Classification problems with imprecise data through separating hyperplanes *

Emilio Carrizosa, José Gordillo
Universidad de Sevilla (Spain)
{ecarrizosa,jgordillo}@us.es

Frank Plastria
Vrije Universiteit Brussel (Belgium)
Frank.Plastria@vub.ac.be

10th September 2007

Abstract

We consider a supervised classification problem in which the elements to be classified are sets with certain geometrical properties. In particular, this model can be applied to deal with data affected by some kind of noise and in the case of interval-valued data. Two classification rules, a fuzzy one and a crisp one, are defined in terms of a separating hyperplane, and a formulation of the rule identification problem by margin maximization is introduced, extending the standard techniques in Support Vector Machines used for single feature vectors. We study in depth the interval data case and report on several numerical experiments. This methodology is also proved to be useful in practice when handling missing values in a database.

Keywords: Supervised Classification, Interval Data, Missing Values, Robust Classification, Gauges, Support Vector Machines, Quadratic Programming.

*This work has been partially supported by the grants MTM2005-09362-103-01 of MEC, Spain, and FQM-329 of Junta de Andalucía, Spain. Their financial support is gratefully acknowledged.

1 Introduction

In certain situations, data cannot be expressed via single feature vectors, and interval data must be introduced. This way, intervals are used for expressing ranges, for instance, the range of temperature during a day, or age intervals for a group of individuals. Intervals can also be used when several measures of the same variable have been taken from an individual, and one wants to summarize these measurements, for example, the fluctuations of blood pressure or pulse rate of a patient. Intervals also naturally occur in case of imprecise data, or when an estimation of a certain parameter must be performed via a confidence interval, and, in general, whenever uncertainty or vagueness arises in our problem.

Another case of interval data appears in the framework of Symbolic Data Analysis ([5, 7]). When one needs to summarize large databases in such a way that the resulting dataset has a more manageable size and it retains enough knowledge from the original database. Different approaches exist to aggregate the data by using classical variables (single values), multi-valued variables (categorical variables which can have several results), interval-valued variables (the data are aggregated into intervals and this is the case of our interest) or modal variables (a single, interval or nominal variable which can have different values with different probabilities associated).

In this work, we study a classification problem where elements of the dataset are not single points, but sets in $\mathbb{R}^d$. In particular, one interesting case which will be modeled via this methodology is when objects X_i to be classified are defined as a Cartesian product of intervals, $X_i = \prod_{j=1}^{d}[l_{ij}, u_{ij}] \subset \mathbb{R}^d$, with $l_{ij} \leq u_{ij}$, that is, l_{ij} and u_{ij} represent, respectively, the lower and upper bounds of each coordinate, these bounds coinciding in some cases.

Classification with interval data has been studied in the literature by following different strategies. In [13], linear discriminant analysis is applied to this type of classification problems by considering three different techniques: assigning a uniform distribution to each interval, expanding the dataset into the corresponding set of vertices and describing each interval via its center and its range. In [12], a Radial Basis Function kernel is built via a Hausdorff distance between intervals, and is applied to classification of intervals. Other techniques involving Neural Networks are described in [32, 35]. Two methods that allow to use interval data as inputs for a multi-layer perceptron are included in [32] (one of them based on a description of the intervals as single points and the other based on a probabilistic understanding of the intervals). A neural expert system for diagnosis is created in [35], where the knowledge base is a Neural Network which is built automatically through a learning algorithm. Likewise, classification methods for symbolic data are explained in [29].

Another case of interest, which will be modeled with our approach in this paper, is when data are affected by some kind of noise or perturbation. In that case, we must build a robust classifier, insensitive to this noise in the data. One model of robust Support Vector Machines has been studied in [37, 38], where an optimization problem must be solved via Second Order Cone Programming. It will be seen that our model generalizes the formulation proposed in [37, 38].

One can find another approach to robust classification in [14], where a binary classification problem is stated in which the data are unknown, but are bounded in hyper-rectangles.

In that paper, the authors design a robust classifier by minimizing the worst-case value of a given loss function. Three different functions are considered, including the linear Hinge loss (see [10]) for SVMs, which provides an upper bound on the number of future expected misclassification errors. In [22], another binary classification problem is formulated where the data are given by the mean and covariance of each class, which are assumed to be known. The objective is minimizing the worst-case (maximum) probability of future misclassified data points, and Second Order Cone Programming techniques must be used to solve it. Geometrically, it can be seen as minimizing the maximum of the Mahalanobis distances between the two classes. With a similar strategy, in [4] the values of a data point are described by a data uncertainty set, which is defined via a bounded ellipsoid parameterized by its location (expected value or center of the ellipsoid) and its shape (covariance matrix or the matrix of squared axis lengths).

The technique described in our paper is also applied to the case in which there exist missing values (see [24] for a complete study on statistical analysis in datasets with missing values), that is, when the database is formed by feature vectors but some of their coordinates do not appear in the dataset. Different techniques have been used in the literature to handle missing data in classification problems (for a survey on the topic, see [25, 26]). In our case, instead of imputing single values as usual for the missing coordinates, they will be replaced by intervals which will be built by using the non-missing values of the same class in the dataset. Different measurements will be taken into account to perform the construction of these intervals.

This paper is structured as follows. In Section 2, our model is explained, the classification rule is defined and the corresponding optimization problem is derived. A general formulation is given, and it is particularized to the case of interval data and perturbed data. Afterwards, we will focus on the case of interval data. In Section 3, an overview on multi-class classification problems is given, before solving a real interval-valued database via our technique (adapted to the multi-class case). In the following sections, more computational experiments will be performed, firstly for a database where data are considered imprecise and the objects have been transformed into intervals, and secondly by erasing at random some coordinates in a real database and by substituting the missing coordinates by intervals built with the remaining elements of the database. These intervals will be based on different percentiles and on the mean and the deviation. Likewise, we will perform another numerical experiment with a database with missing values, where the blanks will be replaced by these intervals. We will finish in Section 6 with some discussion and concluding remarks.

2 The model

2.1 Defining the classification rule

In the standard approach to classification, each instance in the database Ω is of the form $(x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}$. Now we consider a database $\Omega \subset \mathbb{R}^d \times \mathbb{R}$ formed by objects $i = (X_i, Y_i) \in \Omega$, where Y_i is the corresponding class defined by means of a label +1 or -1 and X_i is the form $X_i = x_i + B_i$, with $x_i \in \mathbb{R}^d$ and with B_i a subset of $\mathbb{R}^d$ with certain geometrical properties, namely, it is convex, symmetric with respect to the origin and contains the origin. In other words, B_i is the unit ball of a symmetric gauge γ_i (see [19]), that is,

$$B_i = \{s \in \mathbb{R}^d : \gamma_i(s) \leq 1\}. \tag{1}$$

Two different particular cases are of main interest:

1. γ_i is given by

$$\gamma_i(s_1, \ldots, s_d) = \max_{j=1,\ldots,d} \frac{2|s_j|}{u_{ij} - l_{ij}}, \text{ for } l_{ij} < u_{ij},\ j = 1, \ldots, d, \tag{2}$$

   and then

$$\begin{aligned} B_i &= \{s \in \mathbb{R}^d : \gamma_i(s) \leq 1\} = \{s \in \mathbb{R}^d : \max_{j=1,\ldots,d} \frac{2|s_j|}{u_{ij} - l_{ij}} \leq 1\} \\ &= \{s \in \mathbb{R}^d : |s_j| \leq \frac{u_{ij} - l_{ij}}{2},\ \forall j = 1, \ldots, d\}. \end{aligned} \tag{3}$$

2. γ_i is given by

$$\gamma_i(s_1, \ldots, s_d) = \frac{1}{r_i} \sum_{j=1}^{d} (|s_j|^p)^{\frac{1}{p}}, \text{ for some } p,\ 1 \leq p \leq \infty, \text{ for } r_i > 0, \tag{4}$$

   and then

$$\begin{aligned} B_i &= \{s \in \mathbb{R}^d : \gamma_i(s) \leq 1\} = \{s \in \mathbb{R}^d : \frac{1}{r_i} \sum_{j=1}^{d} (|s_j|^p)^{\frac{1}{p}} \leq 1\} \\ &= \{s \in \mathbb{R}^d : \|s\|_p \leq r_i\}. \end{aligned} \tag{5}$$

In case of γ_i as defined in (2), taking x_i such that $x_{ij} = \frac{l_{ij}+u_{ij}}{2}$, $j = 1, \ldots, d$, one has that $X_i = x_i + B_i$, with B_i as in (3), and thus is a Cartesian product of intervals, that is, $X_i = \prod_{j=1}^{d} [l_{ij}, u_{ij}]$. Interval data can be used, for example, when the data have been aggregated or summarized in an interval (like in Symbolic Data Analysis, see [5, 7, 13]) or there exist missing values in our database and the missing coordinates are replaced by intervals built with the remaining instances of the same group.

The second case, with γ_i of the form (4), is interesting to model the case of data affected by some kind of noise. This is the idea of the so-called robust SVMs, which were introduced in [37, 38] to classifiers, via Support Vector Machines, when the data have suffered some perturbations. These perturbations are supposed to be unknown, but a bound of them is known, for a chosen norm in the input space. Robust Support Vector Machines must remain insensitive to these bounded perturbations of the data.

In that case, we can write $X_i = x_i + B_i$, with x_i the original value of the instance and B_i, as defined in (5), a ball representing the unknown perturbation and r_i being a positive constant which bounds the perturbation in p-norm, since $x \in X_i$ iff $x = x_i + s$, with $\gamma_i(s) \leq 1$, or equivalently, $\|s\|_p \leq r_i$, for each $i \in \Omega$.

The classification rule is defined in terms of a separating hyperplane, whose parameters $\omega \in \mathbb{R}^d$, $\beta \in \mathbb{R}$ must be computed, as in the case of single points where, given a point $x \in \mathbb{R}^d$, it is evaluated in the function $f(x) = \omega^\top x + \beta$, and it is assigned to the group G_{+1} or G_{-1} according to the sign of this evaluation (see [8, 9, 10]).
In our problem, once ω, β are determined, objects are classified according to a rule, which extends naturally the one in which x is a singleton. Two variants are considered:

Crisp classification :
Given an object $X \subset \mathbb{R}^d$, classify X in G_{+1} if $\max_{x\in X}(\omega^\top x+\beta) > -\min_{x\in X}(\omega^\top x+\beta)$, and in G_{-1} otherwise.

Fuzzy classification :
Given an object $X \subset \mathbb{R}^d$, compute $I(X) := [\min\limits_{x\in X}(\omega^\top x + \beta), \max\limits_{x\in X}(\omega^\top x + \beta)]$,

- if $0 \notin I(X)$,

$$\begin{aligned} &\bullet \text{ classify in } G_{+1} \text{ (with intensity equal to 1), if } \min_{x\in X}(\omega^\top x + \beta) > 0, \\ &\bullet \text{ classify in } G_{-1} \text{ (with intensity equal to 1), if } \max_{x\in X}(\omega^\top x + \beta) < 0, \end{aligned} \tag{6}$$

- if $0 \in I(X)$,

$$\begin{aligned} &\bullet \text{ classify in } G_{+1}, \text{ with intensity } \frac{\max\limits_{x\in X}(\omega^\top x + \beta)}{\max\limits_{x\in X}(\omega^\top x + \beta) - \min\limits_{x\in X}(\omega^\top x + \beta)}, \\ &\bullet \text{ classify in } G_{-1}, \text{ with intensity } \frac{-\min\limits_{x\in X}(\omega^\top x + \beta)}{\max\limits_{x\in X}(\omega^\top x + \beta) - \min\limits_{x\in X}(\omega^\top x + \beta)}. \end{aligned} \tag{7}$$

We can simplify the fuzzy rule via the following *clamp* function. Given three numbers x, y and z, we define the *clamp* function for these three values as $clamp(x, y, z) = \min(\max(x, y), z)$.
Then, given an object X, the intensity for classifying it in G_{+1} is

$$intensity_{+1} := clamp(0, fuzzy\ value, 1), \tag{8}$$

where

$$fuzzy\ value := \frac{\max\limits_{x\in X}(\omega^\top x + \beta)}{\max\limits_{x\in X}(\omega^\top x + \beta) - \min\limits_{x\in X}(\omega^\top x + \beta)}. \tag{9}$$

Alternatively, we classify X in G_{-1} with $intensity_{-1} := 1 - intensity_{+1}$.

Thus the fuzzy rule allows for a degree of uncertainty in the classification, while the crisp rule does not.
Observe that the crisp rule simply classifies the object in the group with the highest fuzzy intensity, since the crisp rule includes expression (6).

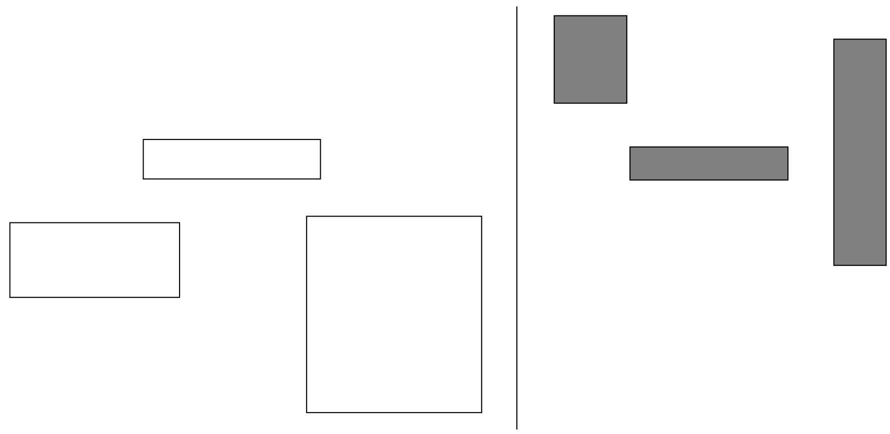


Figure 1: A separating hyperplane

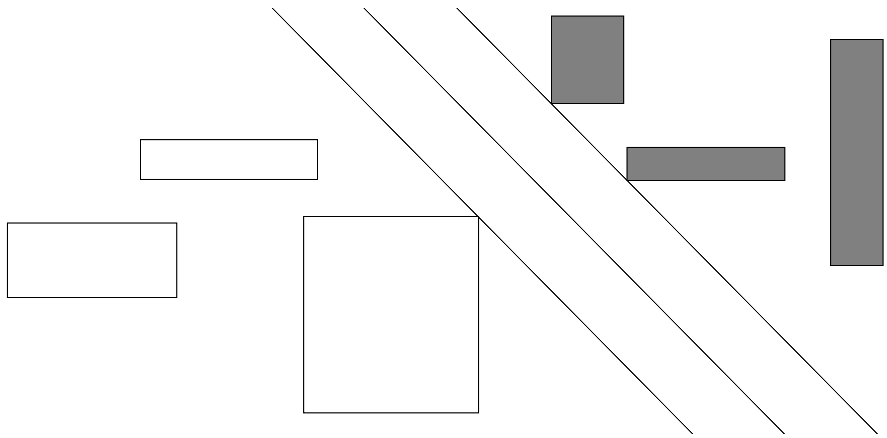


Figure 2: Maximizing the margin

Figure 1 shows an example in dimension 2 of the problem with boxes (Cartesian product of intervals). The white boxes represent the objects of one group, whereas the grey boxes represent the objects of the other group. Our problem is to build a hyperplane separating the two sets of boxes. Observe that when the two sets are linearly separable (as in Figure 1), that is, when a hyperplane can be constructed which separates strictly the two sets, the fuzzy part of the classification rule (expression 7) is not necessary.
Different hyperplanes separating the two groups may exist. In order to choose the hyperplane, we maximize a margin (see Figure 2) as performed in Support Vector Machines in case of databases composed by single feature vectors (see [10, 27, 41]).
Then, given a training sample $I = \{(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)\} \subseteq \Omega$, we will solve the optimization problem to obtain the optimal parameters ω and β (optimal in the sense that the margin is maximized) for constructing the classification rule.

2.2 The optimization problem

Let $\|.\|$ be a norm in $\mathbb{R}^d$. Let us first consider the separable case, i.e., let us assume first that ω, β exist such that the two groups can be separated via a hyperplane $H = \{x \in \mathbb{R}^d : \omega^\top x + \beta = 0\}$.
According to the linear classification rule defined in (6), given $X_i \in I$, it is assigned to the group

$$\begin{array}{lll} G_{+1} & : \text{ if } \min_{x\in X_i} (\omega^\top x+\beta) > 0, & \\ G_{-1} & : \text{ if } \max_{x\in X_i} (\omega^\top x+\beta) < 0 & \leftrightarrow \quad \min_{x\in X_i} -(\omega^\top x+\beta) > 0. \end{array} \tag{10}$$

The distance from a point $x^* \in \mathbb{R}^d$ to a hyperplane $H = \{x \in \mathbb{R}^d : \ \omega^\top x + \beta = 0\}$ is given by $d(x^*, H) = \dfrac{|\omega^\top x^* + \beta|}{\|w\|^0}$, where $\|\omega\|^0$ represents the dual norm of the normal vector ω to the hyperplane H (see [30]). Hence, given the training sample I, the optimization problem to solve is

$$\max_{\omega,\beta} \min\left\{ \min_{i\in G_{+1}} \min_{x\in X_i} \frac{\omega^\top x+\beta}{\|\omega\|^0}, \min_{k\in G_{-1}} \min_{x\in X_k} \frac{-(\omega^\top x+\beta)}{\|\omega\|^0} \right\}. \tag{11}$$

The objective in (11) is homogeneous in its variables. Hence, one can assume without loss of generality that

$$\min\left\{ \min_{i\in G_{+1}} \min_{x\in X_i} (\omega^\top x+\beta), \min_{k\in G_{-1}} \min_{x\in X_k} -(\omega^\top x+\beta) \right\} \geq 1, \tag{12}$$

and the problem can be expressed as

$$\begin{array}{ll} \max_{\omega,\beta} & \dfrac{1}{\|\omega\|^0} \\ \text{s.t.} & \min_{x\in X_i} \ \omega^\top x+\beta \geq 1, \ \forall i \in G_{+1} \\ & \min_{x\in X_k} \ -(\omega^\top x+\beta) \geq 1, \ \forall k \in G_{-1}. \end{array} \tag{13}$$

Taking as $\|.\|$ the Euclidean norm, the problem is equivalent to

$$\begin{array}{ll} \min_{\omega,\beta} & \dfrac{1}{2}\,\omega^\top\omega \\ \text{s.t.} & \min_{x\in X_i} \ \omega^\top x+\beta \geq 1, \ \forall i \in G_{+1} \\ & \min_{x\in X_k} \ -(\omega^\top x+\beta) \geq 1, \ \forall k \in G_{-1}. \end{array} \tag{14}$$

A test to study if there exists a hyperplane separating the two groups will be given later, in Theorem 2.2. Anyway, Problem (14) can be adapted to the one in which the two groups are not linearly separable. Indeed, in case of non-separability, we introduce one slack variable per object, and we penalize the objective function,

$$\begin{array}{ll} \min_{\omega,\beta,\xi,\eta} & \dfrac{1}{2}\,\omega^\top\omega + \dfrac{C_{+1}}{n} \displaystyle\sum_{i\in G_{+1}} \xi_i + \dfrac{C_{-1}}{m} \sum_{k\in G_{-1}} \eta_k \\ \text{s.t.} & \min_{x\in X_i} \ \omega^\top x+\beta \geq 1-\xi_i, \ \forall i \in G_{+1} \\ & \min_{x\in X_k} \ -(\omega^\top x+\beta) \geq 1-\eta_k, \ \forall k \in G_{-1} \\ & \xi_i, \eta_k \geq 0, \ \forall i \in G_{+1}, \ \forall k \in G_{-1}, \end{array} \tag{15}$$

where we denote by n the cardinal of G_{+1} and by m the cardinal of G_{-1}, and C_{+1}, C_{-1} are constants.

2.3 Obtaining an equivalent formulation

Problem (15) has a convex quadratic objective and nonlinear constraints. In the following result, we give an equivalent formulation of Problem (15) by building the dual of the problems appearing in the constraints of Problem (15). Recall that the dual gauge of γ_i in ω is defined by $\gamma_i^0(\omega) = \max\limits_{\gamma_i(u)\leq 1} (\omega^\top u)$.

Theorem 2.1. *Problem (15) admits the following equivalent formulation,*

$$\begin{array}{ll} \min\limits_{\omega,\beta,\xi,\eta} & \frac{1}{2}\sum\limits_{j=1}^{d}\omega_j^2 + \frac{C_{+1}}{n}\sum\limits_{i\in G_{+1}}\xi_i + \frac{C_{-1}}{m}\sum\limits_{k\in G_{-1}}\eta_k \\ s.t. & \omega^\top x_i - \gamma_i^0(\omega) + \beta \geq 1-\xi_i,\ \forall i\in G_{+1} \\ & -\omega^\top x_k - \gamma_k^0(\omega) - \beta \geq 1-\eta_k,\ \forall k\in G_{-1} \\ & \xi_i,\eta_k\geq 0,\ \forall i\in G_{+1},\ \forall k\in G_{-1}, \end{array} \tag{16}$$

where γ_i is the gauge associated to the object $i \in I$ and γ_i^0 is its dual gauge.

Proof.
To prove the result, we change the constraints in Problem (15) by using that $X_i = x_i + B_i$, with B_i the unit ball induced by gauge γ_i for each X_i, and by changing to its dual gauge. One has that

$$\min_{x\in x_i+B_i}\omega^\top x = \min_{\gamma_i(u)\leq 1}\omega^\top(x_i+u) = \omega^\top x_i + \min_{\gamma_i(u)\leq 1}\omega^\top u = \omega^\top x_i - \max_{\gamma_i(u)\leq 1}(-\omega^\top u).$$

By using that $\gamma_i^0(-\omega) = \max\limits_{\gamma_i(u)\leq 1}(-\omega^\top u)$ (with γ_i^0 the dual gauge of γ_i), and since $\gamma_i^0(-\omega) = \gamma_i^0(\omega)$, one obtains that

$$\min_{x\in x_i+B_i}\omega^\top x = \omega^\top x_i - \gamma_i^0(-\omega) = \omega^\top x_i - \gamma_i^0(\omega). \tag{17}$$

Then, we change the constraints for G_{+1} in Problem (15) by using (17),

$$\min_{x\in x_i+B_i}\ \omega^\top x + \beta \geq 1-\xi_i \ \leftrightarrow\ \omega^\top x_i - \gamma_i^0(\omega) + \beta \geq 1-\xi_i,\ \forall i\in G_{+1}. \tag{18}$$

We proceed analogously for the set of constraints for the group G_{-1} in Problem (15), and by using (17) we obtain

$$\begin{aligned} \min_{x\in x_k+B_k}\ -(\omega^\top x+\beta)\geq 1-\eta_k \ &\leftrightarrow\ -\omega^\top x_k - \gamma_k^0(-\omega) - \beta \geq 1-\eta_k \\ &\leftrightarrow\ -\omega^\top x_k - \gamma_k^0(\omega) - \beta \geq 1-\eta_k,\ \forall k\in G_{-1}, \end{aligned} \tag{19}$$

since γ_k^0 is a symmetric gauge. □

Below, we consider the two cases of interest for the two definitions of γ_i in (2) and (4). The first one is the particular case in which the elements of the database are boxes, that is, $X_i = \prod_{j=1}^{d}[l_{ij}, u_{ij}]$, for every $i \in I$.

Corollary 2.1. *Let γ_i be the gauge defined in (2). Then, Problem (15) admits the following equivalent formulation as a convex quadratic problem*

$$\begin{array}{lll} \min\limits_{\sigma,\tau,\beta,\xi,\eta} & \frac{1}{2}\sum\limits_{j=1}^{d}(\sigma_j-\tau_j)^2+\frac{C_{+1}}{n}\sum\limits_{i\in G_{+1}}\xi_i+\frac{C_{-1}}{m}\sum\limits_{k\in G_{-1}}\eta_k & \\ s.t. & \sum\limits_{j=1}^{d}\sigma_j l_{ij}-\sum\limits_{j=1}^{d}\tau_j u_{ij}+\beta\geq 1-\xi_i,\ \forall i\in G_{+1} & \\ & \sum\limits_{j=1}^{d}\tau_j l_{kj}-\sum\limits_{j=1}^{d}\sigma_j u_{kj}-\beta\geq 1-\eta_k,\ \forall k\in G_{-1} & \\ & \xi_i,\eta_k,\sigma_j,\tau_j\geq 0,\ \forall i\in G_{+1},\ \forall k\in G_{-1},\ j=1,\ldots,d. & \end{array} \tag{20}$$

Proof.
Firstly, observe that, if $\gamma_i(s)=\max\limits_{j=1,\ldots,d}\frac{2|s_j|}{u_{ij}-l_{ij}}$, then its dual gauge is

$$\gamma_i^0(s)=\sum_{j=1}^{d}\frac{u_{ij}-l_{ij}}{2}|s_j|. \tag{21}$$

And now, it is sufficient to replace the particular values of x_i and $\gamma_i^0(\omega)$ in the constraints of Problem (16).
In the first set of constraints,

$$\omega^\top x_i-\gamma_i^0(\omega)+\beta\geq 1-\xi_i,\ \forall i\in G_{+1},$$

we replace $x_{ij}=\frac{l_{ij}+u_{ij}}{2}$, $j=1,\ldots,d$ and $\gamma_i^0(\omega)=\sum_{j=1}^{d}|\omega_j|\frac{u_{ij}-l_{ij}}{2}$, and we obtain the constraint

$$\sum_{j=1}^{d}\omega_j(\frac{l_{ij}+u_{ij}}{2})-\sum_{j=1}^{d}|\omega_j|(\frac{u_{ij}-l_{ij}}{2})+\beta\geq 1-\xi_i,\ \forall i\in G_{+1}.$$

Let us define $\sigma_j=\max\{0,\omega_j\}$ and $\tau_j=\max\{0,-\omega_j\}$, for $j=1,\ldots,d$. One has that $\omega_j=\sigma_j-\tau_j$ and $|\omega_j|=\sigma_j+\tau_j$, and the set of constraints can be written as

$$\sum_{j=1}^{d}[(\sigma_j-\tau_j)(\frac{l_{ij}+u_{ij}}{2})-(\sigma_j+\tau_j)(\frac{u_{ij}-l_{ij}}{2})]+\beta\geq 1-\xi_i,\ \forall i\in G_{+1}.$$

After some calculations, we obtain the first set of constraints in Problem (20).
Now, we must proceed analogously for the set of constraints

$$-\omega^\top x_k-\gamma_k^0(\omega)-\beta\geq 1-\eta_k,\ \forall k\in G_{-1}.$$

Replacing the values of x_k and $\gamma_k^0(\omega)$ in these constraints, and by introducing the variables σ and τ, one obtains

$$-\sum_{j=1}^{d}[(\sigma_j-\tau_j)(\frac{l_{kj}+u_{kj}}{2})+(\sigma_j+\tau_j)(\frac{u_{kj}-l_{kj}}{2})]-\beta\geq 1-\eta_k,\ \forall k\in G_{-1},$$

which, after arranging terms, leads to the second set of constraints in Problem (20).

□

Remark 2.1. *When we defined* γ_i *in (2), we assumed that* $l_{ij} < u_{ij}$, $\forall j = 1, \dots, d$. *In the case of degenerate boxes (that is,* $l_{ij} = u_{ij}$ *for some coordinates), denote by* J_F *the set of indexes with* $l_{ij} = u_{ij}$ *and denote by* J_V *the set of indexes with* $l_{ij} < u_{ij}$. *Let us define* γ_i *as*

$$\gamma_i(s_1, \dots, s_d) = \begin{cases} \max\limits_{j \in J_V} \dfrac{2|s_j|}{u_{ij} - l_{ij}}, & \text{if } s_j = 0, \ \forall j \in J_F \\ +\infty, & \text{otherwise.} \end{cases} \tag{22}$$

One has that $\gamma_i^0(s)$ *has the same form as (21) and then, formulation (20) remains valid.*

The problem of interval data will be studied more deeply in the following sections.
The second case of interest is when the data are affected by some kind of perturbations. Then, as a straightforward consequence of Theorem 2.1, we obtain the result previously derived by [37, 38]:

Corollary 2.2. *Let* γ_i *be the gauge defined in (4). Then, Problem (15) can be written as follows,*

$$\begin{array}{ll} \min\limits_{\omega,\beta,\xi} & \dfrac{1}{2}\sum\limits_{j=1}^{d}\omega_j^2 + \dfrac{C_{+1}}{n}\sum\limits_{i\in G_{+1}}\xi_i + \dfrac{C_{-1}}{m}\sum\limits_{k\in G_{-1}}\eta_k \\ s.t. & \omega^\top x_i - r_i\|\omega\|_q + \beta \geq 1 - \xi_i, \ \forall i \in G_{+1} \\ & -\omega^\top x_k - r_k\|\omega\|_q - \beta \geq 1 - \eta_k, \ \forall k \in G_{-1} \\ & \xi_i, \eta_k \geq 0, \ \forall i \in G_{+1}, \ \forall k \in G_{-1}, \end{array} \tag{23}$$

where $\|.\|_q$ *is the dual norm of* $\|.\|_p$*, i.e.,* $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof.
It is sufficient to observe that, if $\gamma_i = \dfrac{1}{r_i}\|.\|_p$, then its dual gauge is $\gamma_i^0 = r_i\|.\|_q$, with $\|.\|_q$ the dual norm of $\|.\|_p$, p and q satisfying that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

□

Problem (23) is equivalent to the formulation given in [37, 38], which was obtained by building the robust counterpart of the problem (by following robust optimization methods, [2, 3]). Hence, formulation (15) for any kind of gauge γ_i is more general than that obtained for robust SVMs.

2.4 Test of linear separability

From now on, we will only consider the case of interval data. Given an interval-valued database, a test of linear separability can be applied to the dataset by solving the linear program proposed in the following result.

Theorem 2.2. *Given $I \subseteq \Omega$ a training sample of interval data, that is, $X_i = \prod_{j=1}^{d}[l_{ij}, u_{ij}]$, for every $i \in I$, there exists a hyperplane separating the two groups, G_{+1} and G_{-1}, iff the problem*

$$\begin{array}{ll} \max\limits_{\lambda,\mu} & \sum\limits_{i\in G_{+1}} \lambda_i + \sum\limits_{k\in G_{-1}} \mu_k \\ s.t. & \sum\limits_{i\in G_{+1}} \lambda_i = \sum\limits_{k\in G_{-1}} \mu_k \\ & \sum\limits_{i\in G_{+1}} \lambda_i l_{ij} - \sum\limits_{k\in G_{-1}} \mu_k u_{kj} \leq 0, \ j = 1,\ldots,d \\ & \sum\limits_{k\in G_{-1}} \mu_k l_{kj} - \sum\limits_{i\in G_{+1}} \lambda_i u_{ij} \leq 0, \ j = 1,\ldots,d \\ & \lambda_i, \mu_k \geq 0, \ \forall i \in G_{+1}, \ \forall k \in G_{-1} \end{array} \tag{24}$$

is feasible with optimal solution equal to 0.

Proof.
Given the training sample I, according to the classification rule (6), the two groups G_{+1} and G_{-1} are linearly separable, that is, there exist ω, β, the parameters of a hyperplane separating the two groups, iff

$$\begin{array}{ll} \min\limits_{x\in X_i} (\omega^\top x + \beta) > 0, & \forall i \in G_{+1} \\ \max\limits_{x\in X_k} (\omega^\top x + \beta) < 0, & \forall k \in G_{-1}. \end{array} \tag{25}$$

By homogeneity, expression (25) is equivalent to say that ω, β exist with

$$\begin{array}{ll} \min\limits_{x\in X_i} \omega^\top x + \beta \geq 1, & \forall i \in G_{+1} \\ \min\limits_{x\in X_k} (-\omega)^\top x - \beta \geq 1, & \forall k \in G_{-1}. \end{array} \tag{26}$$

Now, by using expression (17) and with an analogous reasoning to that used in the proof of Theorem 2.1, we can rewrite (26) as

$$\begin{array}{l} \omega^\top x_i - \gamma_i^0(\omega) + \beta \geq 1, \ \forall i \in G_{+1} \\ -\omega^\top x_k - \gamma_k^0(\omega) - \beta \geq 1, \ \forall k \in G_{-1}, \end{array} \tag{27}$$

and since our database is interval-valued, we consider γ_i the gauge defined in 2, we rewrite expression (27) and we obtain that the two groups G_{+1} and G_{-1} are linearly separable iff the problem

$$\begin{array}{ll} \min\limits_{\sigma,\tau,\beta} & 0^\top \sigma + 0^\top \tau \\ \text{s.t.} & \sum\limits_{j=1}^{d} \sigma_j l_{ij} - \sum\limits_{j=1}^{d} \tau_j u_{ij} + \beta \geq 1, \ \forall i \in G_{+1} \\ & \sum\limits_{j=1}^{d} \tau_j l_{kj} - \sum\limits_{j=1}^{d} \sigma_j u_{kj} - \beta \geq 1, \ \forall k \in G_{-1} \\ & \sigma_j, \tau_j \geq 0, \ j = 1,\ldots,d, \ \beta \text{ s.r.} \end{array} \tag{28}$$

is feasible with optimal solution equal to 0.
And, by using duality properties of linear programming, one can state that the two groups are linearly separable iff the dual of problem (28) is feasible with optimal solution equal to 0. But the dual problem is just the one formulated in (24), and the result follows.

□

3 A multi-class classification experiment with interval data

3.1 The multi-class classification problem for the 'car' dataset

In the previous sections, we have discussed the classification problem for two groups with interval data. However, in many real situations, classification problems arise with data belonging to more than two groups. Solving a multi-class classification problem (see [18, 34, 41]) is, in general, a more difficult task than solving a two-class classification problem. Different strategies have been proposed. Most of these suggest to transform the multi-class problem in a series of two-class problems to be solved (see e.g., [15, 17, 20, 31, 36, 42]).
In this section, we solve a classification problem with interval data belonging to four different groups. Our methodology for classification has been applied to the 'car' dataset, which is a database with 33 car models described by 8 interval variables (explaining the following characteristics of each car model: *price, engine capacity, top speed, acceleration, step, length, width and height*) and one nominal variable which represents one of the four following possible categories: *utilitarian* (U), *berlina* (B), *sportive* (S) or *luxury* (L) (see [13] for more details).
Then, we have a database with four different groups (multiclass classification problem), where the data are Cartesian products of intervals in dimension 8. We have performed several computational experiments, by considering three possible techniques for multiclass classification (1-v-r, 1-v-1 and DDAG).
In order to measure the probability of misclassification, we have used the leave-one-out (LOO) strategy (see e.g. [16, 21]), that is, in turns, we consider only one element in the test sample, we train the model with the remaining elements and we test this model with the unitary test sample. We repeat the process for every element of the database.
Before showing the results of the experiment, the multi-class classification techniques used in the experiments are explained.

One-versus-rest (1-v-r) : N classifiers are constructed, the i-th classifier is the result of solving the corresponding problem (20) where the elements of the i-th group have a label +1 associated and the rest of elements have a label -1 associated.

For every element X of the database, we consider the problem where this element is the only one belonging to the test sample and the rest belong to the training sample and we build the corresponding $N = 4$ classifiers (for the i-th problem, G_{+1} is formed by all the elements of the i-th group). Once ω_i, β_i (the i-th classifier) are obtained, we compute, for X in the test sample, the values $\min_{x \in X}(\omega_i^\top x + \beta_i)$ and

$\max_{x \in X}(\omega_i^\top x + \beta_i)$, and we obtain the intensity of classifying X in the i-th group via (8)-(9).

We assign X to the group with the highest output of the intensity. In case of tie, X is assigned to the group i whose $\min_{x \in X}(\omega_i^\top x + \beta_i)$ is the highest value.

One-versus-one (1-v-1) : In this case, we build a classifier for every possible pair of groups i and j, with $i \neq j$. In total, we need to construct $N(N-1)/2$ classifiers.

Given the test sample with a single element X, we compute every classifier separating two groups i and j. Once ω_{ij}, β_{ij} are obtained, we give one vote to group i if the intensity is higher for i than for j (following the same reasoning as for the 1-v-r case), or equivalently, if $\max_{x \in X}(\omega_{ij}^\top x + \beta_{ij}) \geq -\min_{x \in X}(\omega_{ij}^\top x + \beta_{ij})$. And X is assigned to the group with the highest number of votes (following the Max Wins algorithm, [15]).

In case of tie between two groups i and j, we go back to the classifier separating those groups and we assign to the group with the highest output for the intensity. We only need to build $N(N-1)/2$ comparisons since, for two groups i and j, with $i \neq j$, one obtains that $\omega_{ij} = -\omega_{ji}$ and $\beta_{ij} = -\beta_{ji}$, that is, the classifier is the same (the parameters are opposite).

Decision Directed Acyclic Graphs (DDAG) : This method, proposed in [31], is a modification of the standard 1-v-1 method, which is introduced to reduce the number of classifiers to be built (only $N-1$ classifiers must be computed instead of $N(N-1)/2$ in the 1-v-1 method).

A Directed Acyclic Graph, DAG for short, is a graph with oriented edges and without cycles. In each node of this DAG, one constructs the classifier for two groups i and j and, given an element of the dataset, we assign it to the group with the highest output, after applying the classifier. If we assign the i-th group, we can eliminate the label j as a candidate group for this element, we pass to the next node to compare the group i against another one. The label at the end of this process (after $N-1$ comparisons) will be the assigned group.

An ordering must be imposed to decide which comparisons must be done in each step, but the selection of this ordering is arbitrary. In our case, we follow the DAG depicted in Figure 3 (several experiments done for another ordering yielded very similar results, and in [31] it is said that different orderings did not yield significant changes in terms of accuracy).

We start with the complete list of groups and we build the classifier for the groups 1 and 4. Depending on the assigned group, we take the left or right edge and consequently we eliminate one group. In each node, we build the classifier for the first and the last groups in the new list of groups. And we continue until the end of the graph, where the label is finally assigned.

3.2 Numerical results

All the computational experiments of this paper have been implemented by using AMPL as the modeling language and have been solved via CPLEX or via LOQO, [40], (by using

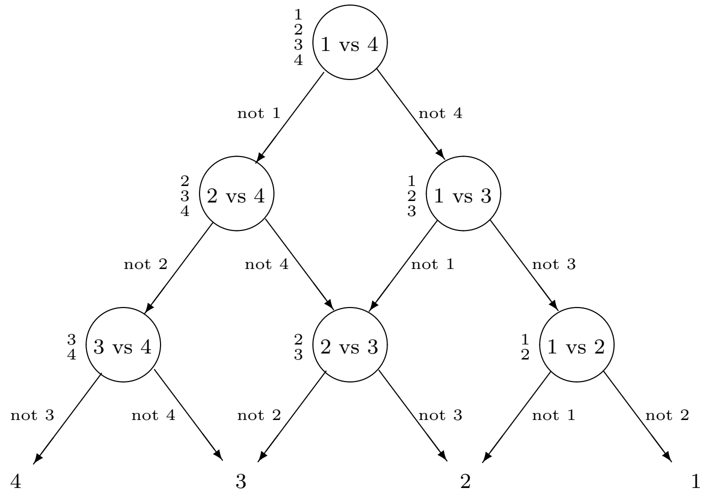


Figure 3: Directed Acyclic Graph

the NEOS server, [28]).
We have solved the classification problem (via leave-one-out) with the three multiclass techniques (1vr, 1v1, DDAG), for several values of the constants, $C_{+1}, C_{-1} \in \{0.001,\ 0.01,\ 0.1,\ 1,\ 10,\ 100,\ 1000\}$ and the results are given in the following tables. In Table 1, we show how many elements from the database of cars are misclassified, that is, how many are given a label different from its own category.
Likewise, we have studied the behaviour of the classifier via a resubstitution strategy (see [11]), that is, when we use the complete set of instances as training sample to build the classifier (no test sample is used) and we apply it later to assign, in turn, a label to each instance. This way, we can study the quality of the classifier as a separator of the database. It is only done in the 1vr and 1v1 method (not in the DDAG method, since this is basically a 1v1 method where not all the comparisons between groups must be done).
In general, the best results for this dataset via leave-one-out and for different values of the constants have been obtained via the 1vr method. In fact, the worst number of misclassified elements in this case is equal to eight (which is quite good in comparison with the other two techniques). The lowest number of misclassified elements for leave-one-out and for resubstitution, through the three multiclass techniques, are shown in the tables in bold. For the 1vr method, there are different combinations of C_{+1} and C_{-1} which give only four misclassified elements for leave-one-out and only two misclassified elements for resubstitution. For high values of C_{+1} and C_{-1}, one obtains good values of well-classified elements, nevertheless, although biggest values of the constants were considered, the results could not be improved more.
On the other hand, although we obtain some high numbers of misclassified elements for the 1v1 method (it is especially remarkable the case when $C_{+1} = 0.001$ or $C_{-1} = 0.001$), the lowest numbers of misclassified elements are obtained with this method. For $C_{+1} = C_{-1} = 0.1$, we obtain only three misclassified elements via leave-one-out, and we got only one misclassified element in the training sample for several combinations of the constants. The results obtained via DDAG are quite similar to those obtained via 1v1.

	C_{-1}	0.001		0.01		0.1		1		10		100		1000	
C_{+1}		loo	rs	loo	rs	loo	rs	loo	rs	loo	rs	loo	rs	loo	rs
0.001	1vr	8	6	**4**	4	7	4	7	4	7	4	7	4	7	5
	1v1	6	5	17	17	17	17	17	17	17	17	17	17	17	18
	ddag	6	-	17	-	17	-	17	-	17	-	17	-	17	
0.01	1vr	7	5	5	5	5	3	6	4	6	3	6	3	6	3
	1v1	13	11	7	4	7	5	7	5	7	5	7	5	7	5
	ddag	11	-	7	-	7	-	7	-	7	-	7	-	7	
0.1	1vr	7	5	5	4	**4**	4	7	5	6	3	6	4	6	4
	1v1	11	9	7	6	**3**	3	4	2	4	2	4	2	4	2
	ddag	11	-	8	-	**4**	-	**4**	-	**4**	-	**4**	-	**4**	
1	1vr	7	5	5	4	5	4	**4**	3	8	5	7	3	7	3
	1v1	11	9	8	6	5	4	5	2	5	2	5	2	5	2
	ddag	11	-	8	-	6	-	5	-	5	-	5	-	5	
10	1vr	7	5	5	4	5	4	**4**	**2**	**4**	**2**	8	4	7	3
	1v1	11	9	8	6	6	4	7	4	6	2	6	**1**	6	**1**
	ddag	11	-	8	-	6	-	7	-	6	-	6	-	6	
100	1vr	7	5	5	4	5	4	**4**	**2**	5	**2**	6	**2**	8	4
	1v1	11	9	8	6	6	4	7	4	8	2	6	**1**	6	**1**
	ddag	9	-	8	-	6	-	7	-	8	-	6	-	6	
1000	1vr	7	5	5	3	5	4	**4**	**2**	5	**2**	7	4	6	**2**
	1v1	12	9	8	6	6	4	7	4	8	2	8	**1**	6	**1**
	ddag	11	-	8	-	6	-	7	-	8	-	8	-	6	

Table 1: Misclassified elements for the 'car dataset' (loo: leave-one-out, rs: resubstitution)

In the following tables, we show the detailed results for accuracy with each technique, for every combination of C_{+1} and C_{-1}. The rows of each cell represent the original category of the car and the columns represent the label which has been assigned to the model through leave-one-out. In bold, we show again the best results for the accuracy in the classification. We can observe that, in general, it is easier to distinguish the utilitarian and the sportive cars, and most of the difficulties arise while trying to discriminate between berlina and luxury cars.

1vr	C_{-1}	0.001				0.01				0.1				1				10				100				1000			
C_{+1}	Cars	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S
0.001	U	7	3	0	0	**10**	**0**	**0**	**0**	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0
	B	1	6	1	0	**1**	**5**	**2**	**0**	1	6	1	0	1	6	1	0	1	6	1	0	1	6	1	0	1	6	1	0
	L	0	3	5	0	**0**	**0**	**8**	**0**	0	3	5	0	0	3	5	0	0	3	5	0	0	3	5	0	0	3	5	0
	S	0	0	0	7	**0**	**1**	**0**	**6**	0	1	0	6	0	1	0	6	0	1	0	6	0	1	0	6	0	1	0	6
0.01	U	9	1	0	0	10	0	0	0	10	0	0	0	10	0	0	0	10	0	0	0	10	0	0	0	10	0	0	0
	B	1	6	1	0	1	6	1	0	1	5	2	0	1	6	1	0	1	6	1	0	1	6	1	0	1	6	1	0
	L	0	3	5	0	0	3	5	0	0	1	7	0	0	3	5	0	0	3	5	0	0	3	5	0	0	3	5	0
	S	0	1	0	6	0	0	0	7	0	1	0	6	0	1	0	6	0	1	0	6	0	1	0	6	0	1	0	6
0.1	U	9	1	0	0	9	1	0	0	**10**	**0**	**0**	**0**	10	0	0	0	10	0	0	0	10	0	0	0	10	0	0	0
	B	1	6	1	0	1	7	0	0	**1**	**5**	**2**	**0**	1	5	2	0	1	6	1	0	1	6	1	0	1	6	1	0
	L	0	3	5	0	0	3	5	0	**0**	**1**	**7**	**0**	0	3	5	0	0	3	5	0	0	3	5	0	0	3	5	0
	S	0	1	0	6	0	0	0	7	**0**	**0**	**0**	**7**	0	0	1	6	0	1	0	6	0	1	0	6	0	1	0	6
1	U	9	1	0	0	9	1	0	0	10	0	0	0	**10**	**0**	**0**	**0**	10	0	0	0	10	0	0	0	10	0	0	0
	B	1	6	1	0	1	7	0	0	1	6	1	0	**1**	**5**	**2**	**0**	1	4	3	0	1	5	2	0	1	5	2	0
	L	0	3	5	0	0	3	5	0	0	3	5	0	**0**	**1**	**7**	**0**	0	3	5	0	0	3	5	0	0	3	5	0
	S	0	1	0	6	0	0	0	7	0	0	0	7	**0**	**0**	**0**	**7**	0	1	0	6	0	1	0	6	0	1	0	6
10	U	9	1	0	0	9	1	0	0	10	0	0	0	**10**	**0**	**0**	**0**	**10**	**0**	**0**	**0**	10	0	0	0	10	0	0	0
	B	1	6	1	0	1	7	0	0	1	6	1	0	**1**	**6**	**1**	**0**	**1**	**6**	**1**	**0**	1	4	3	0	1	5	2	0
	L	0	3	5	0	0	3	5	0	0	3	5	0	**0**	**2**	**6**	**0**	**0**	**2**	**6**	**0**	0	3	5	0	0	3	5	0
	S	0	1	0	6	0	0	0	7	0	0	0	7	**0**	**0**	**0**	**7**	**0**	**0**	**0**	**7**	0	1	0	6	0	1	0	6
100	U	9	1	0	0	9	1	0	0	10	0	0	0	**10**	**0**	**0**	**0**	9	1	0	0	10	0	0	0	10	0	0	0
	B	1	6	1	0	1	7	0	0	1	6	1	0	**1**	**6**	**1**	**0**	1	6	1	0	1	6	1	0	1	4	3	0
	L	0	3	5	0	0	3	5	0	0	3	5	0	**0**	**2**	**6**	**0**	0	2	6	0	0	4	4	0	0	3	5	0
	S	0	1	0	6	0	0	0	7	0	0	0	7	**0**	**0**	**0**	**7**	0	0	0	7	0	0	0	7	0	1	0	6
1000	U	9	1	0	0	9	1	0	0	10	0	0	0	**10**	**0**	**0**	**0**	9	1	0	0	9	1	0	0	10	0	0	0
	B	1	6	1	0	1	7	0	0	1	6	1	0	**1**	**6**	**1**	**0**	1	6	1	0	1	6	1	0	1	6	1	0
	L	0	3	5	0	0	3	5	0	0	3	5	0	**0**	**2**	**6**	**0**	0	2	6	0	0	4	4	0	0	4	4	0
	S	0	1	0	6	0	0	0	7	0	0	0	7	**0**	**0**	**0**	**7**	0	0	0	7	0	0	0	7	0	0	0	7

1v1	C_{-1}	0.001				0.01				0.1				1				10				100				1000			
C_{+1}	Cars	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S
0.001	U	9	1	0	0	1	8	1	0	1	8	1	0	1	8	1	0	1	8	1	0	1	8	1	0	1	8	1	0
	B	1	5	2	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	6	1
	L	0	2	6	0	0	0	7	1	0	0	7	1	0	0	7	1	0	0	7	1	0	0	7	1	0	0	7	1
	S	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
0.01	U	9	1	0	0	9	1	0	0	8	2	0	0	8	2	0	0	8	2	0	0	8	2	0	0	8	2	0	0
	B	0	3	5	0	1	5	2	0	1	3	4	0	1	3	4	0	1	3	4	0	1	3	4	0	1	3	4	0
	L	0	5	3	0	0	3	5	0	0	0	8	0	0	0	8	0	0	0	8	0	0	0	8	0	0	0	8	0
	S	0	1	1	5	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
0.1	U	9	1	0	0	10	0	0	0	**10**	**0**	**0**	**0**	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0
	B	3	5	0	0	1	7	0	0	**1**	**5**	**2**	**0**	1	5	2	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	5	3	0	0	6	2	0	**0**	**0**	**8**	**0**	0	0	8	0	0	0	8	0	0	0	8	0	0	0	8	0
	S	0	1	1	5	0	0	0	7	**0**	**0**	**0**	**7**	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
1	U	9	1	0	0	9	1	0	0	10	0	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	5	2	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	5	3	0	0	6	2	0	0	4	4	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	7	0
	S	0	1	1	5	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
10	U	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	6	1	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	5	3	0	0	6	2	0	0	4	4	0	0	4	4	0	0	2	6	0	0	2	6	0	0	2	6	0
	S	0	1	1	5	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
100	U	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	0	0	1	9	0	0	1
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	6	1	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	5	3	0	0	6	2	0	0	4	4	0	0	4	4	0	0	4	4	0	0	2	6	0	0	2	6	0
	S	0	1	1	5	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
1000	U	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	6	1	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	5	3	0	0	6	2	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	2	6	0
	S	0	1	2	4	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7

DDAG	C_{-1}	0.001				0.01				0.1				1				10				100				1000			
C_{+1}	Cars	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S	U	B	L	S
0.001	U	9	1	0	0	1	8	1	0	1	8	1	0	1	8	1	0	1	8	1	0	1	8	1	0	1	8	1	0
	B	1	5	2	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	7	0
	L	0	2	6	0	0	0	7	1	0	0	7	1	0	0	7	1	0	0	7	1	0	0	7	1	0	0	7	1
	S	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
0.01	U	9	1	0	0	9	1	0	0	8	2	0	0	8	2	0	0	8	2	0	0	8	2	0	0	8	2	0	0
	B	3	5	0	0	1	5	2	0	1	3	4	0	1	3	4	0	1	3	4	0	1	3	4	0	1	3	4	0
	L	0	5	3	0	0	3	5	0	0	0	8	0	0	0	8	0	0	0	8	0	0	0	8	0	0	0	8	0
	S	0	1	1	5	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
0.1	U	9	1	0	0	9	1	0	0	**9**	**1**	**0**	**0**	**9**	**1**	**0**	**0**	**9**	**1**	**0**	**0**	**9**	**1**	**0**	**0**	**9**	**1**	**0**	**0**
	B	3	5	0	0	1	7	0	0	**1**	**5**	**2**	**0**	**1**	**5**	**2**	**0**	**1**	**5**	**2**	**0**	**1**	**5**	**2**	**0**	**1**	**5**	**2**	**0**
	L	0	5	3	0	0	6	2	0	**0**	**0**	**8**	**0**	**0**	**0**	**8**	**0**	**0**	**0**	**8**	**0**	**0**	**0**	**8**	**0**	**0**	**0**	**8**	**0**
	S	0	1	1	5	0	0	0	7	**0**	**0**	**0**	**7**	**0**	**0**	**0**	**7**	**0**	**0**	**0**	**7**	**0**	**0**	**0**	**7**	**0**	**0**	**0**	**7**
1	U	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	5	2	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	5	3	0	0	6	2	0	0	4	4	0	0	1	7	0	0	1	7	0	0	1	7	0	0	1	7	0
	S	0	1	1	5	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
10	U	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	6	1	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	5	3	0	0	6	2	0	0	4	4	0	0	4	4	0	0	2	6	0	0	2	6	0	0	2	6	0
	S	0	1	1	5	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
100	U	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	0	1	0	9	0	1	0
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	6	1	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	4	4	0	0	6	2	0	0	4	4	0	0	4	4	0	0	4	4	0	0	2	6	0	0	2	6	0
	S	0	0	1	6	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7
1000	U	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	1	0	0	9	0	1	0
	B	3	5	0	0	1	7	0	0	1	7	0	0	1	6	1	0	1	5	2	0	1	5	2	0	1	5	2	0
	L	0	4	4	0	0	6	2	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	2	6	0
	S	0	0	3	4	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7	0	0	0	7

original	loo results U	B	L	S	rs results U	B	L	S
U	10	0	0	0	10	0	0	0
B	3	3	2	0	2	4	2	0
L	0	2	6	0	0	1	7	0
S	0	2	0	5	0	0	0	7

Table 2: Best results of accuracy in [13]

Finally, we compare our results with those obtained in [13]. In that paper, several strategies are applied to classify the elements of this database, and the best results are obtained for a distributional approach and are shown in Table 2.
We can observe that the number of misclassified elements is equal to nine (with five misclassified elements on the training sample), while our lowest number of misclassified elements (obtained for 1v1) is equal to three (with three misclassified elements in the resubstitution process). In fact, for 1vr we had that for every combination of the constants, the number of misclassified elements was always smaller than or equal to eight. In [13], it is explained that their methods had tendency to overfit the data. In fact in some of their experiments, they obtained two misclassified elements on the training sample, but higher numbers for the test sample. Our method gets even better results in some cases for the training sample (in 1v1, we got only one mistake) and much better results for the test sample in most of the cases. Then, the results have been clearly improved and hence, one can say that our method is competitive.

4 Computational experiment with uncertain values

4.1 Computational experiment

We have applied our model to a database where the single instances have been transformed into intervals to represent uncertainty on the data. We have used the 'breast-cancer' dataset, which can be downloaded from the UCI Machine Learning Repository [6]. This dataset is composed of 699 instances, each one representing an individual affected by breast cancer, and 9 measurements (represented via a number between 1 and 10) have been taken from each individual. The instances are classified as benign (group G_{+1}) or malignant (G_{-1}). In the database, there are 16 instances with missing values (all of them for the sixth variable), which have been erased for the study (then, there are 444 instances in G_{+1} and 239 instances in G_{-1}, 683 in total).
In order to construct the interval, we have computed the standard deviation σ_{x_j} in each coordinate j from 1 to d of each group of the dataset (G_{+1} and G_{-1}). Then, each coordinate x_{ij} is replaced by the interval $[x_{ij} - k\sigma_{x_j}, x_{ij} + k\sigma_{x_j}]$, with σ_{x_j} the standard deviation of the corresponding group. The values for k are 0, 0.01, 0.05, 0.1, 0.2, 0.5, 0.75 and 1. The higher the value of k, the larger the uncertainty about the data. Observe that, when

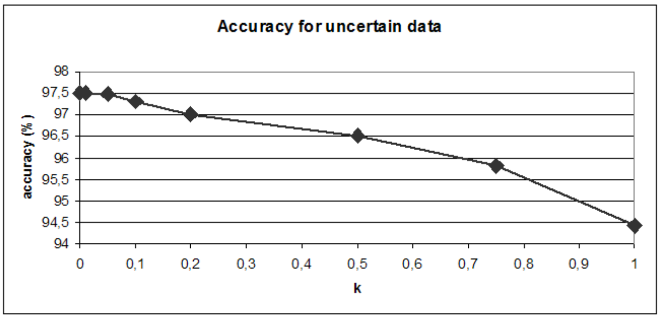


Figure 4: Results for uncertain data

$k = 0$, we obtain the original database.
We have solved the corresponding classification problem with the interval data through 10-fold cross validation, that is, the instances of the database are grouped in 10 sets (these sets forming a partition) and, each one has been used in turn as test set against all 9 others taken together as training set, that is, the process is repeated ten times (see [21]). The optimization problem (20) to compute the parameters of each classifier has been solved via AMPL+CPLEX, for different pairs of C_{+1} and C_{-1}. And we have computed the classification accuracy when the fuzzy rule is considered.

4.2 Numerical results

In the following tables, we present the results for the interval dataset, depending on the value of the parameter k. In Figure 4, the best results in the test sample for each value of k are depicted.
One can observe that, although in general the results get worse as the value of k is increasing, this process is quite smooth, and the results for the accuracy are quite similar in the first tables. In fact, the best results for accuracy continue being quite good when k increases (the accuracy is over 94% in each case).
We can also observe that the best values of accuracy are obtained for similar or almost similar values of the constants C_{+1} and C_{-1}. In fact, as k increases, the accuracy in the training and test sample for different values of C_{+1} and C_{-1} is very bad (in some cases, we obtain degenerate solutions for a pair of values (C_{+1}, C_{-1}), with $C_{+1} \neq C_{-1}$), while the accuracy in the diagonal of the table continues being very acceptable (around 95%).

		Train							Test						
k	$C_{+1} \backslash C_{-1}$	0.001	0.01	0.1	1	10	100	1000	0.001	0.01	0.1	1	10	100	1000
0	0.001	96.88	97.49	95.40	92.61	91.93	91.80	91.80	96.64	**97.50**	95.30	92.26	91.81	91.81	91.81
	0.01	94.05	97.32	97.46	95.67	94.78	94.61	94.60	94.14	96.63	**97.50**	95.59	94.72	94.58	94.58
	0.1	82.37	94.16	97.25	97.43	95.75	95.12	94.92	81.56	93.40	96.78	97.36	95.74	94.87	94.72
	1	73.34	84.48	94.35	97.20	97.43	95.77	95.17	72.76	83.46	93.55	96.78	97.21	95.74	94.87
	10	72.51	78.93	85.15	94.35	97.20	97.41	95.77	72.47	78.04	84.20	93.55	96.78	97.21	95.74
	100	72.41	77.96	80.48	85.25	94.35	97.20	97.41	72.47	77.17	79.21	84.20	93.55	96.78	97.21
	1000	72.41	77.91	79.63	80.66	85.25	94.35	97.20	72.47	77.17	78.77	79.21	84.20	93.55	96.78
0.01	0.001	96.90	97.48	95.16	92.31	91.55	91.50	91.50	96.64	**97.50**	95.20	92.16	91.58	91.53	91.52
	0.01	94.03	97.32	97.44	95.55	94.55	94.31	94.26	94.11	96.61	**97.50**	95.52	94.35	94.02	94.04
	0.1	82.18	94.24	97.26	97.43	95.69	94.99	94.86	81.43	93.63	96.78	97.40	95.52	94.72	94.56
	1	73.00	84.42	94.30	97.24	97.43	95.71	95.03	72.76	83.19	93.66	96.78	97.31	95.53	94.71
	10	72.19	78.28	84.78	94.32	97.23	97.43	95.71	72.04	77.55	84.00	93.68	96.78	97.31	95.53
	100	72.13	77.47	79.70	84.85	94.32	97.23	97.43	72.04	77.04	79.08	84.02	93.68	96.78	97.31
	1000	72.12	77.39	78.87	79.85	84.86	94.32	97.23	72.05	77.01	78.49	79.16	84.02	93.68	96.78
0.05	0.001	96.85	97.42	94.32	90.53	89.88	89.80	89.80	96.60	**97.48**	93.93	90.29	89.35	89.28	89.28
	0.01	94.09	97.29	97.38	95.18	93.64	93.36	93.34	93.92	96.73	97.35	95.00	93.40	93.13	93.10
	0.1	81.05	94.43	97.29	97.36	95.28	94.06	93.80	80.47	93.89	96.67	97.30	95.01	93.58	93.44
	1	71.97	83.39	94.48	97.29	97.35	95.29	94.11	71.89	82.24	93.87	96.69	97.30	95.02	93.64
	10	71.35	75.84	83.71	94.49	97.29	97.35	95.29	71.32	75.86	82.58	93.87	96.69	97.30	95.02
	100	71.27	75.07	76.76	83.74	94.49	97.29	97.35	71.26	75.03	76.64	82.62	93.87	96.69	97.30
	1000	71.27	75.00	76.10	76.86	83.75	94.49	97.29	71.25	74.98	75.90	76.73	82.62	93.87	96.69
0.1	0.001	96.81	97.26	93.12	87.78	86.97	86.89	86.88	96.55	97.30	92.79	87.33	86.68	86.61	86.59
	0.01	94.12	97.26	97.28	94.42	92.25	91.95	91.91	93.77	96.77	**97.31**	94.01	91.70	91.46	91.42
	0.1	79.45	94.52	97.30	97.29	94.56	92.64	92.36	78.87	93.84	96.81	97.26	94.20	92.27	92.01
	1	70.70	81.87	94.57	97.30	97.28	94.58	92.68	70.41	80.97	93.85	96.81	97.26	94.22	92.34
	10	69.76	72.86	82.19	94.58	97.30	97.28	94.58	69.75	72.73	81.30	93.85	96.81	97.26	94.22
	100	69.69	72.35	73.34	82.22	94.58	97.30	97.28	69.68	72.35	73.18	81.33	93.85	96.81	97.26
	1000	69.69	72.30	72.84	73.40	82.22	94.58	97.30	69.67	72.29	72.89	73.24	81.34	93.85	96.81

		Train							Test						
k	$C_{+1} \backslash C_{-1}$	0.001	0.01	0.1	1	10	100	1000	0.001	0.01	0.1	1	10	100	1000
0.2	0.001	96.83	96.98	88.59	74.72	72.08	71.75	71.72	96.63	**97.00**	88.27	74.22	71.57	71.28	71.25
	0.01	93.89	97.13	96.94	91.33	85.65	85.26	85.19	93.56	96.71	96.99	90.96	84.82	84.48	84.41
	0.1	75.63	94.33	97.20	96.93	91.62	86.68	86.39	75.08	93.72	96.73	96.94	91.20	85.81	85.57
	1	67.44	78.22	94.39	97.21	96.93	91.65	86.79	67.16	77.80	93.69	96.73	96.93	91.23	85.92
	10	66.65	68.83	78.52	94.39	97.21	96.93	91.65	66.47	68.62	78.14	93.69	96.73	96.93	91.23
	100	66.64	67.69	69.01	78.55	94.39	97.21	96.93	66.45	67.30	68.82	78.17	93.69	96.73	96.93
	1000	66.64	67.65	67.88	69.03	78.55	94.39	97.21	66.45	67.26	67.48	68.84	78.17	93.69	96.73
0.5	0.001	96.55	95.00	35.21	34.99	34.99	34.99	34.99	96.42	94.79	35.18	34.99	34.99	34.99	34.99
	0.01	91.42	96.67	95.19	36.32	34.99	34.99	34.99	91.04	**96.50**	94.99	36.31	34.99	34.99	34.99
	0.1	65.55	92.03	96.67	95.19	36.51	34.99	34.99	65.58	91.63	96.49	95.00	36.50	34.99	34.99
	1	65.01	65.86	92.10	96.67	95.19	36.53	34.99	65.01	65.84	91.68	96.49	95.00	36.52	34.99
	10	65.01	65.01	65.90	92.10	96.67	95.19	36.54	65.01	65.01	65.88	91.69	96.49	95.00	36.52
	100	65.01	65.01	65.01	65.90	92.10	96.67	95.19	65.01	65.01	65.01	65.88	91.69	96.49	95.00
	1000	65.01	65.01	65.01	65.01	65.90	92.10	96.67	65.01	65.01	65.01	65.01	65.88	91.69	96.49
0.75	0.001	95.86	77.29	34.99	34.99	34.99	34.99	34.99	95.72	77.19	34.99	34.99	34.99	34.99	34.99
	0.01	86.13	96.02	81.72	34.99	34.99	34.99	34.99	85.64	**95.81**	81.59	34.99	34.99	34.99	34.99
	0.1	65.01	87.23	96.03	82.13	34.99	34.99	34.99	65.01	86.80	95.81	82.00	34.99	34.99	34.99
	1	65.01	65.01	87.35	96.03	82.17	34.99	34.99	65.01	65.01	86.91	95.81	82.04	34.99	34.99
	10	65.01	65.01	65.01	87.36	96.03	82.18	34.99	65.01	65.01	65.01	86.92	95.81	82.04	34.99
	100	65.01	65.01	65.01	65.01	87.36	96.03	82.18	65.01	65.01	65.01	65.01	86.92	95.81	82.04
	1000	65.01	65.01	65.01	65.01	65.01	87.36	96.03	65.01	65.01	65.01	65.01	65.01	86.92	95.81
1	0.001	94.08	34.99	34.99	34.99	34.99	34.99	34.99	93.96	34.98	34.99	34.99	34.99	34.99	34.99
	0.01	77.70	94.55	34.99	34.99	34.99	34.99	34.99	77.27	94.37	34.98	34.99	34.99	34.99	34.99
	0.1	65.01	79.66	94.62	34.99	34.99	34.99	34.99	65.01	79.21	94.42	34.97	34.99	34.99	34.99
	1	65.01	65.01	79.87	94.62	34.99	34.99	34.99	65.01	65.01	79.41	94.43	34.97	34.99	34.99
	10	65.01	65.01	65.01	79.89	94.63	34.99	34.99	65.01	65.01	65.01	79.43	**94.43**	34.97	34.99
	100	65.01	65.01	65.01	65.01	79.89	94.63	34.99	65.01	65.01	65.01	65.01	79.43	**94.43**	34.97
	1000	65.01	65.01	65.01	65.01	65.01	79.89	94.63	65.01	65.01	65.01	65.01	65.01	79.43	**94.43**

5 Computational experiment with missing values

5.1 Imputation for missing values via intervals

In the task of editing survey data, the term ‘missing data’ is used to denote invalid blanks in an entry of any field of the survey (invalid in the sense that this value should appear in the dataset) or inconsistent entries.
One of the most widely-used strategies to deal with missing data is the imputation for given records, which means to replace the missing values of a database by other plausible values in such a way that the data must remain consistent.
Different methodologies for imputation have been studied in the literature (see [23, 24, 26] for a list of them), like using the mean (for quantitative variables) or the mode (for qualitative variables) of the non-missing values of the database (see e.g. [1]). One of the drawbacks of this method is that the standard deviation of the sample is ignored and it can contain relevant information which should be taken into account during the imputation process (see [33]).
Our idea to impute the missing values is to replace each blank by an interval (instead of a single value) built with the non-missing values of the dataset. This way, when a blank appears in the j-th variable of an observation, we use the non-missing values in the j-th variable to construct the interval.
Two different strategies will be followed to impute the missing values, by transforming our database into an interval-valued dataset. The first one is to build an interval based on the mean and deviation of the remaining values. This way, the standard deviation is taken into account for the imputation process. For a missing value in the j-th variable of an instance of the training sample, we fill it in by computing the mean $\overline{x}_j$ and the standard deviation σ_{x_j} for the values in this column of the remaining values, and afterwards, we substitute the blank by the interval $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$.
The second strategy is not based on the mean and the deviation, but on the quantiles. We consider the interval which is defined as $[Q_a, Q_{1-a}]$, where Q_a represents the a-th quantile, and thus the interval contains all but a fraction $2a$ of the all non-missing values.

5.2 Computational experiment with missing data completely at random

Our model for classification with interval data has been applied to a database for dealing with missing values. We have used the ‘breast-cancer’ dataset (UCI Machine Learning Repository [6]), with 683 instances in total.
The missing data have been generated in the database completely at random. A parameter p has been defined as the probability of replacing the value of a variable in the database by a missing value. The following values of p have been incorporated in the computational experiment: 0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4 and 0.5. That is, we have performed numerical experiments where very few values have been erased ($p = 0.01, 0.05$) and where around half of the database is missing ($p = 0.5$).
With this modified database, for each value of p, we have solved the corresponding classification problem through 10-fold cross validation (see [21]).

Before solving the corresponding optimization problem (20), we have used the two different strategies explained before for imputation. In the first strategy, we replace the blank by the interval $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$, where $\overline{x}_j$ and σ_{x_j} have been computed with the non-missing values in the j-th column of the instances of the corresponding group (G_{+1} or G_{-1}). In the case of a missing value in any coordinate of an instance in the test sample, we compute the mean and the deviation of the corresponding variable for the remaining instances of the database (since the group which it belongs to is unknown). The values for k are 0, 0.01, 0.05, 0.1, 0.2, 0.5, 0.75 and 1. Observe that, when $k = 0$, the interval is a single point and we are considering the imputation to the mean.
In the second strategy, the blank is replaced by the interval $[Q_a, Q_{1-a}]$, Q_a being the a-th quantile. The values studied for $2a$ are 0, 0.01, 0.05, 0.1, 0.2, 0.5 and 1. Observe that, when $2a = 0$, the interval is the range of the variable, when $2a = 0.5$, we obtain the interquartile range, and when $2a = 1$, the interval is reduced to a singleton which is the median of the variable.
Furthermore, another standard interval based on quantiles is included in this analysis. It is the case of the interval defined by the inner fences (see [39]), which is an interval used in the analysis of outliers (every observation which is not contained between the inner fences is considered as an outlier). This interval is based on the quartiles of the sample,

$$IF = [Q_{0.25} - 1.5 \cdot (Q_{0.75} - Q_{0.25}), Q_{0.75} + 1.5 \cdot (Q_{0.75} - Q_{0.25})]. \tag{29}$$

All these previous modifications of the database have been performed with Matlab 6.5.
Then, we solve the optimization problem (20) to compute the parameters of the classifier with LOQO, [40], for different values of C, where $C_{+1} = C_{-1} = C$ (previous experiments showed that the best results were obtained when the two constants were close to each other). The two classification rules (fuzzy and crisp) have been considered, and in the following tables, we present the results of the accuracy in each case.

5.3 Numerical results

The accuracy, following the crisp or the fuzzy rule, for the different datasets obtained as a result of erasing values with a probability p and replacing those missing values by intervals depending on the parameter k or a, are presented in the following tables. In the tables, we show the accuracy in the training sample, in each group of the test sample, and the average in the test sample. The best results of accuracy for each combination of (p, k) or $(p, 2a)$ (including the IF-interval, defined in (29)), are shown in the tables in bold and are depicted in Figures 5-9.
In the case of replacing the missing values by the interval $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$, one can observe that, when the percentage of introduced missing values is not too high (less than 30%), we obtain better results (or at least similar results) with interval imputation than with only mean imputation (case $k = 0$). In general, the best results are obtained for small intervals (k equal to 0.01 or 0.05). When the parameter p increases, the accuracy for high values of k decreases because, although the results for the training sample continue being very good, the accuracy in the test sample is worse, because the classifier overfits the parameters for the training sample.
In the case of intervals based on quantiles (the different values of $2a$ and the IF-interval), one can observe that, for the crisp classification rule, the interquartile range ($2a = 0.5$)

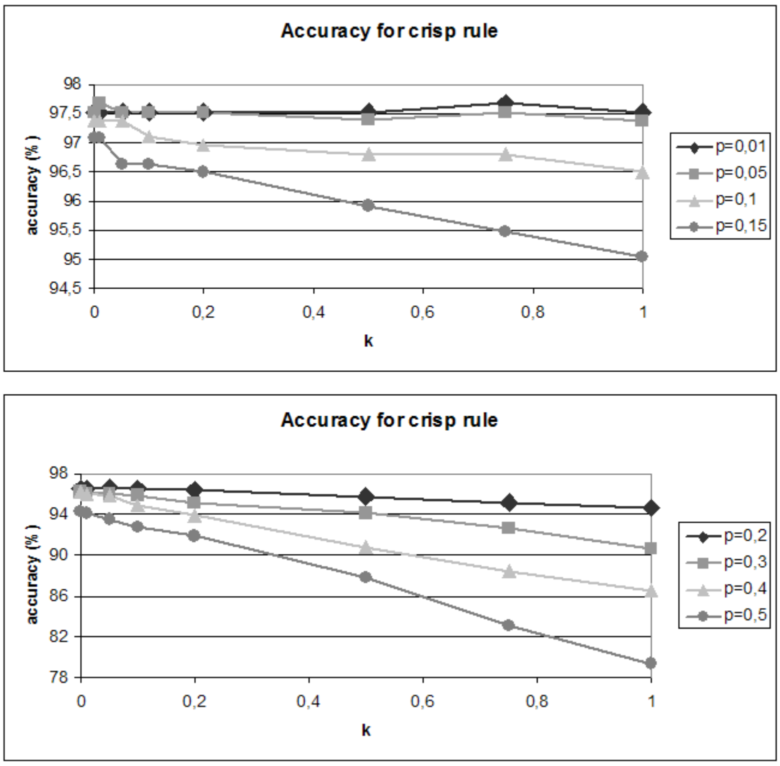


Figure 5: Accuracy for the crisp rule when using the interval $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$

seems to be a better imputation than the median ($2a = 1$), because the results are better or equal in every case for $p \leq 0.3$. This is not so clear when using the fuzzy rule. And when p increases, the results obtained for small values of a (bigger intervals) are not very good, due to the overfitting.
Likewise, we must say that, in general, the results obtained for the intervals based on the mean and the deviation are better than those obtained for intervals based on quantiles.

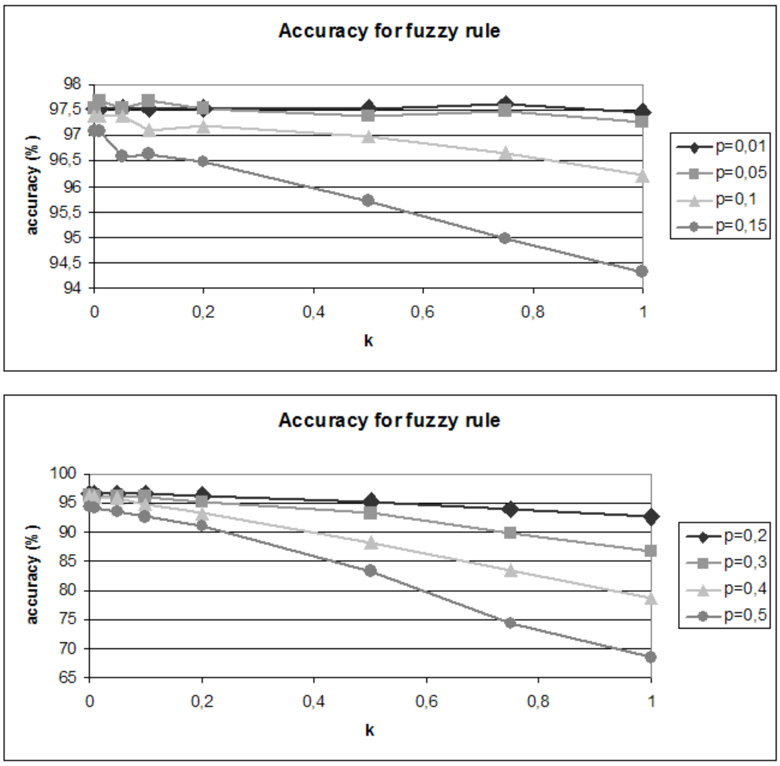


Figure 6: Accuracy for the fuzzy rule when using the interval $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$

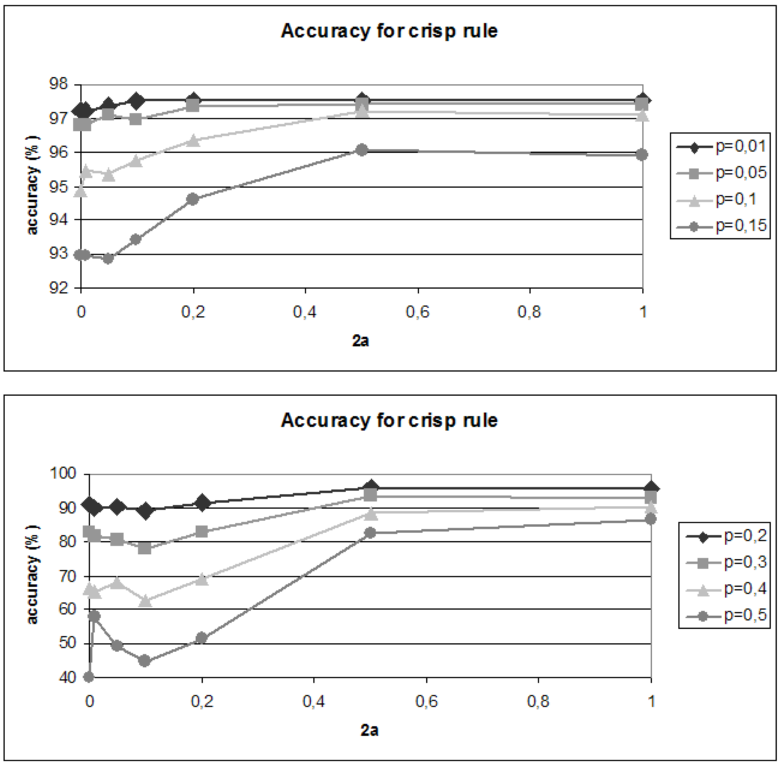
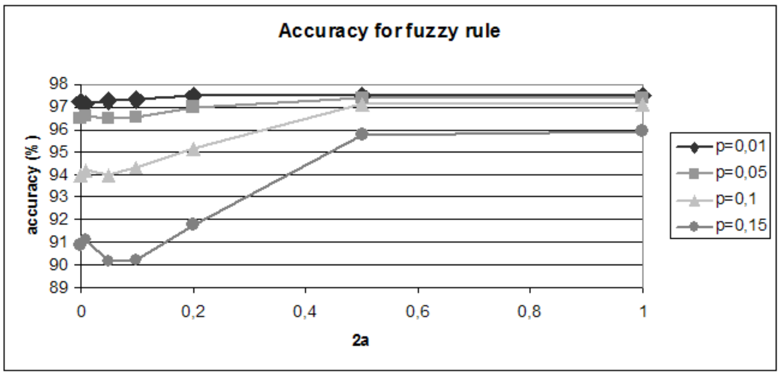


Figure 7: Accuracy for the crisp rule when using the interval $[Q_a, Q_{1-a}]$



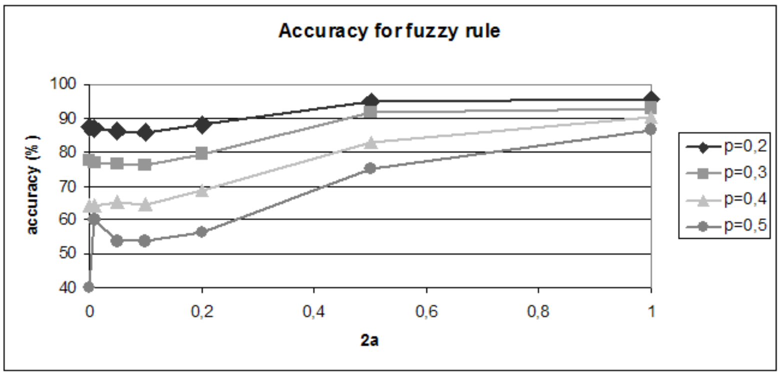


Figure 8: Accuracy for the fuzzy rule when using the interval $[Q_a, Q_{1-a}]$

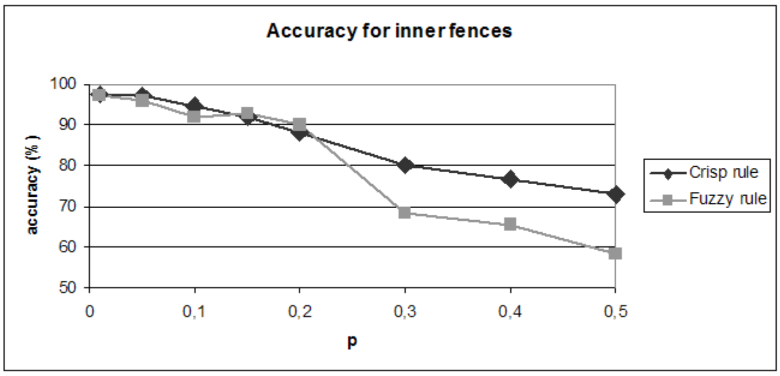


Figure 9: Accuracy for the two rules when using the inner fences

CRISP	k	0				0.01				0.05				0.1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.49	97.31	97.50	97.38	97.49	97.31	97.50	97.38	97.49	97.31	97.50	97.38	97.49	97.31	97.08	97.23
	10	97.53	97.10	97.92	97.39	97.51	97.10	97.92	97.39	97.49	97.10	97.92	97.39	97.49	97.10	97.92	97.39
	10^2	97.61	97.31	97.92	**97.52**	97.62	97.31	97.92	**97.52**	97.59	97.31	97.92	**97.52**	97.62	97.31	97.92	**97.52**
	10^3	97.64	97.33	97.08	97.24	97.67	97.10	97.50	97.24	97.64	97.10	97.92	97.39	97.67	97.10	97.92	97.39
	10^4	97.67	97.33	97.08	97.24	97.64	97.10	97.50	97.24	97.66	97.54	96.67	97.23	97.66	97.10	97.92	97.39
	10^5	97.66	97.33	97.50	97.39	97.64	97.54	96.67	97.23	97.64	97.31	97.92	**97.52**	97.67	97.54	96.67	97.23
	10^6	97.71	97.33	96.67	97.10	97.71	97.33	96.67	97.10	97.62	97.54	97.08	97.38	97.61	97.33	97.08	97.24
0.05	1	97.61	97.10	96.25	96.80	97.63	97.10	96.25	96.80	97.63	97.10	96.25	96.80	97.64	97.10	96.25	96.80
	10	97.61	97.10	97.91	97.39	97.63	97.10	97.92	97.39	97.61	97.10	97.92	97.39	97.62	97.10	97.92	97.39
	10^2	97.72	97.33	97.50	97.39	97.67	97.31	97.92	97.52	97.72	97.10	97.92	97.39	97.69	97.10	97.92	97.39
	10^3	97.74	97.54	97.50	**97.52**	97.67	97.54	97.92	**97.67**	97.67	97.33	97.92	**97.53**	97.67	97.33	97.92	**97.53**
	10^4	97.71	97.54	97.50	**97.52**	97.71	97.54	97.50	97.52	97.74	97.54	97.50	97.52	97.77	97.33	97.92	**97.53**
	10^5	97.74	97.54	97.50	**97.52**	97.71	97.54	97.50	97.52	97.74	97.54	97.50	97.52	97.74	97.33	97.08	97.24
	10^6	97.72	97.54	97.50	**97.52**	97.76	97.54	97.08	97.38	97.72	97.55	97.08	97.38	97.74	97.54	93.74	96.21
0.10	1	97.87	97.10	95.83	96.66	97.87	97.10	95.83	96.66	97.85	97.10	95.42	96.51	97.89	97.10	95.42	96.51
	10	98.03	97.31	97.50	**97.38**	98.02	97.31	97.50	**97.38**	98.00	97.31	97.50	**97.38**	98.00	97.10	97.08	**97.10**
	10^2	98.11	97.33	97.08	97.24	98.15	97.08	97.08	97.08	98.10	97.08	97.08	97.08	98.13	97.31	96.25	96.94
	10^3	98.18	97.54	95.83	96.94	98.10	97.54	96.67	97.23	98.03	97.54	96.67	97.23	98,05	97.33	95.42	96.66
	10^4	98.13	97.54	95.83	96.94	98.11	97.54	96.25	97.09	98.16	97.54	96.25	97.09	98.13	97.33	95.83	96.81
	10^5	98.11	97.54	96.25	97.09	98.13	97.54	95.83	96.94	98.13	97.54	95.83	96.94	98.03	97.33	96.25	96.95
	10^6	98.13	97.54	95.83	96.94	98.13	97.54	96.25	97.09	98.08	97.54	95.83	96.94	98.10	97.33	96.25	96.95
0.15	1	98.10	97.31	94.57	96.35	98.08	97.31	94.57	96.35	98.06	97.31	94.57	96.35	98.00	97.54	94.57	96.50
	10	98.15	97.54	95.40	96.79	98.15	97.31	95.40	96.64	98.11	97.31	95.40	**96.64**	98.10	97.31	94.98	96.50
	10^2	98.29	97.54	96.23	**97.08**	98.34	97.54	96.23	**97.08**	98.32	97.31	94.98	96.50	98.29	97.31	95.40	**96.64**
	10^3	98.52	97.54	93.73	96.21	98.60	97.54	93.73	96.21	98.52	97.54	94.57	96.50	98.50	97.54	94.15	96.35
	10^4	98.62	97.54	93.32	96.06	98.65	97.54	93.73	96.21	98.62	97.54	92.48	95.77	98.57	97.54	93.32	96.06
	10^5	98.62	97.54	92.48	95.77	98.65	97.54	93.32	96.06	98.63	97.54	92.90	95.91	98.60	97.54	92.90	95.91
	10^6	98.62	97.54	92.90	95.91	98.54	97.54	92.90	95.92	98.45	97.54	94.57	96.50	98.57	97.54	92.90	95.91

CRISP	k	0.2				0.5				0.75				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.49	97.31	97.08	97.23	97.51	97.31	97.08	97.23	97.51	97.31	97.08	97.23	97.48	97.31	97.08	97.23
	10	97.49	97.31	97.92	97.52	97.49	97.10	97.92	97.39	97.53	97.10	97.92	97.39	97.54	97.10	97.92	97.39
	10^2	97.59	97.31	97.92	97.52	97.58	97.31	97.50	97.38	97.62	97.54	97.92	**97.67**	97.58	97.31	97.92	**97.52**
	10^3	97.64	97.54	97.50	**97.52**	97.66	97.31	97.50	97.38	97.66	97.54	97.50	97.52	97.67	97.54	97.08	97.38
	10^4	97.66	97.54	97.50	**97.52**	97.72	97.54	97.50	**97.52**	97.72	97.54	97.08	97.38	97.66	97.54	96.67	97.23
	10^5	97.67	97.31	97.50	97.38	97.71	97.33	97.08	97.24	97.67	97.54	95.83	96.94	97.69	97.54	96.67	97.23
	10^6	97.74	97.33	97.08	97.24	97.66	97.54	97.08	97.38	97.72	97.54	96.25	97.09	97.67	97.54	96.25	97.09
0.05	1	97.64	97.10	96.25	96.80	97.54	97.31	95.83	96.79	97.45	97.31	95.83	96.79	97.32	97.31	96.25	96.94
	10	97.62	97.10	97.92	97.39	97.62	97.10	97.08	97.10	97.61	97.10	97.08	97.10	97.53	97.10	97.08	97.10
	10^2	97.72	97.10	97.50	97.24	97.72	97.10	97.50	97.24	97.72	97.31	97.92	**97.52**	97.69	97.54	97.08	**97.38**
	10^3	97.72	97.33	97.50	97.39	97.72	97.33	97.50	**97.39**	97.71	97.33	97.50	97.39	97.77	97.31	97.08	97.23
	10^4	97.72	97.33	97.92	**97.53**	97.79	97.33	97.08	97.24	97.75	97.10	96.67	96.95	97.77	97.31	96.67	97.09
	10^5	97.72	97.33	97.92	**97.53**	97.79	97.33	97.08	97.24	97.74	97.10	97.08	97.10	97.72	97.31	96.67	97.09
	10^6	97.75	97.33	97.08	97.24	97.77	97.33	97.50	**97.39**	97.79	97.54	97.08	97.38	97.79	97.31	96.67	97.09
0.10	1	97.87	97.10	95.00	96.37	97.64	97.31	95.42	96.65	97.64	97.31	95.42	96.65	97.67	97.10	93.73	95.92
	10	97.87	97.10	96.67	96.95	97.74	97.10	96.25	**96.80**	97.74	97.10	96.25	**96.80**	97.64	97.10	95.42	**96.51**
	10^2	97.93	96.87	97.08	96.95	97.76	96.87	95.00	96.22	97.76	96.87	95.00	96.22	97.62	96.65	94.17	95.78
	10^3	97.95	97.31	95.83	96.79	97.79	97.31	95.42	96.65	97.79	97.31	95.42	96.65	97.67	96.87	94.17	95.93
	10^4	97.82	97.33	96.25	**96.95**	97.82	97.31	95.83	96.79	97.82	97.31	95.83	96.79	97.67	97.08	93.73	95.91
	10^5	97.85	97.10	96.67	96.95	97.84	97.08	96.25	96.79	97.84	97.08	96.25	96.79	97.70	97.08	94.17	96.06
	10^6	97.89	97.08	96.67	96.94	97.82	96.86	96.25	96.64	97.82	96.86	96.25	96.64	97.71	97.08	94.17	96.06
0.15	1	98.02	97.54	93.73	96.21	97.84	97.31	92.90	95.77	97.56	97.31	91.23	95.18	97.22	97.31	90.82	**95.04**
	10	98.08	97.54	94.57	**96.50**	98.03	97.08	93.73	**95.91**	97.93	96.86	92.48	95.33	97.54	97.10	90.40	94.76
	10^2	98.26	97.31	94.57	96.35	98.08	97.08	93.32	95.76	97.93	96.86	92.07	95.18	97.85	96.86	90.40	94.60
	10^3	98.32	97.54	94.57	**96.50**	98.21	97.08	93.32	95.76	97.97	97.08	91.65	95.18	97.85	97.08	90.82	94.89
	10^4	98.54	97.54	94.15	96.35	98.39	97.54	92.48	95.77	97.97	97.08	92.48	**95.47**	97.85	96.86	90.82	94.74
	10^5	98.49	97.54	93.73	96.21	98.34	97.54	92.48	95.77	97.95	96.86	92.07	95.18	97.92	96.86	90.40	94.60
	10^6	98.55	97.54	93.32	96.06	98.29	97.31	92.48	95.62	97.98	96.86	92.48	95.33	97.89	97.08	90.40	94.74

CRISP	k	0				0.01				0.05				0.1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	98.24	97.54	94.57	**96.50**	98.24	97.54	94.57	**96.50**	98.24	97.77	94.57	**96.65**	98.24	97.54	94.57	**96.50**
	10	98.42	97.54	94.57	**96.50**	98.42	97.54	94.57	**96.50**	98.44	97.33	94.15	96.22	98.39	97.33	94.57	96.36
	10^2	98.73	97.54	92.90	95.91	98.70	97.54	92.07	95.62	98.70	97.54	93.32	96.06	98.68	97.54	93.32	96.06
	10^3	98.93	97.29	91.21	95.16	98.86	97.31	92.90	95.77	98.91	96.86	91.23	94.89	98.84	97.06	91.21	95.02
	10^4	99.01	97.52	90.80	95.17	98.99	97.52	91.63	95.46	99.02	97.75	90.80	95.31	98.94	97.52	90.80	95.17
	10^5	99.06	97.75	90.38	95.17	99.02	97.52	91.63	95.46	99.01	97.31	90.40	94.89	99.04	97.52	89.96	94.88
	10^6	99.02	97.75	92.46	95.90	99.02	97.75	91.63	95.61	99.02	97.54	91.23	95.33	99.01	97.29	90.80	95.02
0.3	1	98.98	97.33	93.73	96.07	98.98	97.33	93.73	**96.07**	98.98	97.33	93.73	**96.07**	98.98	97.33	93.32	**95.92**
	10	98.98	97.77	93.32	**96.21**	98.93	97.77	92.48	95.92	98.93	97.54	92.07	95.62	98.93	97.54	90.40	95.04
	10^2	99.46	97.97	89.98	95.18	99.43	97.75	90.82	95.32	99.35	97.75	90.82	95.32	99.32	97.75	91.23	95.47
	10^3	99.58	97.97	89.98	95.18	99.58	97.97	90.82	95.47	99.58	97.97	90.40	95.32	99.54	97.97	89.55	95.02
	10^4	99.58	97.97	89.11	94.87	99.58	97.97	89.13	94.88	99.58	98.20	89.11	95.02	99.58	98.20	87.45	94.44
	10^5	99.59	97.97	89.11	94.87	99.59	97.97	89.11	94.87	99.58	98.20	88.70	94.87	99.58	98.20	88.28	94.73
	10^6	99.59	97.97	89.11	94.87	99.59	98.20	88.70	94.87	99.56	97.97	88.70	94.73	99.56	97.97	87.86	94.44
0.4	1	99.17	97.77	93.32	**96.21**	99.17	97.77	92.90	**96.06**	99.17	97.77	92.46	**95.91**	99.17	97.77	89.51	**94.88**
	10	99.33	98.22	88.68	94.88	99.33	97.99	89.95	95.18	99.30	97.77	89.11	94.74	99.22	97.77	88.26	94.44
	10^2	99.50	98.20	88.26	94.72	99.48	97.99	89.11	94.88	99.48	97.54	87.86	94.15	99.50	97.97	85.78	93.71
	10^3	99.77	98.88	82.01	92.98	99.77	98.66	81.58	92.68	99.67	98.66	84.08	93.55	99.63	98.20	83.22	92.96
	10^4	99.76	98.66	82.01	92.83	99.77	98.66	80.72	92.38	99.71	98.66	81.14	92.53	99.66	98.66	81.99	92.82
	10^5	99.80	98.86	81.58	92.81	99.82	98.66	81.16	92.53	99.79	98.66	80.74	92.39	99.74	98.66	79.04	91.79
	10^6	99.84	98.86	81.99	92.96	99.82	98.86	81.58	92.81	99.79	98.66	79.89	92.09	99.69	98.66	81.12	92.52
0.5	1	99.40	97.77	87.88	**94.31**	99.40	97.77	87.45	**94.15**	99.38	97.77	85.76	**93.56**	99.38	97.54	84.08	**92.83**
	10	99.51	97.77	85.76	93.56	99.51	97.77	85.76	93.56	99.50	97.99	84.09	93.13	99.46	97.99	82.39	92.53
	10^2	99.85	97.80	80.36	91.70	99.85	98.01	79.93	91.68	99.85	97.80	79.11	91.26	99.85	97.58	77.83	90.66
	10^3	99.89	97.33	75.33	89.63	99.89	97.56	74.91	89.63	99.89	97.56	72.37	88.74	99.87	97.78	72.37	88.89
	10^4	100	97.54	73.66	89.18	100	97.54	72.83	88.89	100	97.33	71.56	88.31	100	97.78	69.86	88.01
	10^5	100	97.54	74.08	89.33	100	97.54	72.83	88.89	100	97.33	71.56	88.31	100	97.56	69.86	87.86
	10^6	100	97.54	73.66	89.18	100	97.54	72.83	88.89	100	97.33	71.56	88.31	100	97.56	69.86	87.86

CRISP	k	0.2				0.5				0.75				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	98.24	97.54	93.73	96.21	98.02	97.54	91.65	95.48	97.77	97.54	89.96	94.89	97.30	97.54	87.84	94.15
	10	98.31	97.33	94.15	96.22	98.13	97.33	92.90	**95.78**	98.11	97.31	91.23	**95.18**	97.79	97.31	89.57	**94.60**
	10^2	98.62	97.54	94.15	**96.35**	98.39	97.10	91.65	95.19	98.18	97.08	89.96	94.59	97.97	96.86	87.88	93.72
	10^3	98.83	97.08	92.07	95.33	98.42	96.86	90.78	94.73	98.24	96.86	88.28	93.85	97.95	96.65	87.86	93.57
	10^4	98.80	97.54	90.38	95.03	98.55	97.31	88.70	94.30	98.32	96.65	88.30	93.73	97.98	96.65	87.03	93.28
	10^5	98.83	97.54	91.65	95.48	98.54	96.86	89.53	94.29	98.26	96.65	88.30	93.73	98.00	96.65	87.45	93.43
	10^6	98.81	97.54	90.80	95.18	98.52	97.31	88.70	94.30	98.32	96.65	87.88	93.58	98.03	96.42	85.78	92.70
0.3	1	98.91	97.33	91.23	**95.20**	98.67	97.33	87.90	94.03	98.42	97.33	82.01	91.97	97.75	97.33	78.21	90.64
	10	98.86	97.54	90.40	95.04	98.68	97.31	88.32	**94.16**	98.50	97.31	84.13	**92.70**	98.34	96.19	80.34	**90.65**
	10^2	99.11	97.75	89.98	95.03	98.89	96.63	86.63	93.13	98.47	96.63	82.41	91.65	98.28	96.17	79.06	90.18
	10^3	99.46	97.54	88.32	94.31	98.93	96.63	85.38	92.69	98.55	96.42	82.83	91.66	98.32	96.17	77.39	89.60
	10^4	99.56	97.97	87.03	94.14	98.96	96.86	85.78	92.98	98.54	96.40	81.58	91.21	98.23	95.95	78.22	89.75
	10^5	99.58	97.97	86.61	94.00	99.02	97.29	85.78	93.26	98.63	95.97	81.16	90.78	98.34	96.17	76.97	89.46
	10^6	99.56	97.97	86.20	93.85	98.96	97.08	84.95	92.84	98.49	95.97	81.99	91.08	98.36	96.19	78.64	90.05
0.4	1	99.17	97.77	86.59	**93.86**	99.11	97.56	77.83	90.65	98.81	97.33	71.96	**88.45**	97.95	97.54	66.11	**86.54**
	10	99.22	97.54	85.74	93.41	99.17	97.08	79.06	90.76	98.96	95.72	73.22	87.85	98.73	94.58	68.21	85.35
	10^2	99.45	97.75	84.93	93.26	99.22	96.87	79.47	**90.79**	99.04	94.83	74.87	87.85	98.67	92.99	69.44	84.75
	10^3	99.53	97.97	83.22	92.81	99.32	97.52	76.54	90.18	99.15	95.51	72.79	87.56	98.68	93.24	68.62	84.62
	10^4	99.63	98.20	81.97	92.52	99.43	97.75	75.29	89.89	99.14	95.95	72.37	87.70	98.75	93.24	69.04	84.77
	10^5	99.61	98.20	81.99	92.53	99.48	97.97	74.46	89.74	99.15	95.95	71.96	87.55	98.67	93.47	69.04	84.92
	10^6	99.66	98.45	80.33	92.11	99.46	97.75	75.31	89.89	99.20	96.19	72.79	88.00	98.70	93.24	69.04	84.77
0.5	1	99.30	97.77	79.89	91.51	99.27	97.08	69.00	87.26	99.11	96.63	57.30	82.87	98.39	95.49	49.37	**79.35**
	10	99.43	97.99	80.71	**91.94**	99.38	97.08	70.69	**87.85**	99.28	95.04	61.07	**83.15**	98.68	91.86	49.78	77.13
	10^2	99.84	97.35	75.71	89.78	99.50	96.44	67.34	86.26	99.35	94.39	58.59	81.86	98.85	91.44	50.20	77.01
	10^3	99.85	97.35	72.37	88.61	99.77	96.67	64.00	85.24	99.30	94.41	58.57	81.87	98.94	91.89	51.05	77.60
	10^4	99.95	98.01	66.47	86.97	99.84	97.35	61.92	84.95	99.48	95.55	58.57	82.61	98.91	91.91	51.90	77.91
	10^5	100	98.01	66.05	86.83	99.84	96.89	61.92	84.66	99.48	95.32	58.97	82.60	98.94	91.23	51.90	77.47
	10^6	100	98.01	66.05	86.83	99.85	97.12	61.09	84.51	99.59	95.55	59.40	82.90	98.93	92.10	51.07	77.74

FUZZY	k	0				0.01				0.05				0.1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.50	97.31	97.50	97.38	97.50	97.31	97.50	97.38	97.50	97.31	97.50	97.38	97.50	97.31	97.08	97.23
	10	97.53	97.10	97.92	97.39	97.51	97.10	97.92	97.39	97.50	97.10	97.92	97.39	97.50	97.10	97.92	97.39
	10^2	97.61	97.31	97.92	**97.52**	97.62	97.31	97.92	**97.52**	97.59	97.31	97.92	**97.52**	97.62	97.31	97.92	**97.52**
	10^3	97.64	97.33	97.08	97.24	97.67	97.10	97.50	97.24	97.64	97.10	97.92	97.39	97.67	97.10	97.92	97.39
	10^4	97.67	97.33	97.08	97.24	97.64	97.10	97.50	97.24	97.66	97.54	96.67	97.23	97.66	97.10	97.92	97.39
	10^5	97.66	97.33	97.50	97.39	97.64	97.54	96.67	97.23	97.64	97.31	97.92	**97.52**	97.67	97.54	96.67	97.23
	10^6	97.71	97.33	96.67	97.10	97.71	97.33	96.67	97.10	97.62	97.54	97.08	97.38	97.61	97.33	97.08	97.24
0.05	1	97.61	97.10	96.25	96.80	97.63	97.10	96.25	96.80	97.63	97.10	96.25	96.80	97.64	97.10	96.25	96.80
	10	97.61	97.10	97.92	97.39	97.63	97.10	97.92	97.39	97.61	97.10	97.92	97.39	97.63	97.10	97.92	97.39
	10^2	97.72	97.33	97.50	97.39	97.67	97.31	97.92	97.52	97.72	97.10	97.92	97.39	97.69	97.10	97.92	97.39
	10^3	97.74	97.54	97.50	**97.52**	97.67	97.54	97.92	**97.67**	97.67	97.33	97.92	**97.54**	97.67	97.33	97.92	97.54
	10^4	97.71	97.54	97.50	**97.52**	97.71	97.54	97.50	97.52	97.74	97.54	97.50	97.52	97.77	97.33	97.92	97.54
	10^5	97.74	97.54	97.50	**97.52**	97.71	97.54	97.50	97.52	97.74	97.54	97.50	97.52	97.74	97.33	97.08	97.24
	10^6	97.72	97.54	97.50	**97.52**	97.76	97.54	97.08	97.38	97.72	97.54	97.08	97.38	97.74	97.54	97.92	**97.67**
0.10	1	97.87	97.10	95.83	96.66	97.87	97.10	95.83	96.66	97.86	97.10	95.42	96.51	97.89	97.10	95.42	96.51
	10	98.03	97.31	97.50	**97.38**	98.02	97.31	97.50	**97.38**	98.00	97.31	97.50	**97.38**	98.00	97.10	97.08	**97.10**
	10^2	98.11	97.33	97.08	97.24	98.15	97.14	97.08	97.12	98.10	97.15	97.08	97.12	98.13	97.38	96.25	96.98
	10^3	98.18	97.54	95.83	96.94	98.10	97.54	96.67	97.23	98.03	97.54	96.67	97.23	98.05	97.33	95.42	96.66
	10^4	98.13	97.54	95.83	96.94	98.11	97.54	96.25	97.09	98.16	97.54	96.25	97.09	98.13	97.33	95.83	96.81
	10^5	98.11	97.54	96.25	97.09	98.13	97.54	95.83	96.94	98.13	97.54	95.83	96.94	98.03	97.33	96.25	96.95
	10^6	98.13	97.54	95.83	96.94	98.11	97.54	96.25	97.09	98.08	97.54	95.83	96.94	98.10	97.33	96.25	96.95
0.15	1	98.10	97.31	94.57	96.35	98.08	97.31	94.57	96.35	98.06	97.39	94.57	96.40	98.01	97.49	94.57	96.46
	10	98.15	97.54	95.40	96.79	98.15	97.31	95.40	96.64	98.11	97.23	95.40	**96.59**	98.10	97.42	94.98	96.57
	10^2	98.29	97.54	96.23	**97.08**	98.34	97.54	96.23	**97.08**	98.34	97.31	94.98	96.50	98.29	97.31	95.40	**96.64**
	10^3	98.52	97.54	93.73	96.21	98.59	97.54	93.73	96.21	98.53	97.54	94.57	96.50	98.50	97.50	94.15	96.33
	10^4	98.62	97.54	93.32	96.06	98.65	97.54	93.73	96.21	98.63	97.54	92.48	95.77	98.58	97.53	93.32	96.05
	10^5	98.62	97.54	92.48	95.77	98.65	97.54	93.32	96.06	98.63	97.54	92.90	95.91	98.61	97.54	92.90	95.91
	10^6	98.62	97.54	92.90	95.91	98.54	97.54	92.90	95.91	98.45	97.54	94.57	96.50	98.57	97.54	92.90	95.91

FUZZY	k	0.2				0.5				0.75				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.50	97.31	97.08	97.23	97.51	97.31	97.08	97.23	97.51	97.31	97.08	97.23	97.48	97.31	97.08	97.23
	10	97.50	97.31	97.92	97.52	97.49	97.06	97.92	97.36	97.53	97.03	97.92	97.34	97.54	97.01	97.92	97.33
	10^2	97.59	97.30	97.92	97.51	97.58	97.29	97.50	97.37	97.63	97.44	97.92	**97.60**	97.58	97.21	97.92	**97.45**
	10^3	97.64	97.54	97.50	**97.52**	97.66	97.27	97.50	97.35	97.66	97.44	97.50	97.46	97.67	97.44	97.08	97.32
	10^4	97.66	97.52	97.50	97.51	97.72	97.52	97.50	**97.52**	97.72	97.49	97.08	97.35	97.66	97.47	96.67	97.19
	10^5	97.67	97.31	97.50	97.38	97.71	97.27	97.08	97.20	97.67	97.50	95.83	96.91	97.69	97.45	96.67	97.18
	10^6	97.74	97.33	97.08	97.24	97.66	97.52	97.08	97.37	97.72	97.50	96.25	97.06	97.67	97.45	96.25	97.03
0.05	1	97.64	97.10	96.25	96.80	97.54	97.31	95.83	96.79	97.45	97.29	95.83	96.78	97.32	97.27	96.25	96.91
	10	97.63	97.10	97.92	97.39	97.63	97.10	97.08	97.10	97.61	97.05	97.08	97.06	97.53	96.99	97.08	97.03
	10^2	97.72	97.10	97.50	97.24	97.72	97.10	97.50	97.24	97.72	97.22	97.92	**97.47**	97.69	97.36	97.08	**97.26**
	10^3	97.72	97.33	97.50	97.39	97.72	97.31	97.50	**97.38**	97.71	97.24	97.50	97.33	97.77	97.09	97.08	97.09
	10^4	97.72	97.33	97.92	**97.54**	97.79	97.31	97.08	97.23	97.76	96.95	96.67	96.85	97.77	97.11	96.67	96.96
	10^5	97.72	97.33	97.92	**97.54**	97.79	97.30	97.08	97.22	97.74	97.00	97.08	97.03	97.72	97.09	96.67	96.94
	10^6	97.76	97.33	97.08	97.24	97.77	97.31	97.50	97.38	97.79	97.45	97.08	97.32	97.79	97.10	96.67	96.95
0.10	1	97.91	97.10	95.42	96.51	97.89	97.09	95.00	96.36	97.65	97.16	95.42	96.55	97.65	96.79	93.73	95.72
	10	97.98	97.30	96.67	97.08	97.86	96.99	96.67	96.88	97.73	96.86	96.25	**96.65**	97.64	96.63	95.42	**96.21**
	10^2	98.02	97.18	96.25	96.85	97.93	96.94	97.08	**96.99**	97.75	96.86	95.00	96.21	97.62	96.39	94.17	95.61
	10^3	98.06	97.25	95.83	96.75	97.96	97.34	95.83	96.81	97.79	97.09	95.42	96.51	97.67	96.37	94.17	95.60
	10^4	98.03	97.45	96.67	**97.18**	97.82	97.16	96.25	96.84	97.82	96.98	95.83	96.58	97.67	96.51	93.73	95.54
	10^5	98.02	97.46	96.25	97.04	97.85	97.13	96.67	96.97	97.83	96.82	96.25	96.62	97.71	96.48	94.17	95.67
	10^6	98.02	97.44	95.83	96.88	97.89	97.13	96.67	96.97	97.82	96.58	96.25	96.47	97.70	96.46	94.17	95.65
0.15	1	98.01	97.44	93.73	96.14	97.85	97.22	92.90	95.71	97.57	96.75	91.23	94.82	97.21	96.23	90.82	**94.33**
	10	98.08	97.51	94.57	**96.48**	98.05	96.79	93.73	**95.72**	97.94	96.29	92.48	**94.96**	97.55	95.92	90.40	93.99
	10^2	98.24	97.26	94.57	96.32	98.12	96.91	93.32	95.65	97.95	95.76	92.07	94.47	97.90	94.91	90.40	93.33
	10^3	98.33	97.45	94.57	96.44	98.22	96.90	93.32	95.65	98.01	96.16	91.65	94.58	97.89	94.91	90.82	93.48
	10^4	98.51	97.53	94.15	96.35	98.37	97.14	92.48	95.51	98.03	95.97	92.48	94.75	97.87	94.48	90.82	93.20
	10^5	98.50	97.54	93.73	96.21	98.31	97.12	92.48	95.50	98.02	95.90	92.07	94.56	97.92	94.52	90.40	93.07
	10^6	98.54	97.52	93.32	96.05	98.29	96.70	92.48	95.22	98.02	95.80	92.48	94.64	97.90	94.74	90.40	93.22

FUZZY	k	0				0.01				0.05				0.1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	98.24	97.54	94.57	**96.50**	98.24	97.57	94.57	**96.52**	98.24	97.68	94.57	**96.59**	98.24	97.60	94.57	**96.54**
	10	98.42	97.54	94.57	**96.50**	98.43	97.54	94.57	96.50	98.43	97.33	94.15	96.22	98.39	97.33	94.57	96.36
	10^2	98.73	97.54	92.90	95.91	98.70	97.49	92.07	95.59	98.69	97.54	93.32	96.06	98.68	97.42	93.32	95.98
	10^3	98.93	97.29	91.21	95.16	98.86	97.31	92.90	95.77	98.91	97.11	91.23	95.05	98.85	97.15	91.21	95.07
	10^4	99.01	97.52	90.80	95.17	98.99	97.52	91.63	95.46	99.02	97.66	90.80	95.26	98.94	97.38	90.80	95.08
	10^5	99.06	97.75	90.38	95.17	99.02	97.52	91.63	95.46	99.01	97.31	90.40	94.89	99.02	97.48	89.96	94.85
	10^6	99.02	97.75	92.46	95.90	99.03	97.75	91.63	95.61	99.02	97.43	91.23	95.26	99.00	97.41	90.80	95.10
0.3	1	98.98	97.33	93.73	96.07	98.98	97.33	93.73	**96.07**	98.98	97.38	93.73	**96.10**	98.98	97.42	93.32	**95.98**
	10	98.98	97.77	93.32	**96.21**	98.93	97.77	92.48	95.92	98.92	97.54	92.07	95.62	98.94	97.59	90.40	95.07
	10^2	99.46	97.97	89.98	95.18	99.43	97.84	90.82	95.38	99.34	97.84	90.82	95.38	99.33	97.84	91.23	95.53
	10^3	99.58	97.97	89.98	95.18	99.58	97.97	90.82	95.47	99.58	97.97	90.40	95.32	99.55	97.86	89.55	94.95
	10^4	99.58	97.97	89.11	94.87	99.58	97.97	89.13	94.88	99.58	98.17	89.11	95.00	99.58	98.09	87.45	94.37
	10^5	99.59	97.97	89.11	94.87	99.59	97.97	89.11	94.87	99.58	98.14	88.70	94.83	99.58	98.12	88.28	94.68
	10^6	99.59	97.97	89.11	94.87	99.59	98.20	88.70	94.87	99.56	97.97	88.70	94.73	99.56	97.96	87.86	94.43
0.4	1	99.17	97.77	93.32	**96.21**	99.17	97.77	92.90	**96.06**	99.18	97.76	92.46	**95.91**	99.17	97.57	89.51	**94.75**
	10	99.33	98.22	88.68	94.88	99.33	97.96	89.95	95.15	99.28	97.85	89.11	94.79	99.21	97.86	88.26	94.50
	10^2	99.50	98.20	88.26	94.72	99.48	97.99	89.11	94.89	99.48	97.72	87.86	94.27	99.50	97.92	85.78	93.67
	10^3	99.77	98.88	82.01	92.98	99.77	98.72	81.58	92.72	99.67	98.66	84.08	93.55	99.63	98.25	83.22	92.99
	10^4	99.76	98.66	82.01	92.83	99.77	98.66	80.72	92.38	99.71	98.66	81.14	92.53	99.66	98.56	81.99	92.77
	10^5	99.80	98.86	81.58	92.81	99.82	98.66	81.16	92.53	99.79	98.66	80.74	92.39	99.74	98.66	79.04	91.80
	10^6	99.84	98.86	81.99	92.96	99.82	98.86	81.58	92.81	99.80	98.66	79.89	92.09	99.70	98.62	81.12	92.50
0.5	1	99.40	97.77	87.88	**94.31**	99.40	97.77	87.45	**94.15**	99.38	97.68	85.76	**93.51**	99.38	97.33	84.08	**92.69**
	10	99.51	97.77	85.76	93.56	99.51	97.77	85.76	93.56	99.49	97.96	84.09	93.11	99.46	97.71	82.39	92.35
	10^2	99.85	97.80	80.36	91.70	99.85	97.98	79.93	91.66	99.85	97.56	79.11	91.11	99.85	97.40	77.83	90.55
	10^3	99.89	97.33	75.33	89.63	99.89	97.56	74.91	89.63	99.89	97.77	72.37	88.88	99.87	97.54	72.37	88.74
	10^4	100	97.54	73.66	89.18	100	97.54	72.83	88.89	100	97.39	71.56	88.35	100	97.41	69.86	87.77
	10^5	100	97.54	74.08	89.33	100	97.54	72.83	88.89	100	97.39	71.56	88.35	100	97.14	69.86	87.59
	10^6	100	97.54	73.66	89.18	100	97.54	72.83	88.89	100	97.39	71.56	88.35	100	97.14	69.86	87.59

FUZZY	k	0.2				0.5				0.75				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	98.24	97.55	93.73	96.22	98.03	96.98	91.65	**95.12**	97.80	96.07	89.96	**93.93**	97.32	95.03	87.84	**92.51**
	10	98.32	97.24	94.15	96.16	98.17	96.17	92.90	95.03	98.12	95.23	91.23	93.83	97.78	94.10	89.57	92.51
	10^2	98.63	97.39	94.15	**96.26**	98.37	96.03	91.65	94.50	98.22	94.63	89.96	93.00	98.02	93.25	87.88	91.37
	10^3	98.84	97.13	92.07	95.36	98.47	95.62	90.78	93.93	98.25	94.28	88.28	92.18	97.95	92.91	87.86	91.14
	10^4	98.80	97.39	90.38	94.94	98.55	95.98	88.70	93.43	98.30	94.04	88.30	92.03	97.97	92.94	87.03	90.87
	10^5	98.82	97.33	91.65	95.34	98.55	95.73	89.53	93.56	98.29	93.90	88.30	91.94	98.00	92.83	87.45	90.94
	10^6	98.82	97.27	90.80	95.00	98.52	95.98	88.70	93.43	98.31	94.09	87.88	91.91	98.04	92.60	85.78	90.21
0.3	1	98.91	97.29	91.23	**95.17**	98.64	96.01	87.90	**93.17**	98.32	94.03	82.01	**89.82**	97.67	91.47	78.21	**86.83**
	10	98.87	97.54	90.40	95.04	98.70	95.65	88.32	93.08	98.51	92.78	84.13	89.75	98.30	88.14	80.34	85.42
	10^2	99.14	97.61	89.98	94.94	98.90	95.02	86.63	92.09	98.54	92.03	82.41	88.66	98.26	88.20	79.06	85.00
	10^3	99.47	97.62	88.32	94.36	98.93	95.16	85.38	91.74	98.59	92.02	82.83	88.80	98.36	87.96	77.39	84.26
	10^4	99.56	98.03	87.03	94.18	98.97	95.35	85.78	92.00	98.59	91.75	81.58	88.19	98.28	88.14	78.22	84.67
	10^5	99.57	97.91	86.61	93.96	99.01	95.73	85.78	92.25	98.65	91.68	81.16	88.00	98.34	88.45	76.97	84.44
	10^6	99.55	97.84	86.20	93.77	98.97	95.53	84.95	91.83	98.54	91.79	81.99	88.36	98.40	87.99	78.64	84.72
0.4	1	99.17	96.85	86.59	93.26	99.12	93.66	77.83	88.12	98.82	89.56	71.96	**83.40**	97.91	85.64	66.11	**78.80**
	10	99.23	97.36	85.74	**93.29**	99.20	93.35	79.06	**88.35**	98.95	87.47	73.22	82.48	98.69	82.95	68.21	77.79
	10^2	99.45	97.21	84.93	92.91	99.21	93.04	79.47	88.29	99.05	87.12	74.87	82.84	98.61	81.68	69.44	77.39
	10^3	99.52	97.65	83.22	92.60	99.31	93.27	76.54	87.41	99.14	87.74	72.79	82.51	98.64	81.83	68.62	77.21
	10^4	99.61	97.82	81.97	92.27	99.43	93.94	75.29	87.41	99.17	88.21	72.37	82.67	98.68	81.75	69.04	77.30
	10^5	99.61	97.89	81.99	92.33	99.46	93.95	74.46	87.13	99.17	87.99	71.96	82.38	98.61	82.00	69.04	77.46
	10^6	99.66	98.28	80.33	92.00	99.45	93.90	75.31	87.39	99.18	88.23	72.79	82.82	98.65	81.78	69.04	77.32
0.5	1	99.30	96.64	79.89	90.78	99.27	89.68	69.00	82.44	99.07	83.50	57.30	**74.33**	98.33	78.66	49.37	**68.41**
	10	99.44	96.65	80.71	**91.07**	99.39	89.90	70.69	**83.18**	99.25	81.21	61.07	74.16	98.55	75.29	49.78	66.37
	10^2	99.82	96.44	75.71	89.18	99.48	89.24	67.34	81.58	99.34	81.79	58.59	73.67	98.73	74.78	50.20	66.18
	10^3	99.85	96.06	72.37	87.77	99.76	88.84	64.00	80.15	99.34	82.08	58.57	73.86	98.83	74.71	51.05	66.43
	10^4	99.95	96.26	66.47	85.84	99.83	88.91	61.92	79.47	99.55	81.39	58.57	73.41	98.83	74.81	51.90	66.79
	10^5	100	96.28	66.05	85.70	99.84	88.29	61.92	79.07	99.53	81.39	58.97	73.54	98.87	75.01	51.90	66.92
	10^6	100	96.28	66.05	85.70	99.85	88.57	61.09	78.95	99.59	81.36	59.40	73.68	98.84	74.99	51.07	66.62

CRISP	2a	Inner fences				0				0.01				0.05			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.40	97.31	97.08	97.23	97.19	97.54	96.25	97.09	97.17	97.54	96.67	**97.23**	97.36	97.10	97.08	97.10
	10	97.46	97.31	97.50	**97.38**	97.38	97.08	97.50	**97.23**	97.35	96.88	97.50	97.09	97.40	96.88	97.50	97.09
	10^2	97.54	97.31	97.08	97.23	97.46	97.08	97.50	**97.23**	97.51	97.08	97.08	97.08	97.53	97.08	97.92	**97.37**
	10^3	97.59	97.54	96.25	97.09	97.59	97.31	96.67	97.09	97.58	97.08	97.08	97.08	97.62	97.08	97.50	97.23
	10^4	97.76	97.54	95.83	96.94	97.58	97.08	96.67	96.94	97.56	97.08	96.67	96.94	97.71	97.31	97.08	97.23
	10^5	97.67	97.54	95.83	96.94	97.54	97.08	96.67	96.94	97.54	97.08	97.08	97.08	97.66	97.08	97.08	97.08
	10^6	97.69	97.54	96.25	97.09	97.46	97.08	97.08	97.08	97.62	97.08	95.83	96.65	97.66	97.08	97.08	97.08
0.05	1	97.06	97.08	95.83	96.65	96.84	97.54	95.42	**96.80**	96.86	97.31	95.83	96.79	97.07	96.88	95.42	96.36
	10	97.28	97.31	96.67	**97.09**	97.01	97.08	96.25	96.79	97.02	96.88	96.25	96.66	97.36	96.67	96.67	96.67
	10^2	97.54	97.31	96.25	96.94	97.10	97.08	96.25	96.79	97.17	96.88	96.67	**96.80**	97.49	96.65	96.67	96.65
	10^3	97.51	97.31	96.67	**97.09**	97.02	97.08	96.25	96.79	97.17	96.88	96.67	**96.80**	97.48	96.65	97.08	96.80
	10^4	97.59	97.08	96.67	96.94	97.06	97.08	96.25	96.79	97.22	96.88	95.83	96.51	97.56	97.10	96.25	96.80
	10^5	97.56	97.31	96.25	96.94	97.09	97.08	96.25	96.79	97.17	96.88	96.67	**96.80**	97.56	97.10	97.08	**97.10**
	10^6	97.63	97.31	96.67	**97.09**	97.06	97.08	96.25	96.79	97.19	96.88	95.42	96.36	97.61	97.10	97.08	**97.10**
0.10	1	94.94	96.88	88.30	93.87	95.45	97.08	90.80	**94.88**	95.62	96.40	92.05	94.88	96.62	95.76	94.17	95.20
	10	95.48	96.42	90.40	94.31	95.74	96.63	91.21	94.73	95.95	96.40	92.48	95.03	97.22	95.30	95.42	**95.34**
	10^2	95.64	96.65	90.82	**94.61**	95.92	96.63	91.63	94.88	96.03	96.40	92.90	95.18	97.20	94.62	95.83	95.05
	10^3	95.71	96.65	89.98	94.32	95.95	96.63	91.63	94.88	96.06	96.40	92.90	95.18	97.19	94.39	95.83	94.90
	10^4	95.67	96.65	89.98	94.32	95.95	96.63	91.63	94.88	96.08	96.63	93.32	**95.47**	97.23	95.08	95.83	95.34
	10^5	95.66	96.42	90.40	94.31	95.95	96.63	91.63	94.88	96.06	96.40	93.32	95.32	97.20	95.08	95.42	95.20
	10^6	95.64	96.65	90.82	**94.61**	95.95	96.63	91.63	94.88	96.05	96.63	92.90	95.32	97.22	95.30	95.42	**95.34**
0.15	1	92.31	96.88	81.14	91.37	93.51	97.08	85.36	**92.98**	93.51	97.08	85.36	**92.98**	95.87	94.38	89.98	**92.84**
	10	93.56	95.97	83.68	91.67	93.36	96.86	85.78	92.98	93.36	96.86	85.78	92.98	96.21	91.69	91.23	91.53
	10^2	93.61	95.51	85.36	**91.96**	93.38	97.08	85.36	**92.98**	93.38	97.08	85.36	**92.98**	96.19	90.55	91.23	90.79
	10^3	93.57	95.51	84.95	91.81	93.36	97.08	85.36	**92.98**	93.36	97.08	85.36	**92.98**	96.24	89.64	91.65	90.34
	10^4	93.57	95.28	85.36	91.81	93.36	97.08	85.36	**92.98**	93.36	97.08	85.36	**92.98**	96.21	89.64	91.65	90.34
	10^5	93.57	95.28	85.36	91.81	93.36	97.08	85.36	**92.98**	93.36	97.08	85.36	**92.98**	96.29	89.64	91.23	90.20
	10^6	93.57	95.28	85.36	91.81	93.36	97.08	85.36	**92.98**	93.36	97.08	85.36	**92.98**	96.27	89.64	92.07	90.49

CRISP	2a	0.1				0.2				0.5				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.36	97.31	97.08	97.23	97.53	97.31	97.08	97.23	97.49	97.31	97.08	97.23	97.48	97.31	97.08	97.23
	10	97.49	96.88	97.50	97.09	97.53	97.31	97.92	**97.52**	97.53	97.31	97.92	**97.52**	97.48	97.10	97.92	97.39
	10^2	97.58	97.08	97.92	97.37	97.59	97.31	97.92	**97.52**	97.59	97.31	97.50	97.38	97.64	97.31	97.92	**97.52**
	10^3	97.69	97.54	97.50	**97.52**	97.72	97.54	97.08	97.38	97.66	97.54	97.08	97.38	97.64	97.33	97.50	97.39
	10^4	97.69	97.54	97.50	**97.52**	97.71	97.54	97.08	97.38	97.71	97.54	96.67	97.23	97.66	97.33	97.08	97.24
	10^5	97.67	97.54	97.50	**97.52**	97.72	97.54	96.67	97.23	97.72	97.54	97.08	97.38	97.67	97.10	97.50	97.24
	10^6	97.71	97.54	96.67	97.23	97.71	97.31	97.08	97.23	97.76	97.54	97.08	97.38	97.64	97.33	97.08	97.24
0.05	1	97.19	97.10	96.25	96.80	97.25	97.10	96.25	96.80	97.46	97.10	96.25	96.80	97.64	97.10	95.83	96.66
	10	97.56	96.88	97.08	96.95	97.51	96.88	97.08	96.95	97.59	97.31	97.08	97.23	97.61	97.10	97.92	97.39
	10^2	97.66	96.67	97.08	96.81	97.64	96.88	97.50	97.09	97.72	97.33	97.50	**97.39**	97.76	97.31	97.50	97.38
	10^3	97.74	97.10	96.25	96.80	97.71	97.33	97.08	97.24	97.72	97.10	97.08	97.10	97.79	97.33	97.08	97.24
	10^4	97.77	97.10	96.25	96.80	97.64	97.10	97.08	97.10	97.74	97.31	97.50	97.38	97.79	97.10	97.92	97.39
	10^5	97.61	97.10	96.67	**96.95**	97.67	97.31	97.08	97.23	97.76	97.10	97.08	97.10	97.80	97.10	97.08	97.10
	10^6	97.71	96.89	96.25	96.67	97.61	97.31	97.50	**97.38**	97.82	97.10	97.50	97.24	97.74	97.33	97.50	**97.39**
0.10	1	97.06	95.74	95.42	95.63	97.30	96.19	95.83	96.07	97.72	97.10	95.83	96.66	97.95	97.56	95.00	96.66
	10	97.28	95.30	96.67	**95.78**	97.48	95.74	96.25	95.92	97.80	97.31	97.08	**97.23**	98.10	97.33	96.67	**97.10**
	10^2	97.41	95.53	95.83	95.64	97.46	96.42	96.25	96.36	97.84	97.31	95.83	96.79	98.16	97.31	96.25	96.94
	10^3	97.51	95.30	95.83	95.49	97.49	96.19	95.83	96.07	97.82	97.10	97.08	97.10	98.13	97.54	96.25	97.09
	10^4	97.51	95.53	95.42	95.49	97.48	96.65	95.42	96.22	97.90	97.33	96.67	97.10	98.10	97.54	96.25	97.09
	10^5	97.51	94.85	95.42	95.05	97.54	96.21	95.83	96.08	97.84	97.10	95.83	96.66	98.15	97.54	95.83	96.94
	10^6	97.54	95.06	95.83	95.33	97.49	96.65	95.83	**96.36**	97.87	96.88	95.83	96.51	98.18	97.54	96.25	97.09
0.15	1	96.37	93.92	92.48	**93.42**	96.68	95.28	93.32	**94.60**	97.87	97.31	93.73	**96.06**	97.90	97.54	92.48	95.77
	10	96.80	91.69	92.07	91.82	97.43	94.60	92.07	93.71	98.08	96.86	93.32	95.62	98.26	97.54	92.90	**95.91**
	10^2	97.14	89.60	91.65	90.32	97.49	93.69	92.07	93.12	98.02	97.31	92.90	95.77	98.55	97.77	91.65	95.62
	10^3	97.19	90.06	92.07	90.76	97.51	94.62	91.23	93.44	97.97	97.08	92.48	95.47	98.93	97.99	89.57	95.04
	10^4	97.28	89.38	91.65	90.17	97.41	95.04	91.23	93.71	97.97	97.08	92.48	95.47	98.93	98.20	89.15	95.03
	10^5	97.15	89.83	91.65	90.47	97.43	94.13	91.65	93.26	98.08	97.31	92.90	95.77	98.89	98.20	89.57	95.18
	10^6	97.23	89.38	91.65	90.17	97.38	93.94	91.23	92.99	98.03	97.31	92.48	95.62	98.86	98.20	89.57	95.18

CRISP	$2a$	Inner fences				0				0.01				0.05			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	90.26	95.51	74.46	**88.14**	91.77	96.40	80.74	**90.92**	92.35	95.06	80.31	**89.90**	95.61	91.67	87.46	**90.20**
	10	91.25	93.01	77.37	87.54	91.93	95.49	80.74	90.33	92.68	95.06	79.47	89.60	95.98	87.14	88.73	87.70
	10^2	91.43	92.56	77.37	87.24	91.90	95.27	80.33	90.04	92.83	95.06	79.89	89.75	96.05	85.78	89.57	87.10
	10^3	91.43	92.56	77.79	87.39	91.90	95.27	80.33	90.04	92.78	95.06	79.89	89.75	96.11	84.87	89.98	86.66
	10^4	91.41	92.35	77.79	87.25	91.90	95.27	80.33	90.04	92.78	95.06	79.89	89.75	96.14	85.09	89.15	86.51
	10^5	91.41	92.35	77.79	87.25	91.90	95.27	80.33	90.04	92.78	95.06	79.89	89.75	96.13	85.09	89.15	86.51
	10^6	91.41	92.35	77.79	87.25	91.90	95.27	80.33	90.04	92.78	95.06	79.89	89.75	96.13	84.87	89.57	86.51
0.3	1	84.02	93.90	54.78	**80.21**	86.07	93.45	63.61	**83.00**	86.11	92.33	62.30	81.82	94.21	81.70	79.09	**80.79**
	10	85.20	85.98	57.30	75.95	86.40	91.63	62.34	81.38	86.48	91.89	63.15	**81.84**	94.65	75.59	80.74	77.39
	10^2	85.29	84.17	57.74	74.92	86.45	91.40	61.94	81.09	86.45	91.44	63.15	81.54	94.71	73.33	82.03	76.38
	10^3	85.28	83.94	57.32	74.62	86.35	91.40	61.94	81.09	86.42	91.89	63.15	**81.84**	94.70	73.11	81.20	75.94
	10^4	85.26	83.71	57.32	74.48	86.35	91.40	61.94	81.09	86.42	91.89	63.15	**81.84**	94.70	73.11	81.63	76.09
	10^5	85.26	83.71	57.32	74.48	86.35	91.40	61.94	81.09	86.42	91.89	63.15	**81.84**	94.70	73.11	81.63	76.09
	10^6	85.26	83.71	57.32	74.48	86.35	91.40	61.94	81.09	86.42	91.89	63.15	**81.84**	94.70	73.11	81.63	76.09
0.4	1	80.59	95.25	42.68	**76.85**	77.00	83.69	34.28	**66.40**	78.62	79.92	38.50	**65.43**	89.72	70.83	62.37	**67.87**
	10	80.87	92.29	43.10	75.08	77.31	75.76	37.21	62.27	78.30	74.24	37.66	61.44	90.48	65.68	65.29	65.54
	10^2	80.92	90.93	43.10	74.19	77.14	72.37	37.21	60.06	78.30	73.79	37.25	61.00	90.74	64.55	65.29	64.81
	10^3	80.98	90.93	42.68	74.05	77.14	78.48	37.21	64.04	78.31	74.02	37.25	61.15	90.76	63.86	65.71	64.51
	10^4	80.98	90.93	42.68	74.05	77.14	78.03	37.21	63.75	78.36	73.56	37.25	60.85	90.78	63.86	65.71	64.51
	10^5	80.97	90.93	42.68	74.05	77.14	78.26	37.21	63.89	78.51	74.02	37.66	61.29	90.78	63.86	65.71	64.51
	10^6	80.97	90.93	42.68	74.05	77.14	78.03	37.21	63.75	78.31	74.02	37.25	61.15	90.78	63.86	65.71	64.51
0.5	1	80.74	99.32	23.84	**72.91**	39.31	5.21	93.48	36.10	77.86	55.21	38.06	49.21	84.94	51.04	46.01	**49.28**
	10	82.63	95.91	28.42	72.29	43.27	9.75	87.23	36.86	77.60	52.23	36.39	46.69	86.24	45.13	47.70	46.03
	10^2	82.98	95.45	28.41	71.99	43.27	9.75	87.23	36.86	77.60	52.03	36.39	46.56	86.30	43.56	48.95	45.45
	10^3	82.97	95.45	28.41	71.99	43.27	14.34	87.23	**39.84**	77.60	69.34	36.39	**57.81**	86.27	43.33	48.95	45.30
	10^4	82.98	95.45	28.41	71.99	43.27	14.34	87.23	**39.84**	77.60	63.47	36.39	53.99	86.27	43.33	48.95	45.30
	10^5	82.98	95.45	28.41	71.99	43.27	14.34	87.23	**39.84**	77.60	57.97	36.39	50.42	86.27	43.33	48.95	45.30
	10^6	82.98	95.45	28.41	71.99	43.27	12.25	87.23	38.49	77.60	54.55	36.39	48.19	86.27	43.33	48.95	45.30

CRISP	2a	0.1				0.2				0.5				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	96.23	88.05	91.23	**89.16**	97.06	91.21	91.65	**91.36**	97.97	97.54	92.48	**95.77**	98.28	97.77	90.80	95.33
	10	97.01	84.87	92.07	87.39	97.38	90.30	92.07	90.92	98.18	97.08	92.48	95.47	98.45	97.77	91.21	**95.47**
	10^2	97.32	80.78	92.07	84.73	97.50	89.62	91.23	90.18	98.24	96.65	91.23	94.75	98.89	97.97	87.03	94.14
	10^3	97.32	81.69	92.48	85.46	97.46	90.74	91.65	91.06	98.41	96.19	90.80	94.31	99.09	97.95	84.95	93.40
	10^4	97.43	82.14	90.40	85.03	97.45	90.76	90.40	90.63	98.37	96.65	90.80	94.60	99.12	97.73	85.36	93.40
	10^5	97.33	81.00	91.23	84.58	97.43	90.53	90.40	90.48	98.36	96.65	89.55	94.16	99.15	97.95	82.43	92.52
	10^6	97.32	81.46	91.65	85.02	97.45	90.76	91.23	90.92	98.29	96.65	90.40	94.46	99.09	98.18	84.11	93.26
0.3	1	96.27	73.77	86.21	**78.12**	97.38	80.11	87.88	**82.83**	98.47	97.33	86.63	**93.59**	98.98	98.20	82.83	**92.82**
	10	97.01	67.90	87.46	74.74	97.76	75.13	87.88	79.59	98.41	96.65	87.46	93.43	99.15	98.20	81.99	92.53
	10^2	97.46	60.63	86.21	69.58	97.72	74.43	87.88	79.14	98.39	96.42	87.45	93.28	99.59	97.97	79.93	91.66
	10^3	97.49	61.31	86.21	70.02	97.74	75.11	87.46	79.44	98.57	96.42	86.20	92.84	99.61	98.66	78.22	91.51
	10^4	97.48	61.31	86.21	70.02	97.76	75.57	87.46	79.73	98.62	96.42	85.78	92.70	99.64	98.88	76.54	91.06
	10^5	97.53	61.76	86.21	70.32	97.72	74.89	87.05	79.14	98.62	96.65	85.80	92.85	99.63	98.88	76.99	91.22
	10^6	97.51	61.31	86.21	70.02	97.71	75.80	87.05	79.73	98.67	96.42	86.61	92.99	99.66	98.66	76.97	91.07
0.4	1	94.40	57.77	71.97	**62.74**	96.71	64.53	77.83	**69.18**	98.83	93.01	79.08	88.14	99.22	98.66	74.87	**90.33**
	10	96.34	47.82	76.16	57.74	97.51	52.75	80.34	62.40	98.89	91.19	79.08	86.95	99.48	98.66	72.79	89.60
	10^2	96.66	43.98	74.51	54.66	97.53	50.74	78.68	60.52	99.01	93.71	78.24	**88.30**	99.63	98.88	69.00	88.43
	10^3	96.68	43.98	73.68	54.37	97.46	49.39	77.84	59.35	99.07	92.80	74.08	86.25	99.80	98.88	64.84	86.97
	10^4	96.67	43.75	73.26	54.08	97.48	49.62	77.01	59.21	99.12	93.26	73.66	86.40	99.87	98.88	63.59	86.53
	10^5	96.67	43.52	73.68	54.07	97.49	50.28	78.26	60.07	99.07	92.78	74.06	86.23	99.87	98.88	63.59	86.53
	10^6	96.67	43.52	73.68	54.07	97.50	49.38	77.84	59.34	99.01	92.12	74.08	85.81	99.89	98.88	63.17	86.39
0.5	1	91.04	37.39	58.19	**44.67**	96.08	44.26	65.24	**51.60**	99.07	88.07	69.82	81.68	99.43	99.11	63.15	**86.53**
	10	92.83	27.73	62.37	39.85	97.69	28.94	68.19	42.67	99.28	89.03	70.25	**82.46**	99.66	99.11	60.24	85.51
	10^2	93.56	23.20	62.79	37.05	97.87	31.86	63.15	42.81	99.40	87.48	66.92	80.29	99.85	99.36	56.05	84.20
	10^3	93.66	23.24	61.94	36.78	97.95	32.80	65.65	44.30	99.37	83.39	64.82	76.89	99.89	99.34	53.13	83.17
	10^4	93.70	23.01	61.52	36.49	98.10	34.60	62.30	44.29	99.48	84.24	64.82	77.45	100	99.11	52.72	82.88
	10^5	93.70	22.78	61.52	36.34	97.90	34.15	64.38	44.73	99.53	84.24	65.24	77.59	100	99.11	53.13	83.02
	10^6	93.70	22.78	61.52	36.34	97.89	34.13	64.40	44.72	99.54	84.24	65.24	77.59	100	99.11	53.97	83.31

FUZZY	2a	Inner fences				0				0.01				0.05			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.39	96.89	97.08	96.96	97.13	97.31	96.25	96.94	97.13	97.31	96.67	**97.08**	97.36	96.83	97.08	96.92
	10	97.43	96.94	97.50	**97.14**	97.30	97.00	97.50	97.18	97.28	96.79	97.50	97.04	97.39	96.77	97.50	97.02
	10^2	97.51	96.92	97.08	96.97	97.38	97.05	97.50	**97.21**	97.43	97.01	97.08	97.03	97.52	96.96	97.92	**97.30**
	10^3	97.56	97.18	96.25	96.85	97.52	97.21	96.67	97.02	97.51	97.05	97.08	97.06	97.62	97.01	97.50	97.18
	10^4	97.72	97.17	95.83	96.71	97.50	96.98	96.67	96.87	97.50	97.02	96.67	96.89	97.70	97.03	97.08	97.05
	10^5	97.64	97.15	95.83	96.69	97.47	97.03	96.67	96.91	97.48	97.01	97.08	97.04	97.65	96.99	97.08	97.02
	10^6	97.65	97.20	96.25	96.87	97.39	97.01	97.08	97.03	97.56	97.00	95.83	96.59	97.65	96.96	97.08	97.00
0.05	1	97.05	95.61	95.83	95.69	96.73	97.08	95.42	**96.50**	96.80	97.02	95.83	**96.60**	97.07	96.55	95.42	96.15
	10	97.27	95.81	96.67	**96.11**	96.86	96.59	96.25	96.47	96.88	96.43	96.25	96.36	97.36	96.20	96.67	96.37
	10^2	97.52	95.64	96.25	95.85	96.95	96.62	96.25	96.49	97.03	96.36	96.67	96.47	97.49	96.12	96.67	96.31
	10^3	97.49	95.71	96.67	96.05	96.90	96.50	96.25	96.41	97.05	96.33	96.67	96.45	97.48	95.99	97.08	96.37
	10^4	97.58	95.51	96.67	95.91	96.88	96.51	96.25	96.42	97.11	96.24	95.83	96.10	97.56	96.11	96.25	96.16
	10^5	97.54	95.61	96.25	95.83	96.92	96.53	96.25	96.43	97.05	96.29	96.67	96.42	97.56	96.16	97.08	**96.48**
	10^6	97.61	95.74	96.67	96.06	96.92	96.55	96.25	96.45	97.06	96.29	95.42	95.99	97.61	96.10	97.08	96.44
0.10	1	94.91	93.53	88.30	91.70	95.04	95.64	90.80	**93.95**	95.36	95.38	92.05	**94.21**	96.63	93.83	94.17	93.95
	10	95.42	92.20	90.40	91.57	95.24	95.20	91.21	93.81	95.62	94.80	92.48	93.99	97.20	93.20	95.42	**93.97**
	10^2	95.58	92.30	90.82	**91.78**	95.39	95.10	91.63	93.88	95.68	94.67	92.90	94.05	97.19	92.87	95.83	93.90
	10^3	95.65	92.26	89.98	91.46	95.42	95.07	91.63	93.87	95.70	94.65	92.90	94.04	97.19	92.65	95.83	93.77
	10^4	95.62	92.22	89.98	91.44	95.42	95.07	91.63	93.87	95.73	94.68	93.32	94.20	97.22	92.75	95.83	93.83
	10^5	95.60	91.88	90.40	91.36	95.42	95.07	91.63	93.87	95.70	94.64	93.32	94.18	97.18	92.81	95.42	93.72
	10^6	95.58	92.22	90.82	91.73	95.42	95.07	91.63	93.87	95.71	94.65	92.90	94.04	97.21	92.99	95.42	93.84
0.15	1	92.29	86.31	83.68	85.39	92.67	93.69	85.36	90.77	93.61	93.37	87.01	**91.14**	95.90	90.32	89.98	**90.20**
	10	93.49	85.81	85.36	85.65	92.51	93.67	85.78	**90.91**	93.66	92.99	87.03	90.91	96.19	88.29	91.23	89.32
	10^2	93.53	85.78	84.95	85.49	92.52	93.61	85.36	90.72	93.70	92.96	86.61	90.74	96.16	86.85	91.23	88.38
	10^3	93.51	85.72	85.36	85.60	92.51	93.61	85.36	90.72	93.72	92.94	86.61	90.73	96.22	86.39	91.65	88.23
	10^4	93.51	85.72	85.36	85.60	92.51	93.61	85.36	90.72	93.72	92.94	86.61	90.73	96.17	86.40	91.65	88.24
	10^5	93.51	85.72	85.36	85.60	92.51	93.61	85.36	90.72	93.72	92.94	86.61	90.73	96.26	86.05	91.23	87.86
	10^6	93.51	97.08	85.36	**92.98**	92.51	93.61	85.36	90.72	93.72	92.94	86.61	90.73	96.23	86.34	92.07	88.35

FUZZY	2a	0.1				0.2				0.5				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.01	1	97.36	97.06	97.08	97.07	97.53	97.12	97.08	97.10	97.49	97.31	97.08	97.23	97.48	97.31	97.08	97.23
	10	97.49	96.77	97.50	97.03	97.53	97.08	97.92	97.37	97.53	97.26	97.92	**97.49**	97.48	97.10	97.92	97.39
	10^2	97.58	96.98	97.92	97.31	97.59	97.27	97.92	**97.50**	97.59	97.29	97.50	97.36	97.64	97.31	97.92	**97.52**
	10^3	97.69	97.25	97.50	97.34	97.72	97.30	97.08	97.22	97.66	97.50	97.08	97.35	97.64	97.33	97.50	97.39
	10^4	97.69	97.26	97.50	**97.34**	97.71	97.32	97.08	97.24	97.71	97.51	96.67	97.22	97.66	97.33	97.08	97.24
	10^5	97.67	97.25	97.50	97.34	97.72	97.28	96.67	97.07	97.72	97.53	97.08	97.37	97.67	97.10	97.50	97.24
	10^6	97.71	97.23	96.67	97.03	97.71	97.06	97.08	97.07	97.76	97.52	97.08	97.37	97.64	97.33	97.08	97.24
0.05	1	97.19	96.58	96.25	96.46	97.25	96.58	96.25	96.47	97.46	97.10	96.25	96.80	97.64	97.10	95.83	96.66
	10	97.56	96.29	97.08	**96.57**	97.51	96.48	97.08	96.69	97.59	97.31	97.08	97.23	97.61	97.10	97.92	97.39
	10^2	97.66	96.07	97.08	96.43	97.64	96.31	97.50	96.73	97.72	97.33	97.50	**97.39**	97.76	97.31	97.50	97.38
	10^3	97.74	96.37	96.25	96.33	97.71	96.46	97.08	96.68	97.72	97.10	97.08	97.09	97.79	97.33	97.08	97.24
	10^4	97.77	96.28	96.25	96.27	97.64	96.54	97.08	96.73	97.74	97.31	97.50	97.38	97.79	97.10	97.92	97.39
	10^5	97.61	96.17	96.67	96.34	97.67	96.68	97.08	96.82	97.76	97.10	97.08	97.09	97.80	97.10	97.08	97.10
	10^6	97.71	96.06	96.25	96.13	97.61	96.74	97.50	**97.01**	97.82	97.10	97.50	97.24	97.74	97.33	97.50	**97.39**
0.10	1	97.06	93.66	95.42	94.27	97.30	94.46	95.83	94.94	97.72	97.00	95.83	96.59	97.95	97.56	95.00	96.66
	10	97.28	93.02	96.67	**94.30**	97.48	94.08	96.25	94.84	97.80	97.13	97.08	**97.11**	98.10	97.33	96.67	**97.10**
	10^2	97.42	93.45	95.83	94.29	97.45	94.59	96.25	**95.17**	97.84	97.25	95.83	96.75	98.16	97.31	96.25	96.94
	10^3	97.51	92.89	95.83	93.92	97.49	94.76	95.83	95.14	97.82	97.08	97.08	97.08	98.13	97.54	96.25	97.09
	10^4	97.52	92.91	95.42	93.79	97.48	94.75	95.42	94.98	97.90	97.12	96.67	96.96	98.10	97.54	96.25	97.09
	10^5	97.50	92.83	95.42	93.74	97.54	94.61	95.83	95.04	97.84	97.04	95.83	96.62	98.15	97.54	95.83	96.94
	10^6	97.53	92.79	95.83	93.86	97.50	94.79	95.83	95.15	97.87	96.78	95.83	96.45	98.18	97.54	96.25	97.09
0.15	1	96.37	89.03	92.48	**90.23**	96.68	90.98	93.32	**91.80**	97.87	96.87	93.73	**95.77**	97.90	97.54	92.48	95.77
	10	96.79	87.11	92.07	88.85	97.43	90.21	92.07	90.86	98.08	96.67	93.32	95.49	98.26	97.54	92.90	**95.91**
	10^2	97.13	85.71	91.65	87.79	97.51	89.29	92.07	90.26	98.07	96.86	92.90	95.47	98.55	97.77	91.65	95.62
	10^3	97.17	85.56	92.07	87.84	97.52	89.80	91.23	90.30	97.99	96.61	92.48	95.17	98.93	97.99	89.57	95.04
	10^4	97.29	85.56	91.65	87.69	97.46	89.98	91.23	90.42	97.98	96.68	92.48	95.21	98.93	98.20	89.15	95.03
	10^5	97.15	85.97	91.65	87.96	97.49	89.66	91.65	90.35	98.10	96.74	92.90	95.39	98.89	98.20	89.57	95.18
	10^6	97.30	85.49	91.65	87.65	97.43	89.70	91.23	90.23	98.05	96.72	92.48	95.24	98.86	98.20	89.57	95.18

FUZZY	2a	Inner fences				0				0.01				0.05			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	90.22	78.99	77.37	78.42	89.87	90.71	80.74	**87.22**	91.22	90.32	80.31	**86.82**	95.63	85.72	87.46	**86.33**
	10	91.19	78.55	77.37	78.14	89.85	90.01	80.74	86.76	91.39	89.67	79.47	86.10	95.88	83.04	88.73	85.03
	10^2	91.34	78.54	77.79	78.28	89.84	90.05	80.33	86.65	91.53	89.61	79.89	86.21	95.91	81.99	89.57	84.64
	10^3	91.33	78.51	77.79	78.26	89.84	90.08	80.33	86.67	91.48	89.60	79.89	86.20	95.98	80.92	89.98	84.09
	10^4	91.31	78.50	77.79	78.25	89.84	90.08	80.33	86.67	91.48	89.60	79.89	86.20	96.01	81.11	89.15	83.92
	10^5	91.31	78.50	77.79	78.25	89.84	90.08	80.33	86.67	91.48	89.60	79.89	86.20	96.00	80.96	89.15	83.83
	10^6	91.31	95.27	80.33	**90.04**	89.84	90.08	80.33	86.67	91.48	89.60	79.89	86.20	95.99	80.86	89.57	83.91
0.3	1	84.02	75.44	54.78	**68.21**	82.29	84.99	63.61	**77.51**	83.61	84.70	62.30	**76.86**	94.17	75.01	79.09	**76.44**
	10	85.20	70.67	57.30	65.99	82.52	84.46	62.34	76.72	83.80	84.05	63.15	76.74	94.53	71.34	80.74	74.63
	10^2	85.29	70.06	57.74	65.75	82.57	84.39	61.94	76.53	83.77	84.05	63.15	76.73	94.53	69.99	82.03	74.20
	10^3	85.28	70.02	57.32	65.57	82.48	84.43	61.94	76.56	83.75	84.05	63.15	76.74	94.50	69.73	81.20	73.74
	10^4	85.26	69.95	57.32	65.53	82.48	84.43	61.94	76.56	83.75	84.05	63.15	76.74	94.52	69.72	81.63	73.89
	10^5	85.26	69.95	57.32	65.53	82.48	84.43	61.94	76.56	83.75	84.05	63.15	76.74	94.52	69.72	81.63	73.89
	10^6	85.26	69.95	57.32	65.53	82.48	84.43	61.94	76.56	83.75	84.05	63.15	76.74	94.52	69.72	81.63	73.89
0.4	1	80.60	77.92	42.68	**65.59**	70.88	80.61	34.28	**64.40**	72.89	78.15	38.50	**64.27**	89.26	66.83	62.37	**65.27**
	10	80.87	76.00	43.10	64.49	68.63	78.11	37.21	63.80	71.80	77.44	37.66	63.52	89.89	63.75	65.29	64.29
	10^2	80.91	75.82	43.10	64.37	68.37	78.05	37.21	63.76	71.47	77.16	37.25	63.19	90.11	62.94	65.29	63.76
	10^3	80.98	75.94	42.68	64.30	68.37	78.05	37.21	63.76	71.46	77.12	37.25	63.17	90.13	62.74	65.71	63.78
	10^4	80.98	75.94	42.68	64.30	68.37	78.05	37.21	63.76	71.50	77.12	37.25	63.17	90.15	62.73	65.71	63.77
	10^5	80.96	75.94	42.68	64.30	68.37	78.05	37.21	63.76	71.63	76.95	37.66	63.20	90.15	62.72	65.71	63.77
	10^6	80.96	75.94	42.68	64.30	68.37	78.05	37.21	63.76	71.48	77.13	37.25	63.17	90.15	62.72	65.71	63.77
0.5	1	80.74	77.14	23.84	**58.49**	39.31	7.50	93.48	37.59	69.99	71.83	38.06	60.01	84.22	58.17	46.01	**53.92**
	10	82.63	74.09	28.42	58.11	42.46	14.66	87.23	40.05	71.62	72.95	36.39	60.16	85.28	55.34	47.70	52.67
	10^2	82.98	73.66	28.41	57.83	42.46	14.66	87.23	40.05	71.62	72.95	36.39	60.16	85.35	54.96	48.95	52.86
	10^3	82.97	73.67	28.41	57.83	42.46	14.66	87.23	**40.05**	71.62	72.95	36.39	**60.16**	85.32	55.02	48.95	52.89
	10^4	82.98	73.67	28.41	57.83	42.46	14.66	87.23	40.05	71.62	72.95	36.39	60.16	85.32	55.01	48.95	52.89
	10^5	82.98	73.67	28.41	57.83	42.46	14.66	87.23	40.05	71.62	72.95	36.39	60.16	85.32	55.01	48.95	52.89
	10^6	82.98	73.67	28.41	57.83	42.46	14.66	87.23	**40.05**	71.62	72.95	36.39	60.16	85.32	55.01	48.95	52.89

FUZZY	$2a$	0.1				0.2				0.5				1			
		Train	Test			Train	Test			Train	Test			Train	Test		
p	C	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
0.2	1	96.24	83.15	91.23	**85.98**	97.04	86.19	91.65	**88.10**	97.99	96.15	92.48	**94.87**	98.28	97.77	90.80	95.33
	10	96.98	80.75	92.07	84.71	97.36	85.00	92.07	87.47	98.17	95.28	92.48	94.30	98.45	97.77	91.21	**95.47**
	10^2	97.35	77.70	92.07	82.73	97.51	84.38	91.23	86.78	98.25	94.88	91.23	93.61	98.89	97.97	87.03	94.14
	10^3	97.36	78.19	92.48	83.19	97.50	84.82	91.65	87.21	98.36	94.47	90.80	93.18	99.09	97.95	84.95	93.40
	10^4	97.49	79.19	90.40	83.12	97.53	85.34	90.40	87.11	98.37	94.68	90.80	93.32	99.12	97.73	85.36	93.40
	10^5	97.37	78.26	91.23	82.80	97.50	85.58	90.40	87.26	98.37	94.52	89.55	92.78	99.15	97.95	82.43	92.52
	10^6	97.37	78.09	91.65	82.83	97.50	85.43	91.23	87.46	98.29	94.42	90.40	93.01	99.09	98.18	84.11	93.26
0.3	1	96.15	70.98	86.21	**76.31**	97.34	75.19	87.88	**79.63**	98.49	94.38	86.63	**91.67**	98.98	98.20	82.83	**92.82**
	10	96.87	67.27	87.46	74.34	97.72	71.24	87.88	77.06	98.39	92.58	87.46	90.79	99.15	98.20	81.99	92.53
	10^2	97.39	64.01	86.21	71.78	97.72	71.37	87.88	77.15	98.42	93.39	87.45	91.31	99.59	97.97	79.93	91.66
	10^3	97.42	64.20	86.21	71.90	97.75	71.59	87.46	77.14	98.58	93.42	86.20	90.89	99.61	98.66	78.22	91.51
	10^4	97.42	64.10	86.21	71.84	97.76	71.86	87.46	77.32	98.65	93.58	85.78	90.85	99.64	98.88	76.54	91.06
	10^5	97.45	64.45	86.21	72.07	97.71	71.53	87.05	76.96	98.67	93.69	85.80	90.93	99.63	98.88	76.99	91.22
	10^6	97.43	64.22	86.21	71.91	97.75	71.76	87.05	77.11	98.68	93.60	86.61	91.16	99.66	98.66	76.97	91.07
0.4	1	94.38	60.74	71.97	**64.67**	96.69	63.86	77.83	**68.75**	98.81	84.94	79.08	**82.89**	99.22	98.66	74.87	**90.33**
	10	96.20	54.92	76.16	62.36	97.43	57.79	80.34	65.68	98.87	83.99	79.08	82.27	99.48	98.66	72.79	89.60
	10^2	96.41	53.30	74.51	60.72	97.49	58.58	78.68	65.61	98.99	85.37	78.24	82.88	99.63	98.88	69.00	88.43
	10^3	96.48	53.28	73.68	60.42	97.45	58.46	77.84	65.24	99.08	84.97	74.08	81.16	99.80	98.88	64.84	86.97
	10^4	96.48	53.42	73.26	60.36	97.46	58.37	77.01	64.89	99.09	85.23	73.66	81.18	99.87	98.88	63.59	86.53
	10^5	96.47	53.36	73.68	60.47	97.45	58.25	78.26	65.25	99.07	85.25	74.06	81.33	99.87	98.88	63.59	86.53
	10^6	96.47	53.36	73.68	60.47	97.49	58.32	77.84	65.16	99.02	84.33	74.08	80.74	99.89	98.88	63.17	86.39
0.5	1	84.22	58.17	46.01	**53.92**	96.06	51.86	65.24	**56.54**	99.05	75.91	69.82	73.78	99.43	99.11	63.15	**86.53**
	10	85.28	55.34	47.70	52.67	97.60	43.95	68.19	52.43	99.28	77.54	70.25	**74.99**	99.66	99.11	60.24	85.51
	10^2	85.35	54.96	48.95	52.86	97.79	44.76	63.15	51.19	99.38	76.77	66.92	73.32	99.85	99.36	56.05	84.20
	10^3	85.32	55.02	48.95	52.89	97.85	45.35	65.65	52.45	99.38	75.40	64.82	71.70	99.89	99.34	53.13	83.17
	10^4	85.32	55.01	48.95	52.89	97.99	46.82	62.30	52.23	99.49	76.46	64.82	72.39	100	99.11	52.72	82.88
	10^5	85.32	55.01	48.95	52.89	97.84	46.33	64.38	52.65	99.55	76.77	65.24	72.73	100	99.11	53.13	83.02
	10^6	85.32	55.01	48.95	52.89	97.81	45.95	64.40	52.40	99.53	76.64	65.24	72.65	100	99.11	53.97	83.31

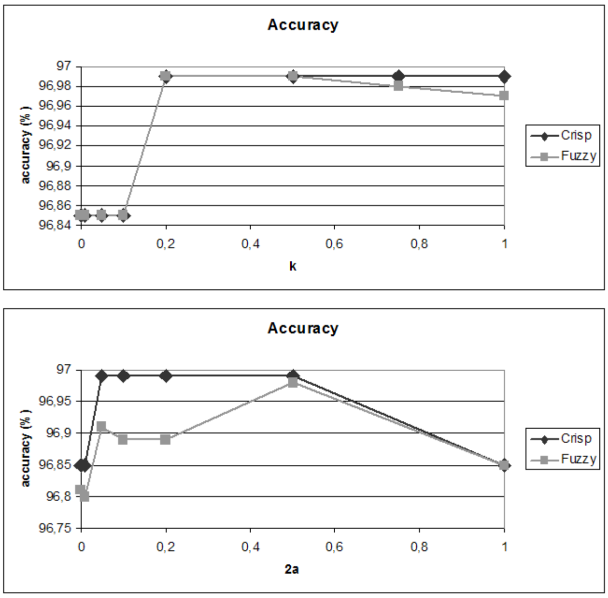


Figure 10: Accuracy for the database with its missing values. Up: interval $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$. Down: interval $[Q_a, Q_{1-a}]$

5.4 Computational experiment for the database with its missing values

In this experiment, we consider the complete 'breast-cancer' dataset (699 instances), including the 16 instances with missing values. All the missing values appear in the sixth variable.

To impute the missing values, we have built an interval following the two strategies explained for the previous experiments, i.e., the blank is replaced by the interval $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$ or by $[Q_a, Q_{1-a}]$, for the same values of k and $2a$ as those used in the other experiments.

The classification problem via 10-fold cross validation (see [21]) has been posed and each optimization problem (20), for different values of C, with $C_{+1} = C_{-1} = C$, has been solved with LOQO, [40]. The accuracy following the crisp and the fuzzy rule are presented in the following tables, and the best results for each k and a are shown in bold and are depicted in Figure 10.

In this case, we can observe that, for the two types of intervals (and the two classification rules), the best results are obtained for non-degenerate intervals. In the case of imputing by $[\overline{x}_j - k\sigma_{x_j}, \overline{x}_j + k\sigma_{x_j}]$, one observes that we obtain the best results in terms of accuracy for high values of k. It means that, in this case, it is better to take into account the value of the standard deviation when imputing the missing value than only using the mean.

Likewise, in the case of imputing by $[Q_a, Q_{1-a}]$, we obtain better results for the accuracy when using intermediate values of $2a$ than when using $2a = 1$ (the case of imputing by the median). In particular, as happened in the previous experiment, the interquartile range seems to yield a better imputation than only considering the median.

Then, we conclude that imputing via intervals seems to be a good strategy when dealing with missing values.

		Train	Test			Train	Test			Train	Test			Train	Test		
		Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
	$C\backslash k$	0				0.01				0.05				0.1			
CRISP	0.01	95.17	98.02	89.20	94.98	95.17	98.02	89.20	94.98	95.18	98.02	88.78	94.84	95.17	98.02	89.20	94.98
	0.1	96.36	97.59	93.77	96.27	96.36	97.59	93.77	96.27	96.36	97.59	93.77	96.27	96.36	97.59	93.77	96.27
	1	97.14	96.94	96.68	**96.85**	97.14	96.94	96.68	**96.85**	97.14	96.94	96.68	**96.85**	97.14	96.94	96.68	**96.85**
	10	97.25	96.72	96.68	96.71	97.23	96.94	96.68	**96.85**	97.22	96.94	96.68	**96.85**	97.22	96.72	96.68	96.71
	100	97.31	96.72	96.27	96.56	97.23	96.50	96.68	96.56	97.30	96.94	95.43	96.42	97.28	96.72	96.27	96.56
	1000	97.33	96.72	95.43	96.28	97.35	96.72	96.68	96.71	97.25	96.94	96.27	96.71	97.35	96.94	95.85	96.56
FUZZY	0.01	95.17	98.02	89.20	94.98	95.17	98.02	89.20	94.98	95.18	98.02	88.78	94.84	95.17	98.02	89.20	94.98
	0.1	96.36	97.59	93.77	96.27	96.36	97.59	93.77	96.27	96.36	97.59	93.77	96.27	96.36	97.59	93.77	96.27
	1	97.14	96.94	96.68	**96.85**	97.14	96.94	96.68	**96.85**	97.14	96.94	96.68	**96.85**	97.14	96.94	96.68	**96.85**
	10	97.25	96.72	96.68	96.71	97.23	96.94	96.68	**96.85**	97.22	96.94	96.68	**96.85**	97.22	96.72	96.68	96.71
	100	97.31	96.72	96.27	96.56	97.23	96.50	96.68	96.56	97.30	96.94	95.43	96.42	97.28	96.72	96.27	96.56
	1000	97.33	96.72	95.43	96.28	97.35	96.72	96.68	96.71	97.25	96.94	96.27	96.71	97.35	96.94	95.85	96.56
	$C\backslash k$	0.2				0.5				0.75				1			
CRISP	0.01	95.22	98.02	88.78	94.84	95.18	98.02	89.20	94.98	95.17	98.02	88.78	94.84	95.07	98.02	89.20	94.98
	0.1	96.36	97.59	93.77	96.27	96.36	97.59	93.35	96.13	96.34	97.59	93.35	96.13	96.41	97.59	93.35	96.13
	1	97.14	96.94	97.10	**96.99**	97.15	96.94	97.10	**96.99**	97.14	96.94	97.10	**96.99**	97.14	97.15	96.68	**96.99**
	10	97.25	96.72	96.68	96.71	97.20	96.94	96.68	96.85	97.20	96.72	97.10	96.85	97.19	96.94	97.10	**96.99**
	100	97.23	96.50	96.68	96.56	97.27	96.50	96.27	96.42	97.30	96.72	95.85	96.42	97.23	96.94	95.43	96.42
	1000	97.33	96.50	95.85	96.28	97.31	96.72	95.43	96.28	97.33	96.94	95.85	96.56	97.33	96.94	95.02	96.27
FUZZY	0.01	95.22	98.02	88.78	94.84	95.18	98.02	89.20	94.98	95.17	98.02	88.78	94.84	95.07	98.02	89.20	94.98
	0.1	96.36	97.59	93.77	96.27	96.36	97.59	93.35	96.13	96.34	97.59	93.35	96.13	96.41	97.59	93.35	96.13
	1	97.14	96.94	97.10	**96.99**	97.15	96.94	97.10	**96.99**	97.14	96.91	97.10	**96.98**	97.14	97.11	96.68	96.96
	10	97.25	96.72	96.68	96.71	97.20	96.94	96.68	96.85	97.20	96.70	97.10	96.84	97.19	96.90	97.10	**96.97**
	100	97.23	96.50	96.68	96.56	97.27	96.50	96.27	96.42	97.30	96.72	95.85	96.42	97.23	96.92	95.43	96.40
	1000	97.33	96.50	95.85	96.28	97.31	96.72	95.43	96.28	97.33	96.94	95.85	96.56	97.33	96.93	95.02	96.27

		Train	Test			Train	Test			Train	Test			Train	Test		
		Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av	Av	G_{+1}	G_{-1}	Av
	$C\backslash 2a$	Inner fences				0				0.01				0.05			
CRISP	0.01	95.06	98.02	88.37	94.69	95.04	97.81	89.20	94.84	95.07	97.81	89.20	94.84	95.02	98.02	88.78	94.84
	0.1	96.31	97.59	92.93	95.98	96.26	97.59	93.77	96.27	96.26	97.59	93.77	96.27	96.28	97.59	93.77	96.27
	1	97.04	97.15	96.68	**96.99**	97.06	97.15	96.27	**96.85**	97.08	97.15	96.27	**96.85**	97.14	97.15	96.68	**96.99**
	10	97.11	96.50	96.68	96.56	97.00	96.72	97.10	**96.85**	96.96	96.72	96.68	96.71	97.11	96.50	97.10	96.71
	100	97.23	96.94	96.27	96.71	97.19	96.72	95.85	96.42	97.17	96.72	96.68	96.71	97.28	96.72	95.85	96.42
	1000	97.27	96.72	95.43	96.28	97.22	97.15	95.85	96.70	97.28	96.94	96.27	96.71	97.28	96.72	96.27	96.56
FUZZY	0.01	95.06	98.06	88.37	94.72	95.04	97.81	89.20	94.84	95.07	97.81	89.20	94.84	95.02	98.02	88.78	94.84
	0.1	96.31	97.36	92.93	95.83	96.23	97.54	93.77	96.24	96.23	97.54	93.77	96.24	96.28	97.53	93.77	96.23
	1	97.04	96.78	96.68	**96.74**	97.02	97.10	96.27	**96.81**	97.04	97.09	96.27	**96.80**	97.14	97.02	96.68	**96.91**
	10	97.11	96.22	96.68	96.38	96.97	96.65	97.10	**96.81**	96.94	96.65	96.68	96.66	97.11	96.42	97.10	96.65
	100	97.23	96.68	96.27	96.53	97.17	96.68	95.85	96.39	97.16	96.68	96.68	96.68	97.28	96.60	95.85	96.34
	1000	97.27	96.43	95.43	96.08	97.21	97.15	95.85	96.70	97.27	96.88	96.27	96.67	97.28	96.60	96.27	96.48
	$C\backslash 2a$	0.1				0.2				0.5				1			
CRISP	0.01	95.06	98.02	88.78	94.84	95.06	98.02	89.20	94.98	95.12	98.02	89.20	94.98	95.17	98.02	89.20	94.98
	0.1	96.30	97.59	93.77	96.27	96.36	97.59	93.35	96.13	96.41	97.59	93.35	96.13	96.36	97.59	93.35	96.13
	1	97.14	97.15	96.68	**96.99**	97.14	97.15	96.68	**96.99**	97.17	96.94	97.10	**96.99**	97.14	96.94	96.68	**96.85**
	10	97.11	96.50	96.68	96.56	97.15	96.72	96.68	96.71	97.27	96.72	96.68	96.71	97.27	96.94	96.68	**96.85**
	100	97.27	96.72	97.10	96.85	97.30	96.72	96.27	96.56	97.30	96.50	95.85	96.28	97.30	96.72	96.68	96.71
	1000	97.31	96.94	95.43	96.42	97.33	96.72	96.68	96.71	97.38	96.72	95.43	96.28	97.36	96.72	95.43	96.28
FUZZY	0.01	95.06	98.02	88.78	94.84	95.06	98.02	89.20	94.98	95.12	98.02	89.20	94.98	95.17	98.02	89.20	94.98
	0.1	96.30	97.53	93.77	96.23	96.36	97.53	93.35	96.09	96.41	97.59	93.35	96.13	96.36	97.59	93.35	96.13
	1	97.14	97.00	96.68	**96.89**	97.14	96.99	96.68	**96.89**	97.17	96.92	97.10	**96.98**	97.14	96.94	96.68	**96.85**
	10	97.11	96.41	96.68	96.50	97.15	96.59	96.68	96.62	97.27	96.70	96.68	96.69	97.27	96.94	96.68	**96.85**
	100	97.27	96.55	97.10	96.74	97.30	96.57	96.27	96.46	97.30	96.50	95.85	96.28	97.30	96.72	96.68	96.71
	1000	97.31	96.78	95.43	96.32	97.33	96.54	96.68	96.59	97.38	96.72	95.43	96.28	97.36	96.72	95.43	96.28

6 Concluding remarks

In this work, a classification problem based on Support Vector Machines has been described, where the elements to be classified are sets with certain geometrical properties. A fuzzy and a crisp classification rule have been defined in terms of a separating hyperplane which is found via solving a margin maximization problem.

Our model generalizes the formulation given in [37, 38] for data affected by some kind of perturbations, which are supposed to be unknown but bounded for a given norm. Likewise, our problem is applied to the case of interval-valued data, where a convex quadratic problem is obtained.

Several experiments have been performed. In particular, our model gets to improve the results published in [13] for the 'car' dataset, in a multi-class problem. This tool also shows that imputation based on intervals (by using the mean and deviation of the data or by using the interquartile range) obtains good results for the case of having missing data in a classification problem.

The extension of this model to regression looks like a promising field of application to be explored. Moreover, the introduction of different kernels in the model is another topic which deserves further studies.

Acknowledgements

The authors would like to thank Prof. Paula Brito for having provided the 'car' dataset which was used in one of the computational experiments.

References

[1] A. A. Afifi and R. M. Elashoff. Missing Observations in Multivariate Statistics: II. Point Estimation in Simple Linear Regression. *Journal of the American Statistical Association*, 62(317):10–29, (1967).

[2] A. Ben-Tal and A. Nemirovski. Robust Convex Optimization. *Mathematics of Operations Research*, 23(4):769–805, (1996).

[3] A. Ben-Tal and A. Nemirovski. Robust Solutions of Uncertain Linear Programs. *Operations Research Letters*, 25:1–13, (1998).

[4] S. Bhattacharyya, L. Grate, S. Mian, L. El Ghaoui, and M. Jordan. Robust Sparse Hyperplane Classifiers: Application to Uncertain Molecular Profiling Data. *Journal of Computational Biology*, 11(6):1073–1089, (2004).

[5] L. Billard and E. Diday. From the Statistics of Data to the Statistics of Knowledge: Symbolic Data Analysis. *Journal of the American Statistical Association*, 98(462):470–487, (2003).

[6] C. L. Blake and C. J. Merz. UCI Repository of Machine Learning Databases. http://www.ics.uci.edu/ mlearn/MLRepository.html, (1998).

[7] H-H. Bock and E. Diday, editors. *Analysis of Symbolic Data: Exploratory Methods for Extracting Statistical Information from Complex Data.* Springer-Verlag, Berlin, (2000).

[8] C. J. C. Burges. A Tutorial on Support Vector Machines for Pattern Recognition. *Data Mining and Knowledge Discovery*, 2(2):121–167, (1998).

[9] C. Cortes and V. Vapnik. Support Vector Networks. *Machine Learning*, 20:273–297, (1995).

[10] N. Cristianini and J. Shawe-Taylor. *An Introduction to Support Vector Machines and other Kernel-based Learning Methods.* Cambridge University Press, Cambridge, (2000).

[11] L. Devroye and T. J. Wagner. Distribution-free Performance Bounds with the Resubstitution Error Estimate. *IEEE Transactions on Information Theory*, 25(2):208–210, (1979).

[12] T-N. Do and F. Poulet. Kernel Methods and Visualization for Interval Data Mining. In *Proc. of the Conference on Applied Stochastic Models and Data Analysis*, pages 345–354, (2005).

[13] A. P. Duarte-Silva and P. Brito. Linear Discriminant Analysis for Interval Data. *Computational Statistics*, 21:289–308, (2006).

[14] L. El Ghaoui, G.R.G. Lanckriet, and G. Natsoulis. Robust Classification with Interval Data. Technical Report CSD-03-1279, Division of Computer Science, University of California, Berkeley.

[15] J. H. Friedman. Another Approach to Polychotomous Classification. Technical report, Stanford University, (1996).

[16] D. J. Hand, H. Mannila, and P. Smyth. *Principles of Data Mining.* MIT Press, Cambridge, (2001).

[17] T. Hastie and R. Tibshirani. Classification by Pairwise Coupling. *The Annals of Statistics*, 26(2):451–471, (1998).

[18] R. Herbrich. *Learning Kernel Classifiers: Theory and Algorithms.* MIT Press, Cambridge, (2002).

[19] J-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms.* Springer-Verlag, Berlin, (1993).

[20] C-W. Hsu and C-J. Lin. A Comparison of Methods for Multi-class Support Vector Machines. *IEEE Transactions on Neural Networks*, 13(2):415–425, (2002).

[21] R. Kohavi. A Study of Cross-Validation and Bootstrap for Accuracy Estimation and Model Selection. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence*, pages 1137–1143, (1995).

[22] G. R. G. Lanckriet, L. El Ghaoui, C. Bhattacharyya, and M. I. Jordan. A Robust Minimax Approach to Classification. *Journal of Machine Learning Research*, 3:555–582, (2002).

[23] R. J. A. Little. Regression with Missing X's. *Journal of the American Statistical Association*, 87(420):1227–1237, (1992).

[24] R. J. A. Little and D. B. Rubin. *Statistical Analysis with Missing Data.* John Wiley & Sons, New Jersey, (2002).

[25] W. Z. Liu, A. P. White, S. G. White, S. G. Thompson, and M. A. Bramer. Techniques for Dealing with Missing Values in Classification. In *Proc. of the Second International Symposium on Intelligent Data Analysis*, pages 527–536, (1997).

[26] M. Magnani. Techniques for Dealing with Missing Data in Knowledge Discovery Tasks. http://magnanim.web.cs.unibo.it/data/pdf/missingdata.pdf, (2004).

[27] J. M. Moguerza and A. Muñoz. Support Vector Machines with Applications. *Statistical Science*, 21(3):322–336, (2006).

[28] NEOS. Server for Optimization. http://www-neos.mcs.anl.gov/.

[29] E. Périnel and Y. Lechevallier. Symbolic Discrimination Rules. In H-H. Bock and E. Diday, editors, *Analysis of Symbolic Data: Exploratory Methods for Extracting Statistical Information from Complex Data*, pages 244–265. Springer-Verlag, Berlin, (2000).

[30] F. Plastria and E. Carrizosa. Gauge-Distances and Median Hyperplanes. *Journal of Optimization Theory and Applications*, 110:173–182, (2001).

[31] J. C. Platt, N. Cristianini, and J. Shawe-Taylor. Large Margin DAGs for Multiclass Classification. In *Advances in Neural Information Processing Systems 12*, pages 547–553, (2000).

[32] F. Rossi and B. Conan-Guez. Multi-layer Perceptron on Interval Data. In K. Jajuga, A. Sokolowski, and H-H. Bock, editors, *Classification, Clustering and Data Analysis*, pages 427–434. Springer-Verlag, Berlin, (2002).

[33] J. Scheffer. Dealing with Missing Data. *Research Letters in the Information and Mathematical Sciences*, 3:153–160, (2002).

[34] B. Schölkopf and A. Smola. *Learning with Kernels: Support Vector Machines, Regularization, Optimization and Beyond.* MIT Press, Cambridge, (2002).

[35] J. Síma. Neural Expert Systems. *Neural Networks*, 8(2):261–271, (1995).

[36] D. M. J. Tax and R. P. W. Duin. Using Two-class Classifiers for Multiclass Classification. In *Proceedings of the 16th International Conference on Pattern Recognition*, pages 124–127, (2002).

[37] T. B. Trafalis and R. C. Gilbert. Robust Classification and Regression Using Support Vector Machines. *European Journal of Operational Research*, 173(3):893–909, (2006).

[38] T. B. Trafalis and R. C. Gilbert. Robust Support Vector Machines for Classification and Computational Issues. *Optimization Methods and Software*, 22(1):187–198, (2007).

[39] J. W. Tukey. *Exploratory Data Analysis.* Addison-Wesley, Reading, MA, (1977).

[40] R. J. Vanderbei. LOQO User's Manual - Version 4.05. Technical Report ORFE-99, Operations Research and Financial Engineering, Princeton University, New Jersey, (2000).

[41] V. N. Vapnik. *Statistical Learning Theory.* John Wiley and Sons, New York, (1998).

[42] J. Weston and C. Watkins. Multi-class Support Vector Machines. Technical Report CSD-TR-98-04, Department of Computer Science, Royal Holloway, University of London, (1998).