On the solution of fuzzy bilevel programming problems

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Abstract

In this paper we formulate the fuzzy bilevel programming problem and describe one possible approach for formulating a crisp optimization problem being attached to it. Due to the nature of fuzzy bilevel programming this is a crisp bilevel programming problem. We compare our approach with one using multicriterial optimization and show, that both approaches are basically different. The use of multicriterial optimization rests on reformulating the original problem into a multicriterial fuzzy optimization problem and thus in deleting the hierarchical structure of the problem.

Keywords: Bilevel Programming; Fuzzy Programming; Membership function; Fuzzy goal programming

1 Introduction

Bilevel programming problems are hierarchical problems of two decision makers in which one, the so-called leader, has the first choice and the other one, the so-called follower reacts optimally on the leader's selection. More formally, let the leader select a solution $x_1 \in X_1 \subseteq \mathbb{R}^{n_1}$. Then, the follower's task is to solve the problem

$$\min_{x_2}\{Z_2(x_1, x_2) : x_2 \in X_2(x_1)\}. \tag{1}$$

Here, $X_2(x_1) \subseteq \mathbb{R}^{n_2}$ for all $x_1 \in X_1 \subseteq \mathbb{R}^{n_1}$, $Z_i : X_1 \times X_2 \to \mathbb{R}, i = 1, 2$. Let $\Psi(x_1)$ denote the set of optimal solutions of the problem (1) for fixed $x_1 \in X_1$. Then, the leader's aim is to minimize his/her objective function $Z_1(x_1, x_2)$ subject to $x_2 \in \Psi(x_1)$ and $x_1 \in X_1$.

Assume that the set $\Psi(x_1)$ consists of no more than one element for each $x_1 \in X_1$. Then, the bilevel programming problem can be formulated as

$$\begin{aligned} Z_1(x_1, x_2) \quad &\to \quad \min_{x_1 \in X_1} \\ \text{subject to} \quad & Z_2(x_1, x_2) \to \min_{x_2} \\ & \text{subject to } x_2 \in X_2(x_1). \end{aligned} \tag{2}$$

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If the assumption that the set of optimal solutions of problem (1) reduces to a singleton is not satisfied, then e.g. the optimistic (or weak, or cooperative) resp. the pessimistic (or strong, or non-cooperative) approaches can be used [4]. Bilevel programming problems have been investigated in the monographs [1, 4] and the edited volumes [6, 14]. The annotated bibliography [5] gives an overview of the large number of references on this topic. One of the surprising facts in bilevel programming is that, if a constraint is moved from the upper to the lower level problem, optimality of a feasible solution can be lost [12, 13]. The following point is of main importance in all models of bi- and multilevel programming and also in Stackelberg games: all the decision makers at all levels of hierarchy follow only their own aims and do not consider the objective functions of the decision makers at other levels of hierarchy. The interaction between the decision makers is only by submitting selected solutions which are then used as parameters in lower respectively as responses on the parameters in higher levels of hierarchy. Hence, in bilevel programming, the lower level decision maker uses the selection x_1 as a parameter and the upper level decision maker has to accept the response $x_2 = x_2(x_1) \in \Psi(x_1)$ of the follower as fixed.

In this paper we investigate the bilevel programming problem where both the upper and the lower level decision makers solve fuzzy (parametric) optimization problems. We will use an approach by Zimmermann [23] to transform the problem into a crisp bilevel programming problem in Section 2. Then we propose an idea to solve such problems. In Section 3 we repeat the multicriterial programming approach used e.g. in [11, 15, 19, 21] and show that this approach is basically different from that in Section 2. It is important to note that the multi-criteria approach violates the above mentioned basic statement of bilevel programming. A different problem is solved which realizes an hierarchical nature by control of distances to best solutions and objective function values. Hence, the user needs to be carefully in selecting the correct approach.

The relations between bilevel programming and multicriterial programming have been investigated in the papers [9, 10].

2 The fuzzy bilevel programming problem

Real world problems are often not deterministic ones but need to be described using fuzzy or stochastic goals and/or constraints, see [2] and the references therein. In this paper we are interested in a fuzzy linear bilevel programming problem in its optimistic formulation

$$\begin{array}{rcl} Z_1(x_1,x_2) & \rightarrow & \widetilde{\min}_{x_1,x_2} \\ \text{subject to} & & Ax_1 \,\tilde{\leq}\, a \\ \text{and } x_2 \text{ solves} & & Z_2(x_1,x_2) \rightarrow \widetilde{\min}_{x_2} \\ & & \quad \text{subject to } B_2x_2 \,\tilde{\leq}\, b - B_1x_1. \end{array} \qquad (3)$$

Here, $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$, A, B_1, B_2 are matrices and a, b are vectors of appropriate dimensions, and $Z_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}, i = 1, 2$. Note, that not the parameters (data) of the problem are assumed to be fuzzy numbers but rather the aims to minimize the objective functions as well as to satisfy the constraints. The notation $\widetilde{\min}$ is used to indicate that the decision maker intends to reach a small objective function value and his/her degree of satisfaction is measured using a membership function for the fuzzy set of "good" objective function values. Also, the sign $\widetilde{\leq}$ represents the fuzzy aim that the left-hand side should be "not larger" than the right hand one. Again this is modeled as a fuzzy set and is described using a membership function.

In problem (3), the upper level decision maker (or leader) selects x_1 satisfying the fuzzy constraints $Ax_1 \widetilde{\leq} a$ and informs the lower level decision maker (follower) about this selection. Then, the lower level decision maker computes an optimal response, which is here a solution of a fuzzy programming problem parameterized in x_1 and gives this decision back to the upper level decision maker. The latter is now able to evaluate his/her objective function value. Only in the case that the lower level decision maker has no unique optimal response, he/she asks the upper level one which solution out of the set of optimal responses should be selected. There is no possibility of a different influence of the leader to the follower's selection than by the leader's solution x_1.

Next, determine the mentioned membership functions μ_0^L resp. μ_i^L for the objective function and the constraints of the leader as well as μ_0^F resp. μ_i^F for the objective function and the constraints of the follower. For this, assume that there are given aspiration levels Z_0^L and Z_0^F as well as permissable tolerances p_0^L, p_0^F for the leader's and the follower's objective functions, as well as aspiration levels a_i and $b_i - B_1^i x_1$ and permissable tolerances p_i^L, p_i^F for the leader's and the follower's constraints. The values Z_0^F, Z_0^L, p_0^L and p_0^F mainly reflect the decision maker's aims and wishes. Sometimes, minimization and maximization of the objective functions over the feasible set is used to determine them. Then, the following trapezoidal membership functions can be considered:

$$\mu_0^L(x_1,x_2) = \begin{cases} 1 & \text{if } Z_1(x_1,x_2) \leq Z_0^L \\ 1 - \dfrac{Z_1(x_1,x_2) - Z_0^L}{p_0^L} & \text{if } Z_0^L \leq Z_1(x_1,x_2) \leq Z_0^L + p_0^L \\ 0 & \text{otherwise} \end{cases}$$

for the leader's objective function,

$$\mu_i^L(x_1,x_2) = \begin{cases} 1 & \text{if } A^i x_1 \leq a_i \\ 1 - \dfrac{A^i x_1 - a_i}{p_i^L} & \text{if } a_i \leq A^i x_1 \leq a_i + p_i^L \\ 0 & \text{otherwise} \end{cases}$$

for the leader's i-th constraint, where A^i is the i-th row in the matrix A,

$$\mu_0^F(x_1,x_2) = \begin{cases} 1 & \text{if } Z_2(x_1,x_2) \leq Z_0^F \\ 1 - \dfrac{Z_2(x_1,x_2) - Z_0^F}{p_0^F} & \text{if } Z_0^F \leq Z_2(x_1,x_2) \leq Z_0^F + p_0^F \\ 0 & \text{otherwise} \end{cases}$$

for the follower's objective function, and

$$\mu_i^F(x_1,x_2) = \begin{cases} 1 & \text{if } B_2^i x_2 \leq b_i - B_1^i x_1 \\ 1 - \dfrac{B_1^i x_1 + B_2^i x_2 - b_i}{p_i^F} & \text{if } b_i - B_1^i x_1 \leq B_2^i x_2 \leq b_i - B_1^i x_1 + p_i^F \\ 0 & \text{otherwise} \end{cases}$$

for the follower's i-th constraint, where B_1^i, B_2^i are the i-th row in the matrices B_1, B_2. We remark that the follower's membership function values for the constraints depend also on the leader's selection. This is in accordance with the idea of bilevel programming that the leader determines the "environment" for the follower's selection. Clearly, these piecewise linear membership functions can be replaced by other ones. In general, the membership functions reflect the information of the decision makers about the conditions for the optimization as well as their aims. We use these trapezoidal membership functions since this leads to a crisp linear bilevel programming problem.

Now, following Zimmermann [23] we obtain the following crisp bilevel programming problem, where the optimization task of both decision makers is replaced by maximizing the minimum value of all the membership functions in the respective problems:

$$\begin{array}{rcl} \alpha_1 & \rightarrow & \max\limits_{\alpha_1, x_1, x_2} \\ \mu_i^L(x_1, x_2) & \geq & \alpha_1, \ i = 0, \ldots, p \\ \alpha_1 & \in & [0, 1] \\ \text{and } x_2 \text{ solves} & & \alpha_2 \rightarrow \max\limits_{\alpha_2, x_2} \\ & & \mu_i^F(x_1, x_2) \geq \alpha_2, \ i = 0, \ldots, q \\ & & \alpha_2 \in [0, 1]. \end{array}$$

This problem can be reformulated as

$$\alpha_1 \rightarrow \max_{\alpha_1, x_1, x_2} \tag{4}$$

$$\left.\begin{array}{r} Z_1(x_1, x_2) - (1 - \alpha_1) p_0^L \leq Z_0^L \\ A^i x_1 - (1 - \alpha_1) p_i^L \leq a_i, \ i = 1, \ldots, p \\ \alpha_1 \in [0, 1] \end{array}\right\} \tag{5}$$

and x_2 solves

$$\alpha_2 \rightarrow \max_{\alpha_2, x_2} \tag{6}$$

$$\left.\begin{array}{r} Z_2(x_1, x_2) - (1 - \alpha_2) p_0^F \leq Z_0^F \\ B_2^i x_2 - (1 - \alpha_2) p_i^F \leq b_i - B_1^i x_1, \ \forall i \\ \alpha_2 \in [0, 1]. \end{array}\right\} \tag{7}$$

This is a crisp bilevel programming problem. Such problems are $\mathcal{NP}$-hard [8] nonconvex optimization problems [4]. To solve them, the lower level problem (6)–(7) needs to be transformed using e.g. the necessary and sufficient Karush-Kuhn-Tucker optimality conditions or the optimal value function of this parametric problem. A local optimal solution of problem (4)–(7) is also a local optimal solution of the transformed problem but, unfortunately, the converse implication is not true in general [7].

Now, let for simplicity $Z_1(x_1, x_2)$ and $Z_2(x_1, x_2)$ be linear functions. All what follows in this chapter works also for nonlinear functions, it is only more cumbersome to write it down. In the following we will describe one promising solution approach which uses the simplified linear bilevel programming problem:

$$\begin{array}{rcl} c^\top y_1 & \to & \max\limits_{y_1,y_2} \\ \overline{A}^1 y_1 + \overline{A}^2 y_2 & \leq & \overline{a} \\ \text{and } y_2 \text{ solving} & & d^\top y_2 \to \max\limits_{y_2} \\ & & \overline{B}_1 y_1 + \overline{B}_2 y_2 \leq \overline{b} \end{array} \tag{8}$$

Here, the objectives $c^\top y_1 \to \max$, $d^\top y_2 \to \max$ represent (4) and (6), the constraints are abbreviations of (5) and (7). Moreover, the variables y_1, y_2 combine x_1, α_1 and x_2, α_2, respectively. In other words, one of the components of the vectors y_i corresponds to α_i and c, d are unit vectors such that $c^\top y_1 = \alpha_1$ as well as $d^\top y_2 = \alpha_2$.

To obtain an ordinary optimization problem replace the lower level programming problem $\max\{d^\top y_2 : \overline{B}_1 y_1 + \overline{B}_2 y_2 \leq \overline{b}\}$ by its necessary and sufficient optimality conditions (the Karush-Kuhn-Tucker conditions). The complementarity constraints in the resulting problem make this problem difficult and lead to violation of regularity conditions as the Mangasarian-Fromowitz constraint qualification at every feasible point [16]. One promising solution approach for this problem is the interior point approach from [3, 17], which is based on a linear perturbation of the complementarity conditions. We use this approach and obtain

$$\begin{array}{rcl} c^\top y_1 & \to & \max\limits_{y_1,y_2} \\ \overline{A}^1 y_1 + \overline{A}^2 y_2 & \leq & \overline{a} \\ \overline{B}_1 y_1 + \overline{B}_2 y_2 & \leq & \overline{b} \\ \overline{B}_2^\top z & = & d \\ z & \geq & 0 \\ z^\top(\overline{B}_1 y_1 + \overline{B}_2 y_2 - \overline{b}) & \geq & -\delta. \end{array} \tag{9}$$

This is a one-level optimization problem with the parameter $\delta > 0$ on the right-hand side of the complementarity condition. For $\delta = 0$ we obtain the problem in which the lower level problem in (8) is replaced by the Karush-Kuhn-Tucker conditions.

In the mentioned solution approach the problem (9) is solved iteratively for $\delta \downarrow 0$ by means of standard nonlinear optimization procedures. Using this iterative process a sequence of stationary points $\{y_1^k, y_2^k\}_{k=1}^\infty$ is constructed whose accumulation points are stationary solutions for problem (9) [3, 17]. In these papers the so-called MPEC-LICQ is assumed to be satisfied which is a generic assumption [18] and which also implies that the computed accumulation point is stationary for the bilevel programming problem (8).

3 Multicriterial optimization approach

In the papers [11, 15, 19, 21] fuzzy bi- and multi-level programming problems have been considered. The authors consider cooperative [22] or satisfactory [11] approaches in the sense that the lower level decision makers accept restrictions of their feasible sets by fuzzy constraints bounding the objective function value of higher level problems. In some papers, membership functions derived from the aim to stay close to higher level solutions are added, and non-fuzzy constraints are allowed.

Consider the bilevel programming problem (3) and assume that the inequality constraints are non-fuzzy constraints. In the papers [11, 15, 20] the approach outlined above results in the following problem:

$$\begin{array}{rcl} \alpha & \to & \max\limits_{\alpha,x_1,x_2} \\ \text{subject to} & & Ax_1 \leq a \\ & & B_1x_1 + B_2x_2 \leq b \\ & & \mu_0^L(x_1,x_2) \geq \alpha \\ & & \mu_0^F(x_1,x_2) \geq \alpha \\ & & \mu_{x_1}(x_1) \geq \alpha. \end{array} \tag{10}$$

Here, a fuzzy constraint for the upper level variable is added. This constraint demands that x_1 is "close to a $\widetilde{\text{best}}$" solution. In problem (10), both fuzzy objective functions $Z_i(x_1,x_2) \to \widetilde{\min}$ and the fuzzy constraint on x_1 are replaced by their membership functions. Using the min-operator for the intersection of fuzzy sets, $\min\{\mu_0^L(x_1,x_2), \mu_0^F(x_1,x_2), \mu_{x_1}(x_1)\}$ is to be maximized over the set of points satisfying the crisp constraints. This problem can be reformulated as problem (10). In some papers compensatory operators are used for the intersection of the fuzzy sets.

The following example illustrates the differences between this approach and the one used in Section 2.

Example 1 Consider the problem of minimizing the upper level objective function

$$x_1 + 3x_2 \to \widetilde{\min}$$

subject to $1 \leq x_1 \leq 6$ and x_2 solving the lower level problem

$$\widetilde{\min_{x_2}}\{-x_2 : x_1 + x_2 \leq 8,\ x_1 + 4x_2 \geq 8,\ x_1 + 2x_2 \leq 13\}. \tag{11}$$

First, consider this problem as a non-fuzzy one and replace $\widetilde{\min}$ with min. The set of all pairs $(x_1,x_2)^\top$ satisfying the constraints of both the lower and the upper level problems, denoted by M, as well as the minimization directions of both objective functions, marked by the arrows, are depicted in Figure 1.

The feasible set of the lower level problem for a fixed value of x_1 is just the intersection of the set M with the set of all points above the point $(x_1, 0)$ on the x_1-axis. Now, if the function $f(x_1,x_2) = -x_2$ is minimized on this set, we arrive at a point on the thick dotted line which thus is the optimal solution of the lower level problem. The (unique) optimal solution of the lower level problem depending on the parameter x_1 is

$$x_2(x_1) = \begin{cases} 6.5 - 0.5x_1 & \text{for } 1 \leq x_1 \leq 3, \\ 8 - x_1 & \text{for } 3 \leq x_1 \leq 6, \end{cases} \tag{12}$$

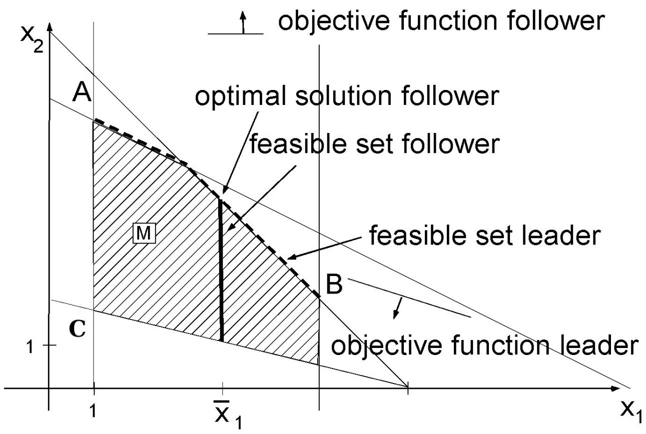

Figure 1: Illustration of the example

These solutions are situated on the thick dotted line, which is the feasible set of the bilevel programming problem and which is often called inducible set in the literature [1]. Then, the upper level objective function $f_L(x) = x_1 + 3x_2$ is minimized on this set and the unique optimal solution at point B is obtained.

It is easy to see that the set of Pareto optimal points of the problem (if both objective functions of the lower and the upper level problems are minimized simultaneously) is the line between the points $A = (1, 6)^\top$ and $C = (1, 7/4)^\top$. The intersection of the set of Pareto optimal solutions with the inducible set is the point A which is the worst solution in the inducible region with respect to the leader's objective function.

Now, if we apply the approach in (10) to the problem in Example 1 we first have to fix membership functions. The objective function values in the lower level problem range between zero and -6. Hence, we may use the membership function

$$\mu_{f_F}(x) = \begin{cases} x_2/6 & \text{for } 0 \geq -x_2 \geq -6 \\ 0 & \text{for } 0 \leq -x_2 \\ 1 & \text{for } -6 \geq -x_2 \end{cases}$$

The upper level objective function ranges between 25/4 and 19. Remember that both objective functions are to be minimized. The membership function may be

$$\mu_{f_L}(x) = \begin{cases} \dfrac{19 - x_1 - 3x_2}{19 - 25/4} & \text{for } 25/4 \leq x_1 + 3x_2 \leq 19 \\ 0 & \text{for } x_1 + 3x_2 \geq 19 \\ 1 & \text{for } x_1 + 3x_2 \leq 25/4 \end{cases}$$

Now, if we replace the lower level problem (11) by its crisp reformulation maximizing the membership function, we again obtain the function $x_2(x_1)$ in (12) as optimal solution. Replacing the upper level problem in Example 1 by its crisp reformulation and dropping the lower level problem for a moment we obtain the problem

$$\max\{\mu_{f_L}(x_1, x_2(x_1)) : 1 \leq x_1 \leq 6\}$$

having again $(x_1, x_2) = (1, 7/4)$ as the unique optimal solution.

Now, put that all together and use the approach in problem (10). Then, the following problem is obtained:

$$\begin{array}{rcl}
\lambda & \rightarrow & \max \\
\dfrac{19 - x_1 - 3x_2}{19 - 25/4} & \geq & \lambda \\
x_2/6 & \geq & \lambda \\
\dfrac{x_1 - 0.5}{0.5} & \geq & \lambda \\
\dfrac{6 - x_1}{5} & \geq & \lambda \\
0 \leq \lambda & \leq & 1 \\
x_1 + x_2 & \leq & 8 \\
x_1 + 4x_2 & \geq & 8 \\
x_1 + 2x_2 & \leq & 13 \\
1 \leq & x_1 & \leq 6
\end{array}$$

Here, the membership function

$$\mu_{x_1}(x_1) = \begin{cases} \dfrac{x_1 - 0.5}{0.5} & \text{for } 0,5 \leq x_1 \leq 1 \\ \dfrac{6 - x_1}{5} & 1 \leq x_1 \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

is used to describe the aim that x_1 should be near 1. The optimal solution of this problem is $(x_1, x_2) = (1, 3.5122)$. Note that this solution is Pareto optimal. A modification of the membership functions in the sense that the aspiration levels and the tolerances are changed will again lead to optimal solutions of problem (10) located in the set of Pareto optimal solutions. This very simple example explains the differences between the fuzzy bilevel programming problem as formulated in (3) and the approach using multicriterial optimization. While in fuzzy bilevel programming the follower does not care about the leader's objective function and takes the selected upper level solution as a parameter, the approach via fuzzy multicriterial optimization starts with the basic assumption that the follower is interested in computing a solution for which the leader's objective function value is close to an aspiration level. This is supplemented by the allowance that the follower can modify the leader's selection. Hence, both approaches are different and have different aims.

As formulated in problem (10) this approach is hierarchical in the sense that the upper level variable (without considering the follower's objective function) is more important than the follower one's. This is realized by including the membership function μ_{x_1} into the maximization.

4 Conclusion

In this paper we considered fuzzy bilevel programming problems. We used an approach in [23] to derive a crisp mathematical programming problem replacing it. This idea can be substituted by other approaches known from literature. For the resulting problem we suggested a promising solution approach known for mathematical programs with equilibrium constraints (MPECs).

In the second part, we compared our approach with one used frequently in literature which is based on the application of multicriterial optimization. We explained by a small example that both methods lead to very different results.

It is easier to compute one Pareto optimal solution of problem (10) than to solve problem (9). But, whether problem (10) can be used to treat the investigated fuzzy programming problem depends strongly on the hierarchical relations between the follower and the leader. If the hierarchical structure is as in [4, 5] this is not possible.

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