Robust Efficient Frontier Analysis with a Separable Uncertainty Model

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Abstract

Mean-variance (MV) analysis is often sensitive to model mis-specification or uncertainty, meaning that the MV efficient portfolios constructed with an estimate of the model parameters (*i.e.*, the expected return vector and covariance of asset returns) can give very poor performance for another set of parameters that is similar and statistically hard to distinguish from the one used in the analysis. Robust MV analysis attempts to systematically alleviate the sensitivity problem, by explicitly incorporating an uncertainty model on the parameters in a portfolio selection problem and carrying out the analysis for the worst-case scenario under the model.

This paper concerns robust MV analysis with a separable uncertainty model, in which uncertainty in the mean return vector is independent of that in the covariance matrix, a model which has been widely used in the literature. The main focus is on the (worst-case) robust efficient frontier, *i.e.*, the optimal trade-off curve in terms of worst-case MV preference, as the extension of the efficient frontier to the worst-case MV analysis setting. We establish some basic properties of the robust efficient frontier, describe a method for computing it, and give several computationally tractable uncertainty models. We also establish a fundamental relation between the robust efficient frontier and the infimum of all efficient frontiers consistent with the assumptions made on the model parameters. The robust efficient frontier analysis method is illustrated with a numerical example.

1 Introduction

We consider MV analysis with n risky assets held over a period of time. Their (percentage) returns over the period are modeled as a random vector $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ with mean $\mu = \mathbf{E}\, a$ and covariance $\Sigma = \mathbf{E}\, (a-\mu)(a-\mu)^T$ of $a = (a_1, \ldots, a_n)$, where $\mathbf{E}$ denotes the expectation operation [48]. We assume that Σ is positive definite. We let w_i denote the

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amount of asset i held throughout the period. The return of a portfolio $w = (w_1, \ldots, w_n)$ is a (scalar) random variable $w^T a = \sum_{i=1}^n w_i a_i$. The mean return of w is $w^T \mu$, and the risk, measured by the standard deviation, is $(w^T \Sigma w)^{1/2}$. We assume that an admissible portfolio $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ is constrained to lie in a closed convex subset $\mathcal{W}$ of $\mathbb{R}^n$. Each portfolio $w \in \mathcal{W}$ must satisfy the budget constraint $\mathbf{1}^T w = 1$, where $\mathbf{1}$ is the vector of all ones. The convex set $\mathcal{W}$ can represent a wide variety of convex asset allocation constraints (beyond the budget constraint) including portfolio diversification, short-selling constraints, long/short constraints, market impact constraints with convex impact costs, transaction cost constraints with convex transaction costs, and bound constraints; see, *e.g.*, [20, 42, 43, 53].

1.1 Efficient frontier analysis

We give a brief review of EF analysis to set up our notation, and to compare it the extension we describe in this paper.

Optimal trade-off between risk and return

The choice of a portfolio involves a trade-off between risk and return [48]. To describe the optimal trade-off, we consider the portfolio optimization problem

$$\begin{array}{ll} \text{maximize} & w^T \mu \\ \text{subject to} & w \in \mathcal{W}, \quad \sqrt{w^T \Sigma w} \leq \sigma, \end{array} \tag{1}$$

where the variable is w and the problem data or parameters are μ and Σ. In this problem we find the portfolio that maximizes the expected return subject to a maximum acceptable volatility level σ (associated with the standard deviation of the return), and satisfying the asset allocation and portfolio budget constraints.

As σ varies over $(0, \infty)$, the trajectory of the optimal solution defines the curve

$$f_{\mu,\Sigma}(\sigma) = \sup_{w \in \mathcal{W},\ \sqrt{w^T \Sigma w} \leq \sigma} w^T \mu. \tag{2}$$

The curve $f_{\mu,\Sigma}(\sigma)$ is concave and increasing over $\sigma \geq \inf_{w \in \mathcal{W}} \sqrt{w^T \Sigma w}$. Since the covariance Σ is positive definite (by assumption), $f_{\mu,\Sigma}(\sigma)$ is strictly concave over the interval $[\sigma_{\text{inf}}, \sigma_{\text{sup}})$, where

$$\sigma_{\text{sup}} = \inf \left\{ \sqrt{w^T \Sigma w} \;\middle|\; w \in \mathcal{W},\ w^T \mu = \sup_{w \in \mathcal{W}} w^T \mu \right\}.$$

The strictly concave portion is the optimal risk-return trade-off curve of the assets $a_1, \ldots, a_n$ and called the (mean-variance or Markowitz) efficient frontier (EF). A portfolio w is called (MV) efficient if its risk and return are on the EF. Any portfolio with the same return as w would have a higher risk. In the sequel, the curve $f_{\mu,\Sigma}(\sigma)$ is often called the efficient frontier, although strictly speaking only the strictly concave portion is.

These definitions are illustrated in Figure 1.

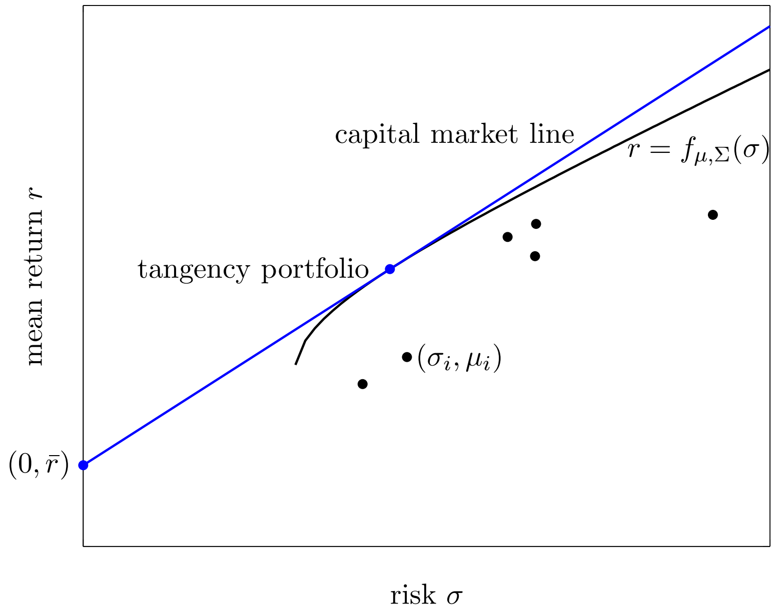

Figure 1: The efficient frontier of risky assets, the tangency portfolio, and the capital market line.

Efficient frontier analysis via Sharpe ratio maximization

When there is no asset allocation constraint (except for the portfolio budget constraint), the two fund theorem tells us that the EF is a hyperbola and two efficient funds (portfolios) can be established so that any efficient portfolio can be duplicated, in terms of mean and variance as a combination of these two [45, 50]. With linear asset allocation constraints, the portfolio optimization problem (1), and hence the EF, can be computed efficiently using a variety of methods including the critical line method and its extensions [33, 49, 66]. With general convex asset allocation constraints, the problem of computing $f_{\mu,\Sigma}(\sigma)$ is a convex optimization problem, so we can compute the EF efficiently using standard methods of convex optimization. Using the idea behind Roy's safety-first approach to portfolio selection [59, 58], the EF can be computed, as shown below.

We use $S_{\bar{r}}(w, \mu, \Sigma)$ to denote the ratio of the excess expected return of a portfolio w relative to the (hypothetical) risk-free return $\bar{r}$ (with zero return variance) to the return volatility:

$$S_{\bar{r}}(w, \mu, \Sigma) = \frac{w^T\mu - \bar{r}}{\sqrt{w^T \Sigma w}}.$$

This ratio is called the reward-to-variability or Sharpe ratio (SR) of w (when the risk-free return is $\bar{r}$).

The problem of finding the admissible portfolio that maximizes the SR can be written as

$$\begin{array}{ll} \text{maximize} & S_{\bar{r}}(w, \mu, \Sigma) \\ \text{subject to} & w \in \mathcal{W}, \end{array} \qquad (3)$$

with variable $w \in \mathbb{R}^n$ and problem data μ and Σ. The optimal value is called the market price of risk. This problem is related to Roy's safety-first approach through the Chebyshev inequality [59].

The line

$$r = \bar{r} + \sup_{w \in \mathcal{W}} S_{\bar{r}}(w, \mu, \Sigma)\sigma, \tag{4}$$

is called the optimal capital allocation line (CAL) or capital market line (CML) (when the risk-free return is $\bar{r}$) [8]. We can see from the concavity of $f_{\mu,\Sigma}$ that

$$\sup_{w \in \mathcal{W}} S_{\bar{r}}(w, \mu, \Sigma) = \sup_{\sigma > 0} \frac{f_{\mu,\Sigma}(\sigma) - \bar{r}}{\sigma}, \quad \bar{r} < \sup_{w \in \mathcal{W}} w^T\mu. \tag{5}$$

(The supremum over the empty set is $-\infty$, so $f_{\mu,\Sigma}(\sigma) = -\infty$ whenever $\sigma < \inf_{w \in \mathcal{W}}(w^T\Sigma w)^{1/2}$.) Therefore, if there is an admissible portfolio that achieves the maximum SR, *i.e.*, the optimal value of (3), the CML is tangential to the EF at the risk and return of the portfolio. This portfolio is called the tangency portfolio (TP). (See figure 1 for an illustration; the filled circle corresponds to the risk and return of the TP.) If there is no such portfolio, then the robust optimal CAL lies entirely above the EF:

$$f_{\mu,\Sigma}(\sigma) < \bar{r} + \sup_{w \in \mathcal{W}} S_{\bar{r}}(w, \mu, \Sigma)\sigma, \quad \sigma > 0.$$

This case can arise only when the EF has an (upper) asymptote and the CML is parallel to the asymptote.

As $\bar{r}$ varies, the tangential point $(\sqrt{w^{\star T}\Sigma w^{\star}}, w^{\star T}\mu)$ moves along the EF. The EF can therefore be computed as the trajectory of the point, as $\bar{r}$ varies. From (5) and the concavity of the efficient frontier, the curve $f_{\mu,\Sigma}$ can be expressed as the infimum of the CMLs as $\bar{r}$ varies over the interval $(-\infty, \sup_{w \in \mathcal{W}} w^T\mu)$:

$$f_{\mu,\Sigma}(\sigma) = \inf_{\bar{r} < \sup_{w \in \mathcal{W}} w^T\mu} \left(\bar{r} + \sup_{w \in \mathcal{W}} S_{\bar{r}}(w, \mu, \Sigma)\sigma \right), \quad \sigma \geq \inf_{w \in \mathcal{W}} \sqrt{w^T\Sigma w}. \tag{6}$$

1.2 MV analysis with a separable uncertainty model

In standard or conventional MV portfolio analysis, we assume that the input parameters, *i.e.*, the mean vector and covariance matrix of asset returns, are known. In practice, however, the input parameters are estimated with error. Standard MV analysis is often sensitive to uncertainty or estimation error in the parameters, meaning that MV efficient portfolios computed with an estimate of the parameters can give very poor performance for another set of parameters that is similar and statistically hard to distinguish from the assumed one.

There has been a growing interest in (worst-case) robust MV analysis and optimization as a systematic way of finding portfolio weights that performs reasonably well despite estimation error or model uncertainty. (A brief review of the literature on robust portfolio optimization will be given in the next section.) In this paper we consider robust MV analysis with a product form or separable uncertainty model

$$\mathcal{U} = \mathcal{M} \times \mathcal{S} \subseteq \mathbb{R}^n \times \mathbb{S}^n_{++}, \tag{7}$$

where $\mathcal{M}$ is the set of possible expected return vectors and $\mathcal{S}$ is the set of possible covariances. Here we use $\mathbb{S}^n_{++}$ to denote the set of all $n \times n$ symmetric positive definite matrices. In this model, the uncertainties in the mean return vector and the covariance are independent of each other. We assume that $\mathcal{M}$ and $\mathcal{S}$ are compact (*i.e.*, bounded and closed). Separable uncertainty models have been widely used in the literature on robust portfolio optimization; see, *e.g.*, [27, 30, 42, 51].

With model uncertainty, the performance of a portfolio w is described by the set of risk-return pairs computed with model parameters over the set $\mathcal{U}$:

$$\mathcal{P}(w) = \left\{ \left((w^T \Sigma w)^{1/2}, w^T \mu\right) \in \mathbb{R}^2 \;\middle|\; (\mu, \Sigma) \in \mathcal{U} \right\}.$$

Assuming $\mathcal{U}$ is connected, the set $\mathcal{P}(w)$ is a box:

$$\mathcal{P}(w) = \left\{ (\sigma, r) \in \mathbb{R}^2 \;\middle|\; \inf_{\Sigma \in \mathcal{S}} \sqrt{w^T \Sigma w} \le \sigma \le \sup_{\Sigma \in \mathcal{S}} \sqrt{w^T \Sigma w},\ \inf_{\mu \in \mathcal{M}} w^T \mu \le r \le \sup_{\mu \in \mathcal{M}} w^T \mu \right\}. \tag{8}$$

The worst-case scenario arises in the lower right corner of the box $\mathcal{P}(w)$, since it has the highest risk and the lowest return. The worst-case risk of w, where 'worst' means largest, is

$$\sigma_{\rm wc}(w) = \sup_{\Sigma \in \mathcal{S}} \sqrt{w^T \Sigma w}.$$

The worst-case return with the (given) portfolio w, where 'worst' means smallest, is

$$r_{\rm wc}(w) = \inf_{\mu \in \mathcal{M}} w^T \mu.$$

A mean vector $\mu \in \mathcal{M}$ is called a worst-case expected return vector for w if $w^T \mu = r_{\rm wc}(w)$, and a covariance $\Sigma \in \mathcal{S}$ is called a worst-case covariance of w if $(w^T \Sigma w)^{1/2} = \sigma_{\rm wc}(w)$.

Since the risk-return set of a portfolio has an obvious worst-case corner, it is straightforward to extend the notion of portfolio preference to robust MV analysis with a separable uncertainty model: A portfolio $\bar{w}$ is preferred to $\hat{w}$ if $\bar{w}$ has a lower or equal worst-case risk, and a higher or equal worst-case return than $\hat{w}$. (If they are both equal, then $\hat{w}$ and $\bar{w}$ are equivalent, in terms of worst-case risk and return.)

Figure 2 illustrates worst-case risk-return analysis. The shaded rectangles in the figure correspond to the risk-return sets of three portfolios $w^{(1)}, w^{(2)}, w^{(3)}$. The worst-case corners are shown as filled circles. We can easily see that $w^{(2)}$ is preferred to $w^{(3)}$ in the worst-case MV sense. There is no worst-case MV portfolio preference relation among the other two pairs (*i.e.*, $w^{(1)}$ and $w^{(2)}$, and $w^{(1)}$ and $w^{(3)}$).

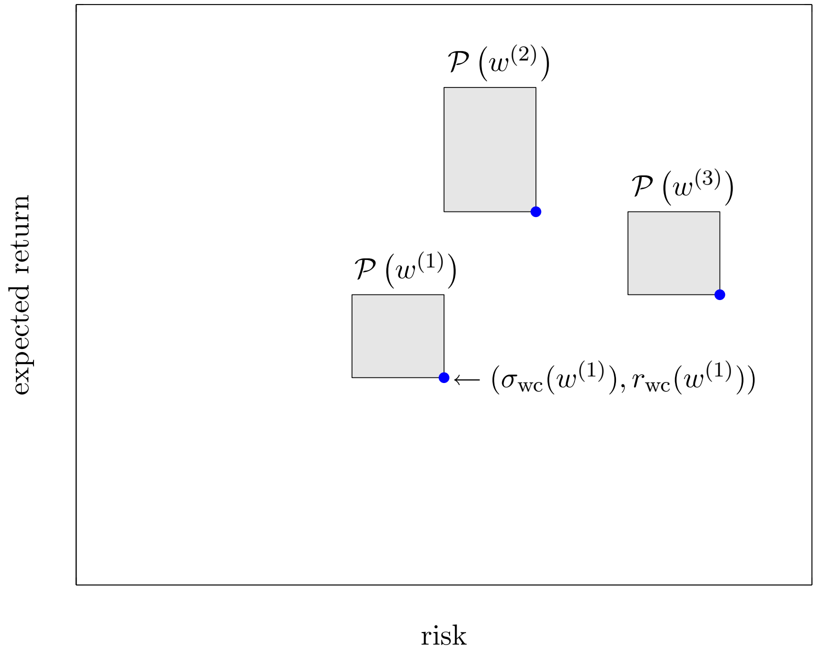

Figure 2: Risk-return sets of three portfolios with a separable uncertainty model. For each portfolio, the worst-case is shown as a filled circle at lower right.

Before proceeding, we show how to efficiently carry out risk-return analysis with a convex separable uncertainty model. The maximum and minimum risk of a portfolio w over the set $\mathcal{S}$ can be written as

$$\inf_{\Sigma\in\mathcal{S}}(w^T\Sigma w)^{1/2} = \left(\inf_{\Sigma\in\mathcal{S}} w^T\Sigma w\right)^{1/2}, \qquad \sup_{\Sigma\in\mathcal{S}}(w^T\Sigma w)^{1/2} = \left(\sup_{\Sigma\in\mathcal{S}} w^T\Sigma w\right)^{1/2}.$$

When the sets $\mathcal{M}$ and $\mathcal{S}$ are convex, computing these two quantities as well as the minimum return $\inf_{\mu\in\mathcal{M}} w^T\mu$ and the maximum return $\sup_{\mu\in\mathcal{M}} w^T\mu$ requires us to minimize or maximize a linear function over a convex set. Therefore, the four corners of the set $\mathcal{P}(w)$, and hence the risk-return set, can be computed efficiently using convex optimization.

1.3 Robust EF analysis with a separable uncertainty model

In this paper, we are interested in optimal trade-off of risk and return in the worst-case sense. To describe the trade-off, we consider the (worst-case) robust counterpart of the portfolio optimization problem (1)

$$\begin{array}{ll} \text{maximize} & r_{\rm wc}(w) \\ \text{subject to} & w \in \mathcal{W}, \quad \sigma_{\rm wc}(w) \le \sigma. \end{array} \tag{9}$$

Here we find the portfolio that maximizes the worst-case expected return subject to a maximum acceptable worst-case volatility level σ, and satisfying the asset allocation and portfolio budget constraints.

As σ varies over $(0, \infty)$, the trajectory of the optimal solution of this problem defines the curve

$$f_{\mathrm{rob}}(\sigma) = \sup_{w \in \mathcal{W},\, \sigma_{\mathrm{wc}}(w) \le \sigma} r_{\mathrm{wc}}(w). \tag{10}$$

The curve $f_{\mathrm{rob}}(\sigma)$ is increasing and concave over $\sigma \ge \sigma_{\mathrm{inf}}$, where σ_{inf} is the minimum worst-case risk level,

$$\sigma_{\mathrm{inf}} = \inf_{w \in \mathcal{W}} \sigma_{\mathrm{wc}}(w).$$

The curve $f_{\mathrm{rob}}(\sigma)$ is strictly concave over the interval $[\sigma_{\mathrm{inf}}, \sigma_{\mathrm{sup}})$, where

$$\sigma_{\mathrm{sup}} = \inf \left\{ \sigma_{\mathrm{wc}}(w) \;\middle|\; w \in \mathcal{W},\; r_{\mathrm{wc}}(w) = \sup_{w \in \mathcal{W}} r_{\mathrm{wc}}(w) \right\}.$$

(Whenever $\sigma \ge \sigma_{\mathrm{sup}}$, $f_{\mathrm{rob}}(\sigma) = \sup_{w \in \mathcal{W}} r_{\mathrm{wc}}(w)$.) The proof of the strict concavity property is given in Appendix A.1.

The strictly concave portion of the curve $f_{\mathrm{rob}}(\sigma)$ is the optimal trade-off curve between worst-case risk and return, and is called the (worst-case) robust EF. (This definition should be distinguished from the EF computed with an estimate of the parameters obtained using a robust statistical estimation procedure, which is often called the robust EF in the literature; see, *e.g.*, [57].) An admissible portfolio $w \in \mathcal{W}$ is called (worst-case) robust MV efficient if its worst-case risk and return lie on the strictly increasing portion. Any portfolio with the same worst-case return would have a higher worst-case risk. In the sequel, the curve f_{rob} is often called the robust EF, although strictly speaking only the strictly increasing portion describes the optimal trade-off.

Figure 3 illustrates the definitions given above. The portfolios $w^{(1)}$, $w^{(2)}$, and $w^{(3)}$ are robust MV efficient, whereas $\hat{w}$ is not.

1.4 Summary of the paper

We give a summary of the main results of this paper.

Robust EF and sampled EFs

There is an interesting relation between the robust EF and the infimum of all EFs consistent with the assumptions made on the model parameters. Since the infimum of concave curves is concave, the infimum of the curves is concave.

Theorem 1. *For any $\mathcal{M} \subseteq \mathbb{R}^n$ and any $\mathcal{S} \subseteq \mathbb{S}^n_{++}$,*

$$f_{\mathrm{rob}}(\sigma) \le \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} f_{\mu,\Sigma}(\sigma), \quad \sigma > 0. \tag{11}$$

When $\mathcal{M}$ and $\mathcal{S}$ are convex,

$$f_{\mathrm{rob}}(\sigma) = \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} f_{\mu,\Sigma}(\sigma), \quad \sigma > 0. \tag{12}$$

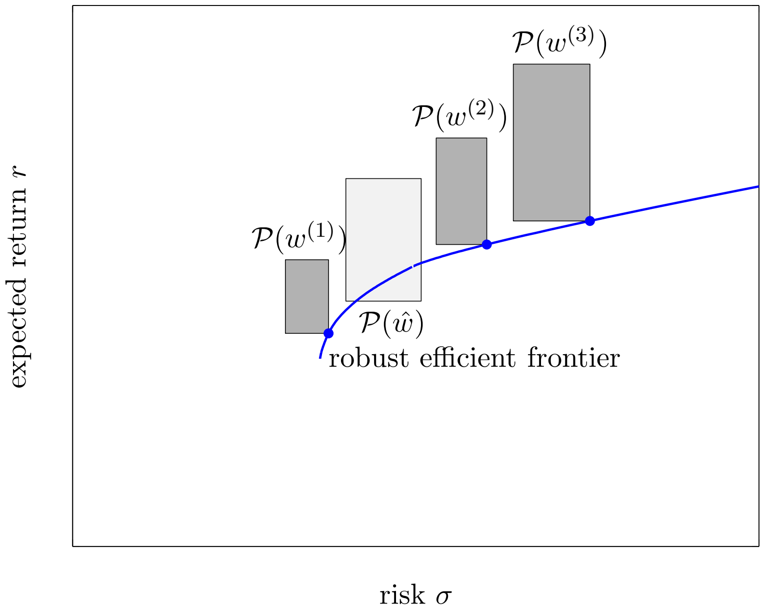

Figure 3: Robust efficient frontier analysis with a separable uncertainty model. The portfolios $w^{(1)}$, $w^{(2)}$, and $w^{(3)}$ are robust MV efficient; the portfolio $\hat{w}$ is not.

The proof is deferred to Appendix A.2.

This theorem has an important implication for the sampling based approach to (approximate) robust EF analysis. Suppose we uniformly sample a finite number of pairs $(\mu^{(i)}, \Sigma^{(i)})$ for $i = 1, \ldots, m$ from $\mathcal{U}$ and compute the corresponding EFs, $f_{\mu^{(i)},\Sigma^{(i)}}$ for $i = 1, \ldots, m$. In the convex case, as the number of samples m increases, the infimum $\inf_i f_{\mu^{(i)},\Sigma^{(i)}}$ of the sampled EFs converges to the robust EF. By contrast, in the non-convex case, the infimum need not give a good approximation of the robust EF, even when m is large.

Robust EF analysis via worst-case SR maximization

The objective of the robust portfolio optimization problem (9) is a concave function of w, since it is the pointwise infimum of a family of linear functions. The constraint set $\mathcal{C} = \left\{ w \in \mathbb{R}^n \;\middle|\; w \in \mathcal{W},\ \sigma_{\mathrm{wc}}(w) \leq \sigma \right\}$ is convex, since the constraints consist of a family of convex quadratic constraints, parameterized by $\Sigma \in \mathcal{S}$ and the convex asset allocation constraint $w \in \mathcal{W}$. The robust portfolio optimization problem (9) is a convex semi-infinite program, so robust EF analysis with a general uncertainty model can be carried out via semi-infinite programming. However, semi-infinite programs are difficult to solve in the absence of special structure. The reader is referred to [31] for more on general approximate solution methods for semi-infinite programs.

For computational tractability, we are interested in uncertainty models with which robust EF analysis can be carried out efficiently using convex optimization. We show how the idea behind the method for computing the EF via SR maximization can be generalized to MV analysis with a separable uncertainty model. Specifically, we show that the robust EF can be computed from the trajectory of the solution to the robust counterpart of the SRMP (3), called the worst-case SRMP, as the hypothetical risk-free return $\bar{r}$ varies. With general convex uncertainty models, the corresponding robust counterparts can be solved using the minimax result for the SR proved in [39] and convex optimization. As a consequence of the minimax result, the robust EF can be computed as the trajectory of the saddle point of the zero-sum game of choosing w from $\mathcal{W}$, to maximize the SR, and choosing (μ, Σ) from $\mathcal{U} = \mathcal{M} \times \mathcal{S}$, to minimize the SR, as $\bar{r}$ varies. For certain types of non-convex models including those considered in [18, 27], the corresponding robust counterparts can be reformulated as convex problems. We conclude that robust EF analysis with these models can be carried out efficiently.

Robust portfolio selection problems

In standard MV analysis, once the EF is computed, a variety of MV portfolio optimization problems reduce to simple one-dimensional search problems over the family of MV efficient portfolios. The list includes minimum variance portfolio selection problems, value-at-risk (VaR) minimization problems, and expected quadratic utility maximization problems. Once the robust EF is found, the robust counterparts of the portfolio selection problems listed above reduce to simple one-dimensional search problems over the set of MV efficient portfolios and hence can be solved efficiently.

Outline

In the remainder of this section, we give a brief review of related literature. In §2-§4, we give the details of the results summarized above. We illustrate the main results with a numerical example in §5. We give our conclusions in §6. The appendix contains the proofs that are omitted from the main text.

1.5 Related literature review

MV efficient portfolios computed with an estimate of the parameters often contain extremely long and short positions (when the constraint set allows such positions), which are difficult to implement. As a result, the optimal portfolios are typically very sensitive to variations in the estimated mean and covariance matrix. The sensitivity problem is often called the *estimation risk*. The sensitivity problem has been well documented in the literature; see, *e.g.*, [5, 7, 11, 16, 23, 29, 35, 52]. Relatively recent work on estimation risk includes [12, 15, 63].

Over the past several decades, a variety of approaches to mitigating the sensitivity problem or accounting for estimation risk in MV analysis have been proposed. The list includes imposing constraints such as no short-sales constraints [34], the resampling approach [52], a non-Bayesian adjustment method [63], Bayesian approaches [2, 13, 40, 55, 56], the shrinkage approach [69], the empirical Bayes approach [24], and the Black-Litterman approach [7] (which incorporates ideas from economics).

Recently, many researchers have paid attention to alleviating the sensitivity problem using the idea of (worst-case) robust optimization [4], which is often called *(worst-case) robust MV analysis*. The key idea is to explicitly incorporate a model of data uncertainty in the formulation of a portfolio optimization problem, and to optimize for the worst-case scenario under the model; see, *e.g.*, [19, 18, 27, 25, 30, 32, 37, 36, 42, 67, 60, 62]. The reader is referred to a recent survey in [22] and recent books on asset allocation methods [14, 21, 51, 61]. Several numerical studies support the expectation that robust portfolio optimization can be a valuable tool in quantitative asset allocation [6, 62]. The idea of robust optimization has been used in other financial optimization problems including robust hedging [47], multi-stage portfolio selection [3], and robust portfolio optimization via worst-case regret minimization [46, 41].

Several researchers have considered the worst-case SRMP and robust EF analysis with specific types of structured uncertainty models. The main focus has been on formulating the problem as a tractable convex problem. In [27], the authors show that the worst-case SRMP with a certain type of covariance model which is not convex can be cast as a second-order cone program (SOCP). They also show that several other robust portfolio selection problems can be cast as SOCPs. The main results of this paper show that once the robust EF is computed, these problems can be solved at no additional computational cost. In [67], the authors consider robust EF analysis with a certain type of separable uncertainty model in which the elements of the covariance are subject to lower and upper bound constraints. The authors show that the worst-case analysis can be carried out, by using a special-purpose interior-point method, developed in [30], for a specific class of saddle-point problems. They define the robust EF as the trajectory of saddle points in a family of certain parameterized zero-sum games. In this paper, we describe a method for computing the robust EF with a more general class of uncertainty models.

2 Robust EF analysis via worst-case SR maximization

We show how the idea behind the method for computing the EF $f_{\mu,\Sigma}$ reviewed above can be generalized to robust EF analysis with a separable uncertainty model $\mathcal{U} = \mathcal{M} \times \mathcal{S}$.

2.1 Worst-case SR analysis and optimization

In the presence of model mis-specification or uncertainty, the SR of a portfolio becomes uncertain. The worst-case SR of a portfolio w (over the uncertainty model $\mathcal{M} \times \mathcal{S}$) is

$$S_{\rm wc}(w, \bar{r}) = \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} S_{\bar{r}}(w, \mu, \Sigma).$$

The worst-case scenario for the SR arises in one of the two lower corners of the risk-return set $\mathcal{P}(w)$, depending on the sign of the worst-case excess return $r_{\rm wc}(w) - \bar{r}$. When $\mathcal{M}$ and $\mathcal{S}$ are convex, the worst-case SR can be readily computed using convex optimization, since the (four) corners of $\mathcal{P}(w)$ can be.

The problem of finding an admissible portfolio that maximizes the worst-case SR can be formulated as

$$\begin{array}{ll} \text{maximize} & \inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} S_{\bar{r}}(w,\mu,\Sigma) \\ \text{subject to} & w \in \mathcal{W}, \end{array} \tag{13}$$

which is called the worst-case SR maximization problem (SRMP) (when the risk-free return is $\bar{r}$). If this problem has a solution, then it has a unique solution [38]. We use the following shorthand notation for the optimal value of this problem:

$$S^\star(\bar{r}) = \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} S_{\bar{r}}(w,\mu,\Sigma).$$

As the robust counterpart of (4), we consider the line

$$r = \bar{r} + S^\star(\bar{r})\sigma \tag{14}$$

which is called the (worst-case) robust optimal CAL. When lending or borrowing at the risk-free rate $\bar{r}$ is allowed and the slope of the robust optimal CAL is positive, the robust optimal CAL describes the fundamental limitations of asset allocation in the worst-case sense (see [38]). In this paper, we consider the case when there is no risk-free asset available.

Another problem of interest is to find least favorable asset return statistics over the set $\mathcal{U}$, with portfolio weights chosen optimally for the asset return statistics:

$$\begin{array}{ll} \text{minimize} & \sup_{w\in\mathcal{W}} S_{\bar{r}}(w,\mu,\Sigma) \\ \text{subject to} & \mu \in \mathcal{M}, \quad \Sigma \in \mathcal{S}. \end{array} \tag{15}$$

Here, the hypothetical risk-free return $\bar{r}$ is fixed. The optimal value is called the worst-case market price of risk (when the risk-free return is $\bar{r}$). This problem is called the worst-case market price of risk analysis problem (MPRAP).

The two problems described above are related to each other via minimax properties. For any $\mathcal{M} \subseteq \mathbb{R}^n$ and any $\mathcal{S} \subseteq \mathbb{S}^n_{++}$, the minimax inequality or weak minimax property

$$\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} \sup_{w\in\mathcal{W}} S_{\bar{r}}(w,\mu,\Sigma) \le \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} S_{\bar{r}}(w,\mu,\Sigma)$$

holds. In other words, the slope of the robust optimal CAL is not greater than the worst-case market price of risk. When $\mathcal{M}$ and $\mathcal{S}$ are convex and compact and the left-hand side is positive, we have the following minimax equality or strong minimax property [38]:

$$\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} \sup_{w\in\mathcal{W}} S_{\bar{r}}(w,\mu,\Sigma) = \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} S_{\bar{r}}(w,\mu,\Sigma). \tag{16}$$

2.2 Existence and uniqueness

We address the basic question in the two problems given above: the existence and uniqueness of solutions. The worst-case MPRAP (15) always has a solution, which is not necessarily unique. The worst-case SRMP (13) does not always have a solution. When (13) has a solution, it has a unique solution [38].

The unique solution of (13) can be found by solving the convex semi-infinite program

$$\begin{array}{ll} \text{maximize} & \inf_{\mu \in \mathcal{M}} x^T(\mu - \bar{r}\mathbf{1}) \\ \text{subject to} & x \in \mathcal{X}, \quad \sup_{\Sigma \in \mathcal{S}} x^T \Sigma x \leq 1, \end{array} \tag{17}$$

where $w \in \mathbb{R}^n$ is the variable and $\mathcal{X}$ is the convex cone defined by

$$\mathcal{X} = \mathbf{cl}\,\{tw \in \mathbb{R}^n \mid w \in \mathcal{W},\, t > 0\} \backslash \{0\}.$$

Here $\mathbf{cl}\,A$ is the closure of the set A and $A \backslash B$ is the complement of B in A.

Proposition 1. *Suppose that there is an admissible portfolio $\bar{w} \in \mathcal{W}$ with*

$$\inf_{\mu \in \mathcal{M}} \bar{w}^T \mu > \bar{r}. \tag{18}$$

Then, the semi-infinite program (17) has a unique solution, say $x^\star$ with $\mathbf{1}^T x^\star \geq 0$. If $\mathbf{1}^T x^\star > 0$, then $w^\star = (1/\mathbf{1}^T x^\star) x^\star$ is the unique solution to the worst-case SRMP (13), and if $\mathbf{1}^T x^\star = 0$, then no admissible portfolio can achieve the optimal value of (13).

The proof is deferred to Appendix A.3.

We can see from the concavity of the robust EF and this proposition that the robust portfolio optimization problem (9) has a unique solution.

2.3 Robust EF analysis via worst-case SR maximization

From the concavity of f_{rob} and the definition of the robust EF, the slope of the robust optimal CAL (14) can be written as

$$\sup_{\sigma > 0} \frac{f_{\text{rob}}(\sigma) - \bar{r}}{\sigma} = \sup_{w \in \mathcal{W}} \frac{r_{\text{wc}}(w) - \bar{r}}{\sigma_{\text{wc}}(w)} = \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} S_{\bar{r}}(w, \mu, \Sigma), \tag{19}$$

whenever $\bar{r} < \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu$. Therefore, if the worst-case SRMP (13) has a (unique) solution $w^\star$, then it is robust MV efficient, *i.e.*,

$$f_{\text{rob}}(\sigma_{\text{wc}}(w^\star)) = r_{\text{wc}}(w^\star),$$

and the robust optimal CAL (14) is tangential to the robust EF at the point $(\sigma_{\text{wc}}(w^\star), r_{\text{wc}}(w^\star))$. Otherwise, the robust optimal CAL lies entirely above the robust EF:

$$f_{\text{rob}}(\sigma) < \bar{r} + S^\star(\bar{r})\sigma, \quad \sigma > 0,$$

which can arise only when the robust EF has an (upper) asymptote and the robust optimal CAL is parallel to the asymptote.

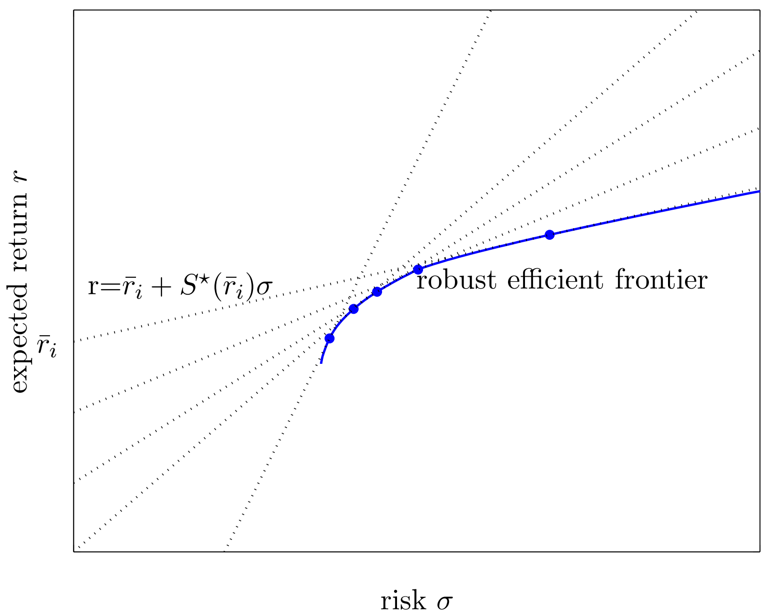

Figure 4: Robust efficient frontier analysis via worst-case SR maximization.

Using the results given above, we can generalize the method for computing the EF with a fixed pair (μ, Σ) described in §1.1 to computing the robust EF. As $\bar{r}$ varies, the worst-case risk-return pair $(\sigma_{\rm wc}(w^\star), r_{\rm wc}(w^\star))$ of the solution to the worst-case SRMP (13) moves along the robust EF. We can now see that there is $r_{\rm thrs} \in \mathbb{R}$ such that the robust EF can therefore be computed as the trajectory of the worst-case risk-return pair, as $\bar{r}$ varies over $(-\infty, r_{\rm thrs})$. In particular when $\mathcal{W}$ is bounded, $r_{\rm thrs}$ is equal to the maximum worst-case return $r_{\rm sup} = \sup_{w \in \mathcal{W}} w^T \mu$.

The expression of the EF given in (6) can be generalized to robust MV analysis.

Proposition 2. *The curve defined in (10) can be expressed as*

$$f_{\rm rob}(\sigma) = \inf_{\bar{r} < \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M}} w^T\mu} \bar{r} + S^\star(\bar{r})\sigma, \quad \sigma \geq \sigma_{\rm inf}. \tag{20}$$

The proof is deferred to Appendix A.4.

We can approximate the robust EF as

$$f_{\rm rob}(\sigma) \approx \inf_{i=1,\dots,M} \bar{r}_+ \gamma(\bar{r}_i)\sigma, \quad \sigma \geq \sigma_{\rm inf},$$

from the solutions to worst-case SRMPs with hypothetical risk-free returns $\bar{r}_1, \dots, \bar{r}_M$ drawn from the interval $(-\infty, \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M}} w^T\mu)$. Figure 4 illustrates the approximation method. Each point in circle corresponds to the point $(\sigma_i, f_{\rm rob}(\sigma_i))$, the worst-case risk and return of the solution of the worst-case SRMP (13) with $\bar{r} = r_i$.

3 Robust EF analysis via convex optimization

In this section, we show how the generalization given above allows us to carry out robust EF analysis with certain types of uncertainty models using convex optimization.

3.1 Convex separable uncertainty models

When the sets $\mathcal{M}$ and $\mathcal{S}$ are convex, we can solve the worst-case SRMP (13) efficiently, using a minimax result for the SR [39]. Here no structural assumptions (*e.g.*, bound constraints on the entries of μ and Σ) other than the convexity are needed. Several types of our approximate prior knowledge on the asset returns can be represent by convex uncertainty models. The list includes known variances of certain portfolios, ellipsoidal confidence region, information about the order of expected returns and variances of assets, and information about correlation coefficients, which have been considered in the prior work [1, 10, 18, 42, 51].

We restrict ourselves to the case of interest when the optimal vale of (15) is positive. We have shown in [38] that the worst-case MPRA problem (15) is equivalent to the convex problem

$$\begin{array}{ll} \text{minimize} & (\mu - \bar{r}\mathbf{1} + \lambda)^T \Sigma^{-1} (\mu - \bar{r}\mathbf{1} + \lambda) \\ \text{subject to} & \mu \in \mathcal{M}, \quad \Sigma \in \mathcal{S}, \quad \lambda \in \mathcal{W}^{\oplus}, \end{array} \tag{21}$$

with variables $\mu \in \mathbb{R}^n$, $\Sigma = \Sigma^T \in \mathbb{R}^{n \times n}$, and $\lambda \in \mathbb{R}^n$, where $\mathcal{W}^{\oplus}$ is the positive conjugate cone of $\mathcal{W}$, *i.e.*,

$$W^{\oplus} = \{\lambda \mid \lambda^T w \geq 0, \ \forall\, w \in \mathcal{X}\} \subseteq \mathbb{R}^n.$$

The objective function is convex and the constraints are convex. This problem always has a solution, say $(\mu^\star, \Sigma^\star, \lambda^\star)$, which satisfies

$$\mathbf{1}^T (\Sigma^{\star -1} (\mu^\star - \bar{r}\mathbf{1} + \lambda^\star)) \geq 0.$$

We then have the following:

- The pair $(\mu^\star, \Sigma^\star)$ is least favorable, *i.e.*, it solves (15).
- The worst-case SRMP (13) has a solution if and only if

  $$\mathbf{1}^T (\Sigma^{\star -1} (\mu^\star - \bar{r}\mathbf{1} + \lambda^\star)) > 0.$$

  If this inequality holds, then the least favorable model $(\mu^\star, \Sigma^\star)$ has the TP

  $$w^\star = \frac{1}{\mathbf{1}^T \Sigma^{\star -1} (\mu^\star - \bar{r}\mathbf{1} + \lambda^\star)} \Sigma^{\star -1} (\mu^\star - \bar{r}\mathbf{1} + \lambda^\star), \tag{22}$$

  which is the unique solution to (13) although there can be multiple least favorable models. Moreover, the triple $(w^\star, \mu^\star, \Sigma^\star)$ satisfies the saddle-point property:

  $$S(w, \mu^\star, \Sigma^\star) \leq S(w^\star, \mu^\star, \Sigma^\star) \leq S(w^\star, \mu, \Sigma), \quad \forall w \in \mathcal{W}, \quad \forall (\mu, \Sigma) \in \mathcal{U}, \tag{23}$$

When the constraints $\mu \in \mathcal{M}$ and $\Sigma \in \mathcal{S}$ can be represented by linear matrix inequalities, the convex optimization problem (21) can be cast as a semidefinite program (SDP) [54], which can be solved efficiently by several high quality interior-point solvers including SeDuMi [64] and SDPT3 [65]. The reader is referred to [68] for more on semidefinite programming.

We can establish the following proposition, using (12) and the minimax equality (16).

Proposition 3. *Suppose that $\mathcal{M}$ and $\mathcal{S}$ are convex and compact. Let $(\mu^\star, \Sigma^\star) \in \mathcal{M} \times \mathcal{S}$ be a solution to the worst-case MPRA problem (15). If the worst-case SRMP (13) has a solution, say $w^\star$, then*

$$f_{\mathrm{rob}}\left(\sqrt{w^{\star T}\Sigma^\star w^\star}\right) = f_{\mu^\star,\Sigma^\star}\left(\sqrt{w^{\star T}\Sigma^\star w^\star}\right) = w^{\star T}\mu^\star,$$

and the robust optimal CAL (14) is tangential to the two curves $r = f_{\mathrm{rob}}(\sigma)$ and $r = f_{\mu^\star,\Sigma^\star}(\sigma)$ at the same point $((w^{\star T}\Sigma^\star w^\star)^{1/2}, w^{\star T}\mu^\star)$.

The proof is deferred to Appendix A.5.

Figure 5 illustrates the assertion of Proposition 3. The solid curve is the robust EF, and the dashed curve is the EF of $(\mu^\star, \Sigma^\star)$, the least favorable asset return statistics when the risk-free rate is $\bar{r}$.

The preceding proposition along with the saddle-point property (23) means that the robust EF can be computed from the trajectory of the saddle point of the zero-sum game of choosing w from $\mathcal{W}$, to maximize the SR, and choosing (μ, Σ) from $\mathcal{U}$, to minimize the SR, as the hypothetical risk-free return $\bar{r}$ varies. (With a special type of convex model, a similar observation has been made in [67].)

3.2 Factor covariance models with uncertain factor loading

Suppose that the asset returns follow a model of the form

$$a = \mu + Vz + u,$$

where z is a random vector (the underlying factors that affect the asset returns), V is an $n \times k$ matrix containing the sensitivities of the asset returns with respect to the various factors, and u_i are independent (additional volatility of each asset return). The mean return vector is μ, and the covariance matrix has the factor structure

$$\Sigma = V\Sigma_{\mathrm{factor}}V^T + D,$$

where $\Sigma_{\mathrm{factor}} \in \mathbb{R}^{k\times k}$ is the factor covariance and D is the diagonal covariance matrix D of the residuals uncorrelated with the factors.

When the factor loading matrix is known for certain and the factor covariance is uncertain but known to belong to a convex subset of $\mathbb{S}^k_{++}$, the uncertainty set for the covariance is convex, so the corresponding worst-case SRMP can be solved using the method described above. We will consider the opposite case in which the factor covariance matrix is known for certain but the factor loading matrix V is uncertain. In this case, the covariance model is not convex.

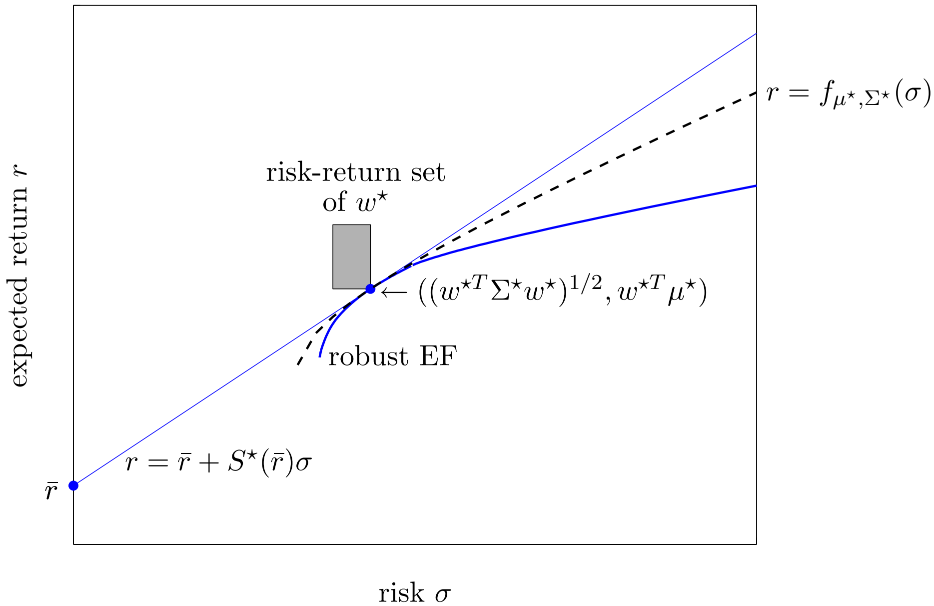

Figure 5: The robust efficient frontier and the efficient frontier of the least favorable model when the uncertainty model is convex.

We assume that the uncertainty in the factor loading matrix V is independent of that in the residual covariance D. We consider a covariance uncertainty model of the form

$$\mathcal{S} = \{D + V\Sigma_{\text{factor}}V^T \in \mathbb{R}^{n\times n} \mid V \in \mathcal{V},\ D \in \mathcal{D}\}, \tag{24}$$

where $\mathcal{V}$ is the set of uncertain sensitivity matrices and $\mathcal{D}$ is the set of uncertain covariance matrices. Here, we assume that it has the form

$$\mathcal{D} = \{D = \text{diag}(d_1,\ldots,d_n) \in \mathbb{R}^{n\times n} \mid \underline{d}_i \le d_i \le \overline{d}_i,\ i = 1,\ldots,n\}$$

and $\underline{d}_i$ and $\overline{d}_i$ are positive known lower and upper bounds on the variances of residuals.

We resort to the semi-infinite formulation given in (17). We note that

$$\sup_{\Sigma\in\mathcal{S}} (w^T\Sigma w)^{1/2} = \left(\sup_{V\in\mathcal{V}} w^TV\Sigma_{\text{factor}}V^Tw + w^T\overline{D}w\right)^{1/2}.$$

Therefore, we can solve the worst-case SRMP by solving the semi-infinite program

$$\begin{array}{ll} \text{maximize} & \inf_{\mu\in\mathcal{M}} w^T(\mu - \bar{r}\mathbf{1}) \\ \text{subject to} & w \in \mathcal{X}, \quad t + w^T\overline{D}w \le 1, \quad h(w) \le t, \end{array} \tag{25}$$

where $w \in \mathbb{R}^n$ and $t \in \mathbb{R}$ are the variables and

$$h(w) = \sup_{V\in\mathcal{V}} w^TV\Sigma_{\text{factor}}V^Tw, \qquad \overline{D} = \text{diag}(\overline{d}_1,\ldots,\overline{d}_n) \in \mathbb{R}^{n\times n}.$$

We will show that the semi-infinite program (25) can be reformulated as an SDP (or sometimes SOCP), when the inequality $h(w) \le t$ in (25) can be represented by linear matrix inequalities (LMIs) and the set $\mathcal{M}$ is a polyhedron or an ellipsoid.

Tractable factor covariance models

We describe several types of structural assumptions on $\mathcal{M}$ that lead to the LMI representability of $h(w) \le t$. (The LMI representability has applications in a variety of robust optimization problems including robust control and robust least squares; see, *e.g.*, [9, 17].)

As the first tractable case, we consider an ellipsoidal uncertainty model for the factor loading matrix

$$\mathcal{V} = \left\{ \bar{V} + \sum_{i=1}^{p} u_i V_i \;\middle|\; \|u\| \le 1 \right\},$$

where $\bar{V} \in \mathbb{R}^{n\times k}$ is the nominal factor loading matrix and the given matrices $V_i \in \mathbb{R}^{n\times k}$, $k = 1, \dots, p$ determine the ellipsoid (in the space of $n \times r$ matrices). As shown in [18], the constraint $h(w) \le t$ is LMI representable.

The constraint $h(w) \le t$ is also LMI representable, with the set

$$\mathcal{V} = \{\bar{V} + L\Delta R \mid \Delta \in \mathbb{R}^{l\times r},\ \|\Delta\| \le 1\},$$

where $\bar{V} \in \mathbb{R}^{n\times k}$, $L \in \mathbb{R}^{n\times l}$, and $R \in \mathbb{R}^{r\times k}$ are given, and $\|\Delta\|$ is the spectral norm of Δ. This set can model unstructured uncertainties in some blocks of the factor loading matrix, with the matrices L and R specifying which blocks are uncertain; the reader is referred to [9] for more on this model.

As another tractable case, we consider the case in which the factor covariance matrix is diagonal, *i.e.*,

$$F = \mathrm{diag}(\sigma_1^2, \dots, \sigma_r^2),$$

and the columns of V are uncertain but known to belong to ellipsoids

$$v_i \in \mathcal{E}_i = \{\bar{v}_i + P_i u \mid \|u\| \le 1\},$$

where $\bar{v}_i$ is the nominal factor loading of the ith factor and the matrices $P_i \in \mathbb{R}^{n\times p}$ determine the shapes of these ellipsoids. (In this case, the factors are uncorrelated.) We can see from the Cauchy-Schwartz inequality that

$$h(w) = \sup_{v_i \in \mathcal{E}_i,\ i=1,\dots,r} w^T \left(\sum_{i=1}^{k} \sigma_i^2 v_i v_i^T \right) w = \sum_{i=1}^{k} \left(\sigma_i^2 \bar{v}_i \bar{v}_i^T + \|P_i w\| \right)^2.$$

We can easily express the constraint $h(w) \le t$ as a set of second-order cone constraints, special types of LMIs; the reader is referred to [44] for more on second-order cone programming.

As the final tractable case, we consider the following model which has been studied in [27] and used in robust active portfolio management in [19]:

$$\mathcal{V} = \{V = [v_1 \ \cdots \ v_n]^T \in \mathbb{R}^{n\times r} \mid v_i \in \mathcal{E}_i\}, \tag{26}$$

where v_i is the ith column of V^T and $\mathcal{E}_i$ are the ellipsoids

$$\mathcal{E}_i = \{v_i \in \mathbb{R}^k \mid (v_i - \bar{v}_i)^T G (v_i - \bar{v}_i) \le \rho_i\}, \quad i = 1, \dots, n.$$

Here G determines the common shape of the ellipsoids whose sizes are controlled by the parameter $\rho_i \geq 0$, and $\bar{V} = [\bar{v}_1 \ \cdots \ \bar{v}_n]^T \in \mathbb{R}^{n\times r}$ is the nominal factor loading matrix. It is important to point out that unlike the previous models, the uncertainty ellipsoids are the same shape although their centers and sizes can be different and the uncertainty is on the row vectors of the factor loading matrix. Using the S-procedure [10, App. B], the authors of [19] show how to equivalently express the constraint $h(w) \leq t$ as second-order cone constraints. The authors give a justification of this uncertainty model in the context of linear regression.

Convex formulation examples

When $\mathcal{M}$ is an ellipsoid of the form

$$\mathcal{M} = \{\bar{\mu} + Pu \mid \|u\| \leq 1\},$$

where the matrix $P \in \mathbb{R}^{n\times p}$ determines the shapes of the ellipsoid, we know from the Cauchy-Schwartz inequality that

$$\inf_{\mu\in\mathcal{M}} w^T(\mu - \bar{r}\mathbf{1}) = w^T(\bar{\mu} - \bar{r}) - \|Pw\|.$$

Therefore, the equivalent formulation (25) can be expressed as

$$\begin{array}{ll} \text{minimize} & -w^T(\bar{\mu} - \bar{r}\mathbf{1}) + \|Pw\| \\ \text{subject to} & w \in \mathcal{X}, \quad t + w^T\bar{D}w \leq 1, \quad h(w) \leq t, \end{array} \tag{27}$$

where the variables are $w \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $P \in \mathbb{R}^n$.

When $\mathcal{M}$ is a polyhedron of the form

$$\mathcal{M} = \{w \mid Aw \leq b\},$$

with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, the equivalent formulation (25) is equivalent to

$$\begin{array}{ll} \text{minimize} & b^T\lambda - w^T\mathbf{1} \\ \text{subject to} & w \in \mathcal{X}, \quad A^T\lambda + w = 0, \quad t + w^T\bar{D}w \leq 1, \quad h(w) \leq t, \end{array} \tag{28}$$

where the variables are $w \in \mathbb{R}^n$, $t \in \mathbb{R}$, $P \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}^m$. (Here $a \leq b$ is componentwise inequality, *i.e.*, $a_i \leq b_i$ for all i.) To establish the equivalence between the two problems (25) and (28), we find the Lagrange dual problem of a linear program in inequality form

$$\begin{array}{ll} \text{minimize} & w^T\mu \\ \text{subject to} & A\mu \leq b. \end{array} \tag{29}$$

The Lagrange dual of the problem is a linear program (LP) in standard form [10],

$$\begin{array}{ll} \text{maximize} & b^T\lambda \\ \text{subject to} & A^T\lambda + w = 0, \quad \lambda \geq 0. \end{array} \tag{30}$$

The two problems (29) and (30) are equivalent, by strong duality which holds for the primal (29) and the dual (30), which shows the equivalence between (25) and (28).

It is now evident that when the constraint $h(w) \leq t$ can be represented by an LMIs the two problems (27) and (28) are SDPs. In particular, when the constraint $h(w) \leq t$ is compatible with second-order cone programming, these problems are SOCPs.

We close by pointing out that a variety of robust portfolio selection problems with the models given above can be readily solved. In [27], the authors show that several types of robust portfolio selection problems with a structurally uncertain factor covariance model, an ellipsoidal uncertainty model on the mean return vector, and a polyhedral constraint set can be formulated SOCPs. The main results of this paper show that the several robust selection problems considered in [27] are to compute robust MV efficient portfolios.

4 Robust portfolio selection with MV preference

An investor is said to have mean-variance preference if his portfolio preference is based only on the mean return $r = w^T\mu$ and the return volatility $\sigma = \sqrt{w^T\Sigma w}$, that is, he judges the performance by

$$f(w, \mu, \Sigma) = h(w^T\mu, w^T\Sigma w), \tag{31}$$

where h is a function from $\mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}$. Here we assume that $h(r, \sigma)$ is strictly decreasing in σ for fixed r and increasing in r for fixed σ.

An example is an expected quadratic utility function

$$f(x, \mu, \Sigma) = w^T\mu - \frac{\gamma}{2} w^T\Sigma w, \tag{32}$$

which is associated with the mean-variance preference function

$$h(r, \sigma) = r - \frac{\gamma}{2}\sigma^2.$$

Here $\gamma > 0$ is a positive constant related to risk aversion.

The problem of finding a portfolio that maximizes the worst-case value of $f(w, \mu, \Sigma)$ over the uncertainty set $\mathcal{U} = \mathcal{M} \times \mathcal{S}$ can be formulated as

$$\begin{array}{ll} \text{maximize} & \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} h(w^T\mu, w^T\Sigma w) \\ \text{subject to} & w \in \mathcal{W}. \end{array} \tag{33}$$

When h is associated with an expected utility function, this problem is related to max-min utility theory; see, *e.g.*, [26] for more on max-min utility theory.

We can observe from the assumption on h that for any $w \in \mathcal{W}$,

$$h(w^T\mu, w^T\Sigma w) \leq h(r_{\text{wc}}(w), \sigma_{\text{wc}}(w)), \quad \forall \mu \in \mathcal{M}, \quad \forall \Sigma \in \mathcal{S},$$

or equivalently,

$$\inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} h(w^T\mu, w^T\Sigma w) \leq h(r_{\text{wc}}(w), \sigma_{\text{wc}}(w)).$$

The following proposition follows from this observation.

Proposition 4. *Suppose that $h(r,\sigma)$ is strictly decreasing in σ for fixed r and increasing in r for fixed σ. If $w^\star$ solves (33), then it is robust MV efficient.*

The robust portfolio selection problem (33) now reduces to a simple one-dimensional search problem over the family of robust MV efficient portfolios. As specific examples, we consider two robust portfolio selection problems: worst-case utility maximization and worst-case VaR minimization.

4.1 Worst-case VaR minimization

The VaR is defined as

$$V(w,\mu,\Sigma) = \kappa\sqrt{w^T\Sigma w} - w^T\mu,$$

where $\kappa > 0$ is an appropriate risk factor which depends on the prior assumptions on the distribution of returns. For the Gaussian case, $\kappa = \Phi^{-1}(\epsilon)$, where ϵ is a given probability level and Φ is the cumulative distribution function of the standard normal distribution. When the first two moments of asset returns are known ($\mathbf{E}\,a = \mu$, $\mathbf{E}\,(a-\mu)^T(a-\mu) = \Sigma$) but higher moments are otherwise arbitrary, we can take $\kappa = \sqrt{(1-\epsilon)/\epsilon}$ from the Chebyshev bound [18].

The worst-case VaR of w over the uncertainty set $\mathcal{U} = \mathcal{M}\times\mathcal{S}$ is

$$V_{\rm wc}(w) = \sup_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} V(w,\mu,\Sigma) = \kappa\sigma_{\rm wc}(w) - r_{\rm wc}(w).$$

The problem of finding the portfolio that maximizes the worst-case VaR over the set $\mathcal{M}\times\mathcal{S}$ can be cast as

$$\begin{array}{ll} \mbox{minimize} & V_{\rm wc}(w) \\ \mbox{subject to} & w \in \mathcal{W}. \end{array} \qquad (34)$$

This problem has been studied by several authors. If the uncertainty set $\mathcal{U} = \mathcal{M}\times\mathcal{S}$ can be represented by linear matrix inequalities and hence are convex, this problem can be cast as a semidefinite program and then solved efficiently using interior-point methods [18]. In [27], the authors consider worst-case VaR minimization problems with a uncertainty factor covariance model described by (24) and (26) and show that they can be reformulated as SOCPs.

The worst-case VaR minimization problem (34) can be posed as a problem of the form (33) with $h(r,v) = r - \kappa\sqrt{v}$. Proposition 4 tells us that the solution can be found easily via searching the portfolio that maximizes the worst-case VaR among all robust MV efficient portfolios. The search problem can be solved using a simple graphical argument: find the point on the curve $r = f_{\rm rob}(\sigma)$ which is tangent to a line with slope κ, $r = \bar{r} + \kappa s$, where $\bar{r}$ is the return-intercept. The point corresponds to the worst-case risk and return of a robust MV efficient portfolio, which solves maximizes the worst-case VaR. Figure 6 illustrates the solution procedure described above.

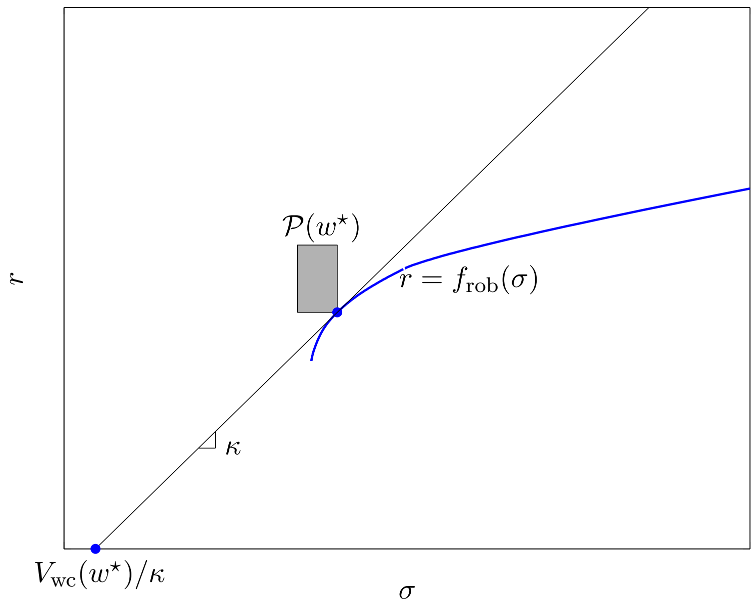

Figure 6: Worst-case VaR minimization via robust efficient frontier analysis.

4.2 Worst-case quadratic utility maximization

The problem of finding the portfolio that maximizes the worst-case (expected) quadratic utility can be expressed as

$$\begin{array}{ll} \text{maximize} & \inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} w^T\mu - (\gamma/2)w^T\Sigma w = r_{\rm wc}(w) - (\gamma/2)(\sigma_{\rm wc}(w))^2 \\ \text{subject to} & w \in \mathcal{W}, \end{array} \tag{35}$$

which has the form (33) with (32). Proposition 4 tells us that the solution to the worst-case expected quadratic utility maximization problem (34) is robust MV efficient.

The portfolio that maximizes the worst-case expected quadratic utility can be easily found in the variance-return space. In this space, the robust EF is transformed into the strictly concave curve $h_{\rm rov}(v) = f_{\rm rob}(\sqrt{v})$, which can be verified easily using basic composition rules for concave functions. There is one and only one line with slope $\gamma/2$, $r = (\gamma/2)v + U^\star$, which is tangential to the strictly concave curve $r = h_{\rm rov}(v)$. Here, $U^\star$ is the return-intercept of the line, that is, the point where the line crosses the axis $v = 0$. The tangential point on the curve $r = h_{\rm rov}(v)$ corresponds to the worst-case variance and return of a robust MV efficient portfolio, say $w^\star$. We can now see that the worst-case quadratic utility is $U^\star$ and the portfolio $w^\star$ is the solution to the worst-case expected utility maximization problem (35). In the risk-return space, the solution procedure amounts to finding the quadratic curve $r = U^\star + (\gamma/2)\sigma^2$ which is tangential to the robust EF. Figures 7 and 8 illustrate the solution procedure described above.

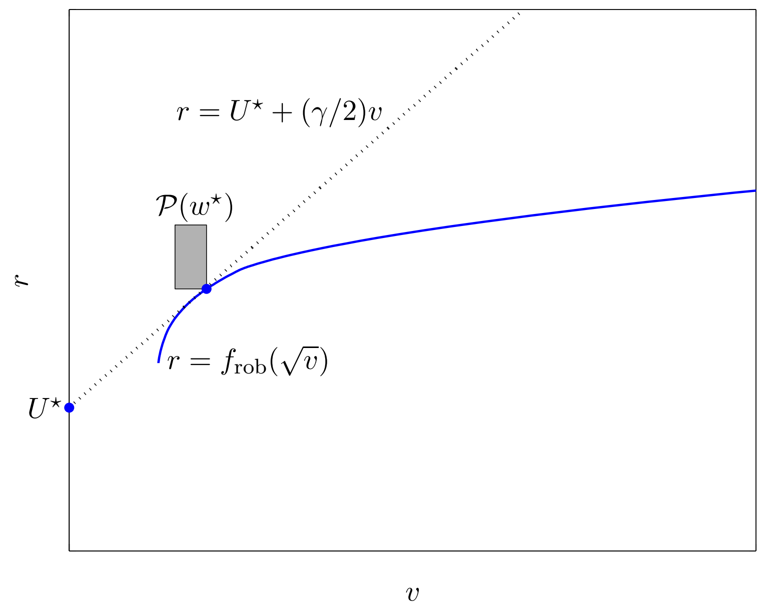

Figure 7: The robust efficient frontier in the variance-return space and quadratic utility maximization under model uncertainty.

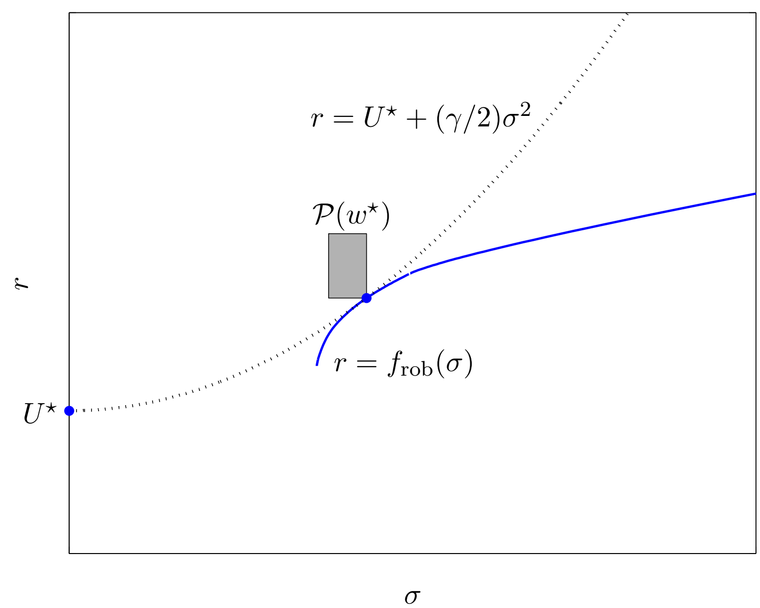

Figure 8: The robust efficient frontier in the risk-return space and quadratic utility maximization under model uncertainty.

5 Numerical example

In this section, we illustrate the results described thus far with a simple example.

5.1 Setup

We illustrate the robust EF analysis method with a simple model with 8 risky assets ($n = 8$). We consider the long-only case, so the asset allocation constraint set is

$$\mathcal{W} = \{w \in \mathbb{R}^8 \mid w \geq 0,\ \mathbf{1}^T w = 1\}.$$

(When short positions are allowed, it is easy to find examples for which the improvement given by robust mean-variance analysis, over nominal mean-variance analysis, is dramatic. With long-only positions, we will see that the improvement is modest, but still quite significant.) The positive conjugate cone of $\mathcal{W}$ is the nonnegative orthant cone:

$$\mathcal{W}^{\oplus} = \mathbb{R}^8_+ = \left\{\lambda \in \mathbb{R}^8 \mid \lambda \geq 0\right\}.$$

Nominal asset return model

We first describe the nominal model with which we compute the nominal EF and MV efficient portfolios. The nominal returns $\bar{\mu}_i$ and nominal variances $\bar{\sigma}_i^2$ of the asset returns are taken as

$$\begin{aligned} \bar{\mu} &= [6.1\ \ 5.9\ \ 12.7\ \ 10.0\ \ 13.99\ \ 9.4\ \ 10.9\ \ 13.7]^T, \\ \bar{\sigma} &= [9.4\ \ 8.1\ \ 19.9\ \ 14.4\ \ 24.6\ \ 15.7\ \ 15.2\ \ 27.8]^T. \end{aligned}$$

All units here are in percentage. The nominal correlation matrix $\bar{\Omega}$ is taken as

$$\bar{\Omega} = \begin{bmatrix} 1 & .41 & .22 & .28 & .11 & .19 & .19 & .02 \\ & 1 & .03 & .06 & .08 & .14 & .39 & .11 \\ & & 1 & .69 & .82 & .58 & .62 & 0.65 \\ & & & 1 & .69 & .81 & .58 & 0.59 \\ & & & & 1 & .86 & .54 & 0.67 \\ & & & & & 1 & .50 & 0.62 \\ & & & & & & 1 & 0.71 \\ & & & & & & & 1 \end{bmatrix} \in \mathbb{R}^{8\times 8}.$$

(Only the upper triangular part is shown because the matrix is symmetric.) The risk-less return is $\bar{r} = 3$. The nominal covariance is

$$\bar{\Sigma} = \mathrm{diag}(\bar{\sigma})\bar{\Omega}\,\mathrm{diag}(\bar{\sigma}),$$

where we use $\mathrm{diag}(z_1, \ldots, z_m)$ to denote the diagonal matrix with diagonal entries $z_1, \ldots, z_m$. The nominal EF is the EF obtained with the nominal model $(\bar{\mu}, \bar{\Sigma})$.

Uncertainty model

We assume that the possible variation in the expected return of each asset is at most 20%:

$$|\mu_i - \bar{\mu}_i| \le 0.2|\bar{\mu}_i|, \quad i = 1, \ldots, 8.$$

We also assume that the possible variation in the expected return of a uniformly weighted portfolio is at most 10%:

$$|\mathbf{1}^T\mu - \mathbf{1}^T\bar{\mu}| \le 0.1|\mathbf{1}^T\bar{\mu}|.$$

These assumptions mean that we know more about the return of the portfolio $w = (1/n)\mathbf{1}$ (in which a fraction $1/n$ of budget is allocated to each asset of the n assets). In summary, the uncertainty model $\mathcal{M}$ for the mean return vector is

$$\mathcal{M} = \{\mu \mid |\mathbf{1}^T\mu - \mathbf{1}^T\bar{\mu}| \le 0.1|\mathbf{1}^T\bar{\mu}|,\ |\mu_i - \bar{\mu}_i| \le 0.2|\bar{\mu}_i|,\ i = 1, \ldots, 8\}.$$

We assume that the possible variation in each component of the covariance matrix is at most 20% and the possible deviation of the covariance from the nominal covariance is at most 10% in terms of the Frobenius norm:

$$|\Sigma_{ij}/\bar{\Sigma}_{ij} - 1| \le 0.2, \quad i,j = 1, \ldots, 8, \qquad \|\Sigma - \bar{\Sigma}\|_F \le 0.1\|\bar{\Sigma}\|_F.$$

(Here, $\|A\|_F$ denotes the Frobenius norm of A, $\|A\|_F = (\sum_{i,j=1}^n A_{ij}^2)^{1/2}$.) Of course, we require $\Sigma \in \mathcal{S}$ to be positive semidefinite. The covariance uncertainty model $\mathcal{S}$ is

$$\mathcal{S} = \left\{ \Sigma = \Sigma^T \in \mathbb{R}^{n\times n} \;\middle|\; \Sigma \ge 0,\ \|\Sigma - \bar{\Sigma}\|_F \le 0.1\|\bar{\Sigma}\|_F,\ |\Sigma_{ij}/\bar{\Sigma}_{ij} - 1| \le 0.2,\ i,j = 1, \ldots, 8 \right\}.$$

5.2 Computation

To compute the robust EF, we first compute the maximum worst-case return

$$r_{\text{sup}} = \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M}} w^T\mu.$$

Using the standard minimax theorem for convex/concave functions, we can show that

$$\sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M}} w^T\mu = \inf_{\mu\in\mathcal{M}} \sup_{w\in\mathcal{W}} w^T\mu.$$

Using linear programming duality, we can show

$$\sup_{w\in\mathcal{W}} w^T\mu = \sup_{w\ge 0,\ \mathbf{1}^T w = 1} w^T\mu = \inf_{\lambda\ge 0,\ \nu\mathbf{1}-\lambda-\mu=0} \nu,$$

which means

$$r_{\text{sup}} = \inf_{\mu\in\mathcal{M}} \sup_{w\in\mathcal{W}} w^T\mu = \inf_{\mu\in\mathcal{M}} \inf_{\lambda\ge 0,\ \nu\mathbf{1}-\lambda-\mu=0} \nu.$$

We can compute the maximum worst-case return $r_{\rm sup}$ as the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & \nu \\ \text{subject to} & |\mu_i - \bar{\mu}_i| \leq 0.2|\bar{\mu}_i|, \quad i = 1, \ldots, 8, \\ & |\mathbf{1}^T\mu - \mathbf{1}^T\bar{\mu}| \leq 0.1|\mathbf{1}^T\bar{\mu}|, \\ & \nu\mathbf{1} - \lambda - \mu = 0, \\ & \lambda \geq 0, \end{array}$$

with variables $\mu \in \mathbb{R}^7$, $\lambda \in \mathbb{R}^8$, and $\nu \in \mathbb{R}$. This problem can be reformulated as an LP.

The general solution procedure described in §3 shows that the robust EF for the model described above can be found via solving the following problem for each $\bar{r} \in (-\infty, r_{\rm sup})$.

$$\begin{array}{ll} \text{minimize} & (\mu - \bar{r}\mathbf{1} + \lambda)^T\Sigma^{-1}(\mu - \bar{r}\mathbf{1} + \lambda) \\ \text{subject to} & F\mu \geq 0, \\ & |\mu_i - \bar{\mu}_i| \leq 0.2|\bar{\mu}_i|, \quad i = 1, \ldots, 8, \\ & |\mathbf{1}^T\mu - \mathbf{1}^T\bar{\mu}| \leq 0.1|\mathbf{1}^T\bar{\mu}|, \\ & |\Sigma_{ij} - \bar{\Sigma}_{ij}| \leq 0.2|\bar{\Sigma}_{ij}|, \quad i, j = 1, \ldots, 8, \\ & \|\Sigma - \bar{\Sigma}\|_F \leq 0.1\|\bar{\Sigma}\|_F, \\ & \Sigma \geq 0, \\ & \lambda \geq 0, \end{array} \tag{36}$$

in which the variables are $\mu \in \mathbb{R}^8$, $\Sigma \in \mathbb{R}^{8\times 8}$, and $\lambda \in \mathbb{R}^8$. Here $A \geq 0$ means that A is positive semidefinite.

To compute the nominal and robust EFs, we discretized the interval $[-20, r_{\rm sup}]$, where $r_{\rm sup}$ is the maximum worst-case return, with 200 grid points, and solved the corresponding portfolio optimization problems of the form (36). Similarly, we computed approximately the nominal efficient frontier. The risk and return computed with the nominal model are called the nominal risk and return. We solved the nominal and worst-case SRMPs using the CVX software package [28], a Matlab-based modeling system for convex optimization. (The CVX package internally transforms the convex problem (36) into an SDP using the Schur complement technique and uses SDPT3 [65] as the solver.)

5.3 Comparison

Performance comparison with the baseline model

Figure 9 compares the robust EF with the nominal EF. The solid curve corresponds to the EF computed with the nominal mean return $\bar{\mu}$ and the nominal covariance $\bar{\Sigma}$. The dotted curve describes the performance of the robust MV efficient portfolios with the baseline model $\bar{\mu}$ and $\bar{\Sigma}$. Robust MV efficient portfolios perform slightly worse than nominal MV efficient portfolios.

Worst-case performance comparison

Figure 10 compares the worst-case performance of nominal and robust MV efficient portfolios. The solid curve corresponds to the robust EF. The dotted curve describes the worst-case

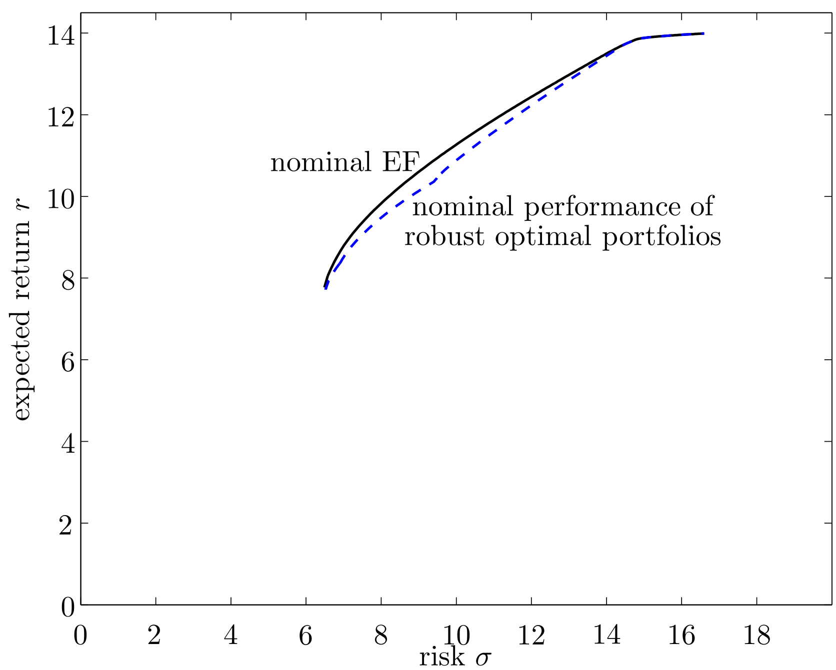

Figure 9: Nominal performance of nominal MV efficient portfolios and robust MV efficient portfolios. The solid curve is the nominal efficient frontier computed with the baseline model. The dotted curve is the trajectory of risk and return of robust MV efficient portfolios computed with the baseline model.

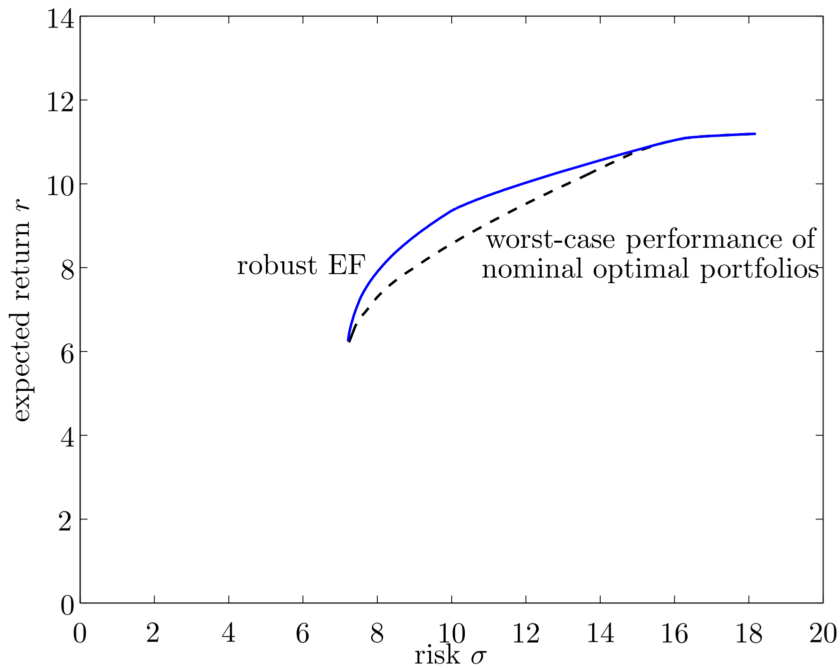

Figure 10: Comparison of nominal and robust MV efficient portfolios. The solid curve is the robust efficient frontier. The dotted curve is the trajectory of worst-case risk and return of nominal MV efficient portfolios.

performance of the nominal MV efficient portfolios; each point on the curve corresponds to the worst-case risk and return of a nominal MV efficient portfolio. The figure shows the robust optimal portfolios are less sensitive to variations in the parameters than the nominal optimal portfolios. (Without short-selling, nominal efficient portfolios are not very sensitive; see, *e.g.*, [34].)

Nominal and robust efficient portfolios

Figure 11 shows the optimal trade-off curve for this portfolio optimization problem. The plot is given in the conventional way, with the horizontal axis showing risk and the vertical axis showing expected return when the asset return statistics are described by the baseline model $(\bar{\mu}, \bar{\Sigma})$. The lower plot shows the optimal asset allocation vector w for each optimal point. The leftmost point corresponds to the risk and return of the nominal minimum variance portfolio (MVP) that minimizes the variance when the asset return statistics are described by the baseline model $(\bar{\mu}, \bar{\Sigma})$. The rightmost point corresponds to the risk and return of the nominal maximum return portfolio (MRP) that maximizes the variance with the baseline model $(\bar{\mu}, \bar{\Sigma})$. Since no short selling is imposed, the portfolios are less sensitive [34]. The results in this example agree with our intuition. For small risk, the optimal allocation consists mostly of the asset with the minimum risk, with a mixture of the other assets in smaller quantities. At the other end of the trade-off curve, we see that aggressive growth portfolios (*i.e.*, those with large mean returns) concentrate the allocation in asset 5, the one with the largest mean return. Overall, the nominal MV efficient portfolios are not diversified well.

Figure 12 shows the worst-case optimal trade-off curve in a similar way, with the horizontal axis showing worst-case risk and the vertical axis showing worst-case expected return. The lower plot shows the robust optimal asset allocation vector w for each point on the curve in the top plot. The leftmost point corresponds to the worst-case risk and return of the robust MVP that minimizes the worst-case variance. The rightmost point corresponds to the risk and return of the robust MRP that maximizes the worst-case variance. One noticeable difference is that robust MV efficient portfolios are more diversified than nominal ones with the same risk levels and so less likely to produce extreme results. We can observe that the nominal minimum variance portfolio and robust minimum variance portfolio perform similarly, although the latter is more diversified. For large risk, the robust optimal allocation consists mostly of the asset with the maximum worst-case return, with a mixture of the other assets in smaller quantities. As in optimal asset allocation with the baseline model, at the other end of the trade-off curve, we see that aggressive growth portfolios (*i.e.*, those with large mean returns) concentrate the allocation in asset 5. As the maximum allowable worst-case risk level increases, the discrepancy between the robust optimal allocation and the nominal optimal allocation tends to diminish, and when it is larger than a threshold, both allocations are the same.

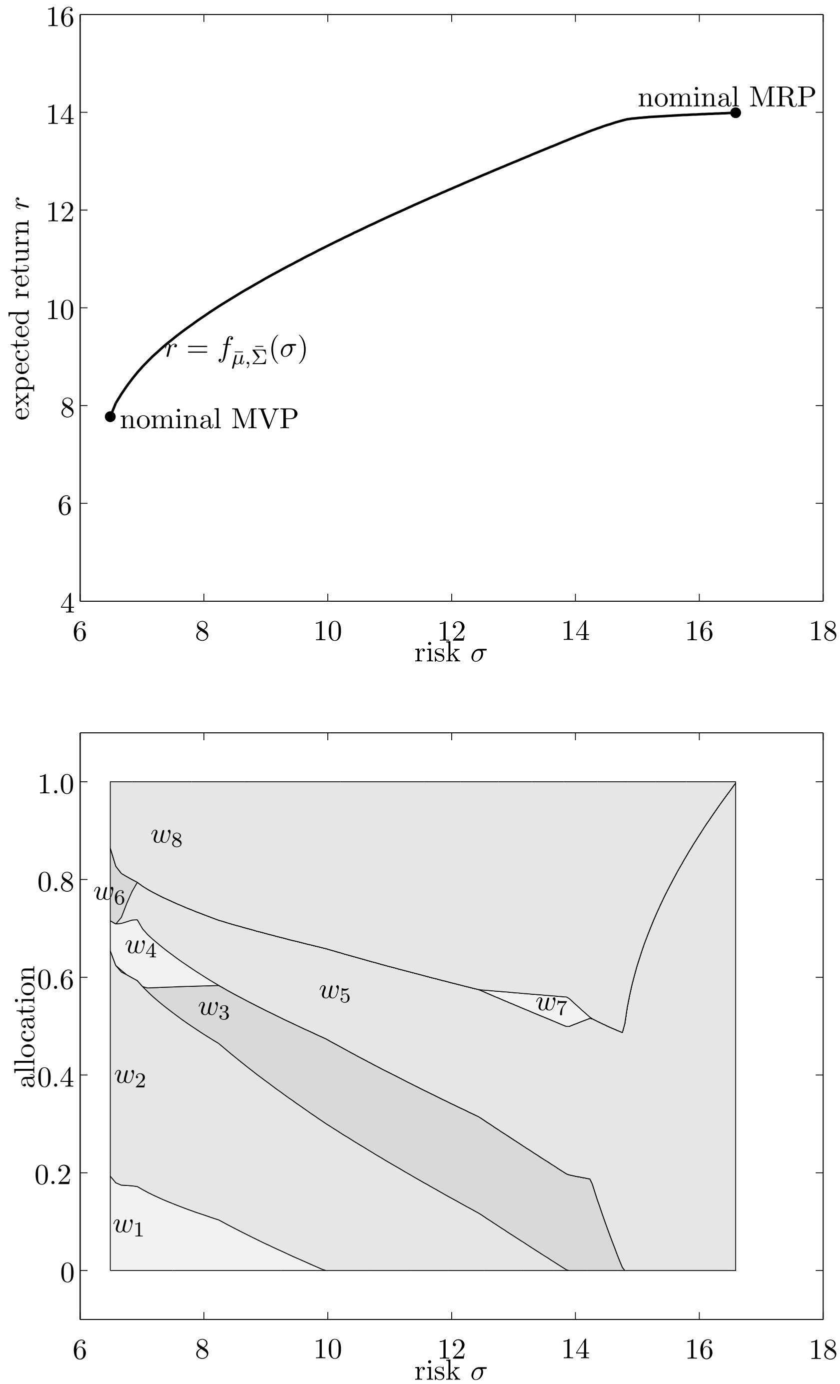

Figure 11: Optimal asset allocation with the baseline model. *Top.* The nominal MV efficient frontier. *Bottom.* Corresponding optimal allocations.

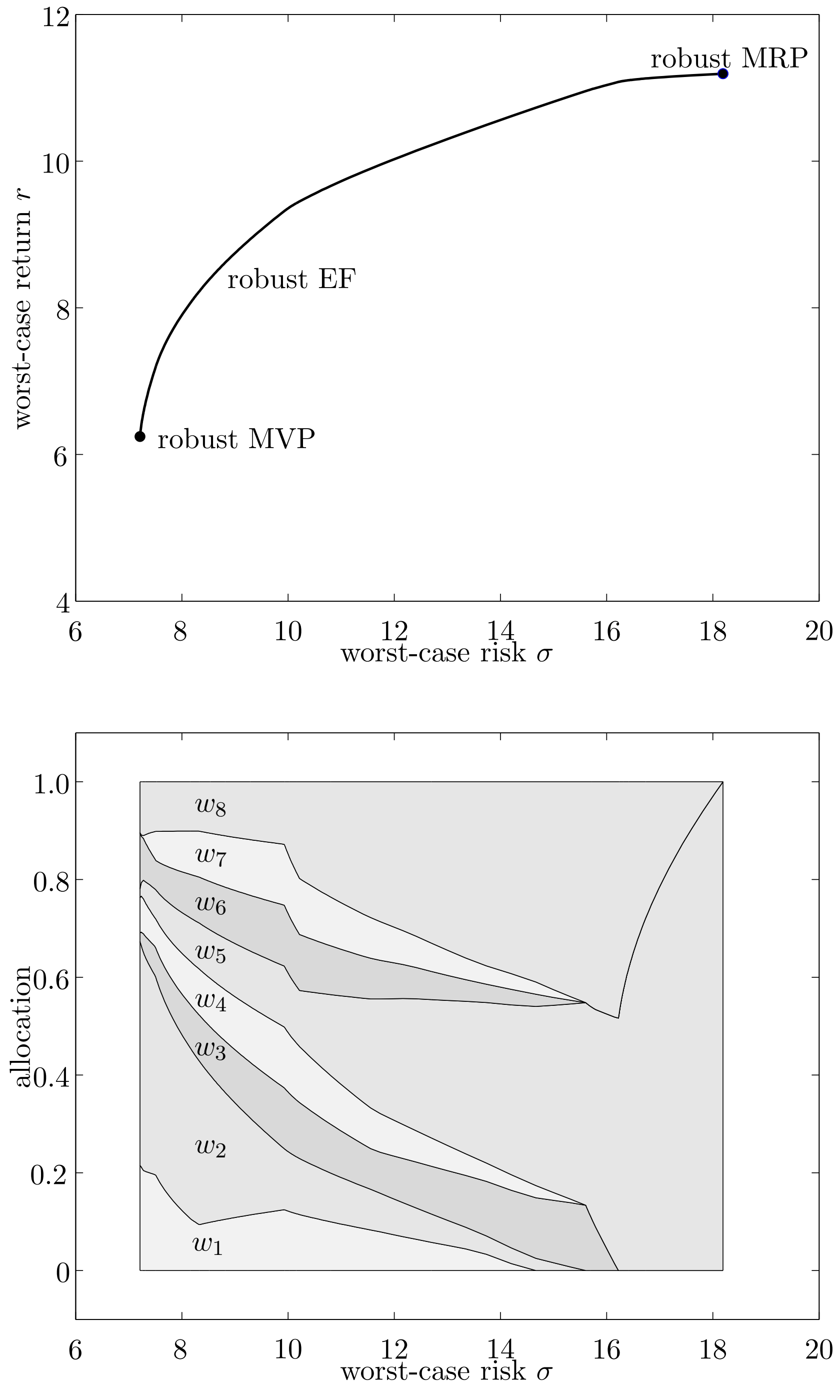

Figure 12: Robust asset allocation. *Top.* The worst-case MV efficient frontier. *Bottom.* Corresponding robust optimal allocations.

6 Conclusions

In this paper, we have considered robust EF analysis with a separable uncertainty model. The extension of the main results of this paper to general nonseparable uncertainty models does not appear to be straightforward; with a general non-separable uncertainty model the risk-return set of a portfolio can have an arbitrary shape, so it is not clear how to define the concept of worst-case preference. We have characterized the robust EF as the trajectory of the solution to the worst-case SRMP. An interesting question is whether this characterization can be generalized to robust EF analysis with a general non-separable uncertainty model.

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A Proofs

A.1 Concavity of the robust EF

It suffices to show that the robust EF is strictly concave over $[\sigma_{\rm inf}, \sigma_{\rm sup})$. If $\sigma_{\rm sup}$ is finite, then $f_{\rm rob}(\sigma) = \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M}} w^T\mu$ whenever $\sigma \geq \sigma_{\rm sup}$. The robust EF is increasing, so it is concave over $[\sigma_{\rm inf}, \infty)$.

Suppose that the robust EF is not strictly concave over $[\sigma_{\rm inf}, \sigma_{\rm sup})$, that is, there are two points, say (σ_1, r_1) and (σ_2, r_2) with $\sigma_{\rm inf} \leq \sigma_1 < \sigma_2 \leq \sigma_{\rm sup}$ and $r_1 < r_2$, lying on the robust EF, such that the line that connects the two points lies on or below the frontier. Let $\bar{w}$ and $\hat{w}$ be two admissible allocations in $\mathcal{W}$ which correspond to the two points (σ_1, r_1) and (σ_2, r_2) in the risk-return space: $(\sigma_{\rm wc}(\bar{w}), r_{\rm wc}(\bar{w}) = (\sigma_1, r_1)$ and $(\sigma_{\rm wc}(\hat{w}), r_{\rm wc}(\hat{w}) = (\sigma_2, r_2)$. The portfolio $\tilde{w} = (\bar{w}+\hat{w})/2$ is admissible, since $\mathcal{W}$ is assumed to be convex. The worst-case return of $\tilde{w}$ satisfies

$$r_{\rm wc}(\tilde{w}) = \inf_{\mu\in\mathcal{M}} \mu^T\tilde{w} \geq \frac{1}{2}\left(\inf_{\mu\in\mathcal{M}} \mu^T\bar{w} + \inf_{\mu\in\mathcal{M}} \mu^T\hat{w}\right) = \frac{1}{2}(r_{\rm wc}(\bar{w}) + r_{\rm wc}(\hat{w})) = \frac{1}{2}(r_1 + r_2).$$

From the compactness of $\mathcal{S}$, any worst-case covariance, say $\Sigma_{\rm wc} \in \mathbb{S}^n_{++}$, that satisfies $(\tilde{w}^T\Sigma_{\rm wc}\tilde{w})^{1/2} = \sigma_{\rm wc}(\tilde{w})$. for $\tilde{w}$, is in $\mathcal{S}$. Then,

$$\sigma_{\rm wc}(\tilde{w}) = \sup_{\Sigma\in\mathcal{S}} \sqrt{\tilde{w}^T\Sigma\tilde{w}} = \sqrt{\tilde{w}^T\Sigma_{\rm wc}\tilde{w}}.$$

Since $\Sigma_{\rm wc}$ is positive definite, $\sqrt{w^T\Sigma_{\rm wc}w}$ is strict concave in w, so

$$\sqrt{\tilde{w}^T\Sigma_{\rm wc}\tilde{w}} < \frac{1}{2}\left(\sqrt{\bar{w}^T\Sigma_{\rm wc}\bar{w}} + \sqrt{\hat{w}^T\Sigma_{\rm wc}\hat{w}}\right).$$

Here,

$$\sqrt{\bar{w}^T\Sigma_{\rm wc}\bar{w}} \leq \sup_{\Sigma\in\mathcal{S}} \sqrt{\bar{w}^T\Sigma\bar{w}}, \qquad \sqrt{\hat{w}^T\Sigma_{\rm wc}\hat{w}} \leq \sup_{\Sigma\in\mathcal{S}} \sqrt{\hat{w}^T\Sigma\hat{w}}.$$

We have thus far shown that

$$\sigma_{\rm wc}(\tilde{w}) < \frac{1}{2}\left(\sup_{\Sigma\in\mathcal{S}} \sqrt{\bar{w}^T\Sigma\bar{w}} + \sup_{\Sigma\in\mathcal{S}} \sqrt{\hat{w}^T\Sigma\hat{w}}\right) = \frac{1}{2}(\sigma_1 + \sigma_2).$$

Taken together, the inequalities on the worst-case risk and return of $\tilde{w}$ mean that the point $(\sigma_{\rm wc}(\tilde{w}), r_{\rm wc}(\tilde{w}))$ lies entirely above the line that connects the two points (σ_1, r_1) and (σ_2, r_2), which is on or above the robust EF. Therefore the point lies entirely above the robust EF, which contradicts the definition of the robust EF.

A.2 Proof of Theorem 1

A.2.1 General uncertainty models

We start by noting that for any $s < \inf_{w\in\mathcal{W}} \sqrt{w^T\Sigma w}$, the set of portfolios that can achieve the risk level σ is empty, so

$$f_{\mu,\Sigma}(\sigma) = \sup_{w\in\mathcal{W},\ \sigma(w,\Sigma)\leq\sigma} w^T\mu = -\infty.$$

Since $\mathcal{W}$ is closed and Σ is positive definite, for any $s \geq \inf_{w \in \mathcal{W}} \sqrt{w^T \Sigma w}$, the set of portfolios that achieve the risk level σ is nonempty, so $f_{\mu,\Sigma}(\sigma)$ is finite. Therefore,

$$\inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} f_{\mu,\Sigma}(\sigma) = -\infty, \quad s < \sup_{\Sigma \in \mathcal{S}} \inf_{w \in \mathcal{W}} \sqrt{w^T \Sigma w}$$

and $\inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} f_{\mu,\Sigma}(\sigma)$ is finite if $s \geq \sup_{\Sigma \in \mathcal{S}} \inf_{w \in \mathcal{W}} \sqrt{w^T \Sigma w}$. Here we use the compactness assumption of $\mathcal{S}$. From the definition of the robust EF, we have

$$f_{\rm rob}(\sigma) = -\infty, \quad s < \inf_{w \in \mathcal{W}} \sigma_{\rm wc}(\sigma) = \inf_{w \in \mathcal{W}} \sup_{\Sigma \in \mathcal{S}} \sqrt{w^T \Sigma w},$$

since no portfolio can achieve the given worst-case risk level σ.

For any $\mathcal{S} \subseteq \mathbb{S}^n_{++}$, the minimax inequality

$$\sup_{\Sigma \in \mathcal{S}} \inf_{w \in \mathcal{W}} \sqrt{w^T \Sigma w} \leq \inf_{w \in \mathcal{W}} \sup_{\Sigma \in \mathcal{S}} \sqrt{w^T \Sigma w}$$

holds. Therefore, $f_{\rm rob}(\sigma) \leq \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} f_{\mu,\Sigma}(\sigma)$, whenever $\sigma < \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{W}} w^T \mu$.

We are now in a position to establish the inequality (11) when $\sigma \geq \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{W}} w^T \mu$. Let $\mathcal{W}_\sigma$ be the set of admissible portfolios whose worst-case risk is equal to σ:

$$\mathcal{W}_\sigma = \{w \in \mathcal{W} \mid \sigma_{\rm wc}(w) \leq \sigma\}.$$

For any w in $\mathcal{W}_\sigma$, it follows from the definition of the EF that

$$w^T \mu \leq f_{\mu,\Sigma}\left(\sqrt{w^T \Sigma w}\right), \quad \forall\, (\mu, \Sigma) \in \mathcal{M} \times \mathcal{S}.$$

Since $f_{\mu,\Sigma}$ is an increasing function, we have

$$w^T \mu \leq f_{\mu,\Sigma}\left(\sup_{\Sigma \in \mathcal{S}} \sqrt{w^T \Sigma w}\right) = f_{\mu,\Sigma}(\sigma_{\rm wc}(w)) \leq f_{\mu,\Sigma}(\sigma), \quad \forall\, w \in \mathcal{W}_\sigma, \quad \forall\, (\mu, \Sigma) \in \mathcal{M} \times \mathcal{S}.$$

Therefore, $\sup_{w \in \mathcal{W}_\sigma} w^T \mu \leq f_{\mu,\Sigma}(\sigma)$ for any $(\mu, \Sigma) \in \mathcal{M} \times \mathcal{S}$, which in turn means that

$$\inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{W}_\sigma} w^T \mu \leq \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} f_{\mu,\Sigma}(\sigma).$$

We use the minimax inequality (which holds for any $\mathcal{W}_\sigma \subset \mathbb{R}^n$ and $\mathcal{M} \subset \mathbb{R}^n$) to obtain

$$\sup_{w \in \mathcal{W}_\sigma} \inf_{\mu \in \mathcal{M}} w^T \mu \leq \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{W}_\sigma} w^T \mu.$$

From the definition of the robust EF, we can write

$$f_{\rm rob}(\sigma) = \sup_{w \in \mathcal{W}_\sigma} \inf_{\mu \in \mathcal{M}} w^T \mu.$$

Taken together, the results established above show that

$$f_{\rm rob}(\sigma) = \sup_{w \in \mathcal{W}_\sigma} \inf_{\mu \in \mathcal{M}} w^T \mu \leq \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{W}_\sigma} w^T \mu \leq \inf_{\mu \in \mathcal{M},\, \Sigma \in \mathcal{S}} f_{\mu,\Sigma}(\sigma), \quad \sigma \geq \sigma_{\rm inf}.$$

A.2.2 Convex uncertainty models

We turn to the convex case. The functions $\sqrt{w^T\Sigma w}$ is convex in w for fixed Σ and concave in Σ for fixed w. Since $\mathcal{S}$ is convex and compact, we can use the standard convex/concave minimax theorem, to obtain

$$\inf_{w\in\mathcal{W}}\sup_{\Sigma\in\mathcal{S}}\sqrt{w^T\Sigma w}=\sup_{\Sigma\in\mathcal{S}}\inf_{w\in\mathcal{W}}\sqrt{w^T\Sigma w}.$$

Therefore,

$$f_{\rm rob}(\sigma)=\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}f_{\mu,\Sigma}(\sigma)=-\infty,\quad \sigma<\sigma_{\rm inf}=\sup_{\Sigma\in\mathcal{S}}\inf_{w\in\mathcal{W}}\sqrt{w^T\Sigma w}=\inf_{w\in\mathcal{W}}\sup_{\Sigma\in\mathcal{S}}\sqrt{w^T\Sigma w}.$$

Moreover, $\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}f_{\mu,\Sigma}(\sigma)$ is finite if $\sigma\geq\sigma_{\rm inf}$.

The functions $w^T\mu$ is convex in w for fixed Σ and concave in μ for fixed w. Since $\mathcal{M}$ is convex and compact, the standard convex/concave minimax theorem tells us that

$$\sup_{w\in\mathcal{W}}\inf_{\mu\in\mathcal{M}}w^T\mu=\inf_{\mu\in\mathcal{M}}\sup_{w\in\mathcal{W}}w^T\mu.$$

We note that

$$f_{\rm rob}(\sigma)=\sup_{w\in\mathcal{W}}r_{\rm wc}(w)=\sup_{w\in\mathcal{W}}\inf_{\mu\in\mathcal{M}}w^T\mu,\quad \sigma\geq\sigma_{\rm sup}.$$

As will be shown soon,

$$f_{\rm rob}(\sigma)=\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}f_{\mu,\Sigma}(\sigma),\quad \sigma\in[\sigma_{\rm inf},\sigma_{\rm sup}). \tag{37}$$

Moreover,

$$\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}f_{\mu,\Sigma}(\sigma)\leq\inf_{\mu\in\mathcal{M}}\sup_{w\in\mathcal{W}}w^T\mu,\quad \forall\,\sigma>0.$$

Since $\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}f_{\mu,\Sigma}(\sigma)$ and $f_{\rm rob}(\sigma)$ are increasing, we have the inequality (12).

Suppose that the line that is tangential to the robust EF at $(s,f_{\rm rob}(s))$ crosses the line $s=0$ at $(0,\bar{r})$. Then,

$$\sup_{\sigma\geq0}\frac{f_{\rm rob}(\sigma)-\bar{r}}{\sigma}=\frac{f_{\rm rob}(s)-\bar{r}}{s}.$$

Here,

$$\sup_{w\in\mathcal{W}}S_{\bar{r}}(w,\mu,\Sigma)\geq\frac{f_{\mu,\Sigma}(\sigma)-\bar{r}}{\sigma},\quad \forall\,(\mu,\Sigma)\in\mathcal{M}\times\mathcal{S},$$

so

$$\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}\sup_{w\in\mathcal{W}}S_{\bar{r}}(w,\mu,\Sigma)\geq\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}\frac{f_{\mu,\Sigma}(\sigma)-\bar{r}}{\sigma}.$$

It follows from (16) and (19) that

$$\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}\frac{f_{\mu,\Sigma}(\sigma)-\bar{r}}{\sigma}\leq\sup_{w\in\mathcal{W}}\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}}S_{\bar{r}}(w,\mu,\Sigma)=\sup_{\sigma\geq0}\frac{f_{\rm rob}(\sigma)-\bar{r}}{\sigma}=\frac{f_{\rm rob}(s)-\bar{r}}{s}.$$

Therefore,

$$\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} \frac{f_{\mu,\Sigma}(\sigma)-r}{\sigma} \le \frac{f_{\rm rob}(\bar\sigma)-r}{\sigma},$$

or equivalently, $\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} f_{\mu,\Sigma}(\sigma) \le f_{\rm rob}(\sigma)$. We can now see from (11) established above that (37) is true.

A.3 Proof of Proposition 1

We start by noting that if the worst-case SR of w is positive, then it can be written as

$$\inf_{\mu\in\mathcal{M},\,\Sigma\in\mathcal{S}} S_{\bar r}(w,\mu,\Sigma) = \frac{\inf_{\mu\in\mathcal{M}} w^T\mu - r}{\sup_{\Sigma\in\mathcal{S}} \sqrt{w^T\Sigma w}}.$$

The worst-case SRMP is equivalent to

$$\begin{array}{ll} \text{maximize} & \dfrac{\inf_{\mu\in\mathcal{M}} w^T\mu - r}{\sup_{\Sigma\in\mathcal{S}} \sqrt{w^T\Sigma w}} \\ \text{subject to} & w\in\mathcal{W}. \end{array}$$

(The optimal value of this equivalent problem is positive, so we can rule out portfolios with negative worst-case SR.) Since $\mathbf{1}^Tw=1$ for each $w\in\mathcal{W}$, this problem is equivalent to

$$\begin{array}{ll} \text{maximize} & \dfrac{\inf_{\mu\in\mathcal{M}} w^T(\mu - r\mathbf{1})}{\sup_{\Sigma\in\mathcal{S}} \sqrt{w^T\Sigma w}} \\ \text{subject to} & w\in\mathcal{W}. \end{array} \tag{38}$$

The objective of this problem is homogeneous, so when it has a solution, say $w^\star$, $w^\star$ also solves the problem

$$\begin{array}{ll} \text{maximize} & \dfrac{\inf_{\mu\in\mathcal{M}} x^T(\mu - r\mathbf{1})}{\sup_{\Sigma\in\mathcal{S}} \sqrt{x^T\Sigma x}} \\ \text{subject to} & x\in\mathcal{X}. \end{array}$$

From (18), we can see that the optimal value of this problem is positive, so this problem is equivalent to

$$\begin{array}{ll} \text{maximize} & \inf_{\mu\in\mathcal{M}} x^T(\mu - r\mathbf{1}) \\ \text{subject to} & x\in\mathcal{X}, \quad \sup_{\Sigma\in\mathcal{S}} \sqrt{x^T\Sigma x}=1, \end{array}$$

which is in turn equivalent to

$$\begin{array}{ll} \text{maximize} & \inf_{\mu\in\mathcal{M}} x^T(\mu - r\mathbf{1}) \\ \text{subject to} & x\in\mathcal{X}, \quad \sup_{\Sigma\in\mathcal{S}} x^T\Sigma x=1. \end{array}$$

We can relax the equality constraint to obtain the equivalent formulation (17). Any solution of (17), say $x^\star$, always satisfies the equality constraint $\sup_{\Sigma\in\mathcal{S}} x^{\star T}\Sigma x^\star = 1$. (If it does not, then there is a positive constant $\alpha>1$ such that $\alpha x^\star$ achieves a higher objective value.) Let

$x^\star$ be the solution to (17). Then, we can see $\mathbf{1}^T x^\star \geq 0$ from the definition of the cone $\mathcal{X}$. (Any point $x \in \mathcal{X}$ satisfies $\mathbf{1}^T x \geq 0$.) If $\mathbf{1}^T x^\star > 0$, then $w^\star = (1/\mathbf{1}^T x^\star) x^\star$ solves (38), which is equivalent to the worst-case SRMP (13). We conclude that $w^\star$ solves (13). Moreover, it is the unique solution to (13), which follows from the uniqueness of the solution to (17), which we will establish shortly.

Suppose that the problem has two solutions $u^\star$ and $v^\star$ which are not identical. The mean $x^\star = (u^\star + v^\star)/2 \in \mathcal{X}$ of the two solutions is feasible for (17), since $\sqrt{x^{\star T} \Sigma x^\star} < 1$. However, $x^\star$ achieves the same objective value as $u^\star$, a contradiction to the fact that any solution is on the intersection of the boundary of the strictly convex set $\{x \in \mathbb{R}^n \mid \sup_{\Sigma \in \mathcal{S}} x^T \Sigma x \leq 1\}$ and the convex cone $\mathcal{X}$. We conclude that (17) has a unique solution.

We next turn to the case of $\mathbf{1}^T x^\star = 0$. If the worst-case SRMP (13) has a solution, say $\bar{w}$, whose entries adds up to one, *i.e.*, $\mathbf{1}^T \bar{w} = 1$, then it is contradictory to the uniqueness of the solution to (17).

A.4 Proof of Proposition 2

Let $\bar{r} < \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu$ be fixed. It follows from (19) that

$$f_{\rm rob}(\sigma) \leq \bar{r} + S^\star(\bar{r})\sigma, \quad \sigma > 0.$$

Therefore,

$$f_{\rm rob}(\sigma) \leq \inf_{\bar{r} < \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu} \bar{r} + S^\star(\bar{r})\sigma, \quad \sigma > 0.$$

From the strict concavity of the robust EF, we can see that

$$f_{\rm rob}(\sigma) = \inf_{\bar{r} < \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu} \bar{r} + S^\star(\bar{r})\sigma, \quad \sigma \in (\sigma_{\rm inf}, \sigma_{\rm sup}).$$

A simple argument shows that both curves are right continuous at $\sigma_{\rm inf}$. Therefore,

$$\inf_{\bar{r} < \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu} \bar{r} + S^\star(\bar{r})\sigma_{\rm inf} = f_{\rm rob}(\sigma_{\rm inf}).$$

By the definition of the robust EF, we have

$$f_{\rm rob}(\sigma) = \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu, \quad \sigma \geq \sigma_{\rm sup}.$$

A simple argument shows that when $\sigma_{\rm sup}$ is finite, $S^\star(\bar{r})$ tends to zero, as $\bar{r}$ tends to $\sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu$. Therefore,

$$f_{\rm rob}(\sigma) = \inf_{\bar{r} < \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu} \bar{r} + S^\star(\bar{r})\sigma = \sup_{w \in \mathcal{W}} \inf_{\mu \in \mathcal{M}} w^T \mu, \quad \sigma \geq \sigma_{\rm sup}.$$

A.5 Proof of Proposition 3

Suppose that the line tangential to the robust EF at $(\bar{\sigma}, f_{\text{rob}}(\bar{\sigma}))$ intercepts the line $s = 0$ at $(0, \bar{r})$. Then,

$$\frac{f_{\text{rob}}(\bar{\sigma}) - \bar{r}}{\bar{\sigma}} = \sup_{\sigma>0} \frac{f_{\text{rob}}(\sigma) - \bar{r}}{\sigma} = \sup_{w\in\mathcal{W}} \inf_{\mu\in\mathcal{M},\, \Sigma\in\mathcal{S}} S_{\bar{r}}(w, \mu, \Sigma).$$

Let $(\mu^\star, \Sigma^\star)$ be least favorable in terms of the market price of risk, *i.e.*,

$$\sup_{w\in\mathcal{W}} S_{\bar{r}}(w, \mu^\star, \Sigma^\star) = \inf_{\mu\in\mathcal{M},\, \Sigma\in\mathcal{S}} \sup_{w\in\mathcal{W}} S_{\bar{r}}(w, \mu, \Sigma).$$

From the concavity of $f_{\mu^\star,\Sigma^\star}$, the left-hand side can be written as

$$\sup_{w\in\mathcal{W}} S_{\bar{r}}(w, \mu^\star, \Sigma^\star) = \sup_{\sigma>0} \frac{f_{\mu^\star,\Sigma^\star}(\sigma) - \bar{r}}{\sigma}.$$

Therefore,

$$\frac{f_{\text{rob}}(\bar{\sigma}) - \bar{r}}{\bar{\sigma}} = \sup_{\sigma>0} \frac{f_{\text{rob}}(\sigma) - \bar{r}}{\sigma} = \sup_{\sigma>0} \frac{f_{\mu^\star,\Sigma^\star}(\sigma) - \bar{r}}{\sigma}.$$

Taken together, the results established above lead to

$$\sup_{\sigma>0} \frac{f_{\text{rob}}(\sigma) - \bar{r}}{\sigma} = \sup_{\sigma>0} \frac{f_{\mu^\star,\Sigma^\star}(\sigma) - \bar{r}}{\sigma}.$$

If $f_{\mu^\star,\Sigma^\star}(\bar{\sigma}) > f_{\text{rob}}(\sigma)$, we have the following contradiction:

$$\sup_{\sigma>0} \frac{f_{\mu^\star,\Sigma^\star}(\sigma) - \bar{r}}{\sigma} \geq \frac{f_{\mu^\star,\Sigma^\star}(\bar{\sigma}) - \bar{r}}{\bar{\sigma}} > \frac{f_{\text{rob}}(\bar{\sigma}) - \bar{r}}{\bar{\sigma}} = \sup_{\sigma>0} \frac{f_{\text{rob}}(\sigma) - \bar{r}}{\sigma}.$$

Therefore, $f_{\mu^\star,\Sigma^\star}(\bar{\sigma}) = f_{\text{rob}}(\sigma)$, since $f_{\text{rob}}(\sigma) = \inf_{\mu\in\mathcal{M},\, \Sigma\in\mathcal{S}} f_{\mu,\Sigma}(\bar{\sigma})$. Thus far we have seen that

$$\sup_{\sigma>0} \frac{f_{\mu^\star,\Sigma^\star}(\sigma) - r}{\sigma} = \frac{f_{\mu^\star,\Sigma^\star}(\bar{\sigma}) - r}{\bar{\sigma}}.$$

In other words, the line tangential to the robust EF at $(\bar{\sigma}, f_{\text{rob}}(\bar{\sigma}))$ intercepts the line $s = 0$ at $(0, \bar{r})$ is also tangential to the curve $r = f_{\mu^\star,\Sigma^\star}(\sigma)$ at the same point.