_{jk} > 0,\ j \in J_0\} = \frac{x_r^0}{z_{rk}}.$$

Let x^0 be a given efficient extreme solution. Below we present a procedure for determining all the efficient extreme points adjacent to x^0 and the efficient extreme rays emanating from x^0. This task seems to be simple as we can obtain adjacent edges by above simplex pivot technique and check whether a point $x^0 \in M$ is an efficient solution for problem (VP) by solving the associating system (3). Namely, this procedure can be described as follows

Procedure EFFI(x^0)
Step 1. Determine the set $J_0 = \{j \in \{1, 2, \cdots, n\} \ : \ x_j^0 > 0\}$. Set $i = 1$ and

$$\{k_1, \cdots, k_{n-m}\} := \{1, \cdots, n\} \setminus J_0.$$

We arrange the data associated to x^0 in a tableau that similar to Tableau 1. Set $k = k_i$.

Step 2. If $A_k \leq 0$, we have $\Gamma(k) := \{x^0 + tz^0(k) \ : \ t \geq 0\}$ where $z^0(k)$ determined by (8) is an extreme ray emanating from x^0, then go to Step 3. Otherwise, go to Step 4.

Step 3. Put $x^1 = x^0 + \theta z^0 \in M$ with a real number $\theta > 0$. It is clear that x^1 is relative interior point of this ray $\Gamma(k)$.

a) If $x^1 \in M_E$, then store the result and go to Step 5. Since x^1 is a relative interior point of the extreme ray $\Gamma(k) = \{x^0 = \theta z^0(k) \ : \theta > 0\}$, by Proposition 2.2, $\Gamma(k)$ is efficient extreme rays.

b) Otherwise, go to Step 5.

Step 4. (*There is* $z_{jk} > 0$, $j \in J_0$) We obtain the new extreme point $x^1 = (x_1^1, \cdots, x_n^1) \in M$, determined by (9), adjacent to the extreme efficient point x^0.

a) If $x^1 \in M_E$, then store it if it has not been stored before and go to Step 5.

b) Otherwise, go to Step 5.

Step 5. Set $i := i + 1$;

If $i \leq n - m$, then set $k = k_i$ and go to Step 2. Otherwise, terminate the procedure: *We obtain all efficient extreme solutions adjacent to a given efficient extreme solution* x^0 *and all efficient extreme rays emanating from* x^0.

Here, we invoke Proposition 2.3 to check whether a point $x^1 \in M$ is an efficient solution to problem (VP). To do this we have to reduce the system (7) to the form of the system (1) first.

Let I_n be a $n \times n$ identity matrix and $0 \in \mathbb{R}^n$ and $\bar{m} = 2m + n$. Let

$$\bar{A} = \begin{pmatrix} A \\ -A \\ -I_n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} h \\ -h \\ 0 \end{pmatrix},$$

denote the $\bar{m} \times n$ matrix and the vector $b \in \mathbb{R}^{\bar{m}}$, respectively. Note that we can now rewrite M in form (1) as follows

$$M = \{x \in \mathbb{R}^n : \bar{A}x - b \leq 0\} = \{x \in \mathbb{R}^n : g_i(x) = \langle a^i, x\rangle - b_i \leq 0, \quad i = 1, \cdots, \bar{m}\}$$

where

$$a^1, \cdots, a^{\bar{m}} \ \text{ are rows of } \bar{A} \tag{10}$$

and

$$b_i = \begin{cases} h_i & \text{if } i = 1, \cdots, m \\ -h_i & \text{if } i = m+1, \cdots, 2m \\ 0 & \text{if } i = 2m+1, \cdots, 2m+n. \end{cases} \tag{11}$$

Let x^1 be a point of M defined by (7). Then, according to Proposition 2.3, if the system (3) with a^i and b^i determined by (10) and (11) has a solution then $x^1 \in M_E$ else $x^1 \notin M_E$.

We conclude this section with some remarks.

Remark 2.1 Several methods can be used in order to find the first efficient extreme solution to problem (VP), see, e.g., [3, 12, 15]. Here we use the Procedure 1 in [15]. This procedure specifies if the efficient set M_E is empty, and in case of non-emptiness provides an efficient extreme point.

Remark 2.2 Note that if $(\lambda, \mu) \in \mathbb{R}^{p+|I(x^1)|}$ is a solution of the system (3) then $(t\lambda, t\mu)$ is also a solution of this system for an arbitrary real number $t > 0$. Therefore, instead of checking the consistency of the system (3), one can use Phase I of simplex algorithm to check the consistency of the following system

$$\begin{cases} \sum_{j=1}^{p} \lambda_j c^j + \sum_{i \in I(x^1)} \mu_i a^i = 0, \\ \lambda_j \geq 1, \forall i = 1, \cdots, p, \\ \mu_i \geq 0, \forall i \in I(x^1). \end{cases}$$

Remark 2.3 Denote the edge connecting two efficient extreme points x^1 and x^2 by $[x^1,\ x^2]$. Notice that the relative interior of the edge $[x^1,\ x^2]$ may not belong to the efficient set M_E. Indeed, if we consider the problem

$$\text{Min} \begin{pmatrix} x_1 + 0x_2 \\ 0x_1 + \ x_2 \end{pmatrix},$$

$$x \in M = \{x \in \mathbb{R}^2 \ : \ 5x_1 + x_2 \geq 10,\ x_1 + 5x_2 \geq 10,\ x_1 + x_2 \leq 10\},$$

two extreme points $(10, 0)$, $(0, 10)$ belong to M_E but the relative interior of the edge connecting $(10, 0)$ and $(0, 10)$ is not efficient.

4 Computational Results

We begin with the following simple example to illustrate our algorithm. Consider the multiple objective linear programming

$$\text{MIN}\{Cx \ : \ Ax = b, x \geq 0\},$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & -1 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and } b = \begin{pmatrix} -2 \\ -2 \\ 6 \end{pmatrix}.$$

In this example, $m = 3$, $n = 5$ and $p = 2$. Using Procedure 1 in [15], we obtain the first efficient extreme solution

$$x^0 = (\frac{2}{3}, \frac{2}{3}, 0, 0, 4\frac{2}{3}).$$

It is clear that

$$J_0 = \{j \in \{1,2,3,4,5\} : x^0_j > 0\} = \{2,1,5\}, \text{ and } \{1,2,3,4,5\} \backslash J_0 = \{k_1, k_2\} = \{3,4\}.$$

The associating data to this efficient extreme solution is shown in Tableau 2.

Tableau 2

A_J	x_J	A_1	A_2	A_3	A_4	A_5
A_2	$\frac{2}{3}$	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0
A_1	$\frac{2}{3}$	1	0	$\frac{1}{3}$	$\frac{2}{3}$	0
A_5	$4\frac{2}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	1

• Consider $k_1 = 3$. We have $\theta_0 = 2,\ r = 1$ and obtain the new extreme point

$$x^1 = (0, 2, 2, 0, 4).$$

By solving the associating system (3), we see that x^1 is efficient solution for the problem (VP). Therefore, we have $x^1 \in M_E \cap M_{ex}$. The associating data to extreme solution x^1 is shown in Tableau 3.

Tableau 3

A_J	x_J	A_1	A_2	A_3	A_4	A_5
A_2	2	2	1	0	-1	0
A_3	2	3	0	1	-2	0
A_5	4	-1	0	0	1	1

• Consider $k_2 = 4$. We have $\theta_0 = 2,\ r = 2$ and obtain the new extreme point

$$x^2 = (2, 0, 0, 2, 4).$$

By solving the associating system (3), we claim that x^2 is efficient solution for the problem (VP). Therefore, we have $x^2 \in M_E \cap M_{ex}$. The associating data to extreme solution x^2 is shown in Tableau 4.

Tableau 4

A_J	x_J	A_1	A_2	A_3	A_4	A_5
A_4	2	0	3	-2	1	0
A_1	2	1	2	-1	0	0
A_5	4	0	-1	1	0	1

Repeat using Procedure EFFI(x^1) and EFFI(x^2) where x^1 and x^2 are two efficient vertices have just determined above. At last, we have obtained 3 efficient extreme solution $x^0 = (\frac{2}{3}, \frac{2}{3}, 0, 0, 4\frac{2}{3})$; $x^1 = (0, 2, 2, 0, 4)$; $x^2 = (2, 0, 0, 2, 4)$ for this problem.

In order to obtain a preliminary evaluation of the performance of the proposed algorithm, we build a test software using C^{++} programming language that implements the algorithm. We notice that for checking whether the system (3) is consistent and determining all edges adjacent the given vertex we used an own code based on the well known simplex method.

The following example introduced by Yu and Zeleny [24], and also considered in [1,2,15]. The problem is stated as follows.

$$\text{MIN}\{Cx \ : \ Ax \leq b, \ x \geq 0\},$$

where

$$C = \begin{pmatrix} -3 & 7 & -4 & -1 & 0 & 1 & 1 & -8 \\ -2 & -5 & -1 & 1 & -6 & -8 & -3 & 2 \\ -5 & 2 & -5 & 0 & -6 & -7 & -2 & -6 \\ 0 & -4 & 1 & 1 & 3 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 3 & -4 & 1 & -1 & 1 & 2 & 4 \\ 5 & 2 & 4 & -1 & 3 & 7 & 2 & 7 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ -3 & -4 & 8 & 2 & 3 & -4 & 5 & -1 \\ 12 & 8 & -1 & 4 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 8 & -12 & -3 & 4 & -1 & 0 & 0 & 0 \\ 15 & -6 & 13 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 40 \\ 84 \\ 18 \\ 100 \\ 40 \\ -12 \\ 30 \\ 100 \end{pmatrix}.$$

Note that each vertex is nondegenerate. We computed and have obtained 29 efficient extreme solutions in 0.031 seconds. This numerical result coincides the result reported in [2] and [15].

Below we present our computational experimentation with the algorithm. For each triple (p, m, n), the algorithm was run on 20 randomly generated test problems having form similar to Yu and Zeleny problem. The elements of constraint matrix A, the right-hand-side vector b and the objective function coefficient matrix C were randomly generated integers belonging to the discrete uniform distribution in the intervals $[-12,\ 15]$, $[-12,\ 100]$ and $[-7,\ 8]$, respectively. Test problems are executed on IBM-PC, chip Intel Celeron PIV 1.7 GHz, RAM 640 MB, C^{++} programming language, Microsoft Visual C^{++} compiler. Numerical results are summarized in Table 5.

In Table 5, it can be observed that computational requirements increase with constraints size (i.e. $m \times n$ size). Another observation is that the number of objectives have a significant effect on number of efficient points, therefore, effect on computation time.

Table 5. Computational Results

p	m	n	$NEPs$	$TIME$
3	17	20	8	0.08
3	20	20	6	0.068
3	25	30	8	0.248
5	17	20	110	1.308
5	20	20	106	1.324
5	25	30	40	1.467
7	17	20	209	3.292
7	20	20	160	2.166
7	25	30	176	7.178
8	17	20	553	14.471
9	17	20	665	23.769
10	17	20	372	7.033

NEPs: Average number of efficient vertices
TIME: Average CPU-Time in seconds.

5 An Application

As an application of above proposed algorithm, we consider the linear multiplicative programming

$$\min\{\prod_{j=1}^{p}\langle c^j, x\rangle \ : \ x \in M\}, \qquad (LMP)$$

where M is the polyhedral convex set defined by (7), $p \geq 2$ is an integer, and for each $j = 1, \cdots, p$, vector $c^j \in \mathbb{R}^n$ satisfies

$$\langle c^j, x\rangle > 0 \ \text{ for all } x \in M. \qquad (12)$$

It is well known that the problem (LMP) is difficult global optimization problem and it has been shown to be NP-hard, even when $p = 2$ [18]. This problem have some important applications in engineering, finance, economics, and other fields (see, e.g., [6]). In recent years, due to the requirement of the practical applications, a resurgence of interest in problem (LMP) occurred (see. e.g., [5, 6, 14, 22]). In this section, we solve the problem (LMP) based on the relationships between this problem and associated multiple objective linear programming problem.

First, we show the existence of solution of the problem (LMP).

Proposition 5.1 *The problem (LMP) always has an optimal solution.*

Proof. It is clear that it is sufficient to treat the case in which M is unbounded. Let C denote the $p \times n$ matrix whose j^{th} row equals c^j, $j = 1, 2, ..., p$. Let Y be defined by

$$Y = \{y \in \mathbb{R}^p | y = Cx, \text{ for some } x \in M\}.$$

It follows readily from definitions that the problem (LMP) is equivalent to the following problem

$$\min\{g(y) = \prod_{j=1}^{p} y_j \; : \; y \in Y\}. \qquad (LMP_Y)$$

Therefore instead of showing the existence of solution of the problem (LMP) we show the existence of solution of the problem (LMP_Y). It can be shown that the set Y is a nonempty polyhedral convex set in $\mathbb{R}^p$, see, e.g., [19]. Denote the set extreme points of Y by $V(Y)$ and the set of extreme directions of Y by $R(Y)$. Then

$$Y = \text{conv}V(Y) + \text{cone}R(Y), \tag{13}$$

where conv$V(Y)$ is the convex hull of $V(Y)$ and cone$R(Y)$ is the cone generated by $R(Y)$ [19]. Taking account of the assumption (12) the set Y must be contained in $\text{int}\mathbb{R}^p_+ = \{u \in \mathbb{R}^p | u \gg 0\}$. It implies that

$$\text{cone}R(Y) \subset \text{int}R^p_+ \cup \{0\}. \tag{14}$$

Since conv$V(Y)$ is a compact set, there is $y^0 \in \text{conv}V(Y)$ such that

$$g(\hat{y}) \geq g(y^0), \text{ for all } \hat{y} \in \text{conv}V(Y). \tag{15}$$

We claim that y^0 must be a global optimal solution for problem (LMP_Y). Indeed, for any $y \in Y$, it follows from (13) and (14) that

$$y = \bar{y} + v \geq \bar{y}, \tag{16}$$

where $\bar{y} \in \text{conv}V(Y)$ and $v \in \text{cone}R(Y)$. Furthermore, it is easily seen that the objective function $g(y) = \prod_{j=1}^{p} y_j$ of problem (LMP_Y) is increasing on int$\mathbb{R}^p_+$, i.e., if $y^1 \geq y^2 \gg 0$ implies that $g(y^1) \geq g(y^2)$. Combining (15), (16) and this fact gives

$$g(y) \geq g(\bar{y}) \geq g(y^0).$$

In other words, y^0 is a minimal optimal solution of problem (LPM_Y). The proof is complete. ■

The multiple objective linear programming problem (VP) associated with the linear multiplicative programming problem (LMP) may be written as

$$\text{MIN}\{Cx,\ x \in M\},$$

where C is the $p \times n$ matrix whose j^{th} row equals c^j, $j = 1, ..., p$.

The next proposition tells us the relationships between problem (VP) and problem (LMP). It is obtained from the definitions and the fact that the objective function $h(x) = \prod_{j=1}^{p} \langle c^j, x\rangle$ of problem (LMP) is a quasiconcave function [5].

Proposition 5.2. *Each global optimal solution of problem (LMP) must belong to the efficient extreme solution set $M_E \cap M_{ex}$ of the problem (VP).*

As a consequence of Proposition 5.2, one can solve problem (LMP) by evaluating the objective function $h(x) = \prod_{j=1}^{p} \langle c^j, x\rangle$ at each efficient extreme solution of problem (VP). More precise, we have the following procedure

Procedure SOLVE(LMP)
Step 1. Determine the set of all efficient extreme solution $M_E \cap M_{ex}$ for the multiple objective linear programming problem (Section 3).

Step 2. Determine the set

$$S^* = \{x^* \in M_E \cap M_{ex}\ :\ h(x^*) \leq h(x)\ \forall\, x \in M_E \cap M_{ex}\}$$

and terminate the procedure: each $x^* \in S^*$ is a global optimal solution to (LMP).

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