On hyperbolicity cones associated with elementary symmetric polynomials

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Abstract

Elementary symmetric polynomials can be thought of as derivative polynomials of $E_n(x) = \prod_{i=1,\ldots,n} x_i$. Their associated hyperbolicity cones give a natural sequence of relaxations for $\mathbb{R}^n_+$. We establish a recursive structure for these cones, namely, that the coordinate projections of these cones are themselves hyperbolicity cones associated with elementary symmetric polynomials. As a consequence of this recursion, we give an alternative characterization of these cones, and give an algebraic characterization for one particular dual cone associated with $E_{n-1}(x) = \sum_{1\le i\le n} \prod_{j\ne i} x_j$ together with its self-concordant barrier functional.

Keywords: hyperbolic polynomials; hyperbolicity cones; elementary symmetric polynomials; positive semi-definite representability.

1 Introduction

Let X ($\equiv \mathbb{R}^n$) be a finite dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$. Denote $\mathbf{1} \in \mathbb{R}^n$ – vector of all ones.

Hyperbolic polynomials and the associated hyperbolicity cones have origins in partial differential equations [12]. Recently, these structures have drawn considerable attention in the optimization community as well [6, 1, 14, 7]. It turns out that most of interior-point methods (IPM) theory [11, 13] applies naturally to the class of conic programs[1] (CP) arising from hyperbolicity cones. In particular, linear programming, second-order conic programming and positive semi-definite programming are instances of conic programs posed over corresponding hyperbolicity cones.

Recall for a cone $K \subseteq \mathbb{R}^n$, the *dual cone* is defined as $K^* = \{y \in \mathbb{R}^n : \forall x \in K, \langle x, y\rangle \ge 0\}$. Often, the dual cone provides much information about the original CP. Indeed, the most successful IPM algorithms are the so-called primal-dual algorithms, which follow central paths in K and K^* simultaneously. Hence, the understanding

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[1] A *conic program* is an optimization problem of the form $\{\inf_x \langle c, x\rangle : Ax = b, x \in K\}$ with $K \subset \mathbb{R}^n$ being a closed convex cone, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$. It is well known that any convex optimization problem can be recast as conic programming problem.

of the structure of both the primal cone and the dual cone for a given conic program usually plays a very important role in achieving greater computational efficiency in solving these optimization problems.

While a simple characterization for the hyperbolicity cones as a set of polynomial inequalities is known, little is known regarding the algebraic structure of their dual cones, with some exceptions. Same applies for computing self-concordant barriers for these cones. That the dual cones can be represented by systems of polynomial inequalities follows from Tarski's establishment of quantifier elimination methods [3]. These methods, however, give little insight into the algebraic structure of the dual cones, because the methods result in extremely complicated systems of polynomial inequalities, even for hyperbolic polynomials in 3 variables.

It has been long hypothesized that the hyperbolicity cones and the cone of positive semi-definite matrices have strong connections. In 1958, Peter Lax conjectured that each hyperbolic polynomial $p(x)$ in 3 variables satisfies $p(x) = \det(x_1 A + x_2 B + x_3 C)$, for some $A, B, C \in \mathbb{S}^d$ where $\mathbb{S}^d$ denotes the space of symmetric $d \times d$ real matrices, consequently each hyperbolicity cone in 3 variables can be realized as the intersection of the cone of $d \times d$ positive semi-definite matrices $\mathbb{S}^d_+$ with an affine subspace of $\mathbb{S}^d$. The conjecture was recently established affirmatively in [10] – as a corollary to work of [8]. It remains open whether similar representations hold for hyperbolicity cones in more than three variables and consequently whether CP's over hyperbolicity cones are any more general than positive semi-definite programming problems, although such representations have been established for important broad family of hyperbolicity cones, the so-called homogeneous cones [4, 5].

We attempt to provide more insight into the structure of hyperbolicity cones associated with elementary symmetric polynomials, which is an important family of hyperbolicity cones. In particular, we show that these cones have a recursive structure similar to that of $\mathbb{R}^n_+$: the coordinate projections of these cones are themselves hyperbolicity cones associated with elementary symmetric polynomials. As a consequence of this observation, we give an alternative characterization of these cones, and provide a simple algebraic characterization of the dual cone associated with $E_{n-1}(x) = \sum_{1 \le i \le n} \prod_{j \ne i} x_j$ demonstrating how one can construct a logarithmic self-concordant barrier functional for this cone. To get this dual cone characterization we rely on the cone representation as an intersection of an affine subspace and a cone of positive semi-definite matrices. Section 2 contains definitions and some preliminary results on cone characterization, in Section 3 we uncover the recursive structure of the studied cones, and in Section 4 we describe one particular dual cone.

2 Hyperbolicity cones and cone characterization

2.1 Preliminaries

Definition 2.1. *Let $p : X \to \mathbb{R}$ be a homogeneous polynomial of degree $m \in \mathbb{N}$, i.e., $p(\alpha x) = \alpha^m p(x)$ for all $\alpha \in \mathbb{R}$ and every $x \in X$, and $d \in X$ is such that $p(d) \neq 0$. Then p is* hyperbolic with respect to d *if the univariate polynomial $t \mapsto p(x + td)$ has all roots real for every $x \in X$.*

Examples:

- $X = \mathbb{R}^n$, $d = \mathbf{1}$. The n^{th} elementary symmetric polynomial, $E_n(x) = \prod_{i=1}^n x_i$, is a hyperbolic polynomial with respect to d, for $E_n(x + t\mathbf{1}) = \prod_{i=1}^n (x_i + t)$,
- $X = \mathbb{S}^k$, the space of real symmetric $k \times k$ matrices, $d = I$, the identity matrix. The determinant, $\det(x)$, is a hyperbolic polynomial in direction d, for the eigenvalues of $x \in \mathbb{S}^k$ are the roots of $\det(x + tI)$ and are real.

The minus roots of $t \mapsto p(x + td)$are called the *eigenvalues* of x (in direction d), terminology motivated by the last example. We denote the eigenvalues by

$$\lambda_1(x) \leq \lambda_2(x) \leq \cdots \lambda_m(x)$$

or simply $\lambda(x) \in \mathbb{R}^m$.

Definition 2.2. *The* hyperbolicity cone of p *with respect to* d*, written* $C(p, d)$*, is the set* $\{x \in X : p(x + td) \neq 0, \forall t \geq 0\}$.

Note that $C(p, d) = \{x \in X : \lambda_1(x) > 0\}$.

Examples:

- $X = \mathbb{R}^n$, $d = \mathbf{1}$, $p(x) = E_n(x)$, then $C(p, d) = \mathbb{R}^n_{++}$, the positive orthant,
- $X = \mathbb{S}^k$, $d = I$, $p(x) = \det(x)$, then $C(p, d) = \mathbb{S}^k_{++}$, the cone of positive definite matrices.

Fact 2.3. [12] Given a pair p, d

(i) $d \in C(p, d)$,

(ii) $C(p, d)$ is an open convex cone,

(iii) the closure of $C(p, d)$, cl $C(p, d) = \{x \in X : \lambda_1(x) \geq 0\}$.

2.2 Derivative polynomials and primal cone characterization

Given a hyperbolic polynomial p of degree m in direction d, denote

$$p'(d; x) = \left. \frac{\partial p(x + td)}{\partial t} \right|_{t=0}$$

We will refer to $p'(d; x)$ as the "derivative polynomial of p with respect to d" and usually will write $p'(x)$ instead of $p'(d; x)$ omitting the direction d when the choice of d is clear. By the root interlacing property for the polynomials with all real roots, i.e., since between any two roots of $t \mapsto p(x + td)$ there is a root of $t \mapsto \frac{\partial}{\partial t} p(x + td)$, it follows that $p'(x)$ is also hyperbolic in direction d.

Similarly, for a fixed hyperbolicity direction d we can define higher derivatives $p'', p''', \ldots, p^{(m)}$. Note that since p was assumed to be of degree m, $p^{(m-1)}(x)$ is linear and $p^{(m)}(x)$ is constant.

Example: $X = \mathbb{R}^n$, $d \in \mathbb{R}^n_{++}$, $p(x) = E_n(x)$. Then by easy computation one can show that

$$E_n^{(k)}(x) = (k!)E_n(d)E_{n-k}\left(\frac{x_1}{d_1}, \frac{x_2}{d_2}, \dots, \frac{x_n}{d_n}\right)$$

where $E_k(x)$ is the k^{th} elementary symmetric polynomial (recall that $E_0(x) = 1$ and $E_k(x) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \prod_{j=1}^{k} x_{i_j}$ for $k > 0$).

Remark 2.4. *It should be noted that the elementary symmetric polynomials in the example above also play an important role in representing the derivative polynomials via the eigenvalues $\lambda(x)$ at a point $x \in X$ [14]. As a consequence of homogeneity it follows that*

$$p^{(k)}(x) = k!\, p(d)E_{m-k}(\lambda(x)).$$

Denote the closure of hyperbolicity cone

$$K_{p,d} := \text{cl}\, C(p, d).$$

When the choice of d is clear we will simply write K_p. We present a well-known result giving one particular characterization of K_p.

Fact 2.5. [14] Suppose p is a hyperbolic polynomial of degree m with respect to d, $p(d) > 0$, and $p', p'', \dots, p^{(m-1)}$ are defined as above. Then

$$K_p = \{x \in \mathbb{R}^n : p(x) \ge 0, p'(x) \ge 0, p''(x) \ge 0, \dots, p^{(m-1)}(x) \ge 0\}$$

As a corollary, we have the following cone inclusion

$$K_p \subseteq K_{p'} \subseteq \cdots \subseteq K_{p^{(m-1)}}.$$

In particular, for $X = \mathbb{R}^n$, $d \in \mathbb{R}^n_{++}$, we have a natural sequence of relaxations of the nonnegative orthant $\mathbb{R}^n_+ = K_{E_n} \subseteq K_{E_n^{(1)}} \subseteq \cdots \subseteq K_{E_n^{(n-1)}}$.

Corollary 2.6. *Given a pair p, d with $p(d) > 0$, the boundary of K_p satisfies*

$$\partial K_p = \{x \in \mathbb{R}^n : p(x) = 0, p'(x) \ge 0, \dots, p^{(m-1)}(x) \ge 0\}$$

The proof follows easily from the root interlacing property for polynomials with all real roots.

Proposition 2.7. *Assume $1 \le r \le m - 2$. If $x \in K_{p^{(r)}}$ and $p^{(r+1)}(x) = 0$, then $x \in K_p$.*

Proof. By the root interlacing property for polynomials with all real roots it follows that $t = 0$ is a multiple root of $t \mapsto p^{(r)}(x + td)$ of multiplicity $l \ge 2$. Therefore, 0 is a root of multiplicity $(l + 1)$ for $t \mapsto p^{(r-1)}(x + td)$, and so on, until we get to p itself. Since 0 is the right-most root for $t \mapsto p^{(r)}(x + td)$ because $x \in K_{p^{(r)}}$, it is also the right-most root $t \mapsto p(x + td)$ by root counting. So $x \in \partial K_p \subset K_p$. □

2.3 Semi-definite representability and the dual cones

Definition 2.8. [2] A convex set $Y \subseteq \mathbb{R}^n$ is said to be *positive semi-definite representable (SDR)* if

$$x \in Y \Leftrightarrow \mathcal{A}\begin{pmatrix} x \\ u \end{pmatrix} + B \succeq 0, \text{ i.e., is positive semi-definite, for some } u \in \mathbb{R}^m,$$

where $B \in \mathbb{S}^k$ and $\mathcal{A} : \mathbb{R}^{n+m} \to \mathbb{S}^k$ can be written as

$$\mathcal{A}\begin{pmatrix} x \\ u \end{pmatrix} = \sum_{i=1}^{n} x_i A_i + \sum_{j=1}^{m} u_j B_j$$

with $A_i, B_j \in \mathbb{S}^k$.

Fact 2.9. [2] If X is SDR then so is an affine image of X.

This can be easily shown by switching to an appropriate basis in $\mathbb{S}^k$.

It turns out that under some mild assumptions this representation also explains the structure of the corresponding dual cone. We give an SDR analogue of a second-order cone representability theorem in [2].

Proposition 2.10. *If $K \subset \mathbb{R}^n$ is a closed convex cone with nonempty interior and*

$$K = \left\{ x \in \mathbb{R}^n : \exists u \quad \textit{such that} \quad \mathcal{A}\begin{pmatrix} x \\ u \end{pmatrix} + B \succeq 0 \right\}$$

with a strictly-feasible point, then its dual satisfies

$$K^* = \left\{ y \in \mathbb{R}^n : \exists \Lambda \quad \textit{such that} \quad \begin{pmatrix} y \\ 0 \end{pmatrix} = \mathcal{A}^*\Lambda, \langle B, \Lambda \rangle \leq 0, \Lambda \succeq 0 \right\}$$

where $\mathcal{A}^* : \mathbb{S}^k \to \mathbb{R}^{n+m}$ is the adjoint of $\mathcal{A}$, defined as

$$\mathcal{A}^*\Lambda = (\langle A_1, \Lambda \rangle, \langle A_2, \Lambda \rangle, \cdots, \langle A_n, \Lambda \rangle, \langle B_1, \Lambda \rangle, \cdots, \langle B_m, \Lambda \rangle)$$

and $\langle A_1, \Lambda \rangle = \text{trace}(A_1 \Lambda)$ is the trace inner product of two matrices.

Proof. Considering the primal-dual pair

$$\inf_{x,u} \left\{ \begin{pmatrix} y \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ u \end{pmatrix} : \mathcal{A}\begin{pmatrix} x \\ u \end{pmatrix} + B \succeq 0 \right\}$$

and

$$\sup_{\Lambda \succeq 0} \{ \langle -B, \Lambda \rangle : \langle A_i, \Lambda \rangle = y_i, i = 1, \ldots, n, \ \langle B_j, \Lambda \rangle = 0, j = 1, \ldots, m \},$$

by the Conic Duality Theorem [2, Theorem 1.7.1. (online version)] we conclude that $y \in K^*$ iff the first problem is bounded below by 0, and hence iff the second has a feasible solution with the value of at least 0. □

3 The recursive structure of the hyperbolicity cones for elementary symmetric polynomials

Throughout this and the next section fix the underlying vector space to be $\mathbb{R}^n$, the hyperbolicity direction $d = \mathbf{1}$.

We will see that the cone K_{E_k} has a recursive structure similar to that of $\mathbb{R}^n_+ = K_{E_n}$: by dropping some of the coordinates of $x \in K_{E_k}$, we obtain a vector in "almost the same" cone with respect to the degree of the underlying polynomial in a lower dimensional space; compare it with the fact that a face of a simplex is a simplex. In turn, this will give us an alternative characterization for K_{E_k}.

For a vector $x \in \mathbb{R}^n$ and an arbitrary index $1 \leq i \leq n$, denote

$$x_{-i} := (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}.$$

For a fixed $2 \leq k \leq n-1$, denote

$$p(t) : t \mapsto E_k(x + t\mathbf{1}),$$

and for $1 \leq i \leq n$ denote

$$p_{-i}(t) : t \mapsto E_k(x_{-i} + t\mathbf{1}_{-i})$$

and

$$p'_{-i}(t) : t \mapsto (n-k)E_{k-1}(x_{-i} + t\mathbf{1}_{-i}).$$

Observe the recursive expression $E_k(x) = x_i E_{k-1}(x_{-i}) + E_k(x_{-i})$ for any $n > k \geq 2$ and an arbitrary index i, where $E_k(\cdot_{-i}) : \mathbb{R}^{n-1} \to \mathbb{R}$ is the k^{th} elementary symmetric polynomial on $\mathbb{R}^{n-1}$.

Theorem 3.1 (Necessary condition for $x \in K_{E_k}$). *Assume $2 \leq k \leq n$. Then $x \in K_{E_k(\cdot)}$ only if $x_{-i} \in K_{E_{k-1}(\cdot_{-i})}$ for any i.*

Proof. If $k = n$ the result is obvious, so assume $k < n$. Fix i. Recall that $x \in K_{E_k(\cdot)}$ iff $p(t) : t \mapsto E_k(x + t\mathbf{1})$ has only non-positive roots. $E_k(\cdot_{-i})$ and $E_{k-1}(\cdot_{-i})$ are both hyperbolic along $\mathbf{1}_{-i} \in \mathbb{R}^{n-1}$, and

$$\lim_{t\uparrow\infty} \tfrac{p'_{-i}(t)}{t^{k-1}} \geq 0 \quad \text{and} \quad \lim_{t\uparrow\infty} \tfrac{p_{-i}(t)}{t^k} \geq 0,$$

for all $x \in \mathbb{R}^n$, since as $t \uparrow \infty$, $E_{k-1}(x_{-i} + t\mathbf{1}_{-i})$ and $E_k(x_{-i} + t\mathbf{1}_{-i})$ will eventually be ≥ 0. Note that we can write

$$p(t) = E_k(x + t\mathbf{1}) = \frac{(x_i + t)}{n-k} p'_{-i}(t) + p_{-i}(t).$$

Suppose $x_{-i} \notin K_{E_{k-1}(\cdot_{-i})}$, so there must be at least one positive root of $p'_{-i}(t)$. We know that roots of $p_{-i}(t)$ and $p'_{-i}(t)$ are interlaced: enumerating all roots, including multiplicities, of $p_{-i}(t)$ as $\{t_i : i = 1, \ldots, k\}$ and roots of $p'_{-i}(t)$ as $\{t'_i : i = 1, \ldots, k-1\}$ in non-decreasing order, we must have $t_1 \leq t'_1 \leq t_2 \leq t'_2 \leq \cdots \leq$

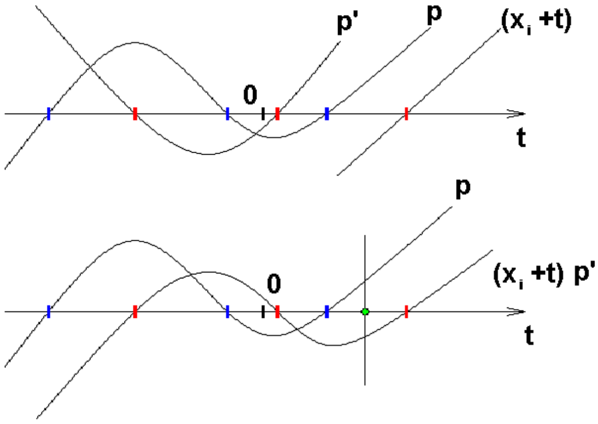

Figure 1: Necessary condition for $x \in K_{E_k}$, root interlacing case 1

$t_{k-1} \le t'_{k-1} \le t_k$, $0 < t'_{k-1} \le t_k$. Combined with the observation about signs of $p_{-i}(t)$ and $p'_{-i}(t)$ as $t \uparrow \infty$ we get that

$$\begin{aligned} p'_{-i}(t) \ge 0 \quad &\text{for} \quad t \ge t'_{k-1}, \\ p_{-i}(t'_{k-1}) \le 0,\ p_{-i}(t) \ge 0 \quad &\text{for} \quad t \ge t_k. \end{aligned}$$

We consider three cases depending on the value x_i.

Case 1. Suppose that $-x_i \le t'_{k-1}$. Then

$$\begin{aligned} p(t'_{k-1}) &= \tfrac{(x_i+t'_{k-1})}{n-k} p'_{-i}(t'_{k-1}) + p_{-i}(t'_{k-1}) \le 0, \\ p(t_k) &= \tfrac{(x_i+t_k)}{n-k} p'_{-i}(t_k) + p_{-i}(t_k) \ge 0, \end{aligned}$$

so by continuity, $p(t)$ must have a root between t'_{k-1} and t_k. Since $0 \le t'_{k-1}$, this root must be positive, hence $x \notin K_{E_k(\cdot)}$ (see Figure 1).

Case 2. Suppose $t'_{k-1} < -x_i \le t_k$. Then we can write

$$\begin{aligned} p(-x_i) &= \tfrac{(x_i+(-x_i))}{n-k} p'_{-i}(-x_i) + p_{-i}(-x_i) \le 0, \\ p(t_k) &= \tfrac{(x_i+t_k)}{n-k} p'_{-i}(t_k) + p_{-i}(t_k) \ge 0, \end{aligned}$$

and, again, by continuity, $p(t)$ must have a positive root, so $x \notin K_{E_k(\cdot)}$.

Case 3. Finally, suppose that $t_k < -x_i$. Then

$$\begin{aligned} p(t_k) &= \tfrac{(x_i+t_k)}{n-k} p'_{-i}(t_k) + p_{-i}(t_k) \le 0, \\ p(-x_i) &= \tfrac{(x_i+(-x_i))}{n-k} p'_{-i}(-x_i) + p_{-i}(-x_i) \ge 0, \end{aligned}$$

so by continuity, $p(t)$ must have a positive root and, therefore, $x \notin K_{E_k(\cdot)}$. □

Corollary 3.2. *Assume* $2 \le k \le n-1$ *and* $x \in K_{E_k(\cdot)}$. *If* $x_i \le 0$ *then* $x_{-i} \in K_{E_k(\cdot_{-i})}$. *Moreover, if* $x \in \partial K_{E_k(\cdot)}$, $x \notin \mathbb{R}^n_+$, *and* $x_i > 0$, *then* $x_{-i} \notin K_{E_k(\cdot_{-i})}$.

Proof. We write $p(t) = E_k(x + t\mathbf{1}) = \frac{(x_i+t)}{n-k} p'_{-i}(t) + p_{-i}(t)$. At $t = 0$ we have $E_k(x) = p(0) = \frac{x_i}{n-k} p'_{-i}(0) + p_{-i}(0) = x_i E_{k-1}(x_{-i}) + E_k(x_{-i})$. Since $x_{-i} \in K_{E_{k-1}(\cdot_{-i})}$ by Theorem 3.1, we have $p'_{-i}(0) = (n-k)E_k(x_{-i}) \ge 0$, and since $x \in K_{E_k(\cdot)}$ by Fact 2.5 we have $p(0) = E_k(x) \ge 0$. We rearrange terms: $p'_{-i}(0)\frac{x_i}{n-k} = E_k(x) - p_{-i}(0)$.

If $x_i \le 0$ we have $E_k(x) - p_{-i}(0) \le 0$, so $p_{-i}(0) = E_k(x_{-i}) \ge 0$ and combined with $x_{-i} \in K_{E_{k-1}(\cdot_{-i})}$, this gives us $x_{-i} \in K_{E_k(\cdot_{-i})}$.

Now let $x \in \partial K_{E_k(\cdot)}$, so that $E_k(x) = 0$, and $x_i > 0$. We have two possibilities here. If $p'_{-i}(0) > 0$, then $-p_{-i}(0) > 0$ and hence $x_{-i} \notin K_{E_k(\cdot_{-i})}$. Alternatively, if $p'_{-i}(0) = 0$, that is $x \in \partial K_{E_{k-1}(\cdot_{-i})}$, then $p_{-i}(0) = 0$ and $x_{-i} \in \partial K_{E_k(\cdot_{-i})}$, so by Proposition 2.7, $x \in K_{E_{n-1}(\cdot_{-i})} = \mathbb{R}^{n-1}_+$. But $x \notin \mathbb{R}^n_+$, we have a contradiction. □

Now we are in position to provide an alternative to Fact 2.5 characterization of the hyperbolicity cones associated with elementary symmetric polynomials. Instead of considering $x \in \mathbb{R}^n$ we confine ourselves to the cone $\mathbb{R}^n_\downarrow := \{x \in \mathbb{R}^n : x_n \le x_{n-1} \le x_{n-2} \le \cdots \le x_1\}$.

Theorem 3.3 (K_{E_k} characterization). *Assume* $2 \le k \le n$ *and* $x \in \mathbb{R}^n_\downarrow$. *Then* $x \in K_{E_k(\cdot)}$ *iff* $x_{-n} \in K_{E_{k-1}(\cdot_{-n})}$ *and* $E_k(x) \ge 0$.

Proof. The conditions are necessary by the previous theorem and Fact 2.5. We need to show sufficiency. The case $k = n$ is trivial, so assume $k < n$. If $x_n \ge 0$, then obviously $x \in \mathbb{R}^n_+ \subseteq K_{E_k(\cdot)}$, so assume $x_n < 0$.

Let the roots of $p_{-n}(t)$ including multiplicities in non-decreasing order be $\{t_i : i = 1, \ldots, k\}$, and the roots of $p'_{-n}(t)$ be $\{t'_i : i = 1, \ldots, (k-1)\}$. Write $p(t) = E_k(x + t\mathbf{1}) = \frac{(x_n+t)}{n-k} p'_{-n}(t) + p_{-n}(t)$ and observe

$$p(t_k) = \frac{(x_n + t_k)}{n-k} p'_{-n}(t_k) \le 0$$

since by the root interlacing $t_k \le -x_n$, and $p'_{-n}(t_k) \ge 0$ recalling that $p'_{-n}(t) \uparrow \infty$ as $t \uparrow \infty$. $p(0) = E_k(x) \ge 0$ by the assumption. Thus the interval $[t_k, 0]$ must contain at least one root of $p(t)$.

Counting the remaining roots of $p(t)$ for $t \le t_k$ by looking at sign patterns at the endpoints of intervals $[t_i, t'_i], i = 1, \ldots, k-1$, we conclude that $[t_k, 0]$ must contain only one rightmost root of $p(t)$, so there could be no other roots to the right of 0, and hence $x \in K_{E_k(\cdot)}$. □

Corollary 3.4. *Assume* $2 \le k \le n-1$ *and* $x \in \mathbb{R}^n_\downarrow$. *Then* $x \in K_{E_k(\cdot)}$ *iff* $x_{-n} \in K_{E_k(\cdot_{-n})}$ *and* $E_k(x) \ge 0$.

4 First derivative cone for $\mathbb{R}^n_+$ and its dual

Following the alternative characterization of the hyperbolicity cones associated with E_k to gain insight into the dual cones $K^*_{E_k,\mathbf{1}}$, we create a suitable decomposition of the

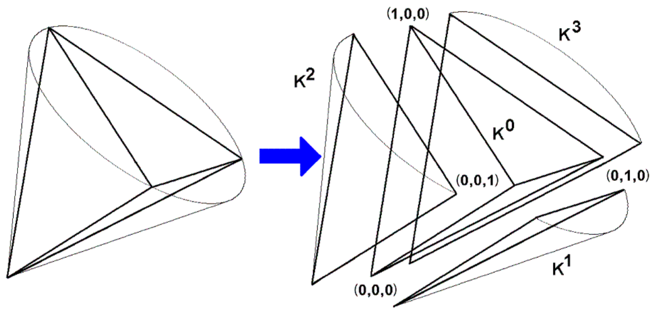

Figure 2: $K_{E_{n-1}} \bigcap \{x \in \mathbb{R}^n : \mathbf{1}^T x = n\}$ schematic decomposition in $\mathbb{R}^3$

cone $K_{E_{n-1},\mathbf{1}}$ into smaller convex cones, in the sense that each of the smaller cones admits a positive semi-definite representation. Relying on the conic duality theory, we then obtain the dual cone for each of the smaller cones as an SDR set in itself, and finally, we reconstruct $K^*_{E_{n-1},\mathbf{1}}$ as the intersection.

Namely, from Corollary 3.4, for $E_n(x)$ and its derivative (with respect to $d = \mathbf{1}$) $E_{n-1}(x)$ we claim that $x \in K_{E_{n-1}}$ iff $E_{n-1}(x) \geq 0$ and at most one $x_i < 0$ with the rest $x_j \geq 0$ for $i \neq j$. We are going to construct a representation of the dual cone to $K_{E_{n-1}}$ using this characterization. We will need the following simple statement that follows immediately from the definition of the dual cone.

Lemma 4.1. *If $K \subseteq \mathbb{R}^n$ is a cone admitting a decomposition into cones $\{K^i\}_{i \in I}$, $K = \bigcup_{i \in I} K^i$, then its dual cone satisfies $K^* = \bigcap_{i \in I} {K^i}^*$.*

Proposition 4.2. *The dual cone $K^*_{E_{n-1}}$ satisfies*

$$K^*_{E_{n-1}} = \bigcap_{i=1}^{n} {K^i}^*$$

where

$$\begin{aligned} {K^i}^* = \{y \in \mathbb{R}^n : \quad & -y_i = \textstyle\sum_{k \neq i} (\Lambda_{k,i} + \Lambda_{i,k}) + \Lambda_{i,i}, \\ & y_j = \Lambda_{j,j} \quad \textit{for} \quad j \neq i, \quad \Lambda \succeq 0\}. \end{aligned}$$

Proof. We form a disjoint-interior partitioning for $K_{E_{n-1}}$ in the following manner: $K_{E_{n-1}} = \bigcup_{i=0}^{n} K^i$ where $K^0 = \mathbb{R}^n_+$ and $K^i = \{x \in \mathbb{R}^n : x_i \leq 0, x_j \geq 0, j \neq i, E_{n-1}(x) \geq 0\}$ for $i \geq 1$, claiming that each of the K^i is SDR with a strictly-feasible solution, see Figure 2. Based on Proposition 2.10 and Lemma 4.1 it is now easy to reconstruct the dual cone. It is left to demonstrate how to represent each K^i via linear matrix inequality. $K^0 = \mathbb{R}^n_+$ is trivial, consider $W_0(x) = \mathrm{Diag}(x) \succeq 0$. Next we show how to do this for K^1.

Consider

$$W_1(x) := \text{Diag}(x) - x_1(\mathbf{1}\cdot(1,0,\dots,0)^T + (1,0,\dots,0)\cdot\mathbf{1}^T) \succeq 0$$

Recall that for a real symmetric matrix to be positive definite it is necessary and sufficient that all its principal minors have positive determinants [9]. Proceed by evaluating the determinants of $W_1(x)$ from the bottom-right corner to get $x_j, j = 2,\dots,n$, $-x_1E_{n-1}(x) \geq 0$, and from the top-left corner $x_1 \leq 0$, thus implying $E_{n-1}(x) \geq 0$. To see that $\det W_1(x) = -x_1E_{n-1}(x)$, evaluate the determinant using algebraic complements of the first row. Therefore, the interior of the cone K^1 coincides with $\{x \in \mathbb{R}^n : W_1 \succ 0\}$, thus the closures of the cones coincide as well. Clearly, the strict feasibility for this linear matrix inequality is insured, e.g., take $x_2 = x_3 = \dots = x_n = 1, x_1 < 0$ with $|x_1|$ small enough. So Proposition 2.10 can be applied to get the dual cone to K^1 as a SDR set:

$$\begin{aligned}\{x \in \mathbb{R}^n : W_1(x) \succeq 0\}^* = \{y \in \mathbb{R}^n : \quad & -y_1 = \textstyle\sum_{i=2}^n(\Lambda_{i,1} + \Lambda_{1,i}) + \Lambda_{1,1},\\ & y_j = \Lambda_{j,j} \text{ for } j \geq 1, \quad \Lambda \succeq 0\}.\end{aligned}$$

Finally, to get the representation of the dual cone to $K_{E_{n-1}}$ take the intersection of the dual cones corresponding to its components, noting that nonnegativity of y, that is, $y \in K^{0^*}$, is readily implied by $y \in \bigcap_{i=1}^n K^{i^*}$. □

For an illustration, consider the characterization of $K^*_{E_2}$ in $\mathbb{R}^3$, which is, perhaps, not the most exciting example (it is just a quadratic cone after all) but is quite an illustrative one since it is easy to appeal to its geometric interpretation.

Example: In $\mathbb{R}^3$, for $E_2(x) = x_1x_2 + x_1x_3 + x_2x_3$, we have

$$W_1(x) : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ -1 & & \\ -1 & & \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}}_{A_3} x_3$$

and thus we get the following representation of $K^{1^*} = \{x \in \mathbb{R}^3 : W_1(x) \succeq 0\}^*$:

$$\begin{gathered} -y_1 = \langle A_1, \Lambda\rangle = \Lambda_{3,1} + \Lambda_{2,1} + \Lambda_{1,1} + \Lambda_{1,2} + \Lambda_{1,3} \\ y_2 = \langle A_2, \Lambda\rangle = \Lambda_{2,2}, \quad y_3 = \langle A_3, \Lambda\rangle = \Lambda_{3,3}, \\ \Lambda \succeq 0. \end{gathered}$$

We can derive similar characterizations for $K^{2^*} = \{x \in \mathbb{R}^3 : W_2 \succeq 0\}^*$ and $K^{3^*} = \{x \in \mathbb{R}^3 : W_3 \succeq 0\}^*$ and reconstruct the dual cone to K_{E_2} as a collection of three sets of linear matrix inequalities, each corresponding to K^{i^*}, $i = 1, 2, 3$, with the same $y \in \mathbb{R}^3$ but distinct matrices $\Lambda \in \mathbb{S}^3_+$.

An interesting question that remains unanswered is this: how would one get the representation of the original cone K_{E_2} in terms of linear matrix inequalities? To do this we take the dual of $K^*_{E_2}$. Firstly, let us switch from the image of a positive semidefinite cone to its affine slice in each of the $\{x \in \mathbb{R}^n : W_i(x) \succeq 0\}^*$, $i = 1, 2, 3$.

Starting with $\{x \in \mathbb{R}^n : W_1(x) \succeq 0\}^*$, fixing a basis in $\mathbb{S}^3$ to be

$$\{B_i\}_{i=1}^6 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

substituting $\Lambda = \sum_{j=1}^6 B_j \lambda_j$, we can write $\langle A_i, \Lambda \rangle = y_i$ as $\sum_{j=1}^6 \langle B_j, A_i \rangle \lambda_j = y_i$ to get

$$y \in K^{1^*} \quad \Leftrightarrow \quad \begin{bmatrix} -y_1 - 2(\lambda_2 + \lambda_3) & \lambda_2 & \lambda_3 \\ \lambda_2 & y_2 & \lambda_5 \\ \lambda_3 & \lambda_5 & y_3 \end{bmatrix} \succeq 0 \quad \text{for some } \lambda_2, \lambda_3, \lambda_5,$$

and similarly for $\{x \in \mathbb{R}^3 : W_2(x) \succeq 0\}^*$ and $\{x \in \mathbb{R}^3 : W_3(x) \succeq 0\}^*$.

Now we can apply the same procedure as before to take the dual of the dual cone to get the primal cone itself. Note again that the resulting linear matrix inequality for $K_{E_2}^*$ is strictly feasible, for example, take $y = \mathbf{1}$, $-1/2 < \lambda_j < -1/3$ in all three sets of contributing matrix inequalities. Writing the constraints corresponding to $K_{E_{n-1}}^*$ in the form $\sum_{i=1}^3 y_i \widetilde{A}_i + \sum_{j=1}^3 (\lambda_{j2} \widetilde{B}_{j2} + \lambda_{j3} \widetilde{B}_{j3} + \lambda_{j5} \widetilde{B}_{j5}) \succeq 0$, we obtain

$$\left[\begin{array}{ccc|ccc|ccc} \mu_{1,11} & \mu_{1,11} & \mu_{1,11} & & & & & & \\ \mu_{1,11} & \mu_{1,22} & & & & & & & \\ \mu_{1,11} & & \mu_{1,33} & & & & & & \\ \hline & & & \mu_{2,11} & \mu_{2,11} & \mu_{2,11} & & & \\ & & & \mu_{2,11} & \mu_{2,22} & & & & \\ & & & \mu_{2,11} & & \mu_{2,33} & & & \\ \hline & & & & & & \mu_{3,11} & \mu_{3,11} & \mu_{3,11} \\ & & & & & & \mu_{3,11} & \mu_{3,22} & \\ & & & & & & \mu_{3,11} & & \mu_{3,33} \end{array}\right] \succeq 0$$

$$x_1 = \mu_{1,33} + \mu_{2,33} - \mu_{3,11}$$
$$x_2 = \mu_{1,22} - \mu_{2,11} + \mu_{3,22}$$
$$x_3 = -\mu_{1,11} + \mu_{2,22} + \mu_{3,33}$$

for $x \in K_{E_2}$, where without loss of generality the off-diagonal blocks may be assumed zeros relying on characterization of positive semi-definite matrices using minors [9]. The last constraint is decomposable into three sub-matrices being positive semi-definite and a set of affine constraints, that together give us the primal variables x_1, x_2, x_3. There is a simple interpretation for this set of constraints. Observe that each of the blocks, $i = 1, 2, 3$,

$$\begin{bmatrix} \mu_{i,11} & \mu_{i,11} & \mu_{i,11} \\ \mu_{i,11} & \mu_{i,22} & \\ \mu_{i,11} & & \mu_{i,33} \end{bmatrix} \succeq 0$$

corresponds precisely to $K^i = \{x \in \mathbb{R}^n : x_i \leq 0, x_j \geq 0, j \neq i, E_{n-1}(x) \geq 0\} = \{x \in \mathbb{R}^n : W_i(x) \succeq 0\}$ but with x's now renamed into $\pm\mu$'s. The remaining affine constraints

$$
\begin{aligned}
x_1 &= \mu_{1,33} + \mu_{2,33} - \mu_{3,11} \\
x_2 &= \mu_{1,22} - \mu_{2,11} + \mu_{3,22} \\
x_3 &= -\mu_{1,11} + \mu_{2,22} + \mu_{3,33}
\end{aligned}
$$

are building a convex combination of the cones K^i.

The last observation is not specific to $\mathbb{R}^3$, that is, any point in $K_{E_{n-1}} \subset \mathbb{R}^n$ can be obtained as a convex combination of the points in $K^i = \{x \in \mathbb{R}^n : x_i \leq 0, x_j \geq 0, j \neq i, E_{n-1}(x) \geq 0\}$, $i = 1, \ldots, n$, and thus we can easily get a positive semi-definite representation of $K_{E_{n-1}}$.

Remark 4.3 (On constructing a self-concordant barrier functional for $K^*_{E_{n-1}}$). *It should be noted that since the dual cone $K^*_{E_{n-1}}$ was constructed as an affine section of $\mathbb{S}^{n^2}_+$, our approach readily provides us with a way to extract a self-concordant barrier functional for this cone.*

5 Conclusion

A new characterization for hyperbolicity cones associated with elementary symmetric polynomials $E_k(x)$, derived from the unveiled recursive structure of these cones, is introduced. This characterization has a potential advantage of each of the $k-1$ polynomials required to describe the cone involving one less variable than the preceding polynomial, compared to a well known polynomial characterization of the hyperbolicity cone where every polynomial has exactly n variables, where n is the number of variables in the original hyperbolic polynomial and k is its degree.

The paper illustrates the idea of finding a positive semi-definite representation of the cone and its dual by first considering the positive semi-definite representable partitioning of the cone. Using similar approach one may derive a partial (non-trivial) description of the cone corresponding to $E_{n-2}(x)$, namely, the following system of linear matrix inequalities

$$
\begin{bmatrix} Z & x_5 B^{1/2} \\ x_5 B^{1/2} & W_4(x_{-5}) \end{bmatrix} \succeq 0,
$$
$$
\text{trace} Z + x_5 \leq 0
$$

where $B := \frac{1}{2}(\text{Diag}(0,1,\ldots,1) + (0,1,\ldots,1)\cdot(0,1,\ldots,1)^T)$, represents $x \in K_{E_{n-2}}$ with $x_5 \leq x_4 \leq 0 \leq x_3 \leq x_2 \leq x_1$, although a complete description of this cone (and all the remaining cones corresponding to $E_k(x)$, $k < n-2$) together with its dual cone is yet to be found.

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