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YOSHIHIRO TANAKA

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Graduate School of Economics and
Business Administration
Hokkaido University
Kita 9, Nishi 7, Kita-Ku
Sapporo 060-0809, Japan

E-mail: tanaka@econ.hokudai.ac.jp

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Yoshihiro Tanaka*

*Graduate School of Economics and Business Administration,
Hokkaido University, Sapporo 060-0809, Japan*

Abstract

Production theory needs generalizations so that it can incorporate broader class of production functions. A generalized Shephard's lemma and a generalized Hotelling's lemma in economic theory, which are presented in virtue of nonsmooth analysis under the assumption of upper semicontinuity on production functions.

Continuity of factor inputs with respect to a change of the factor prices is important in the cost minimization model. Locally Lipschitz strongly quasiconcave production functions are then introduced, which are shown to be a necessary and sufficient condition to have global stability and are consistent in the cost minimization model. Finally, a perturbation result of the cost minimization model is presented for locally Lipschitz continuous production functions, and a new simple constraint qualification is considered for quasiconcave production functions.

Keywords: nonsmooth analysis; production theory; continuity; comparative statics

JEL classification: C61; D20

*Tel.: +81-11-706-3175; fax: +81-11-706-4947.

E-mail address: tanaka@econ.hokudai.ac.jp (Y. Tanaka)

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1 Introduction

Economic theory largely depends on mathematical programming in static analysis, and needs generalization of the class of functions since there are few literature (Diewert 1982, Avriel, Diewert, Schaible, Zang 1988) on it. Especially, production theory, similar to consumer theory, has been analyzed in the context of optimization by several researchers, such as Hicks (1946), Shephard (1953, 1970) and Samuelson (1983).

We mainly deal with the two fundamental production models in modern competitive microeconomic theory, which are the profit maximization model and the cost minimization model.

It is well documented in economics literature that

- (A) Shephard sheds light on the relationship between conditional factor demands and the first-order partial derivatives of the cost function with respect to factor prices.
- (B) Hotelling's lemma sheds light on the relationship between an output or inputs and the first-order partial derivatives of the profit function with respect to a price or factor prices.

Especially, several researchers (Samuelson 1953-1954, Shephard 1953) examined the cost minimization model in relation to economics duality theorems under several conditions.

Nonsmooth analysis in optimization theory, on the other hand, has been examined by many researchers (e.g., Clarke 1983) in order to extend the scope of mathematical programming. More recently, Rockafellar and Wets (1998) synthesized variational analysis which aims for broader treatments of continuous functions. We should notice that discrete mathematics such as lattice theory and supermodularity has been introduced to comparative statics in recent years (e.g., Quah 2007), however, in the author's view, methods based on cardinality is still important for theory of production.

In this paper, we will present a generalized Shephard's lemma and a generalized Hotelling's lemma employing the subdifferential and Clarke's subgradient under the framework of nonsmooth optimization. It is noted that there have been so far few references which deal with continuity of solutions both in economics and in mathematical programming. We examine a necessary and sufficient condition that ensures continuity of the conditional factor demand functions $x(w)$ has been presented under the framework of locally Lipschitz programming. Finally, we present complementary

useful results of our sensitivity analysis under less assumptions than the usual.

2 Preliminaries

To begin with, we summarize briefly several definitions in nonsmooth analysis for the convenience's sake.

First, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be upper semicontinuous at x when

$$f(x) \geq \limsup_{y \rightarrow x} f(y).$$

We call $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz continuous if and only if for each $x \in \mathbb{R}^n$ there exist $\delta > 0$ and $c > 0$ such that $|f(y) - f(z)| \leq c \|y - z\|$ whenever $\|y - x\| \leq \delta$ and $\|z - x\| \leq \delta$, where $\|\cdot\|$ denotes any norm in \mathbb{R}^n . If f is locally Lipschitz continuous, the generalized directional derivative $f^\circ(x; d)$ is defined by

$$f^\circ(x; d) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{f(y + td) - f(y)}{t}, \quad (2.1)$$

which is equivalent to

$$f^\circ(x; d) = \max\{\langle \xi, d \rangle \mid \xi \in \partial f(x)\}, \quad \forall d \in \mathbb{R}^n \quad (2.2)$$

in terms of Clarke's subgradient $\partial f(x)$ (see Proposition 2.1.2 in Clarke (1983)). If f is also convex, then $\partial f(x)$ coincides with the (ordinary) subdifferential $\partial^c f(x)$ (cf. Rockafellar 1970), and $f^\circ(x; d)$ coincides with the directional derivative $f'(x; d)$ for each d . We write $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$ and $\mathbb{R}_{++}^n = \{x \in \mathbb{R}^n \mid x_i > 0, i = 1, \dots, n\}$.

We will consider the following two fundamental models in production theory throughout the paper.

Cost minimization model

Find an x which satisfies

$$\begin{aligned} (\text{Cmin}) \quad C(w, y) = & \text{minimize} && \langle w, x \rangle \\ & \text{subject to} && f(x) \geq y, \\ & && x \in \mathbb{R}_+^n, \end{aligned}$$

where $x \in \mathbb{R}_+^n$ is an input vector, $w \in \mathbb{R}_+^n$ is a factor price vector, $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is a production function, and $y \geq 0$ is a given level of output.

Profit maximization model

Find $x(p, w)$ and $y(p, w)$ which satisfy

$$\begin{aligned} (\text{Pmax}) \quad \pi(p, w) = \quad & \text{maximize} \quad py - \langle w, x \rangle \\ & \text{subject to} \quad f(x) \geq y, \\ & \quad \quad \quad x \in \mathbb{R}_+^n, \end{aligned}$$

where $x \in \mathbb{R}_+^n$ is an input vector, $y \geq 0$ is an output, $p \geq 0$ is a price, $w \in \mathbb{R}_+^n$ is a factor price vector, $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is a production function.

We will investigate the above two models under the framework of upper semi-continuous functions and derive the generalized Shepard's lemma and the generalized Hotelling's lemma in section 3.

Consider the case that the isoquant curve of $f(x)$ is depicted as in Figure 1. It is shown that the factor demand function may be discontinuous (see Figure 2) even if the production function is differentiable (in fact, even if the production function is linear (see Example 1 in section 4)). This discontinuity may cause a bad influence to the whole economy, which leads to our motivation to establish a necessary and sufficient condition to rule out such cases under mild assumptions in section 4.

3 Generalized Production Theory

We should consider a general class of production functions so as to pursue qualitative properties of each variables in the fundamental models in production theory in a general setting.

We will impose the following assumption on the production function.

Assumption 1 $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is upper semicontinuous, and $f(x) > 0$ for $\forall x > 0$.

Let V be the feasible set $V \equiv \{(x, y) \mid f(x) \geq y, x \in \mathbb{R}_+^n, y \geq 0\}$, and $U(f, y)$ be the upper level set $U(f, y) \equiv \{x \mid f(x) \geq y, x \in \mathbb{R}_+^n, y \geq 0\}$.

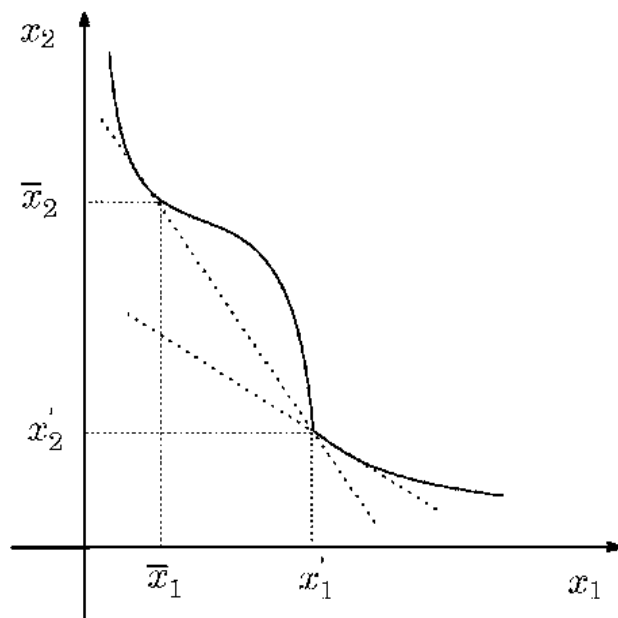


Figure 1: isoquant curve of production function

We state next the results for the cost minimization model (Cmin).

Lemma 1 *Suppose that $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is upper semicontinuous in the cost minimization model (Cmin). Then $C(w, y)$ is homogeneous of degree 1 in w , and has a minimizer in $U(f, y)$ for $w > 0$. Moreover, $C(w, y)$ is concave in w .*

Proof The homogeneousness of degree 1 in w of $C(w, y)$ is immediate from the definition. For $y \geq 0$, the set $\{x \mid f(x) = y\}$ is compact since f is upper semicontinuous. Therefore $C(w, y)$ has a minimum in $U(f, y)$ for $w > 0$. The concavity of $C(w, y)$ in w follows from Theorem 4.1(d) in Avriel, Diewert, Shaible, and Zang (1988). ■

We should notice that the above result does not require even the continuity of f .

We state the generalized Shephard's lemma.

Theorem 3.1 *Suppose that $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is upper semicontinuous in the cost minimization model (Cmin). Let \bar{x} be the solution of (Cmin), namely, $C(w, y) = \langle w, \bar{x} \rangle =$*

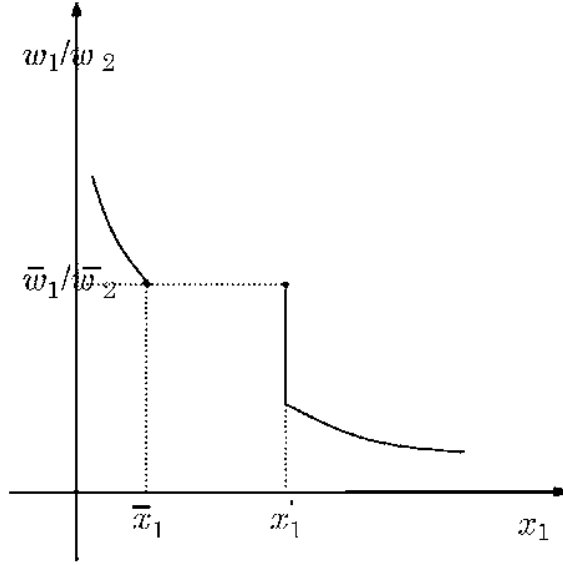


Figure 2: conditional factor demand function

$\min\{(w, x) \mid f(x) \geq y, x \in \mathbb{R}_+^n\}$ for $w > 0$. Then there exists Clarke's subgradient of C in w at (w, y) such that

$$\bar{x} \in -\partial_w^c(-C)(w, y) = \partial_w C(w, y), \quad (3.1)$$

where \bar{x} is a solution of (Cmin) at (w, y) .

Proof We first remark that $C(w, y)$ is concave on w from Lemma 1 without even a continuity assumption on f , whether or not $f(x)$ is quasiconcave.

If we define $\psi(w')$ as

$$\psi(w') = C(w', y) - \langle w', \bar{x} \rangle,$$

then $\psi(w) = 0$. It holds that

$$\psi(w') \leq 0 = \psi(w) \quad \text{for } w' \in \{w' \mid w'_i > 0\},$$

because of the definition of $\psi(w) = 0$. Therefore, since $\psi(w')$ attains the global maximal value at w , it holds from Theorem 23.5 in Rockafellar (1970) that

$$0 \in \partial_w^c(-\psi)(w) = \partial_w^c(-C)(w, y) + \bar{x},$$

in other words,

$$\bar{x} \in -\partial_w^c(-C)(w, y) = \partial_w C(w, y),$$

of which equality is derived since

$$\partial^c(-\varphi)(x) = \partial(-\varphi)(x) = -\partial\varphi(x), \quad \forall x \in \mathbb{R}^n, \quad (3.2)$$

for any locally Lipschitz continuous concave function φ from Proposition 2.2.7 and 2.3.1 in Clarke (1983). ■

We should notice that kinks are likely to be minima in the sense of the Lebesgue measure, since $\partial_w C(w, y)$ is an closed interval at kinks, while it is unique at any point where the cost function is differentiable. In fact, the "lock-in" phenomenon in Arthur (1996) may be explained in a same argument since as an isoquant curve of a cost function of increasing returns is concave, there must be some kinks along the curve. We also note that the above result can be regarded as the sensitivity analysis with respect to the perturbation of the factor prices w . However, as the concavity of $C(w, y)$ in w does not imply the continuity of $x(w)$ in general, we will pursue the continuity property in the next section.

We will show the fundamental result for the profit maximization model (Pmax).

Lemma 2 *Suppose that $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is upper semicontinuous in the profit maximization model (Pmax), and that the value of (Pmax) is bounded above. Then $\pi(p, w)$ is homogeneous of degree 1 in (p, w) and convex.*

Proof The level set $U(f, y)$ is closed from the upper semicontinuity of the production function f for every y . The feasible set V is also bounded above in the direction of x_i for $w_i > 0$ and y for $p > 0$. Then there exists a (x, y) which attains $\pi(p, w)$. The rest of the proof is similar to Section 3.1 of Varian (1992). ■

We state the generalized Hotelling's lemma.

Theorem 3.2 *Suppose that $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is upper semicontinuous in the profit maximization model (Pmax). Let \bar{x}, \bar{y} be internal solutions of (Pmax), namely, $\pi(p, w) = p\bar{y} - \langle w, \bar{x} \rangle = \max\{py - \langle w, x \rangle \mid f(x) \geq y, x \in \mathbb{R}_+^n\}$. Then there exist subdifferentials of*

π on p and w at (p, w) such that

$$\begin{aligned}\bar{y} &\in \partial_p^c \pi(p, w) \\ -\bar{x}_j &\in \partial_{w_j}^c \pi(p, w), \quad j = 1, \dots, n.\end{aligned}\tag{3.3}$$

Proof Let \bar{x}, \bar{y} be a profit maximizer of (Pmax) for (p, w) . If we define $\phi(p', w')$ as

$$\phi(p', w') = \pi(p', w') - (p'\bar{y} - \langle w', \bar{x} \rangle),$$

then $\phi(p, w) = 0$. It holds that

$$\phi(p', w') \geq 0 = \phi(p, w),$$

because of the definition of $\pi(p', w')$. Therefore, it follows that

$$\begin{aligned}0 &\in \partial_p^c \phi = \partial_p^c \pi - \bar{y}, \\ 0 &\in \partial_{w_j}^c \phi = \partial_{w_j}^c \pi + \bar{x}_j, \quad j = 1, \dots, n,\end{aligned}$$

from Theorem 23.5 in Rockafellar (1970) and the convexity of $\pi(p, w)$ in (p, w) from Lemma 2. ■

The following important relations are established.

Theorem 3.3 *Suppose that $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is upper semicontinuous in the profit maximization model (Pmax). Let \bar{x}, \bar{y} be internal solutions of (Pmax). Then the output \bar{y} increases as p increases, and the factor demand \bar{x}_j increases as w_j increases.*

Proof It follows from Lemma 2, Theorem 3.2, and the definition of subdifferentials. ■

4 Sensitivity Analysis

We first consider the optimality condition for the cost minimization model (Cmin).

Constraint Qualification (Q1): $r > 0$ does not exist at the solution \bar{x} of (Cmin) with the exception of $r = 0$ such that

$$0 \in -r\partial f(\bar{x}) + N(\mathbb{R}_+^n; \bar{x}),\tag{4.1}$$

where $N(\mathbb{R}_+^n; \bar{x})$ is a normal cone to \mathbb{R}_+^n at \bar{x} .

We should notice that the Constraint Qualification (Q1) can be regarded as a generalization of the Cottle qualification (cf. Bazaraa, Goode, Shetty 1972), where $f(x)$ is differentiable. If the Constraint Qualification (Q1) holds, the generalized Kuhn-Tucker conditions can be eventually written as follows:

$$0 \in w - r\partial f(\bar{x}) + N(\mathbb{R}_+^n; \bar{x}), \quad f(\bar{x}) = y, \quad \bar{x} \in \mathbb{R}_+^n \quad (4.2)$$

since $f(0) = 0$ (impossibility of free lunch) and f is continuous. We still consider (Cmin) since the production set (x, y') , $y \leq y' \leq f(x)$ is admissible.

Before stating Theorem 4.1, we will prove the following lemmas.

For $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, we introduce the level set S defined by $S \equiv \{x \in \mathbb{R}^n \mid \phi(x) \geq \phi(x_0)\}$, and the distance function $d_S(\cdot) : X \rightarrow \mathbb{R}$ defined by

$$d_S(x_0) \equiv \inf\{\|x_0 - c\| \mid c \in S\},$$

where S is a nonempty subset of X (see Clarke (1983)).

Let $\mathcal{T}(S; x_0)$ be a closed convex cone at x_0 defined by

$$\mathcal{T}(S; x_0) \equiv \{d \in \mathbb{R}^n \mid d_S^0(x_0; d) = 0\},$$

where S is a convex set, and let $N(x_0)$ be a normal cone to $\mathcal{T}(S; x_0)$ at x_0 defined by

$$N(x_0) \equiv \{\zeta \in \mathbb{R}^n \mid \langle \zeta, d \rangle \leq 0, \quad \forall d \in \mathcal{T}(S; x_0)\}. \quad (4.3)$$

We will show the following lemma that holds for locally Lipschitz continuous quasiconcave function at any x_0 which is not necessarily a solution.

Lemma 1 *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, m$, be locally Lipschitz continuous quasiconcave functions. Suppose that $0 \notin \partial\phi(x_0)$. Then one has*

$$N(x_0) \subset -\mathbb{R}_+ \partial\phi(x_0). \quad (4.4)$$

Furthermore, if ϕ is regular, then it holds that

$$N(x_0) = -\mathbb{R}_+ \partial\phi(x_0). \quad (4.5)$$

The proof is given in Appendix.

Consequently, the constraint qualification can be simplified as follows.

Lemma 2 *Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally Lipschitz continuous quasiconcave function in (Cmin). Let $f(x') > y$ and $f(\bar{x}) = y$ for some $x', \bar{x} \geq 0$. If*

$$0 \notin \partial f(\bar{x}) \tag{4.6}$$

holds, then $f(x)$ satisfies the constraint qualification (Hiriart-Urruty 1978):

$$\exists d \in \mathcal{T}(\mathbb{R}_+^n; \bar{x}), \quad (-f)^\circ(\bar{x}; d) < 0.$$

The proof is given in Appendix.

If f is also quasiconcave, namely when it is the case of diminishing returns, the Constraint Qualification (Q1) can be simplified as follows.

Definition. A function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is said to be *pseudoconcave on \mathbb{R}_+^n* , if $\langle \xi, x - x_0 \rangle \leq 0$, $\xi \in \partial f(x_0)$ for any $x, x_0 \in \mathbb{R}_+^n$, then $f(x) \leq f(x_0)$ holds.

Theorem 4.1 *Suppose that the production function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous in the cost minimization model (Cmin). Suppose also that one of the following conditions:*

1. *f is quasiconcave, and $0 \notin \partial f(\bar{x})$,*
2. *f is pseudoconcave, and $f(x') > y$ for some $x' \in \mathbb{R}_+^n$,*

holds. Then \bar{x} is a global minimum if and only if the multiplier rule (4.2) holds at \bar{x} .

Proof If f is quasiconcave or pseudoconcave then \bar{x} is a global minimum, since (Cmin) is a convex programming problem because $U(f, y)$ is convex (cf. Mangasarian (1994)).

Under the Hiriart-Urruty constraint qualification,

$$\begin{aligned} \langle -r\partial f(\bar{x}) + z, d \rangle &= \langle r\partial(-f)(\bar{x}) + z, d \rangle \\ &\leq r(-f)^\circ(\bar{x}; d) + \langle z, d \rangle \leq 0, \end{aligned}$$

for $r \geq 0$, $\forall z \in N(\mathbb{R}_+^n; \bar{x})$, $\exists d \in \mathcal{T}(\mathbb{R}_+^n; \bar{x})$, where the equalities hold if and only if $r = 0 \in \mathbb{R}_+^n$, which implies the Constraint Qualification (Q1).

If f satisfies (b), it follows that for $0 \in \partial f(\bar{x})$, if $\langle 0, x' - \bar{x} \rangle \leq 0$ then $f(x') \leq f(\bar{x})$ from its definition. And hence $0 \notin \partial f(\bar{x})$ by contradiction since $f(x') > f(\bar{x})$, which satisfies (a).

Thus the Constraint Qualification (Q1) holds, and hence (4.2) is a necessary and sufficient rule for \bar{x} to be a global minimum, since (Cmin) is a convex programming problem (cf. Borwein and Lewis (2000)). ■

If the production function f is concave, as is often the case, Theorem 4.1 (in this case, (b) is the Slater condition) can be applied, since f is also quasiconcave and pseudoconcave.

In the cost minimization model (Cmin), concavity of the cost function C does not imply that the production function f is locally Lipschitz. Furthermore, the conditional factor demand functions $x(w)$ can be the discontinuous function of w even if the objective function $C(w, y)$ is a continuous function in w (see Figure 2). This phenomenon corresponds to the drastic variation (or instability) of $x(w)$, which appears in Brian Arthur's (Arthur, 1996) economic theory on the basis of increasing-returns suited to modern high-technology economies, although $x(w)$ tend to lock in once they are chosen. If $x(w)$ discontinuously change greatly with respect to w , it has a bad influence upon the entire economy since $x(w)$ might be regarded as inputs related to other firms.

Let us consider the conditions under which $x(w)$ are stable. It is remarkable that, even if the production function f is linear, this does not always imply stability of $x(w)$. If f is smooth, it is sufficient that the curvature of f is not zero in order to avoid such an unfavorable phenomenon.

Before proceeding further, consider the cost minimization problem defined by

$$C(w, y) = \min_x \{ \langle w, x \rangle \mid f(x) \geq y, x \in \mathbb{R}_+^n \},$$

and

$$f^C(x) \equiv \max_y \{y \geq 0 \mid C(w, y) \leq \langle w, x \rangle, \quad \text{for every } w \in \mathbb{R}_{++}^n\}.$$

Definition. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *consistent* if and only if $f = f^C$.

Theorem 4.2 *Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is upper semicontinuous. Then f is consistent if and only if f is quasiconcave.*

Proof The "if part" follows from Theorem 4.5 in Avriel, Diewert, Schaible, Zang (1988).

The "only if part" follows from the fact consistency equals to convexity of $U(f, y)$ under the hypothesis which ensures closeness of $U(f, y)$ and existence of the tangent line corresponding to the cost function at any point along the boundary of $U(f, y)$ together with convexity. Then, the result follows from Theorem 9.1.3 in Mangasarian (1994). ■

We should consider more restrictive class of f , since if f is upper semicontinuous, then $x(w)$ may be discontinuous. We define a class of locally Lipschitz continuous functions as follows.

Assumption 2 $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is locally Lipschitz continuous and nondecreasing.

We remark that the production function may be typically locally Lipschitz continuous when products are produced by the most efficient (or nonproductive) producer for an amount of inputs, which can be expressed by taking the maximum (or minimum) of production functions. For instance, the Leontief production function is locally Lipschitz continuous (in fact, it is concave). We also remark that the nondecreasing property of f is commonly assumed under the free-disposal assumption.

Definition. A function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is said to be *strongly quasiconcave on \mathbb{R}_+^n* , if there exists $\alpha > 0$ such that

$$f((1 - \lambda)x_1 + \lambda x_2) \geq \min\{f(x_1), f(x_2)\} + (1 - \lambda)\lambda\alpha \|x_2 - x_1\|^2 \quad (4.7)$$

for all x_1 and x_2 such that $\pm(x_2 - x_1) \notin \mathbb{R}_{++}^n$ and all $\lambda \in [0, 1]$.

It is noted that we may assume the function f is quasiconcave since it corresponds to

the ordinary case of diminishing returns and is convenient to construct a comprehensive theory. In fact, it has broad applications in customary production problems.

We can now establish the following theorem, which ensures the continuity of $x(w)$.

Theorem 4.3 *Suppose that the production $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is locally Lipschitz and consistent in the cost minimization model (Cmin). Suppose also that $\bar{x}_1 = x(w_1)$ and $\bar{x}_2 = x(w_2)$ are internal minimizers to the cost minimization problem. If f is strongly quasiconcave on \mathbb{R}_+^n , then the conditional factor demand functions $x(w_1)$ are locally Lipschitz continuous, i.e., for all $y < y_0$, $y_0 = \sup f(x)$ there exists an $M_y > 0$ such that if $\bar{x} \in U(f, y)$, $g \in \partial f(\bar{x})$ then $\|g\| \leq M_y$, and it holds that for $\delta w = w_2 - w_1$, and $w_1 \neq 0$,*

$$\|\delta x\| \leq \frac{M_y \|\delta w\|}{\alpha \|w_1 + \delta w\|} \rightarrow 0, \quad \|\delta w\| \rightarrow 0. \quad (4.8)$$

Furthermore, if f is C^2 and $x(w)$ is not a kink, the converse also holds.

Proof The first half of the assertion is as follows. Let $\bar{x}_2 = \bar{x}_1 + \delta x$, $w_2 = w_1 + \delta w$. Then, by choosing unit inward normals as $p_1 = w_1 / \|w_1\|$, $p_2 = w_2 / \|w_2\|$, similar to the proof of Corollary 1 in Vial (1982),

$$\begin{aligned} \|\delta x\| = \|\bar{x}_2 - \bar{x}_1\| &\leq \frac{M_y}{2\alpha} \cdot \|p_2 - p_1\| \\ &= \frac{M_y}{2\alpha} \cdot \left\| \frac{w_2}{\|w_2\|} - \frac{w_1}{\|w_1\|} \right\| \\ &= \frac{M_y}{2\alpha} \cdot \left\| \frac{\|w_1\| (w_1 + \delta w) - \|w_1 + \delta w\| w_1}{\|w_1\| \|w_1 + \delta w\|} \right\| \\ &= \frac{M_y}{2\alpha} \cdot \frac{\|w_1\| \|\delta w\| - (\|w_1 + \delta w\| - \|w_1\|) \|w_1\|}{\|w_1\| \|w_1 + \delta w\|} \\ &\leq \frac{M_y}{2\alpha} \cdot \frac{\|w_1\| \|\delta w\| + (\|w_1 + \delta w\| - \|w_1\|) \|w_1\|}{\|w_1\| \|w_1 + \delta w\|} \\ &\quad \text{(Schwartz's inequality)} \\ &\leq \frac{M_y}{2\alpha} \cdot \frac{\|w_1\| \|\delta w\| + (\|w_1\| + \|\delta w\| - \|w_1\|) \|w_1\|}{\|w_1\| \|w_1 + \delta w\|} \\ &\quad \text{(Minkowski's inequality)} \\ &= \frac{M_y}{2\alpha} \cdot \frac{2 \|w_1\| \|\delta w\|}{\|w_1\| \|w_1 + \delta w\|} \\ &= \frac{M_y}{\alpha} \cdot \frac{\|\delta w\|}{\|w_1 + \delta w\|} \rightarrow 0, \quad \|\delta w\| \rightarrow 0. \end{aligned}$$

The latter half of the assertion employs the concept in Poliquin and Rockafellar (1998). Actually, as \bar{x} gives a *tilt-stable* local maximum of $f(x)$ over $U(f, y)$ because of locally Lipschitzness of $f(x)$ and $x(w)$, $\nabla^2 f(\bar{x}_1)$ is negative definite in the sense that $\langle \nabla^2 f(\bar{x}_1)v, v \rangle < 0$ for $v \in \mathbb{R}^n$, $\pm v \notin \mathbb{R}_{++}^n$ such that $\langle \nabla f(\bar{x}_1), v \rangle = 0$, $\|v\| = 1$, because $f(\bar{x}_1 + v) = f(\bar{x}_1) + \langle \nabla f(\bar{x}_1), v \rangle + \frac{1}{2} \langle \nabla^2 f(\bar{x}_1)v, v \rangle + o(v^2)$, by which there exists $\alpha > 0$ in (4.1) for any \bar{x}_1 and \bar{x}_2 , since $\langle \nabla f(x')v', v' \rangle < 0$, $v' = \bar{x}_2 - \bar{x}_1 / \|\bar{x}_2 - \bar{x}_1\|$, $x' = (1 - \lambda)\bar{x}_1 + \lambda\bar{x}_2$, $\lambda \in [0, 1]$, because $\langle \nabla f(x)v, v \rangle$ is continuous in x and v and because $\bar{x}_2 \approx \bar{x}_1$ and $v' \approx v$. ■

Nextly, we will consider sensitivity analysis with respect to the perturbation of the output variable y .

We define the perturbed problem (Cmin(q)) as follows:

$$\begin{aligned} \text{(Cmin}(q)) \quad \mathcal{C}(q) = \quad & \text{minimize} \quad \langle w, x \rangle \\ & \text{subject to} \quad f(x) \geq y + q, \\ & \quad \quad \quad x \in \mathbb{R}_+^n. \end{aligned}$$

We may state the assumption of Clarke (1983) so as to establish a formula for the subgradient of \mathcal{C} as follows:

Assumption C *$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous, $\mathcal{C}(0)$ is finite, and there exists a compact subset $\Omega \in \mathbb{R}_+^n$ and an $\epsilon_0 > 0$ such that (Cmin(q)) has its solution in Ω and $\mathcal{C}(q) < \mathcal{C}(0) + \epsilon_0$, $\forall |q| < \epsilon_0$.*

We will show that local Lipschitzness of f implies Assumption C, since the objective function is linear in this case.

Theorem 4.4 *Suppose that $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is locally Lipschitz continuous in (Cmin(q)). Suppose also that (Cmin(q)) for all sufficiently small $|q|$ is solvable. Then Assumption C holds.*

Proof Let \bar{x} be a solution to (Cmin), and \bar{x}_q be a solution to (Cmin(q)). Then, from local Lipschitzness of f , for sufficiently small $|q|$, there exists an x_q and $\delta > 0$ such that $f(x_q) = y + q$, $\|\bar{x} - x_q\| < \delta$. On the other hand, from local Lipschitzness of the

objective function, for all $\varepsilon' > 0$, $|\langle w, \bar{x} \rangle - \langle w, \bar{x}' \rangle| < \delta'$, there exists an \bar{x}' and $\delta' > 0$ such that $f(\bar{x}') = y$, $\|\bar{x}_q - \bar{x}'\| < \delta'$.

We obtain $\langle w, \bar{x} \rangle \leq \langle w, \bar{x}' \rangle$, $\langle w, \bar{x}_q \rangle \leq \langle w, x_q \rangle$ from feasibility, and $|\langle w, \bar{x} \rangle - \langle w, x_q \rangle| \leq \|w\| \|\bar{x} - x_q\| < \|w\| \delta$, $|\langle w, \bar{x}_q \rangle - \langle w, \bar{x}' \rangle| \leq \|w\| \|\bar{x} - x_q\| < \|w\| \delta'$ by Schwarz's inequality.

Then there are six possible cases:

1. $\langle w, \bar{x} \rangle \leq \langle w, \bar{x}' \rangle \leq \langle w, \bar{x}_q \rangle \leq \langle w, x_q \rangle$,
2. $\langle w, \bar{x} \rangle \leq \langle w, \bar{x}_q \rangle \leq \langle w, \bar{x}' \rangle \leq \langle w, x_q \rangle$,
3. $\langle w, \bar{x} \rangle \leq \langle w, \bar{x}_q \rangle \leq \langle w, x_q \rangle \leq \langle w, \bar{x}' \rangle$,
4. $\langle w, \bar{x}_q \rangle \leq \langle w, \bar{x} \rangle \leq \langle w, \bar{x}' \rangle \leq \langle w, x_q \rangle$,
5. $\langle w, \bar{x}_q \rangle \leq \langle w, \bar{x} \rangle \leq \langle w, x_q \rangle \leq \langle w, \bar{x}' \rangle$,
6. $\langle w, \bar{x}_q \rangle \leq \langle w, x_q \rangle \leq \langle w, \bar{x} \rangle \leq \langle w, \bar{x}' \rangle$.

However, in each case,

$$\begin{aligned} |\mathcal{C}(0) - \mathcal{C}(q)| &= |\langle w, \bar{x} \rangle - \langle w, \bar{x}_q \rangle| \\ &\leq \max\{|\langle w, x_q \rangle - \langle w, \bar{x} \rangle|, |\langle w, \bar{x}_q \rangle - \langle w, \bar{x}' \rangle|\} \\ &< \max\{\|w\| \delta, \|w\| \delta'\}. \end{aligned}$$

And hence, Assumption C holds. ■

We should note that $\|\bar{x}_q - \bar{x}\|$, the variation of the optimal solution, may be large, though $|\mathcal{C}(0) - \mathcal{C}(q)|$ is relatively small. If \mathcal{C} is locally Lipschitz continuous, (Cmin) is *calm* (see Clarke (1983)) at \bar{x} from Proposition 6.4.2 in Clarke (1983).

Let Σ denote the solution set of (Cmin) in Ω , and let $M(\Sigma) = \cup_{\bar{x} \in \Sigma} M(\bar{x})$ where $M(\bar{x})$ is the multiplier set corresponding to \bar{x} .

Theorem 4.5 *Suppose that the production function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is locally Lipschitz continuous. If \bar{x} solves (Cmin), and the Constraint Qualification (Q1) holds, then the multiplier rule (4.2) holds at \bar{x} .*

If (Cmin(q)) for all sufficiently small $|q|$ is solvable and that Σ is a singleton \bar{x} , then (4.2) holds at \bar{x} and

$$\begin{aligned}\partial_y C(w, y) &= \partial \mathcal{C}(0) \\ &= \text{cl co } M(\bar{x}) \\ &= \{r \mid w \in r\partial f(\bar{x}) - N(\mathbb{R}_+^n; \bar{x}); \quad f(\bar{x}) = y, \quad \bar{x} \in \mathbb{R}_+^n\}.\end{aligned}\tag{4.9}$$

Proof Under the Constraint Qualification (Q1), the abnormal multiplier set $M_k^0(\bar{x}) = \{0\}$ (for the definition, see Section 6.3 in Clarke (1983)). Then (CM) is calm at \bar{x} from Corollary 5 of Theorem 6.5.2 in Clarke (1983). Hence the multiplier rule (4.2) holds from Proposition 6.4.4 in Clarke (1983). The Clarke's subgradient $\partial_y C(w, y)$ follows from Theorem 4.4 above, Proposition 6.4.5 in Clarke (1983), and Corollary 1 of Theorem 6.5.2 in Clarke (1983) with $r > 0$. ■

We remark that the first half of the assertion does not assume the solvability of (Cmin(q)) for all $|q|$ which is not necessarily indispensable and is hard to verify, though the calmness of (Cmin) at \bar{x} directly follows from Theorem 4.4 if (Cmin(q)) for all $|q|$ is solvable. We should note that the result of Theorem 4.5 is not redundant in the sense that, by (3.2), it does not overestimate the variation of \mathcal{C} when f is concave.

We should remark that quasiconcavity does not suffice to warrant the stability of the optimal solution.

Example 1 Consider the following cost minimization program, which belongs to a linear programming problem.

$$\begin{aligned}\text{minimize} & \quad \langle w, x \rangle \\ \text{subject to} & \quad \langle a, x \rangle \geq y, \\ & \quad x \geq 0,\end{aligned}$$

where $\langle a, x \rangle$, $a \in \mathbb{R}_{++}^n$ is a linear production function.

We can derive the optimal solution as

$$\bar{x}_i = \begin{cases} \frac{w_k}{a_k} = \min_i \frac{w_i}{a_i}, & i = k \\ 0, & i \neq k \end{cases}$$

by simple calculation. Then $\partial_y C(w, y) = w_k/a_k$ ($= {}^t c_B B^{-1}$ conventionally) becomes the *shadow price* of the product since the price is increased by $\partial_y C(w, y)$ per unit increase of the product. In this case, the variation of the objective function is $O(\delta C) = O(\delta y)$ where $O(\cdot)$ stands for Landau's symbol; however, the variation of input is $O(\delta x) > O(\delta w)$ since the production function is not strongly quasiconcave on \mathbb{R}_+^n but linear (and hence, differentiable and convex).

Finally, we show a result for the profit maximization model (Pmax), similar to the cost minimization model (Cmin).

Constraint Qualification (Q2): $r > 0$ does not exist at the solution (\bar{x}, \bar{y}) of (Pmax) with the exception of $r = 0$ such that

$$0 \in -r\partial f(\bar{x}) + N(\mathbb{R}_+^n; \bar{x}), \quad 0 \in r + N(\mathbb{R}_+; \bar{y}), \quad (4.10)$$

where $N(\mathbb{R}_+^n; \bar{x})$ is a normal cone to \mathbb{R}_+^n at \bar{x} , and $N(\mathbb{R}_+; \bar{y})$ is a normal cone to \mathbb{R}_+ at \bar{y} .

If the Constraint Qualification (Q2) holds, the generalized Kuhn-Tucker conditions for the profit maximization model (Pmax) can be eventually written as follows:

$$\begin{aligned} 0 &\in w - r\partial f(\bar{x}) + N(\mathbb{R}_+^n; \bar{x}), \\ 0 &\in -p + r + N(\mathbb{R}_+; \bar{y}), \\ f(\bar{x}) &= \bar{y}, \quad \bar{x} \in \mathbb{R}_+^n, \quad \bar{y} \geq 0. \end{aligned} \quad (4.11)$$

5 Conclusion

We have presented the generalized Shephard's lemma and the generalized Hotelling's lemma in production theory under the framework of upper semicontinuous functions.

We have also presented the results of sensitivity analysis for the cost minimization model under the framework of locally Lipschitz functions. The variation of x in relation to the variation of w is unstable and may change greatly unless strong quasiconcavity of

a problem is assumed. The variation of the objective function value, which is expressed by Lagrange multipliers, can be regarded as the shadow price of a product. The strong quasiconcave production function is an important class which ensures the continuity, feasibility, and stability of the optimal solution.

As for the profit maximization model, a further strongness concept— like a strong concaveness of the production function— might be necessary in order to stabilize the variation of (x, y) in relation to the variation of (p, w) , which is our future task.

Appendix

Proof of Lemma 1 It follows from its definition, the local Lipschitzness of ϕ , and the convexity of S that $T(x_0)$ can be expressed as

$$T(x_0) = \mathcal{T}(S; x_0) = \text{cl} \{ \mathbb{R}_+ (\text{cl } S - x_0) \} \quad (\text{A.1})$$

as in Remark 2 to Theorem 2 in Hiriart-Urruty (1979).

Then, by regarding $-\phi$ as g_i in Remark 1 to Proposition 4 in Hiriart-Urruty (1979), one has

$$N(x_0) \subset -\mathbb{R}_+ \partial\phi(x_0).$$

where $\partial\phi(x_0)$ is nonempty from the local Lipschitzness of ϕ .

Conversely, if ϕ is regular, it follows from Proposition 2.1.1(c) and Theorem 2.4.7 in Clarke (1983) that

$$\begin{aligned} \{d \in \mathbb{R}^n \mid (-\phi)^\circ(x_0; d) \leq 0\} &= \{d \in \mathbb{R}^n \mid \phi^\circ(x_0; -d) \leq 0\} \\ &= \{d \in \mathbb{R}^n \mid \max_{\xi_i \in \partial\phi(x_0)} \langle \xi_i, -d \rangle \leq 0\} \\ &= \{d \in \mathbb{R}^n \mid \max_{\xi_i \in \partial\phi(x_0)} \langle -\xi_i, d \rangle \leq 0\} \\ &= T_S(x_0) = T(x_0) \end{aligned}$$

for $S = \{x \in \mathbb{R}^n \mid (-\phi)(x) \leq (-\phi)(x_0)\} = \{x \in \mathbb{R}^n \mid \phi(x_0) \leq \phi(x)\}$.

Then it holds that

$$\langle -\xi, d \rangle \leq 0, \quad \forall \xi \in \partial\phi(x_0), \quad \forall d \in T(x_0),$$

which implies

$$-\mathbb{R}_+ \partial \phi(x_0) \subset N(x_0).$$

And hence, if ϕ is regular, then it holds that

$$N(x_0) = -\mathbb{R}_+ \partial \phi(x_0).$$

■

Proof of Lemma 2 Let $T(\bar{x})$ be a closed convex cone at \bar{x} . Then if f is a locally Lipschitz continuous quasiconcave function and $0 \notin \partial f(\bar{x})$, then

$$N(\bar{x}) \subset -\mathbb{R}_+ \partial f(\bar{x}), \tag{A.2}$$

where $N(\bar{x}) = \{\zeta \in \mathbb{R}^n \mid \langle \zeta, x - \bar{x} \rangle \leq 0, \forall x \in T(\bar{x})\}$ follows from Lemma 1.

If we take $(0 \neq) x' - \bar{x} \in \text{int } T(\bar{x})$, we can obtain $f^\circ(\bar{x}; x' - \bar{x}) > 0$, since

$$\begin{aligned} f^\circ(\bar{x}; x' - \bar{x}) &= \max\{\langle \xi, x' - \bar{x} \rangle \mid \xi \in \partial f(\bar{x})\} \\ &\geq \langle \xi, x' - \bar{x} \rangle, \quad x' - \bar{x} \in \text{int } T(\bar{x}), \end{aligned}$$

and $\langle \xi^0, x' - \bar{x} \rangle > 0$ for $0 \neq \exists \xi^0 \in \partial f(\bar{x}) \cap -N(\bar{x})$ from (A.2).

On the other hand,

$$\begin{aligned} f^\circ(\bar{x}; \bar{x} - x') &= \max\{\langle \xi, \bar{x} - x' \rangle \mid \xi \in \partial f(\bar{x})\} = \max\{\langle -\xi, x' - \bar{x} \rangle \mid \xi \in \partial f(\bar{x})\} \\ &< 0. \end{aligned}$$

Thus, the Hiriart-Urruty constraint qualification holds since

$$(-f)^\circ(\bar{x}; x' - \bar{x}) = f^\circ(\bar{x}; \bar{x} - x') < 0$$

from Proposition 2.1.1(c) in Clarke (1983).

■

References

- Arthur, W.B. 1996. "Increasing returns and the new world of business," *Harvard Business Review*, July-August, pp. 100–109.
- Avriel, M., Diewert, W.E., Schaible, S., Zang, I. 1988. *Generalized Concavity*, Plenum Press, New York.

- Bazaraa, M.S., Goode, J.J., and Shetty, C.M. 1972. "Constraint qualifications revisited," *Management Science*, 18, pp. 567–573.
- Borwein, J.M. and Lewis, A.S. 2000. *Convex Analysis and Nonlinear Optimization — Theory and Examples*, Springer, New York.
- Clarke, F.H. 1983. *Optimization and Nonsmooth Analysis*, Wiley, New York.
- Cobb, C.W. and Douglas, P.H. 1928. "A theory of production," *American Economic Review*, 18, pp. 139–165.
- Diewert, W.E. 1982. "Duality approaches to microeconomic theory", in: *Handbook of Mathematical Economics, vol. II*, (K.J. Arrow and M.D. Intriligator eds.), North-Holland.
- Hicks, J. 1946. *Value and Capital*, Clarendon Press, England.
- Hiriart-Urruty, J.B. 1978. "On optimality conditions in nondifferentiable programming," *Mathematical Programming* 14, pp. 73–86.
- Hiriart-Urruty, J.B. 1979. "Tangent cones, generalized gradients and mathematical programming in Banach spaces," *Mathematics of Operations Research*, 4, pp. 79–97.
- Intriligator, M.D. 2002. *Mathematical Optimization and Economic Theory*, SIAM, Philadelphia.
- Krishna, V. and Maenner, E. 2001. "Convex potentials with an application to mechanism design," *Econometrica* 69, pp. 1113–1119.
- Mangasarian, O.L. 1994. *Nonlinear Programming*, SIAM, Philadelphia.
- Minkowski, H. 1911. "Theorie der konvexen körper in besondere begründung ihres oberflächenbegriffs," in: *Gesammelte Abhandlungen II*, (B.G. Teubner ed.), Leipzig, Germany.
- Peterson, D.W. 1973. "A review of constraint qualifications in finite-dimensional spaces," *SIAM Review*, 15, pp. 639–654.
- Poliquin, R.A. and Rockafellar, R.T. 1998. "Tilt stability of a local minimum," *SIAM Journal on Optimization* 8, pp. 287–299.
- Quah, J.K.-H. 2007. "The comparative statics of constrained optimization problems," *Econometrica* 75, pp. 401–431.
- Rockafellar, R.T. 1970. *Convex Analysis*, Princeton University Press, Princeton.
- Rockafellar, R.T. and Wets, R.J-B. 1998. *Variational Analysis*, Springer, Berlin.
- Samuelson, P.A. 1953–1954. "Prices of factors and goods in general equilibrium," *Review of Economic Study* 21, pp. 1–20.
- Samuelson, P.A. 1983. *Foundations of Economic Analysis, 2nd ed.*, Harvard University Press, Cambridge.
- Shephard, R.W. 1953. *Cost and Production Functions*, Princeton University Press, Princeton.
- Shephard, R.W. 1970. *The Theory of Cost and Production Functions*, Princeton University Press, Princeton.
- Suzuki, T. 1995. "Nonconvex production economics," *Journal of Economic Theory* 66, pp. 158–177.
- Tanaka, Y. 1990. "Note on generalized convex functions," *Journal of Optimization Theory and Applications* 66, pp. 345–349.

Varian, Hal R. 1992. *Microeconomic Analysis — 3rd Edition*, Norton, New York.

Vial, J.-P. 1982. “Strong convexity of sets and functions,” *Journal of Mathematical Economics* 9, pp. 187–205.