Approximation Algorithms for Linear Fractional-Multiplicative Problems

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Abstract

In this paper we propose a Fully Polynomial Time Approximation Scheme (FPTAS) for a class of optimization problems where the feasible region is a polyhedral one and the objective function is the sum or product of linear ratio functions. The class includes the well known ones of Linear (Sum-of-Ratios) Fractional Programming and Multiplicative Programming.

KEYWORDS: Fractional Programming, Multiplicative Programming, Approximation Problems

1 Introduction

In this paper we consider a class of problems with a polyhedral feasible region and an objective function which is the sum or the product of $p \geq 1$ ratio functions where both the numerator and the denominator are affine functions. We will call it the class of Linear Fractional-Multiplicative Programming (LFMP in what follows) problems. Formally, LFMP problems are defined as follows

$$\begin{array}{ll} \min & \Lambda_{i=1}^{p} \frac{c_i x + c_{0i}}{d_i x + d_{0i}} \\ & Ax \geq b \\ & x \geq 0 \end{array} \tag{1}$$

where $\Lambda \in \{\sum, \prod\}$. We will assume throughout the paper that all data are integer ones, i.e., $c_i \in \mathbb{Z}^n$, $c_{0i} \in \mathbb{Z}$, for all i, $b \in \mathbb{Z}^m$, and $A \in \mathbb{Z}^{m \times n}$. We will also initially assume that all the affine functions are strictly positive over the feasible region, i.e.,

$$c_i x + c_{0i},\ d_i x + d_{0i} > 0 \quad \forall\, x \in P, \quad \forall\, i \in \{1, \ldots, m\}, \tag{2}$$

where

$$P = \{x \in \mathbb{R}^n \ :\ Ax \geq b, x \geq 0\}$$

denotes the feasible region of the problem (the assumption of strict positivity for the munerators will be relaxed in Section 5.1).

The class LFMP includes some well known classes of problems in the field of global optimization. In particular:

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- the case $\Lambda = \sum$ corresponds to the class of Linear (Sum-of-Ratios) Fractional Programming (LFP) for which many applications (see, e.g., [6, 13, 20, 23]) and methods (see, e.g., [11, 17]) have been reported in the literature. For more details about this class we can also refer to different surveys [8, 27, 29, 30];
- the case $\Lambda = \prod$ and $d_i x + d_{0i} \equiv 1$ for all $i = 1, \ldots, p$, corresponds to the class of Linear Multiplicative Programming (LMP), which arises in different applications like multi-objective decision, financial optimization, VLSI chip design, and for which both exact methods (like, e.g., parameter-based [12, 14, 28], branch-and-bound [7, 11, 24, 25], outer-approximation [9, 15], mixed branch-and-bound and outer-approximation [2], vertex enumeration [22], and outcome-space cutting plane ones [4]), and heuristic methods [3, 18] have been proposed in the literature.

Obviously, LP is also a subclass of LFMP, namely the one for which $p = 1$ and $d_1 x + d_{01} \equiv 1$ for all $x \in P$. From the point of view of complexity, the case $p = 1$ (minimization of a ratio of two affine functions) is solvable in polynomial time in view of the pseudoconvexity of the objective function, but for $p \geq 2$ the problem is non-convex with local minima which are not necessarily global ones and in [19] it has been proved that even the case where $p = 2$, $\Lambda = \prod$ and $d_1 x + d_{01}, d_2 x + d_{02} \equiv 1$ for all $x \in P$ (i.e., the problem of minimizing the product of two affine functions over a polyhedron), is NP-hard (but two distinct FPTAS for this case have been proposed in [5, 10]).

Throughout the paper we will also make the following assumption.

Assumption 1 *Each single ratio* $\frac{c_i x + c_{0i}}{d_i x + d_{0i}}$, $i = 1, \ldots, p$, *attains its minimum value* ℓ_i *over* P *. In such case the value must be attained at a vertex of* P *(see, e.g., [26]) and, in view of (2), must be strictly positive, i.e.,*

$$\ell_i > 0 \qquad \forall\, i = 1, \ldots, p. \tag{3}$$

Also note that, as already remarked, each value ℓ_i *can be computed in polynomial time.*

Note that the above assumption is certainly satisfied under some conditions like, e.g., the boundedness of P or, in view of (2), $d_i x + d_{0i} \equiv 1$ for all $i = 1, \ldots, p$ (as already pointed out, the latter condition defines the subclass of LMP problems).

In this paper we want to present a Fully Polynomial Time Approximation Scheme (FPTAS) for LFMP *when* p *is fixed*, i.e., a procedure that solves the ε-approximation problem in polynomial time both with respect to the problem dimension and with respect to $\frac{1}{\varepsilon}$.

First we need to recall the definition of approximation problem related to a global optimization one.

Definition 1 *Let* $\min_{x\in X} \ f(x)$ *be a global optimization problem for which it holds that* $f(x) \geq 0, \ \forall \ x \in X$. *Given* $\varepsilon > 0$, *we say that* $\tilde{x} \in X$ *is an* ε-*approximate solution for the global optimization problem if*

$$f(\tilde{x}) - f_* \leq \varepsilon f_*,$$

where f_* *denotes the optimal value of the problem.*

Actually, in the literature about complexity results for global optimization problems the approximation problem is usually defined in a slightly different way (see, e.g., [31]), i.e., $\tilde{x} \in X$ is called an ε-approximate solution if

$$f(\tilde{x}) - f_* \leq \varepsilon(f^* - f_*),$$

where f^* denotes the maximum value of f over X. However, such definition does not apply here because the objective function of LFMP may be unbounded from above when the feasible region is unbounded.

The paper is structured as follows. In Section 2 we propose a parametric reformulation of the problem. In Section 3 we introduce the approximation algorithm. In Section 4 we prove that class LFMP admits a FPTAS when p is fixed. Finally, in Section 5 we consider some extensions of the complexity results relaxing the assumption of strict positivity of the numerator affine functions into an assumption of nonnegativity (for the product case we also investigate the removal of sign restrictions), and considering the case of maximization problems.

2 A parametric reformulation

First of all we notice that besides the positive lower bounds ℓ_i's for the ratio functions $\frac{c_i x + c_{0i}}{d_i x + d_{0i}}$ over the feasible region P, we can also introduce upper bounds for the same functions. Since P can be unbounded, we can not define a finite upper bound for a ratio function by simply maximizing it over P. However, we can take any vertex in P and evaluate the objective function at it in order to define an upper bound U' of the problem (1) and after that we can impose $\forall \ i \in \{1, \ldots, p\}$

$$\frac{c_i x + c_{0i}}{d_i x + d_{0i}} \leq u_i = \begin{cases} \frac{U'}{\prod_{j=1, \ j\neq i}^{p} \ell_j} & \text{if} \ \ \Lambda = \prod \\ U' & \text{if} \ \ \Lambda = \sum \end{cases} \tag{4}$$

In both cases the values u_i's are not necessarily upper bounds of the ratio functions over P, but any point for which at least one ratio function is above the corresponding u_i value can not be an optimal solution of problem (1) (each point violating the above inequalities has objective function value larger than U').

Problem (1) can be equivalently rewritten as follows by introducing p new variables t_i, $i = 1, \dots, p$

$$\begin{array}{ll} \min & \Lambda_{i=1}^p t_i \\ & Ax \geq b \\ & x \geq 0 \\ & c_i x + c_{0i} = t_i(d_i x + d_{0i}) \quad i = 1, \dots, p \end{array} \tag{5}$$

Now, let us fix the values $t_i = \gamma_i$ for all $i = 1, \dots, p$ and let

$$P(\gamma_1, \dots, \gamma_p) = P \cap \{x \in \mathbb{R}^n \ : \ c_i x + c_{0i} = \gamma_i(d_i x + d_{0i})\}$$

Basically, $P(\gamma_1, \dots, \gamma_p)$ is the level set of all the points in P where the objective function value is equal to $\Lambda_{i=1}^p \gamma_i$. Then, we can define the p-dimensional function $h(\gamma_1, \gamma_2, \dots, \gamma_p)$ as follows

$$h(\gamma_1, \dots, \gamma_p) = \begin{cases} \Lambda_{i=1}^p \gamma_i & \text{if } P(\gamma_1, \dots, \gamma_p) \neq \emptyset \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

Note that problem (1) turns out to be equivalent to the following problem

$$\min_{(\gamma_1, \dots, \gamma_p) \in \prod_{i=1}^p \ [\ell_i, u_i]} h(\gamma_1, \dots, \gamma_p).$$

Now, let us substitute the equality constraints $c_i x + c_{0i} = \gamma_i(d_i x + d_{0i})$ with the inequality ones $c_i x + c_{0i} \leq \gamma_i(d_i x + d_{0i})$ and let us define the region

$$P'(\gamma_1, \dots, \gamma_p) = P \cap \{x \in \mathbb{R}^n \ : \ c_i x + c_{0i} \leq \gamma_i(d_i x + d_{0i})\}.$$

It obviously holds that

$$P'(\gamma_1, \dots, \gamma_p) \supseteq P(\gamma_1, \dots, \gamma_p). \tag{7}$$

We can define a further p-dimensional function

$$f(\gamma_1, \dots, \gamma_p) = \begin{cases} \Lambda_{i=1}^p \gamma_i & \text{if } P'(\gamma_1, \dots, \gamma_p) \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}$$

Note that, in view of (7), for all $(\gamma_1, \dots, \gamma_p) \in \prod_{i=1}^p \ [\ell_i, u_i]$ it holds that

$$f(\gamma_1, \dots, \gamma_p) \leq h(\gamma_1, \dots, \gamma_p). \tag{8}$$

An easy observation is the following.

Observation 1 *For each $(\bar{\gamma}_1, \dots, \bar{\gamma}_p) \geq (\ell_1, \dots, \ell_p)$ and any $\delta \in (0, 1]$ it holds that*

$$\delta \sum_{i=1}^p \bar{\gamma}_i \leq \delta f(\bar{\gamma}_1, \dots, \bar{\gamma}_p) \leq f(\delta\bar{\gamma}_1, \dots, \delta\bar{\gamma}_p) \leq h(\delta\bar{\gamma}_1, \dots, \delta\bar{\gamma}_p) \tag{9}$$

if $\Lambda = \sum$, and

$$\delta^p \prod_{i=1}^p \bar{\gamma}_i \leq \delta^p f(\bar{\gamma}_1, \dots, \bar{\gamma}_p) \leq f(\delta\bar{\gamma}_1, \dots, \delta\bar{\gamma}_p) \leq h(\delta\bar{\gamma}_1, \dots, \delta\bar{\gamma}_p) \tag{10}$$

if $\Lambda = \prod$.

Proof. The result follows immediately from (8) and noticing that, in view of (2),

$$P'(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) \supseteq P'(\delta\bar{\gamma}_1, \ldots, \delta\bar{\gamma}_p)$$

if $\delta \in (0, 1]$ and $\bar{\gamma}_i \geq 0$ for any i. □

Now, let us assume that we have evaluated $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)$. As we will see more precisely in Section 3, if we are interested in an ε-approximate solution, we can discard all solutions whose value is at least as large as

$$(1 - \eta)f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p).$$

where

$$\eta = \frac{\varepsilon}{1 + \varepsilon}. \tag{11}$$

We prove the following observation.

Observation 2 *For any*

$$\delta \geq \begin{cases} (1 - \eta)^{\frac{1}{p}} & \textit{if } \Lambda = \prod \\ (1 - \eta) & \textit{if } \Lambda = \sum \end{cases}$$

it holds that

$$h(\gamma_1, \ldots, \gamma_p) \geq (1 - \eta)f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) \quad \forall\ (\gamma_1, \ldots, \gamma_p) \in \prod_{i=1}^{p} [\delta\bar{\gamma}_i, \bar{\gamma}_i].$$

Proof. It is enough to observe that in view of Observation 1 a lower bound for f over the set $\prod_{i=1}^{p} [\delta\bar{\gamma}_i, \bar{\gamma}_i]$ is $\delta f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)$ if $\Lambda = \sum$, and $\delta^p f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)$ if $\Lambda = \prod$. □

Since our aim is to minimize function h, while function f is just an underestimator of h, the above observation alone would not be enough to discard the set $\prod_{i=1}^{p} [\delta\bar{\gamma}_i, \bar{\gamma}_i]$ once f has been evaluated at point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)$. However, the following simple observation shows that each time we evaluate f at such point, we can always always detect some other point $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p)$ where the value of h is at least as good as the value of f at $(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)$, thus justifying the fact that we can discard the set $\prod_{i=1}^{p} [\delta\bar{\gamma}_i, \bar{\gamma}_i]$.

Observation 3 *For any* $(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) \geq (\ell_1, \ldots, \ell_p)$*, we are always able to find a point* $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p) \geq (\ell_1, \ldots, \ell_p)$ *such that*

$$h(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p) \leq f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p).$$

Proof. The result is trivial if $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) = \infty$. If $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) < \infty$, let $x^* \in P'(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)$. We define the $\tilde{\gamma}_i$'s as follows

$$\tilde{\gamma}_i = \frac{c_i x^* + c_{0i}}{d_i x^* + d_{0i}} \leq \bar{\gamma}_i \quad \forall\ i \in \{1, \ldots, p\}.$$

Then, x^* also belongs to $P(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p)$ and, consequently, it holds that

$$\Lambda_{i=1}^{p} \tilde{\gamma}_i = f(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p) = h(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p) \leq f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) = \Lambda_{i=1}^{p} \bar{\gamma}_i.$$

as we wanted to prove. □

3 An ε-approximation algorithm

In this section we present the ε-approximation algorithm.

Algorithm `AppLFMP`

Initialization Let $\mathcal{F}$ be a collection of p-dimensional points initialized with the single point $(u_1, \ldots, u_p)$, with the u_i's defined in (4). Let $\mathcal{V}$ be another collection of p-dimensional points initialized with the empty set. Set

$$U = f(u_1, \ldots, u_p),$$

and

$$\bar{\delta} = \begin{cases} (1-\eta)^{\frac{1}{p}} & \text{if } \Lambda = \prod \\ 1-\eta & \text{if } \Lambda = \sum \end{cases} \tag{12}$$

Step 0. Select a point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) \in \mathcal{F}$. Set $\mathcal{V} = \mathcal{V} \cup \{(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)\}$. Consider the 2^p points

$$(\xi_1 \bar{\gamma}_1, \ldots, \xi_p \bar{\gamma}_p),$$

where $\xi_i \in \{\bar{\delta}, 1\}$ for all i and discard all such points for which $\xi_i \bar{\gamma}_i < \ell_i$ for at least an index i. Let $\mathcal{G}$ be the set of the remaining points.

Step 1. Evaluate f at all points belonging to $\mathcal{G} \setminus \mathcal{F}$ and set

$$U = \min \left\{ U, \min_{(\gamma_1, \ldots, \gamma_p) \in \mathcal{G} \setminus \mathcal{F}} f(\gamma_1, \ldots, \gamma_p) \right\}.$$

Step 2. Update $\mathcal{F}$ as follows

$$\mathcal{F} = (\mathcal{F} \cup \mathcal{G}) \setminus \mathcal{V}.$$

Step 3. If $\mathcal{F} = \emptyset$, then STOP, otherwise go back to Step 0.

What we can notice is that Algorithm `AppLFMP` evaluates function f at the following points

$$\left(\bar{\delta}^{k_1} u_1, \ldots, \bar{\delta}^{k_p} u_p\right)$$

where the k_i's are integer values ranging between 0 and

$$\bar{k}_i = \max\{k_i \ : \ \bar{\delta}^{k_i} u_i \geq \ell_i\}. \tag{13}$$

For each $(\gamma_1, \ldots, \gamma_p) \in \prod_{i=1}^p [\ell_i, u_i]$, there exists an integer vector $(k_1, \ldots, k_p)$ with $0 \leq k_i \leq \bar{k}_i$ for all i, such that

$$(\gamma_1, \ldots, \gamma_p) \in \prod_{i=1}^p [\bar{\delta}^{k_i+1} u_i, \bar{\delta}^{k_i} u_i].$$

Therefore, in view of Observation 2 and of the definition (12) of $\bar{\delta}$, it holds that

$$f(\gamma_1, \ldots, \gamma_p) \geq (1-\eta) f\left(\bar{\delta}^{k_1} u_1, \ldots, \bar{\delta}^{k_p} u_p\right).$$

Then, we can conclude that

$$\min_{(\gamma_1,\ldots,\gamma_p)\in\prod_{i=1}^p\ [\ell_i,u_i]} f(\gamma_1,\ldots,\gamma_p) \geq (1-\eta) \min_{k_i\in\{0,\ldots,\bar{k}_i\},\ \forall\ i} f\left(\bar{\delta}^{k_1}u_1,\ldots,\bar{\delta}^{k_p}u_p\right).$$

The above inequality is not enough yet. What we would like is to prove something similar but with f replaced by h. In view of (8) it certainly holds that

$$\min_{(\gamma_1,\ldots,\gamma_p)\in\prod_{i=1}^p\ [\ell_i,u_i]} h(\gamma_1,\ldots,\gamma_p) \geq \min_{(\gamma_1,\ldots,\gamma_p)\in\prod_{i=1}^p\ [\ell_i,u_i]} f(\gamma_1,\ldots,\gamma_p)$$

where the left-hand side of the above inequality is also the optimal value of problem (1). In view of Observation 3 a solution $(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_p) \in \prod_{i=1}^p\ [\ell_i,u_i]$ such that

$$\min_{k_i\in\{0,\ldots,\bar{k}_i\},\ \forall\ i} f\left(\bar{\delta}^{k_1}u_1,\ldots,\bar{\delta}^{k_p}u_p\right) \geq h(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_p).$$

can be easily detected once all the points $\left(\bar{\delta}^{k_1}u_1,\ldots,\bar{\delta}^{k_p}u_p\right)$, $k_i\in\{0,\ldots,\bar{k}_i\}$, $\forall\ i$ are known. Therefore, it holds that

$$\begin{array}{rcl} h(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_p) & \leq & \frac{1}{1-\eta}\min_{(\gamma_1,\ldots,\gamma_p)\in\prod_{i=1}^p\ [\ell_i,u_i]}\ h(\gamma_1,\ldots,\gamma_p) = \\ & = & (1+\varepsilon)\min_{(\gamma_1,\ldots,\gamma_p)\in\prod_{i=1}^p\ [\ell_i,u_i]}\ h(\gamma_1,\ldots,\gamma_p), \end{array}$$

where the last equality follows from the definition (11) of η.

4 The complexity of the algorithm

First, we prove a lemma.

Lemma 1 *The number of points $\left(\bar{\delta}^{k_1}u_1,\ldots,\bar{\delta}^{k_p}u_p\right)$ at which function f is evaluated is not larger than*

$$\left\{\begin{array}{ll} \prod_{i=1}^p\left[1+\frac{p\log\left(\frac{u_i}{l_i}\right)}{\eta}\right] & \textit{if } \Lambda=\prod \\ \prod_{i=1}^p\left[1+\frac{\log\left(\frac{u_i}{l_i}\right)}{\eta}\right] & \textit{if } \Lambda=\sum \end{array}\right. \tag{14}$$

Proof. Taking into account the ranges $[0,\bar{k}_i]$ of the k_i values, it obviously holds that the number of points at which f is evaluated is

$$\prod_{i=1}^p(\bar{k}_i+1). \tag{15}$$

Now, if we recall the definition (13) of $\bar{k}_i$, we have that

$$\bar{k}_i \leq \frac{\log\left(\frac{\ell_i}{u_i}\right)}{\log(\bar{\delta})}.$$

In view of the definition (12) of $\bar{\delta}$ we have that the right-hand side of the above inequality is equal to

$$\frac{p \log\left(\frac{u_i}{l_i}\right)}{-\log(1-\eta)},$$

if $\Lambda = \prod$, and is equal to

$$\frac{\log\left(\frac{u_i}{l_i}\right)}{-\log(1-\eta)},$$

if $\Lambda = \sum$. Since $-\log(1-\eta) > \eta$ for all positive and small enough η values, then we can derive the following upper bound

$$\bar{k}_i \leq \begin{cases} \frac{p \log\left(\frac{u_i}{l_i}\right)}{\eta} & \text{if } \Lambda = \prod \\ \frac{\log\left(\frac{u_i}{l_i}\right)}{\eta} & \text{if } \Lambda = \sum \end{cases}$$

Therefore, it follows from the above inequality and from (15) that the number of points at which function f is evaluated by Algorithm `AppLFMP` is bounded from above by

$$\begin{cases} \prod_{i=1}^{p} \left[1 + \frac{p \log\left(\frac{u_i}{l_i}\right)}{\eta}\right] & \text{if } \Lambda = \prod \\ \prod_{i=1}^{p} \left[1 + \frac{\log\left(\frac{u_i}{l_i}\right)}{\eta}\right] & \text{if } \Lambda = \sum \end{cases}$$

as we wanted to prove. □

Now, we recall a well known result from linear algebra. Let $P = \{x \in \mathbb{R}^n : Ax \geq b,\ x \geq 0\}$ be a polyhedron, with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Let

$$\theta_A = \max_{i,j} \mid A_{ij} \mid, \quad \theta_b = \max_{i} \mid b_i \mid .$$

Then, the following theorem holds (see, e.g., [21, Page 123]).

Theorem 2 *Let x^0 be a vertex of P. Then, for $j = 1, \ldots, n$ it holds that*

$$x_j^0 = p_j / q$$

for integers p_j and q such that

$$0 \leq p_j \leq n\theta_b (n\theta_A)^{n-1}, \quad 1 \leq q \leq (n\theta_A)^n.$$

Let us denote by θ the maximum absolute value for all the coefficients defining problem (1). For each i, let x_i^* be the vertex of P at which the minimum value of the ratio $\frac{c_i x + c_{0i}}{d_i x + d_{0i}}$ is attained over P, and y_i^* be the vertex of P at which the positive minimum value of $c_i x + c_{0i}$ over P is attained. Then,

$$\ell_i = \frac{c_i x_i^* + c_{0i}}{d_i x_i^* + d_{0i}} \geq \frac{c_i y_i^* + c_{0i}}{d_i x_i^* + d_{0i}}$$

In view of Theorem 2 and (2), it must hold that

$$c_i y_i^* + c_{0i} \geq \frac{1}{n^n \theta^n},$$

while again in view of Theorem 2 it must hold that

$$d_i x_i^* + d_{0i} \leq n^{n+1}\theta^{n+1} + \theta \leq 2n^{n+1}\theta^{n+1}.$$

Therefore, we can conclude that

$$\ell_i \geq \frac{1}{2n^{2n+1}\theta^{2n+1}}.$$

Next, we search for an upper bound for the u_i values. Taking into account the definition (4), we can exploit again Theorem 2 to derive through simple but tedious computations the following upper bounds for the u_i values

$$u_i \leq \begin{cases} (2)^{2p-1}(n)^{(4p-2)n+2p-1}(\theta)^{(4p-2)n+2p-1} & \text{if } \Lambda = \prod \\ 2p(n)^{2n+1}(\theta)^{2n+1} & \text{if } \Lambda = \sum \end{cases}$$

Therefore, combining the upper bound for u_i and the lower bound for ℓ_i, it holds that

$$\frac{u_i}{l_i} \leq \begin{cases} 2^{2p} n^{4pn+2p}\theta^{4pn+2p} & \text{if } \Lambda = \prod \\ 4pn^{4n+2}\theta^{4n+2} & \text{if } \Lambda = \sum \end{cases}$$

and, consequently,

$$\log\left(\frac{u_i}{l_i}\right) \leq \begin{cases} (2p)\log(2) + (4pn+2p)\log(n) + (4pn+2p)\log(\theta) & \text{if } \Lambda = \prod \\ \log(4p) + (4n+2)\log(n) + (4n+2)\log(\theta) & \text{if } \Lambda = \sum \end{cases}$$

Substituting the right-hand side in (14), we get to the following upper bound for the number of times we need to evaluate f

$$\begin{cases} \prod_{i=1}^p \left[1 + \frac{p[2p\log(2)+(4pn+2p)\log(n)+(4pn+2p)\log(\theta)]}{\eta}\right] & \text{if } \Lambda = \prod \\ \prod_{i=1}^p \left[1 + \frac{\log(4p)+(4n+2)\log(n)+(4n+2)\log(\theta)}{\eta}\right] & \text{if } \Lambda = \sum \end{cases}$$

Now, recalling the definition (11) of η and the fact that each evaluation of f requires the solution of an LP, we end up with the following theorem.

Theorem 3 *The number of operations required by* `AppLFMP` *in order to detect an ε-approximate solution for (1), is bounded from above by*

$$\begin{cases} O\left(T(m,n)\frac{4^p(p)^{2p}n^p(\log(n)+\log(\theta))^p}{\varepsilon^p}\right) & \textit{if } \Lambda = \prod \\ O\left(T(m,n)\frac{4^p n^p(\log(n)+\log(\theta))^p}{\varepsilon^p}\right) & \textit{if } \Lambda = \sum \end{cases}$$

where $T(m,n)$ is a (polynomial) upper bound for the time needed to evaluate f at some point.

In view of the above theorem we can conclude that for *fixed* p, our ε-approximation algorithm `AppLFMP` is a FPTAS (Fully Polynomial Time Approximation Scheme) for problem LFMP. On the other hand, the same theorem evidentiates that we have an exponential increase of the computational time for `AppLFMP` as p increases, which has been often observed also in practical algorithms for LMP or LFP.

5 Some extensions

In this section we discuss some extensions of the results derived above. In particular, we will first discuss the relaxation of the assumption that all the numerator affine functions are strictly positive, and then we will consider the case of maximization problems.

5.1 Nonnegative numerators

Here we substitute the requirement that all the numerator affine functions $c_i x + c_{0i}$ are strictly positive over P, with the milder requirement that such affine functions are nonnegative over P (while we keep the requirement of strict positiveness of the denominator functions in order to guarantee that the objective function is always defined over P). We will deal with the two cases $\Lambda = \prod$ and $\Lambda = \sum$ separately.

Product

The case $\Lambda = \prod$ is quite simple. We can minimize each function $c_i x + c_{0i}$ over P. If all the optimal values are strictly positive, then we are exactly in the situation already described. Otherwise, if the optimal values are all nonnegative and at least one is equal to 0, then the optimal value of problem (1) with $\Lambda = \prod$ is equal to 0 and an optimal solution is any optimal solution of each problem $\min_{x \in P} \; c_i x + c_{0i}$ with optimal value equal to 0.

Sum

Things are a little bit more complicated when $\Lambda = \sum$. The main difficulty when we allow for the numerator functions to attain the zero value, is that the minimum values ℓ_i of the single ratio functions are also equal to 0, so that Algorithm `AppLFMP` does not stop in a finite number of steps. In order to have finite termination we need to modify it a little bit. First of all we substitute Assumption 1 with the following.

Assumption 2 *The minimum value of the single ratio function*

$$\frac{\sum_{i=1}^{p} c_i x + c_{0i}}{\sum_{i=1}^{p} d_i x + d_{0i}}$$

is attained over P *. In such case the value must be attained at a vertex of* P*.*

Note that the following holds

$$\sum_{i=1}^{p} \frac{c_i x + c_{0i}}{d_i x + d_{0i}} \geq \frac{\sum_{i=1}^{p} c_i x + c_{0i}}{\sum_{i=1}^{p} d_i x + d_{0i}}.$$

Then, the minimum value of the single ratio function in Assumption 2, denoted in what follows with L^*, is a lower bound for the optimal value of (1) when $\Lambda = \sum$. If $L^* = 0$, than an optimal solution x^* of this problem satisfies

$$c_i x^* + c_{0i} = 0 \quad \forall\ i$$

and, consequently, x^* is also an optimal solution for (1), whose optimal value is 0.

Otherwise, we propose the following modification of Algorithm `AppLFMP`. First we fix a value

$$\tilde{\ell} = \frac{\eta L^*}{p}. \tag{16}$$

Then

- in Step 0, instead of discarding all points for which $\xi_i \bar{\gamma}_i < \tilde{\ell}$ for at least an index i, we consider all these points but we substitute each coordinate $\xi_i \bar{\gamma}_i < \tilde{\ell}$ with the coordinate value 0;
- we evaluate f also in these points and possibly update U as already done in Step 1;
- we do *not* insert these points in the collection $\mathcal{F}$.

What we would like to show is that for all $(\gamma_1, \ldots, \gamma_p) \in \prod_{i=1}^{p} [s_i, \bar{\gamma}_i]$, where

$$s_i = \begin{cases} 0 & \text{if } \bar{\delta}\bar{\gamma}_i < \tilde{\ell} \\ \bar{\delta}\bar{\gamma}_i & \text{otherwise} \end{cases}$$

it holds that

$$f(\gamma_1, \ldots, \gamma_p) \geq (1 - 2\eta) f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p). \tag{17}$$

This way we can still guarantee that we are still covering all the $\prod_{i=1}^{p} [0, u_i]$ region and, with an appropriate choice for η (more precisely, we need to halve the value reported in (11)), we will return an ε-approximate solution of the problem (1). In order to prove (17) we first observe that if $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) = +\infty$, then it also holds that $f(\gamma_1, \ldots, \gamma_p) = +\infty$ for all $(\gamma_1, \ldots, \gamma_p) \in \prod_{i=1}^{p} [s_i, \bar{\gamma}_i]$ and (17) obviously holds. Otherwise, if $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) < \infty$, then, in view of the definitions of f, $\bar{\delta}$ and $\tilde{\ell}$, we have that

$$f(\gamma_1, \ldots, \gamma_p) \geq (1 - \eta) f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) - \tilde{\ell} p = (1 - \eta) f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) - \eta L^*$$

Since $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p)$ is an upper bound for the optimal value of (1) while L^* is a lower bound for the same problem, then it also holds that

$$f(\gamma_1, \ldots, \gamma_p) \geq (1 - \eta) f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) - \eta f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) = (1 - 2\eta) f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p),$$

as we wanted to prove.

After this modification the complexity is proven in a way completely analogous to what we have already seen in Section 4. We just need to observe that exploiting again Theorem 2 we have that

$$L^* \geq \frac{1}{2pn^{2n+1}\theta^{2n+1}}.$$

and that the number of points at which f is evaluated (i.e., the number of LP problems to be solved) is

$$O\left(\prod_{i=1}^{p} \log\left(\frac{u_i}{\tilde{\ell}}\right)\right).$$

5.2 The maximization problem

We can deal with the maximization problem

$$\begin{array}{ll} \max & \Lambda_{i=1}^{p} \frac{c_i x + c_{0i}}{d_i x + d_{0i}} \\ & Ax \geq b \\ & x \geq 0 \end{array} \tag{18}$$

in a way analogous to the minimization one. But there are a few differences which need to be underlined.

Fits of all we remark that, given the problem $\max_{x \in X}\ f(x)$ and some $\varepsilon > 0$, we say that $\tilde{x} \in X$ is an ε-approximate solution for the problem if

$$f^* - f(\tilde{x}) \leq \varepsilon f^*,$$

where f^* denotes the optimal value of the problem. Next, we substitute Assumption 1 with the following.

Assumption 3 *Each single ratio* $\frac{c_i x + c_{0i}}{d_i x + d_{0i}}$, $i = 1, \ldots, p$, *attains its maximum value* u_i *over* P *. The value must be attained at a vertex of* P *and can be computed in polynomial time.*

Function $h(\gamma_1, \ldots, \gamma_p)$ and the region $P(\gamma_1, \ldots, \gamma_p)$ are defined as before. Region $P'(\gamma_1, \ldots, \gamma_p)$ and, consequently, also function $f(\gamma_1, \ldots, \gamma_p)$ are defined in a slightly different way. More precisely,

$$P'(\gamma_1, \ldots, \gamma_p) = P \cap \{x \in \mathbb{R}^n \ :\ c_i x + c_{0i} \geq \gamma_i (d_i x + d_{0i})\}.$$

and

$$f(\gamma_1, \ldots, \gamma_p) = \begin{cases} \Lambda_{i=1}^{p} \gamma_i & \text{if } P'(\gamma_1, \ldots, \gamma_p) \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

Note that

$$f(\gamma_1, \ldots, \gamma_p) \geq h(\gamma_1, \ldots, \gamma_p).$$

The maximization problem turns out to be equivalent to the following one

$$\max_{(\gamma_1,\ldots,\gamma_p)\in\prod_{i=1}^{p}[\ell_i,u_i]} h(\gamma_1,\ldots,\gamma_p).$$

where the ℓ_i's are the minimum values of the single ratio functions over P. The approximation algorithm has to be slightly modified as follows

Algorithm `AppLFMP-MAX`

Initialization Let $\mathcal{F}$ be a collection of p-dimensional points initialized with the single point $(\ell_1,\ldots,\ell_p)$. Let $\mathcal{V}$ be another collection of p-dimensional points initialized with the empty set. Set

$$L = f(\ell_1,\ldots,\ell_p),$$

and

$$\bar{\delta} = \begin{cases} (1+\eta)^{\frac{1}{p}} & \text{if } \Lambda = \prod \\ 1+\eta & \text{if } \Lambda = \sum \end{cases}$$

Step 0. Select a point $(\bar{\gamma}_1,\ldots,\bar{\gamma}_p)\in\mathcal{F}$. Set $\mathcal{V}=\mathcal{V}\cup\{(\bar{\gamma}_1,\ldots,\bar{\gamma}_p)\}$. Consider the 2^p points

$$(\xi_1\bar{\gamma}_1,\ldots,\xi_p\bar{\gamma}_p),$$

where $\xi_i\in\{\bar{\delta},1\}$ for all i and discard all such points for which $\xi_i\bar{\gamma}_i > u_i$ for at least an index i. Let $\mathcal{G}$ be the set of the remaining points.

Step 1. Evaluate f at all points belonging to $\mathcal{G}\setminus\mathcal{F}$ and set

$$L = \max\left\{L, \max_{(\gamma_1,\ldots,\gamma_p)\in\mathcal{G}\setminus\mathcal{F}} f(\gamma_1,\ldots,\gamma_p)\right\}.$$

Step 2. Update $\mathcal{F}$ as follows

$$\mathcal{F} = (\mathcal{F}\cup\mathcal{G})\setminus\mathcal{V}.$$

Step 3. If $\mathcal{F}=\emptyset$, then STOP, otherwise go back to Step 0.

If the ℓ_i values are all strictly positive the above algorithm works and, in a way completely similar to what already seen in Section 4, we can prove that the algorithm is a FPTAS for this problem. If some ℓ_i is equal to 0 we need to substitute the ℓ_i values with something different. Again we distinguish the case $\Lambda=\sum$ and the case $\Lambda=\prod$.

Product

This is the simpler case. Let us solve the following LP

$$\begin{aligned} t^* = \max \quad & t \\ & Ax \geq b \\ & t \leq c_i x + c_{0i} \quad i = 1, \ldots, p \\ & x \geq 0 \end{aligned}$$

Let (x^*, t^*) be an optimal solution of this LP problem. If $t^* = 0$, then also the optimal value of (18) is equal to 0 and x^* is an optimal solution for (18). Otherwise, let

$$\tilde{L} = \prod_{i=1}^{p} \frac{c_i x^* + c_{0i}}{d_i x^* + d_{0i}} > 0.$$

Then, we can substitute the ℓ_i values in Algorithm `AppLFMP-Max` with the following values

$$\tilde{\ell}_i = \frac{\tilde{L}}{\prod_{j \neq i} u_i}.$$

In view of Assumption 3, with this choice we can guarantee that no solution with function value larger than $\tilde{L}$ will be lost by not considering points with at least one coordinate i lower than $\tilde{\ell}_i$.

Sum

As in Section 5.1, the situation is a bit more complicated for the sum. But we can proceed in a quite similar way. We define $\bar{L} > 0$ as any initial positive lower bound for the maximization problem. E.g., we can take $\bar{L} \geq \max_{i=1,\ldots,p} u_i$ (note that if all the u_i's are equal to 0, then the optimal value of the maximization problem is trivially equal to 0). Next we fix the value

$$\tilde{\ell} = \frac{\eta \bar{L}}{p}.$$

Finally, we would like to show that for all $(\gamma_1, \ldots, \gamma_p) \in \prod_{i=1}^{p} [\bar{\gamma}_i, s_i]$, where

$$s_i = \begin{cases} \tilde{\ell} & \text{if } \bar{\gamma}_i = 0 \\ \bar{\delta}\bar{\gamma}_i & \text{if } \bar{\gamma}_i \geq \tilde{\ell} \end{cases}$$

it holds that

$$f(\gamma_1, \ldots, \gamma_p) \leq (1 + 2\eta) L, \tag{19}$$

where L is the current lower bound for the optimal value of (18) and it obviously holds that $L \geq \bar{L}$. In order to prove (19) we first observe that if $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) = -\infty$, then it also holds that $f(\gamma_1, \ldots, \gamma_p) = -\infty$ for all $(\gamma_1, \ldots, \gamma_p) \in \prod_{i=1}^{p} [\bar{\gamma}_i, s_i]$

and (19) obviously holds. Otherwise, if $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) > -\infty$, then, in view of the definitions of f, $\bar{\delta}$ and $\tilde{\ell}$, we have that

$$f(\gamma_1, \ldots, \gamma_p) \leq (1+\eta) f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) + \tilde{\ell} p = (1+\eta) f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) + \eta \bar{L}$$

Since $f(\bar{\gamma}_1, \ldots, \bar{\gamma}_p) \leq L$ and $\bar{L} \leq L$, then it also holds that

$$f(\gamma_1, \ldots, \gamma_p) \leq (1+\eta)L + \eta L = (1+2\eta)L,$$

as we wanted to prove. Now the complexity result follows in a way completely similar to what already seen for the sum case in Section 5.1.

5.3 No sign requirement for the numerator functions

In this subsection we briefly consider the case where no restriction on the signs of the numerator functions is imposed. We will restrict our attention to the product, which, as we will see, turns out to be quite simple, while we do not investigate here the sum for which a more technical development is required.

As already done for Linear Multiplicative Problems, e.g., in [18], we can split the feasible region into 2^p subdomains:

$$P_r = P \cap \{x \in \mathbb{R}^n \ : \ c_i x + c_{0i} \geq 0 \text{ for } i \notin I_r, \ c_i x + c_{0i} \leq 0 \text{ for } i \in I_r\},$$

where the sets I_r are chosen between all the possible subsets of $\{1, \ldots, p\}$. After that, the original problem (1) can be split into the following 2^p ones: if $\mid I_r \mid$ is odd

$$-\max_{x \in P_r} \prod_{i \in I_r} \frac{-c_i x - c_{0i}}{d_i x + d_{0i}} \prod_{i \notin I_r} \frac{c_i x + c_{0i}}{d_i x + d_{0i}};$$

if $\mid I_r \mid$ is even

$$\min_{x \in P_r} \prod_{i \in I_r} \frac{-c_i x - c_{0i}}{d_i x + d_{0i}} \prod_{i \notin I_r} \frac{c_i x + c_{0i}}{d_i x + d_{0i}}.$$

Then, we can find an ε-approximate solution of the original problem by finding ε-approximate solutions for these 2^p maximization or minimization problems, for all of which the numerator functions satisfy the nonnegativity requirement.

Things are less obvious for the sum. As for the product, the problem can be split into the 2^p subproblems

$$\min_{x \in P_r} \sum_{i \notin I_r} \frac{c_i x + c_{0i}}{d_i x + d_{0i}} - \sum_{i \in I_r} \frac{-c_i x - c_{0i}}{d_i x + d_{0i}},$$

with all nonnegative ratio functions, but some of them are subtracted. We are confident that such problems admit a FPTAS, but a direct extension of the previous theoretical development does not lead to it (or, at least, we were not able to derive it). More work will be needed to deal with this case.

6 Conclusions

In this paper we have shown that the class of Linear Fractional-Multiplicative Programming problems, including the well known classes of Linear Fractional Programming and Linear Multiplicative Programming admits a FPTAS. The result is mainly of theoretical interest. We do not expect that a direct application of the proposed approximation algorithms gives better results than other approaches for these problems, such as the branch-and-bound ones. However, it is possible that more sophisticated variants of these approximation algorithms may be competitive also from the practical point of view and this is a possible subject for future researches.

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